Chapter 4

Applications

Introduction

The previous concepts put an additional structure on locally convex spaces and can therefore be used to extend existing theories related to this particular structure (that exist on Hilbert spaces, Krein spaces or dualities) to locally convex spaces. This is for instance the case for Gaussian measures over locally convex spaces. We moreover go further into the formalism and study also its implication in terms of Krein subspaces and subdualities.

One can also work the other way round: by embedding a duality into a specific locally convex space (or a duality) one can study some objects with the use of the kernel. This is particularly true in the second section that deals with operator theory.

Finally a third section is devoted to the starting point of our investigation: approximation theory.
4.1 From Gaussian measures to Boehmians (generalized distributions) and beyond

Hilbertian subspaces play a great role in the (infinite-dimensional) probability theory since Gaussian measures over a locally convex space may be entirely defined by a positive kernel and its associated Hilbertian subspace. After precisely reviewing the link between Gaussian measures and Hilbertian subspaces we will extend the construction to Krein subspaces which will appear to be strongly linked with some generalization of distributions and finally question the case of subdualities. This will be done through abstract operator algebra theory.

4.1.1 Hilbertian subspaces and Gaussian measures

The Gaussian measures play a fundamental role in probability theory. In infinite-dimensional probability theory at least two approaches are possible, one based after Radon measures theory and the other after cylindrical measures. It is the second we (briefly) study here for we can define Gaussian (cylindrical) measures in terms of Hilbertian subspaces. We refer to [48] for the general theory of Radon, cylindrical and Gaussian measures or to [33] for a more precise study of Gaussian measures.

Gauss measure over a Hilbert space

The Gauss measure over a finite $n$-dimensional Hilbert space $H$ is defined as follows: Let $dx = dx_1dx_2...dx_n$ be the Lebesgue measure on $\mathbb{R}^n$ and $dh$ its image under the isomorphism $(x_i) \mapsto h = \sum_i x_i e_i$ the Gauss measure $\gamma$ on $H$ is $d\gamma_H = exp(-\pi \|h\|^2) dh$. Its variance and Fourier transform (characteristic functional) are given by:

$$\int_H \left( \langle \psi | h \rangle_H \langle \phi | h \rangle_H \right) \gamma_H(dh) = \frac{1}{2\pi} \langle \psi | \phi \rangle_H$$

$$\mathcal{F}_{\gamma_H}(\overline{h}) = exp(-\pi \|\overline{h}\|^2_H)$$
with the identifications $H' \sim \overline{H} \sim H$.

We can now define the Gauss measure over an arbitrary Hilbert space $H$. It is the (unique) cylindrical measure $\gamma$ defined as follows:

**Definition 4.1 (– Gauss measure over a Hilbert space –)** The Gauss measure is the (unique) cylindrical measure $\gamma$ such that for any finite-dimensional vector subspace $G \subset H$

$$p_G(\gamma_H) = \gamma_G$$

where $p_G$ is the orthogonal projection on $G$ and $\gamma_G$ the previously defined Gauss measure over the finite-dimensional Hilbert space $G$.

The previous equations regarding the covariance and Fourier transform remain valid.

**Gaussian measure over a locally convex space (over a duality)**

Based after the definition of the Gauss measure over a Hilbert space, we can define Gaussian measures over a locally convex space (or a duality):

**Definition 4.2 (– Gaussian measure –)** Let $E$ be a locally convex space (resp. $(E,F)$ a duality) and $\mu$ a cylindrical measure on $E$. We say that $\mu$ is a Gaussian measure if there exists a Hilbertian subspace $H$ of $E$ (resp. of $(E,F)$) such that

$$\mu = i(\gamma_H)$$

where $\gamma_H$ is the Gauss measure on $H$ and $i : H \rightarrow E$ the canonical injection.

We note $Gauss((E,F))$ the set of Gaussian measures over a duality $(E,F)$.
Covariance operators, kernels and support

The Hilbertian kernel of $H$ is closely linked with the covariance operators and Fourier transform. Precisely

**Proposition 4.3** Let $(\mathcal{E}, \mathcal{F})$ be a duality and $\mu$ the Gaussian measure associated to $H$. then

$$\int_{\mathcal{E}} \left( (\overline{\psi}, \epsilon)(\mathcal{F}, \mathcal{E}) (\phi, \epsilon)(\mathcal{F}, \mathcal{E}) \right) \mu(d\epsilon) = \frac{1}{2\pi} \left( \overline{\psi}, \mathcal{K}(\phi) \right)_{(\mathcal{F}, \mathcal{E})}$$

$$\mathcal{F}_\mu(\phi) = \exp \left( -\pi \left( \overline{\psi}, \mathcal{K}(\phi) \right)_{(\mathcal{F}, \mathcal{E})} \right) = \exp \left( -\pi \| \mathcal{K}(\phi) \|^2_H \right)$$

The Hilbertian subspace itself is related to sets associated to Radon measures (theorem 6p 97 [33]):

**Proposition 4.4** Suppose $\mu$ is a Radon Gaussian measure on $(\mathcal{E}, \mathcal{F})$ associated to $H$. Then its topological support, its linear support and the closure of its kernel (space of admissible directions) coincide with $H$.

Cone convex structure and functors

The image of a measure by a weakly continuous application is a classical tool in measure theory and we used it to define Gaussian measures over l.c.s. or dualities. As well it is classical to define the convolution $(\ast)$ of two measures (that stands for an addition law) or the external product $(.)$ of a measure by a positive number. A very significant result concerning Gaussian measures and their Hilbertian subspaces is:

**Theorem 4.5** Let $(\mathcal{E}, \mathcal{F})$ be a duality. Then there is a bijection between $\text{Hilb}((\mathcal{E}, \mathcal{F}))$ and $\text{Gauss}((\mathcal{E}, \mathcal{F}))$. Moreover, this bijection is compatible with the operations of addition (resp. convolution) and external multiplication over the two sets. $(\text{Gauss}((\mathcal{E}, \mathcal{F})), \ast, .)$ is a convex cone isomorphic to the convex cone of Hilbertian subspaces.
This bijection is moreover compatible with the effect of weakly continuous linear applications and we can state a theorem regarding categories:

Let \( \mathcal{C} \) be the category of dual systems \( (\mathcal{E}, \mathcal{F}) \) the morphisms being the weakly continuous linear applications. Let \( \mathcal{G} \) be the category of salient and regular convex cones, the morphisms being the applications preserving multiplication by positive scalars and addition (hence order). Then according that to a morphism \( u : \mathcal{E} \to \mathcal{F} \) we associate the morphisms

\[
\tilde{u} : \text{Hilb}( (\mathcal{E}, \mathcal{F}) ) \to \text{Hilb}( (\mathcal{E}, \mathcal{G}) ) \quad H \mapsto u(H)
\]

and

\[
\tilde{\mu} : \text{Gauss}( (\mathcal{E}, \mathcal{F}) ) \to \text{Gauss}( (\mathcal{E}, \mathcal{G}) ) \quad \mu \mapsto u(\mu)
\]

Then

**Theorem 4.6** \( \text{Hilb} : (\mathcal{E}, \mathcal{F}) \mapsto \text{Hilb}( (\mathcal{E}, \mathcal{F}) ) \) and \( \text{Gauss} : (\mathcal{E}, \mathcal{F}) \mapsto \text{Gauss}( (\mathcal{E}, \mathcal{F}) ) \) are isomorphic covariant functors of category \( \mathcal{C} \) into category \( \mathcal{G} \).

### 4.1.2 Krein subspaces and Boehmians

These last two theorems are very important since the regular convex cone of Gaussian measures will generate a vector space \( \mathbb{R} \otimes \text{Gauss}( (\mathcal{E}, \mathcal{F}) ) \) isomorphic to \( \mathbb{R} \otimes \text{Hilb}( (\mathcal{E}, \mathcal{F}) ) \). We will be able to use the theory of Krein subspaces but once again an interpretation of \( \mathbb{R} \otimes \text{Gauss}( (\mathcal{E}, \mathcal{F}) ) \) is needed.

**Boehmians**

The name Boehmians is used for all objects obtained by an abstract algebraic construction similar to the one of the field of quotients, but even if the “multiplication” law has divisors of zero (by using “quotients of sequences” instead of “quotients”). We will not deal with
sequences here since convolution admits no divisor of zero in the set of Gaussian measures but we keep the name since there has been a great study ([38], [39], [30]) of Boehmians based on function spaces (such as distributions).

A precise definition of Boehmians is given in [19]:

Let $G$ be a vector space and $M$ a subspace of $G$, $\star$ a binary operation from $G \times H$ into $G$ and $\Delta$ a family of sequences of elements of $M$ (the binary operation and the family $\Delta$ verifying additional conditions). Then the class of equivalence of quotients sequences $(\frac{g_n}{\phi_n})$ verifying

1. Quotient sequences: $\forall n \in \mathbb{N}$, $g_n \in G$, $\phi_n \in \Delta$, $g_n \star \phi_n = g_m \star \phi_m$;

2. Equivalence: $(\frac{g_n}{\phi_n}) \sim (\frac{f_n}{\psi_n}) \iff g_n \star \psi_n = f_n \star \phi_n$;

is called a Boehmian and we note the space of Boehmians $\mathcal{B}(G, M, \star, \Delta)$.

In general functional Boehmians are defined after the convolution. For instance it is commonly agreed that by Boehmians one means:

1. $G = C(\mathbb{R}^n \rightarrow \mathbb{C})$;

2. $M = D(\mathbb{R}^n \rightarrow \mathbb{C})$;

3. $\star = \ast$ is the standard convolution;

4. $\Delta$ is the set of delta sequences.

The obtained space of Boehmians contains Schwartz’s space of distributions $D'$, but also hyperdistributions, Mikusinski operators, Roumieu ultradistributions or regular operators ([19]).

Among all properties we may cite this interesting result based on the Fourier transform ([39]):

**Theorem 4.7** The Fourier transform is a one-to-one mapping from the space of tempered Boehmians to the space of distributions over $\mathbb{R}^n$. 
Gauss Boehmians

Let \((E, F)\) be a duality. Then we define the following space of Boehmians:

**Definition 4.8 (– Gauss Boehmian space –)** The Gauss Boehmian space (over \((E, F)\)) is the space of Boehmians with

1. \(G = \text{Gauss}((E, F))\);
2. \(M = G\);
3. \(\ast = \ast\) is the standard convolution;
4. \(\Delta\) is the set of constant sequences.

A Gauss Boehmian is of the form \(\gamma_{H_1} \gamma_{H_2}\), and the space of Gauss Boehmians is denoted by \(GB((E, F))\).

**Theorem 4.9** The two spaces \(R \otimes \text{Gauss}((E, F))\) and \(GB((E, F))\) are equal.

**Proof.** – They both are the vector space extension of the convex cone of Gaussian measures \(G((E, F))\) with respect to the equivalence relation induced by the cone. \(\square\)

**Fourier transform, covariance and support**

We can now define the Fourier transform of a Gauss Boehmian \(\gamma_{H_1} \gamma_{H_2}\).

**Proposition 4.10**

\[
\mathcal{F} \left( \frac{\gamma_{H_1}}{\gamma_{H_2}} \right) (\phi) = \exp \left( -\pi (\mathcal{K}(\phi), \mathcal{K}(\phi))_{(F, E)} \right) = \exp \left( -\pi \|\mathcal{K}(\phi)\|^2_{H=H_1 \oplus H_2} \right)
\]
where $\mathcal{X} = \mathcal{X}_1 - \mathcal{X}_2$.

From theorem 4.7 it follows that Gauss Boehmians may be seen as ultradistributions, tempered Boehmians, hyperdistributions etc... when $\mathcal{E} = \mathcal{F}$ is a finite-dimensional space.

We can also state a result concerning “covariance”:

**Proposition 4.11**

$$
\int_{\mathcal{E}} \left( (\psi, \epsilon)(\mathcal{F}, \mathcal{E}) (\phi, \epsilon)(\mathcal{F}, \mathcal{E}) \right) \frac{\gamma_{H_1}}{\gamma_{H_2}} (d\epsilon) = \frac{1}{2\pi} \left( \psi, \mathcal{X}(\phi) \right)(\mathcal{F}, \mathcal{E})
$$

For instance in [30] p 61 they derive the expression of $\frac{\gamma_1}{\gamma_2}$ in a hyperdistribution form where $\sigma_1$ and $\sigma_2$ are the variances of two Gaussian measures over $\mathbb{R}$:

$$
\frac{\gamma_1}{\gamma_2} = \sum_{k=0}^{\infty} \frac{(\sigma_1^2 - \sigma_2^2)^k}{k!} \Delta^{2k} \delta
$$

In terms of support, we can see that in the Pontryagin kernel case the support will be exactly the (unique) Pontryagin space associated to the kernel. The problem arises when speaking of kernel of multiplicity i.e. in the infinite-dimensional case.

This infinite-dimensional case then seems of very peculiar interest but the existing theory on Boehmians, ultradistributions etc... has not been extended to the infinite-dimensional case so far. The example of Gauss Boehmians would certainly raise interesting questions concerning generalized distributions in infinite dimension.

**4.1.3 Interpretation in terms of subdualities: the noncommutative algebra approach?**

The use of symmetry or symmetric structure has always been a crucial tool in mathematics. Symmetry appears to be closely linked with commutativity and the commutativity of the algebra of continuous function over a set $M$ generates Hilbert spaces through the covariance operator of the measure. It then appears “natural” to try to interpret the loss of symmetry.
dealing with subdualities in terms of non-commutative algebras. The main difficulty is that though the Gelfand transform provides a particularly clever formula to link sets or spaces and commutative algebras (of functions), it is generally assumed that non-commutative algebras cannot be interpreted in terms of functions. A solution may then be the use of subdualities.

Precisely we can interpret Gaussian measures this way: let \((\mathcal{E}, \mathcal{F})\) be a duality and \(\varpi\) a positive kernel. Let \(\mathcal{A}\) be the algebra of continuous functions over \(\mathcal{E}\) and define the following involution over \(\mathcal{A}\): \(\psi^*(\varepsilon) = \overline{\psi(\varepsilon)}\) (remark that it implies the following identity: \((\psi^* \phi)^* = \phi^* \psi\)).

Then we can rewrite the covariance equality of Gaussian measures as:

There exists a unique “Gaussian” linear form \(\mu\) over the algebra \(\mathcal{A}\) (equivalently a Gaussian measure on \(\mathcal{E}\)) such that \(\forall (\phi, \psi) \in \mathcal{F}^2\)

\[
\mu(\psi^*, \phi) = \int_{\mathcal{E}} \left( \overline{\psi(\varepsilon)} \left( \phi(\varepsilon) \right) \right) \gamma_H \left( d\varepsilon \right) = \frac{1}{2\pi} \left( \overline{\psi}, \varpi(\phi) \right)_{(\mathcal{F}, \mathcal{E})}
\]

Remark that by the self-adjoint property of the kernel, we get that

\[
\mu((\psi^* \phi)^*) = \overline{\mu(\phi^* \psi)}
\]

or more generally:

\[
\forall \psi \in \mathcal{A}, \quad \mu(\psi^*) = \overline{\mu(\psi)}
\]

Regarding subdualities we then would have to define a generally non-commutative algebra \(\mathcal{A}\) such that \(\mathcal{F} \subset \mathcal{A}\) and a linear form \(\mu\) on this algebra verifying

\[
\mu(\psi^*, \phi) = \frac{1}{2\pi} \left( \psi, \varpi(\phi) \right)_{(\mathcal{F}, \mathcal{E})}
\]

### 4.2 Operator theory

Hilbertian subspaces (and to a lesser extent Pontryagin subspaces) have been widely used in (at least) two directions regarding operator theory. The first concerns operators in repro-
ducing kernel spaces and the second deals with the particular positive differential operators. We then extend these two trends in terms of subdualities and differential kernels of any type. Finally a third application in terms of similarity in Hilbert spaces is given.

4.2.1 Operators in evaluation subdualities

In [3] D. Alpay proves that continuous endomorphisms in reproducing kernel Hilbert spaces are characterized by a function of two variables and up to unitary similarity by actually a function of one single variable called the Berezin symbol (theorem 2.4.1 p 33). This theorem extends naturally to the context of Krein spaces.

In the subduality setting it appears that many morphisms on evaluation subdualities are also characterized by a function of two variables:

**Theorem 4.12** Let \( (E, F) \) be an evaluation subduality on the set \( \Omega \) with reproducing kernel \( K(.,.) \). Then any weakly continuous operator \( S : F \rightarrow E \) and \( T : E \rightarrow E \) (resp. from \( E \) to \( F \) or from \( F \) to \( F \)) can be written as

\[
S(f)(t) = (f, S(t, .))_{(F, E)}
\]

\[
T(e)(s) = (T(., s), e)_{(F, E)}
\]

where \( S(t, .) =^t S[K(., t)] \in E \) and \( T(., s) =^t T[K(., s)] \in F \)

**Proof.** For instance for \( S \):

\[
S(f)(t) = (K(., t), S(f))_{(F, E)} = (f, S[K(., t)])_{(F, E)} = (f, S(t, .))_{(F, E)} \quad \square
\]

The following transposition and composition rules follow:
1. \( t^* S(t,.) = S[K(. , t)] = S(., t) \), \( t^* T(s,.) = T[K(s, .)] = T(s, .) \)

2. \( T \circ S \) is associated to \( [T \circ S](t, s) = (T(., t), S(s, .))_{(F, E)} \)

3. \( T_1 \circ T_2 \) is associated to \( [T_1 \circ T_2](t, s) = (T_1(., s), T_2(t, .))_{(F, E)} \)

**Example:** Consider the previous example of evaluation subduality: \( E = l(\mathbb{N}) \) is the set of sequences endowed with the pointwise convergence,

\[
E = \{(e_i) \in l^1(\mathbb{N}), \ e_0 = 0 \}
\]

the set of absolutely summable sequences starting from zero and

\[
F = \left\{(f_i) \in l^1(\mathbb{N}), \ \sum_{i=0}^{\infty} f_i = 0 \right\}
\]

the set of absolutely summable sequences summing to zero.

These two spaces are in separate duality with respect to the bilinear form

\[
L : F \times E \rightarrow \mathbb{R}
\]

\[
f, e \mapsto \sum_{i=0}^{\infty} f_i (\sum_{j=0}^{i} e_j) = -\sum_{i=0}^{\infty} (\sum_{j=0}^{i} f_j) e_{i+1}
\]

and their kernel is the two dimensional sequence

\[
K(i, j) = \begin{pmatrix}
0 & -1 & 0 & 0 & \ldots \\
0 & 1 & -1 & 0 & \ldots \\
0 & 0 & 1 & -1 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\]

For the following weakly continuous operator

\[
S : E \rightarrow F
\]

\[
e = (e_i) \mapsto f = \left\{-\sum_{j=0}^{\infty} e_j, e_1, e_2, \ldots \right\}
\]
a straightforward calculation gives

\[ t^* S : E \longrightarrow F \]

\[ e = (e_i) \quad \longmapsto \quad f = \left\{ e_1 - \sum_{j=0}^{\infty} e_j, e_2, e_3, \cdots \right\} \]

and finally

\[
S(i,j) = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
1 & -1 & 0 & 0 & \cdots \\
0 & 1 & -1 & 0 & \cdots \\
0 & 0 & 1 & -1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

### 4.2.2 Differential operators and subdualities

Spaces linked with differential theory such as Sobolev spaces are widely used in functional analysis. In particular it is now standard to define Sobolev-Hilbert spaces as Hilbertian subspaces of the space of distributions with a particular differential operator as kernel (see for instance [46]).

Obviously there exist too many useful standard Hilbert spaces (Sobolev spaces, Beppo-Levi spaces, Hardy spaces) to perform a general theory but there exists an interesting result due to L. Schwartz concerning some generalized Sobolev spaces of integer order.

His setting is as follows:

Ω is an open set of \( \mathbb{R}^n \) and for any positive integer \( s \) we define the space \( H^s \) as the equivalent class of functions of \( L^2(\Omega) \) such that their derivatives of any order \( (p_1, p_2, \ldots, p_n) \), \( |p| = p_1 + p_2 + \cdots + p_n \leq s \) are in \( L^2(\Omega) \). This space is endowed with the scalar product

\[
\langle \psi | \phi \rangle_{H^s} = \sum_{|p| \leq s} \int_{\Omega} a_p \Delta^p \psi \Delta^p \phi \, d\mu
\]
that makes it a Hilbert space where the \( a_p \) are strictly positive constant coefficients and \( \Delta^p = \sum_{k=1}^n \frac{\partial^{2p}}{\partial x^p} \).

Moreover we define the Hilbert spaces of distributions \( H_0^s \) closure of \( D(\Omega) \) in \( H^s \) and \( H^{-s} \) dual space of \( H_0^s \) by the transpose of the canonical dense injection \( j : D(\Omega) \rightarrow H_0^s \).

**Proposition 4.13** The kernel of the Hilbertian subspace \( H^{-s} \) of \( (D'(\Omega), D(\Omega)) \) is the positive differential operator

\[
\begin{align*}
\varepsilon D(\Omega) & \rightarrow D'(\Omega) \\
\phi & \mapsto \sum_{|\alpha| \leq s} (-1)^{|\alpha|} a_{\alpha} \Delta^{2p} \phi
\end{align*}
\]

The kernel of \( H_0^s \) is its Green operator \( G_\varepsilon \) and the kernel of \( H^s \) its Neumann operator \( L_\varepsilon \).

This theorem naturally extends to the Krein subspaces setting with non-necessarily positive coefficients \( a_p \). We can associate to any differential operator of even order a Krein space constructed after Sobolev spaces of integer order.

The following question then arises naturally: can we associate a subduality of \( (D'(\Omega), D(\Omega)) \) constructed after (possibly fractional) Sobolev spaces to any differential operator of integer order? The answer is positive and based after the following theorem (see also the examples in chapter 3 concerning Sobolev-Slobodeckij spaces):

**Theorem 4.14** Let \( p = (p_1, p_2, \ldots, p_n) \) be a positive multi-index,

\[
\begin{align*}
\varepsilon D(\Omega) & \rightarrow D'(\Omega) \\
\phi & \mapsto \Delta^p \phi
\end{align*}
\]

\(^1\)the Green operator of a linear elliptic differential operator is the “inverse” operator that yields the solution as a linear map of the data (Courant and Hilbert [18]).
and $H^p$ the Sobolev-Slobodeckij space $W^{0,p}$. Let as before $H_0^p$ be the closure of $D(\Omega)$ in $H^p$ and $H^{-p} = \overset{\rightarrow}{j}[H_0^p]$ be the image in $D'(\Omega)$ of $(H_0^p)'$ dual space of $H_0^p$ by the transpose of the canonical dense injection $j : D(\Omega) \rightarrow H_0^p$.

Then $(H^{-p}, H^{-p})$ is a (inner) subduality of $(D'(\Omega), D(\Omega))$ with kernel $\varpi$ and The Green’s and Neumann’s functions are respectively the kernels of the inner subdualities $(H_0^p, H_0^p)$ and $(H^p, H^p)$

**Proof.** – This result follows directly from the following majoration (derived from [52]):

$$\int_\Omega |\varphi \varpi(\phi)| \leq c \|\varphi\|_{W^{0,p}_2} \|\varphi(\phi)\|_{W^{-0,p}_2} \leq c \|\varphi\|_{W^{0,p}_2} \|\phi\|_{W^{-0,p}_2}$$

Finally any differential operator of integer order can be associated with a generalized Sobolev space via the functor $\mathcal{S}\mathcal{D}$ and the previous results (non constant coefficients will also be handled by the image of a continuous morphism).

### 4.2.3 Similarity in Hilbert spaces

We treat here the problem of similarity for operators in real Hilbert spaces. Let $L$ be an operator on a Hilbert space $H$. Does there exist a self-adjoint operator $A$ and an isomorphism $T$ such that $L = T^{-1}AT$?

We give here an answer in terms of subdualities. Let $(E, F)$ be the primary subduality associated to $L$. Then

**Proposition 4.15** The answer to the similarity problem is positive if and only if exists a positive operator $Q$ on $H$ such that:

1. $Q(E) = F$;
2. \( Q : F \rightarrow E \) is self-adjoint for the duality \((E, F)\) i.e.

\[
(Q(\varepsilon_1), \varepsilon_2)_{(E,F)} = (Q(\varepsilon_2), \varepsilon_1)_{(F,E)}
\]

A choice for \( T \) and \( A \) is then \( T = \sqrt{(Q)} \) and \( A = TLT^{-1} \).

**Proof.** – Suppose the answer is positive. Then one checks easily that

1. \( ^tTT : H \rightarrow H \) is a positive and self-adjoint isomorphism;

2. \( ^tTT(E) = F \);

3. \( ^tTT : F \rightarrow E \) is self-adjoint for the duality \((E, F)\)

Conversely the existence of such an operator \( Q \) gives \( L = \sqrt{(Q)^{-1}} A \sqrt{(Q)} \) with \( A = \sqrt{(Q)} A \sqrt{(Q)^{-1}} \) self-adjoint. □

### 4.3 Approximation theory: the interpolation problem

Positive reproducing kernels are widely used in the learning community and the domain of application is very large. We are interested here in the approximation problem, or more precisely in the interpolation problem. It appears that this problem can easily be solved using positive kernels. One may then wonder if it is possible to solve it without the positivity requirement.

#### 4.3.1 The problem

A way to state the interpolation problem is as follows:

We are given a data set \( \{(s_i, y_i), i \in I\} \) (\( I \) finite integer set) where the \( s_i \in \Omega \) and \( y_i \) in \( \mathbb{R} \)
and we want to find a “good” function $\phi$ in a suitable space $E$ such that

$$\phi(s_i) = y_i \quad \forall i \in I$$

Obviously the following constraints on $E$ follow:

1. $E$ must be a space of genuine functions on $\Omega$ i.e. $E \subset \mathbb{H}^\Omega$

2. The evaluation values must bring some information on the function i.e. the evaluation functionals must be continuous on $E$ which must be continuously embedded in $\mathbb{H}^\Omega$

A classical way to solve the problem is to associate to each function of $E$ an “energy” i.e. to state that $E$ is a Hilbert space. It follows that it is a R.K.H.S. (we note its kernel function $K$) and a possible choice for the “best” interpolating function would be the one with least energy.

Mathematically, one has then to solve the minimization problem:

**Problem 4.16** $\phi = \arg \min_{\phi \in S} \| \phi \|^2$

where $S = \{ f \in E, f(s_i) = y_i \quad \forall i \in I \}$ is the set of interpolating functions.

Remark that this minimization problem always has a unique solution since $S$ is a closed\(^2\) convex set. If $Q = K(s_i, s_j)$ is the “covariance matrix” we get

$$\phi = \sum_{i \in I} \alpha_i K(s_i, \cdot)$$

with

$$A = (\alpha) = Q^{-1}Y$$

**4.3.2 4 equivalent problems: from minimization to projections**

It is usual to interpret the previous minimization problem as a projection problem in the Hilbert space $E$:

\(^2\)by the continuity of the canonical injection
**Problem 4.17** \( \phi = p_S^\perp (0) \)

where \( p_S^\perp \) denotes the orthogonal projection on the closed convex set \( S \). The equivalence of the two problems is clear since by definition, the projection of 0 is the point of \( S \) minimizing the distance to 0.

These two equivalent problems rely however heavily on the Hilbertian structure of \( E \) and it is interesting to state a third and fourth equivalent problems.

Let \( L \) be the vector space spanned by the \( K(.,a_i), i \in I \) (respectively by the \( K(s_i,.)\), \( i \in I \) by the symmetry of the kernel). Then

**Problem 4.18** \( \forall f \in \mathcal{S}, \phi = \arg \min_{\lambda \in L} \| f - \lambda \|^2 \)

But once again, this minimization problem as an interpretation in terms of orthogonal projection:

**Problem 4.19**

\[
\forall f \in \mathcal{S}, \phi = p_L^\perp (f) = \sum_{i \in I} \alpha_i K(s_i,.)
\]

with \( \Lambda = (\alpha) = Q^{-1} Y \).

In other terms, all the interpolating functions have the same orthogonal projections on the subspace \( L \).

These results could mean that the finite-dimensional subspace \( L \) is a good space to summarized interpolating functions for they all have the same orthogonal projection but we will see below that \( L \) defines actually the good direction for projection.

### 4.3.3 Interpolation in evaluation subdualities

The interest of problem 4.19 is that projections on subspaces in dual systems can be defined naturally whereas we cannot generally define projection on convex sets. The main difference
is that we now have to define two subspaces: a support and a direction. Precisely let \((E, F)\) be a dual system with respect to a bilinear form \(B\) and \(T \subset E, L \subset F\) finite subspaces such that \((T, L)\) are in separate duality with respect to \(B\) (and hence \(P\) and \(L\) have the same dimension). Then it follows that

1. \(E = T \oplus L^\perp\)
2. \(F = L \oplus T^\perp\)

and these decompositions define projections \(p_T^L\) and \(p_L^T\) respectively the projection of vectors of \(E\) on \(T\) orthogonally to \(L\) and the projection of vectors of \(F\) on \(L\) orthogonally to \(T\).

**Remark 4.20** If \(E = F\), then any subspace \(T = L\) such that \((T, T)\) are in separate duality is called admissible (see for instance [24]).

Suppose now that \((E, F)\) is an evaluation subduality with kernel \(K\). Fix \(L\) the vector subspace of \(F\) spanned by the \(K(., s_i), i \in I\) and choose a support of the form \(T = \text{Vec} \{K(t_i, .), i \in I\}\) such that the matrix \(Q = K(t_i, s_j)\) is invertible. Then \((T, L)\) is a duality and the projections are well defined.

**Theorem 4.21**

\[
\forall f \in \mathcal{S}, \phi = p_{T}^{L}(f) = \sum_{i \in I} \alpha_i K(t_i, .)
\]

with \(A = (\alpha) = Q^{-1}Y\). \(\phi\) is then independent of the particular interpolating function projected. Moreover \(\phi \in \mathcal{S}\).
Proof. – By the reproduction property the orthogonality of \( f - \phi \) with the \( K(., s_i) \) is just \( (f - \phi)(s_i) = 0 \) and the function \( \phi \) is interpolating. Finally, the invertibility of \( Q \) gives the desired result. \( \square \)

We see that the loss of symmetry and of the norm affects the choice of \( T \): there is now no intrinsic reason to choose a particular set of points \( \{t_i, i \in I\} \) than another. We can however state an interesting result when two data sets are available.

Suppose we are now given two data sets \( \{(t_i, x_i), i \in I\} \{\left(s_i, y_i\right), i \in I\} \) (\( I \) finite integer set) where the \( t_i, x_i \in \Omega \) and \( x_i, y_i \) in \( \mathbb{K} \) such that the matrix \( Q = K(t_i, s_j) \) is invertible.

Define \( S = \{f \in E, f(s_i) = y_i \ \forall i \in I\} \) and \( G = \{g \in F; f(t_i) = x_i \ \forall i \in I\} \) and \( T \) and \( L \) as before. Then

**Theorem 4.22**

1. \( \forall f \in S, \phi = p^{LT}_T(f) = \sum_{i \in I} \alpha_i K(t_i, ..) \in S \) with \( A = (\alpha) = Q^{-1} Y \)
2. \( \forall g \in G, \psi = p^{LT}_L(g) = \sum_{i \in I} \beta_i K(., s_i) \in G \) with \( B = (\beta) = (\psi Q)^{-1} X \)
3. \((\phi, \psi) \) stabilizes the quantity \((g, f)\) with \( f \in S, g \in G \)
4. \((\phi, \psi) \) stabilizes the quantity \((g - \lambda, f - \tau)\) with \( \lambda \in L, \tau \in T \) for all \( f \in S, g \in G \)

Proof. – A straightforward calculation gives the desired result. \( \square \)

Figure 4.1 represents such a stabilization in the case of two different data sets with asymmetric smooth kernel.

In the case of (self)-dualities, we can do orthogonal projections on admissible subspaces and then use a single data set when admissible. In the following examples (figures 4.2 and 4.3), we use an asymmetric (discontinuous then smooth) kernel function based on the Heaviside
Figure 4.1: stabilization

function to detect discontinuity.
Conclusion and comments

We have initiated here three possible fields of applications that show more insights of the general theory of subdualities. The first one concerning “generalized” measure theory remains widely open. The second concerning operator theory shows some very peculiar uses but there probably are many more problems that could gain something at using subdualities. The case of differential operators gives another perspective on Sobolev spaces. Finally, we solve the interpolation problem by using projection in evaluation subdualities but not without difficulties due to the lack of a norm. A solution is then to solve two different interpolation
problems in a single common setting. This can be applied to rupture detection.