On generalized inverses and Green’s relations

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Abstract. We study generalized inverses on semigroups by means of Green’s relations. We first define the notion of pseudo-inverse along an element and study its properties. Then we show that the classical generalized or pseudo-inverses (group inverse, Moore-Penrose inverse and Drazin inverse) belong to this class.

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1. Introduction

When studying regular semigroups, it is sometimes of interest to look for specific generalized inverses, such as inverses in subgroups [4], the group inverse or the Moore-Penrose inverse [1]. It may also be of interest to see if the semigroup admits inverse transversals [2], [14]. And if the semigroup is not regular, one may look for pseudo-inverses (such as the Drazin inverse [3], [1]).

We propose here to define a new type of generalized pseudo-inverse, the inverse along an element, which will encompass the classical notions but is of richer type. This notion is based on Green’s relation $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ [6]. We will also need the less known notion of trace product ([11], [12]): for $a, b \in A \times A$, we say that $ab$ is a trace product if $ab \in \mathcal{R}_a \cap \mathcal{L}_b$, or equivalently (Theorem 3 p. 277 [11]) if $\mathcal{R}_b \cap \mathcal{L}_a$ contains an idempotent
element. As usual $E(A)$ will denote the set of idempotents of the semigroup $A$, and $A^1$ the monoid generated by $A$.

2. A new generalized pseudo-inverse: the inverse along an element

Now, let $a \in A$ and $d \in A$. We define the following notion of pseudo-inverse along a given element:

**Definition 1.** We say that $b$ is a pseudo-inverse of $a$ along $d$ if exists an idempotent $e \in \mathcal{R}_d \cup 1$:

1. $b$ is a generalized inverse of $ae$.
2. $b \in \mathcal{H}_d$.

If $e = 1$ we say simply that $b$ is a (generalized) inverse of $a$ along $d$.

The use of the idempotent is necessary if we want to be able to define (pseudo)-inverses of non regular elements. Moreover, we restrict our attention to idempotents since they act as left identity on $\mathcal{R}_e = \mathcal{R}_d$ (lemma 4 in [11]). In particular it follows that $b$ is an outer inverse of $a$ ($bab = b$).

Remark that this definition is actually symmetric:

**Lemma 2.** $b$ is a pseudo-inverse of $a$ along $d$ if exists an idempotent $f \in \mathcal{L}_d \cup 1$:

1. $b$ is a generalized inverse of $fa$.
2. $b \in \mathcal{H}_d$.

**Proof.** Since $b \in \mathcal{H}_d$, $\exists x, y \in A^1 \times A^1$, $b = dx$, $d = by$. Then one can verify that $f = xae \in L_d \cap E(A)$ ($d = df$), and that $b$ is a generalized inverse of $af$. □

**Example.** A monoid is a group if and only if every element is invertible along $1$. 
Example. Let $A = T_3$ be the full transformation semigroup. Then $\mathcal{H}_{(232)} = \{(232), (323)\}$. It follows that $a = (121)$ is invertible along $d = (232)$ with inverse $b = (323)$, and that $a' = (122)$ and $a'' = (123)$ are pseudo-invertible along $d = (232)$ (with inverse $b = (323)$), take for instance $e = (121)$). Note that (122) and (123) are not invertible along (232), and that (111) is not pseudo-invertible along (232).

Lemma 3. If such an idempotent $e$ exists, then $d$ is regular and $ed = d$.

Proof. Suppose such an idempotent exists. Then $b$ exists and is regular and $d \in \mathcal{H}_b \subset \mathcal{D}_b$ is regular (Theorem 1 in [11]).

Note that $ed = d$ holds when $e \neq 1$ since $e \in \mathcal{R}_d$ is a left identity on $\mathcal{R}_e$. □

We can now state our first theorem, that deals with existence and unicity of pseudo-inverses along an element $d$:

Theorem 4. Let $a, d \in A \times A$

1. $a$ is pseudo-invertible along $d$ if and only if exists $e \in (\mathcal{R}_d \cap E(A)) \cup 1$, $(ae)d$ and $d(ae)$ are trace products.

2. If a pseudo-inverse along $d$ exists, it is unique.

If it exists, we note $a^{\Leftarrow}d$ the pseudo-inverse along $d$ and $a^{\Leftarrow d}$ if it is a generalized inverse ($e = 1$).

Proof. Existence: Suppose $a$ is pseudo-invertible along $d$ with $e \in E(A^1)$. Then (corollary 1 in [11]), the $\mathcal{H}$-classes $\mathcal{R}_{ae} \cap \mathcal{L}_d$ and $\mathcal{L}_{ae} \cap \mathcal{R}_d$ contain idempotents elements and $(ae)d$ and $d(ae)$ are trace products.

Suppose now exists $e \in (\mathcal{R}_d \cap E(A)) \cup 1$, $(ae)d$ and $d(ae)$ are trace products.

Then by the same theorems $ae$ admits a generalized inverse in $\mathcal{H}_d$ and $a$ is pseudo-invertible along $d$. 
UNICITY: First, by corollary 1 in [11], for a given \( e \) the inverse of \( ae \) in \( \mathcal{H}_d \) is unique. Suppose \( e \) and \( e' \) are two idempotents, \( e \in \mathcal{R}_d \cup 1 \) and \( e' \in \mathcal{R}_d \). Then \( \exists x \in A^1, e' = ex \). Let \( b \) be the (unique inverse) of \( ae \) in \( \mathcal{H}_d \). Then \( b \) verifies

\[
\begin{align*}
    a e' b a e' &= a e' b a e x = a e b a e x \quad \text{(from } e' b = b = eb) \\
    &= a e x = a e' \quad \text{ (} b \text{ is an inner inverse of } ae \text{)}
\end{align*}
\]

but it verifies also \( b a e' b = b \) since \( e' b = b \). Finally \( b \in \mathcal{H}_d \) is the unique generalized inverse of \( ae' \) in \( \mathcal{H}_d \) and the pseudo-inverse along \( d \) is unique. \( \square \)

Example. Let \( S \) be the subsemigroup of \( M_3(\mathbb{N}) \) generated by the matrices

\[
    a = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 0 \\
    \end{pmatrix} \quad b = \begin{pmatrix}
    0 & 1 & 1 \\
    1 & 0 & 0 \\
    0 & 0 & 0 \\
    \end{pmatrix} \quad c = \begin{pmatrix}
    0 & 1 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 0 \\
    \end{pmatrix} \quad d = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 1 & 1 \\
    0 & 0 & 0 \\
    \end{pmatrix}
\]

Then \( a R b R c R d \) (the semigroup is right simple), \( a L c \) and \( b L d \). Since \( a \) and \( d \) are nilpotent, each \( L \)-class contains idempotents elements and it follows that any product of two elements is a trace product. Finally any element is invertible along another one, or equivalently any \( \mathcal{H} \)-class \( \{a, c\}, \{b, d\} \) is an inverse transversal (see note 3.16 in [15]).

In the preceding proof, we do not use the fact that \( ae' \) was invertible. Actually:

Corollary 5. The choice if the idempotent is not important. If \( a \) is pseudo-invertible along \( d \), then \( \forall e' \in \mathcal{R}_d \cap E(A) \), \( ae' \) is invertible in \( \mathcal{H}_d \).

Proof. Same proof as above, \( b \in \mathcal{H}_d \) inverse of \( ae \) is an inverse of \( ae' \). \( \square \)

Corollary 6. If \( a \) is pseudo-invertible along \( d \) and \( ad = da \), then \( ad \in \mathcal{H}_d \).

Proof. Let \( (e, f) \in \mathcal{R}_d \times \mathcal{L}_d \) be idempotents, \( ae \) and \( fa \) invertible in \( \mathcal{H}_d \). Then by Theorem 4 \( aed = ad \) and \( df a = da \) are trace products, which implies \( ad = da \in \mathcal{R}_d \cap \mathcal{L}_d = \mathcal{H}_d \).
Theorem 7. Suppose $b = a^{\angle d}$ exists. Then $ba \in \mathcal{R}_d \cap E(A)$ and $ab \in \mathcal{L}_d \cap E(A)$.

In particular, by lemma 4 in [11], $(ba)d = d = d(ab)$.

Proof. First, since $b \mathcal{R} d$, exists $x$, $d = bx$. But exists $e \in \mathcal{R}_d \cap E(A)$, $bae = b$, and it follows that $d = bx = baebx$. Also exists $y$, $dy = b$, and $ba = dya$. Finally $ba \mathcal{R} d$.

Second $baba = baeba = ba$ since $e$ acts as a left identity on $\mathcal{R}_d \ni b$ and $b$ is an outer inverse of $ae$.

We can then prove an interesting result regarding commutativity:

Theorem 8. If $a$ is pseudo-invertible along $d$, then $a^{\angle d} \in \text{comm}^2 (a,d)$.

Proof. Suppose $c \in \text{comm} (a,d)$. Then

$$cd = cbad = cdab = dacb = dc = badc = bcad = dabc$$

hence

$$cbad = bcad, \quad dabc = dabc$$

but $ad \mathcal{R} ae$ by theorem 4, then $ad \mathcal{R}^* ae$ (lemma 1.1 in [5]) and we get

$$cb = cbab = bcaeb = bcab$$

Similarly $da \mathcal{L} fa$ and

$$bc = bfabc = bfacb = bab$$

Finally $bc = cb$ and $b \in \text{comm}^2 (a,d)$.
3. Inverses along $d$ and classical inverses

One of main interest of these notions of pseudo-inverse and inverse along an element is that the classical generalized inverses belong to this class:

**Theorem 9.** Let $a \in A$. ($A$ is a $*$-semigroup in $3$.)

1. $a$ is group invertible if and only if it is pseudo-invertible along $a$. In this case the pseudo-inverse is a true inverse and the two coincide.
2. $a$ admits a Drazin inverse if and only if it is invertible along some $a^m$, $m \in \mathbb{N}$, and in this case the two coincide.
3. $a$ is Moore-Penrose invertible if and only if it is pseudo-invertible along $a^\ast$. In this case the pseudo-inverse is a true inverse and the two coincide.

**Proof.**

**Group inverse:** Suppose $a$ is group invertible. Then $\mathcal{H}_a$ is a group that contains the group inverse $a^\sharp$. It follows that $a$ is invertible along $a$, with inverse $a^{\bot a} = a^\sharp$.

Conversely, if $a^{\bot a}$ exists, then by corollary 6 ($a \in \text{comm}(a)$) $a^2\mathcal{H}_a$ and $\mathcal{H}_a$ is a group ([6], Theorem 7). $a$ is then group invertible with inverse $a^\sharp \in \mathcal{H}_a$ and by unicity of the pseudo-inverse along $a$, $a^{\bot a} = a^\sharp = a^{\bot a}$.

**Drazin inverse:** Suppose a Drazin invertible (with Drazin inverse $a^D$). Then (Theorem 7 in [3]) exists $m \in \mathbb{N}^*$, exists $e$ idempotent in $\mathcal{H}_{a^m}$, $ae = ea \in \mathcal{H}_{a^m}$. Then $\mathcal{H}_{a^m}$ is a group ([6] Theorem 7 and corollary, or [11] corollary 2) and $a$ is pseudo-invertible along $a^m$ (take $e$ for idempotent). Moreover (see the proof of Theorem 7 in [3]), the inverse of $ae$ in $\mathcal{H}_{a^m}$ is precisely the Drazin inverse, hence $a^D = a^{\bot a^m}$.

Conversely, suppose exists $m \in \mathbb{N}^*$, a pseudo-invertible along $a^m$. Then by corollary 6 ($a \in \text{comm}(a^m)$), $a^{m+1}\mathcal{H}_a^m$. But by Theorem 4 p 510 in [3], $a$ is Drazin invertible if and only if it is strongly $\pi$-regular, i.e. exists $m \in \mathbb{N}^*$, $x, y \in A^1$,

$$a^{m+1}x = a^m = ya^{m+1}$$
or using Green’s relation if and only if exists $m \in \mathbb{N}^*, a^{m+1}H(a^m)$. Finally $a$ is Drazin invertible and by the previous result, the two pseudo-inverses coincide.

**M-P inverse:** Suppose $a$ Moore-Penrose invertible with Moore-Penrose inverse $a^+$. Then

$$a^+ = a^+ aa^+ = (a^+ a)^* a^+ = a^* (a^+)^* a^+$$

$$a^+ = a^+ (aa^+) = a^+ (aa^*)^* = a^+ (a^+)^* a^*$$

$$a^* = (aa^+ a)^* = (a^+ a)^* a^* = a^+ a a^*$$

$$a^* = (aa^+ a)^* = a^* (aa^*)^* = a^* a a^+$$

These four relations imply that $a^+ H a^*$, hence $a$ is invertible along $a^*$ with $a^\perp a^* = a^+ = a^\perp a^*$.

Conversely, suppose $a$ is pseudo-invertible along $a^*$, with idempotent $e \in R_{a^*}$. Then $(ae) a^* = aa^*$ is a trace product and $aa^* L a^*$, hence by transposition $aa^* R a$ which implies $aa^* R^* a$.

Now by the properties of the pseudo-inverse along $a^*$ (in particular by theorem 7, $(ba)a^* = a^*$),

$$aa^* b a a^* = aa^* b^* a^* = (baa^*)^* a^* = aa^*$$

But $aa^* R^* a$ hence it follows that $aa^* b^* ba = a$ ($a^* b^* b$ is an inner inverse of $a$).

Then $a = aa^* b^* ba = a(ba)a^* b^* ba = ab(aa^* b^* ba) = aba$, $b$ is actually an inner inverse of $a$, we may take $e = 1$ for idempotent and $a$ is invertible along $a^*$.

But $b\perp a^*$, and exists $x, y \in A^1 \times A^1$, $b = a^* x = ya^*$ and

$$b = a^* x = a^* b^* a^* x = a^* b^* b$$

$$= ya^* = ya^* b^* a^* = bb^* a^*$$
Finally \( b = a^*b^*b = bb^*a^* \in \mathcal{H}_{a^*} \) is a generalized inverse of \( a \) that verifies

\[
(ab)^* = (a b b^*)^* = ab, \quad (ba)^* = (a^* b^* b a)^* = ba
\]

hence it is the Moore-Penrose inverse of \( a \).

\[\square\]

Combining theorem 4 and theorem 9, we then get directly the following existence criteria and commuting relations for the classical inverses [6], [3], [7], [8], [9], [10], [13]:

**Corollary 10.** Let \( a \in A \).

1. A group inverse \( a^\# \) exists if and only if \( a^2 H_a \), and \( a^\# \in \text{comm}^2(a) \).
2. A Drazin inverse \( a^D \) exists if and only if exists \( m \in \mathbb{N}^\ast, a^{m+1} H a^m \), and \( a^D \in \text{comm}^2(a) \).
3. A Moore-Penrose inverse \( a^+ \) exists if and only if \( a a^* \in R_a \) and \( a^* a \in L_a \), and \( a^+ \in \text{comm}^2(a, a^*) \).

**References**


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