PRICING WITH SPLINES

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Abstract

The exponential affine pricing principle is applied to the family of skewed Laplace historical distributions. The risk-neutral distribution is shown to belong to the same family and a closed form pricing formula for European call is derived. This formula is a direct competitor of the Black-Scholes formula, but involves more parameters, that are location and tail parameters. This approach is extended to exponential affine spline conditional probability density function and stochastic discount factor leading to nonparametric pricing approaches. Finally the time coherency is introduced by means of a Markov specification.

Keywords : Stochastic Discount Factor, Moment Generating Function, Laplace Distribution, Derivative Pricing, Splines, Nonparametric Approach, Markov Model.

1 Introduction

The standard for option pricing is the Black-Scholes approach [Black, Scholes (1973)], which assumes i.i.d. Gaussian geometric stock returns, continuous trading and derives an analytical formula for pricing European calls from the arbitrage free restrictions. The derivative prices and the associated risk neutral probability basically depend on the underlying historical volatility (and not on the historical mean). However the misspecification of Black-Scholes approach is largely documented both for return dynamics and for pricing derivatives with different characteristics. Typically the implied Black-Scholes volatility surfaces are not flat and vary with the day and the environment.

Different solutions have been proposed in the literature to reduce the misspecification errors. The extensions of the basic model can be classified according to the assumptions introduced on the two components of a pricing models, which are the historical distribution and the stochastic discount factor (s.d.f.).

i) Parametric historical distribution-parametric sdf

A first direction consists in extending the parametric dynamic model for the underlying asset price and in deriving the new corresponding parametric valuation formulas. For instance the Black-Scholes model has been extended by introducing stochastic volatility [3], or jumps [4]. These models are generally written in continuous time and provide coherent specifications for analyzing return dynamics and cross-sectional derivative pricing. However the introduction of a non traded random factor creates an incomplete market framework. The incompleteness, that is the multiplicity of admissible s.d.f., is solved by imposing a parametric specification of the risk premium corresponding to this unobservable factor. It can be assumed constant (and unknown), but very often is taken equal to zero as in the standard Hull-White formula.

ii) Nonparametric sdf

Alternatively the practitioners often perform a direct nonparametric anal-

[3]See e.g. Hull, White (1987), Hull (1989), Chesney, Scott (1989), Melino, Turnbull (1990), Stein, Stein (1991), Heston (1993), Ball, Roma (1994).

[4]See e.g. Merton (1976), Ball, Torous (1985), Bates (1996).

ysis of the state prices based on derivative prices. For each date they study how the call prices depend on strike and maturity. They can consider directly the price surface or equivalent characteristics. Standard ones are 1) the state price density which provides the Arrow-Debreu prices and is deduced from the second order derivative of the call price with respect to the strike [see Breeden, Litzenberger (1978), Banz, Miller (1978)]; 2) the surface of Black-Scholes implied volatilities obtained by inverting the Black-Scholes formula with respect to the volatility. In practice the state price or implied volatility surfaces are smoothed by nonparametric approaches. For instance Ait-Sahalia, Lo (1998) apply kernel smoothing to observed call prices and deduce the state price density as a by-product[5]. Other authors propose direct approximations of the state price density. For instance the risk neutral distribution can be approximated by mixture of distributions [6], or by means of Hermite expansions [7]s. In the latter approach, it is possible to estimate daily parameters measuring the weights of polynomials of degree one, two, three, four... in this expansion. They are generally interpreted as implied mean, volatility, skewness and kurtosis.

A limitation of these nonparametric approaches comes from the number of liquid derivative assets. To get accurate estimators they require a large number of highly traded derivatives, with an appropriate distribution of the associated strikes. For a given day and a given underlying asset, these numbers are generally between 5 and 20, with a clustering of traded strikes close to moneyness. Thus the cross-sectional asymptotic theory generally developed with these approaches cannot apply.

Finally note that some authors [8] are interested in testing structural equilibrium models. For this purpose they focus on the sdf which is generally estimated by moment methods from data on returns, consumption,... Since they are dealing neither with historical pdf, nor with the state price density, the results are difficult to use for derivative pricing.

iii) Nonparametric historical distribution- parametric sdf.

[5]However their approach assumes that the call prices depend in a deterministic way of the asset price.

[6]See Bahra (1996), Campa, Chang, Reider (1997), Melick, Thomas (1997).

[7]See Jarrow, Ruud (1982), Madan, Milne (1994), Abken, Madan, Ramamurtie (1996)

[8]See Bansal, Hsieh, Viswanathan (1993), Bansal, Viswanathan (1993), Cochrane (1996), Chapman (1997), Dittmar (2002).

A nonparametric specification of the state prices can also be derived with a nonparametric historical distribution and a parametric sdf. The advantage of such a specification is to correspond to available data. The underlying asset is generally liquid, and the associated return data can be used to estimate nonparametrically the historical distribution. Once this distribution is known, the small number of parameters defining the risk correction are calibrated on observed derivative prices. Such approaches have been developed rather early as direct extensions of the standard Black-Scholes formula.

For instance it is possible to consider a continuous time model and to assume that the infinitesimal drift and volatility functions are unknown deterministic functions of time[9]. The model still assumes a complete market framework, Girsanov theorem provides the unique admissible sdf and the risk neutral distribution depends on the volatility only (called local volatility). The local volatility can be estimated directly from the return data on the underlying asset. It can also be estimated from derivative data, by using the interpretation of local volatility from partial derivatives of the call price with respect to strike and time to maturity [see Dupire (1994)]. This second estimation technique is not very accurate due to the small number of liquid derivatives.

Instead of deterministic drift and volatility functions, it is also possible to assume that drift and volatility depend on the return and to still derive the unique sdf by Girsanov theorem. The unknown functional parameters, that are the drift and volatility, can be estimated in various ways. For instance Ait-Sahalia (1996) assumes a linear drift and deduce a nonparametric estimator of the volatility from a kernel estimator of the marginal density. The drift can also be let unconstrained and the two functional parameters estimated by nonparametric nonlinear canonical analysis based on either kernel method [Darolles, Florens, Gourieroux (2003)], or Sieve method [Hansen, Scheinkman, Touzi (1994), Chen, Hansen, Scheinkman (1998), Darolles, Gourieroux (2002)].

However it is known that a (one dimensional) diffusion model implies restrictions on return dynamics, which are not observed on available data, for instance : time reversibility [10], or constraints on tail magnitude due to the normality of the Brownian motion.

[9] See Merton (1973), Dupire (1994).

[10] See Darolles, Florens, Gourieroux (2003).

To avoid the restrictions associated with the usual approaches, we focus in this paper on the historical transition pdf, instead of the local volatility functions. The analysis is performed in discrete time, which implies an incomplete market framework.

The return process (y_t) is such that the conditional distribution of y_t given its past values $\underline{y_{t-1}} = (y_{t-1}, y_{t-2} \ldots)$ belongs to a skewed Laplace family or, more generally, to the family of mixtures of skewed Laplace distribution, which is equivalent to the family with first order spline log-densities. Then the dimension of incompleteness is diminished by considering a parametric family of stochastic discount factors. We restrict the choice by imposing an exponential-affine stochastic discount factor. This allows the use of the Esscher transformation to pass from the historical distribution to the risk-neutral one [11]. Then the s.d.f. parameters are constrained by the arbitrage free restrictions, and the risk neutral condition distribution of y_t given $\underline{y_{t-1}}$ is derived. Closed form expressions of the prices of European calls and puts are finally derived.

An extension to exponential spline sdf is provided as well as a particular Markov specification which gives a simple nonparametric, time coherent pricing methodology valid for any horizon.

The plan of the paper is as follows. In Section 2, the principle of exponential-affine pricing is reviewed. Then this approach is applied to a skewed Laplace conditional historical distribution of geometric returns. The example of the conditional Laplace distribution is interesting as an introductory case for exponential-splines. It is also important in its own, since the price of the European calls admit simple closed form expressions. The pricing formula is a direct competitor of the standard Black-Scholes, and involves two types of parameters, which capture location and tail effects. An extension to exponential affine, spline conditional probability densities is presented in Section 3, when the sdf is either exponential affine, or exponential affine spline. Section 4 presents the dynamic extension by means of Markov models and Section 5 concludes.

[11] See Gerber, Shiu (1994), Buhlman et alii (1996), Shiryaev (1999), Darolles, Gourieroux, Jasiak (2001), Dijkstra, Yao (2002).

2 Pricing with Laplace distribution

For expository purpose let us consider the two period framework and denote by r the riskfree rate between the dates t and $t+1$, by $y = y_{t+1} = \log(S_{t+1}/S_t)$ the geometric return on the risky asset with price S_t. The aim of this section is to explain how to derive the state price density at horizon 1, that is how to price the European derivative written on y_{t+1}. Of course the horizon is fixed at 1 by convention, but the approach can be applied to any horizon. We first recall the principle of exponential-affine pricing initially introduced by Gerber, Shiu (1994)[12]. Then this approach is applied to a skewed Laplace conditional historical distribution of geometric return.

2.1 Exponential-affine pricing

Let us introduce the truncated Laplace transform (or moment generating function) of the conditional distribution of the geometric return. It is defined by :

$$\psi(u, \gamma) = E[\exp(uy)\mathbb{1}_{y>\gamma}], \tag{2.1}$$

where the notation above means :

$$\psi(u, \gamma) = E[\exp\{u \log(S_{t+1}/S_t)\}\mathbb{1}_{\log(S_{t+1}/S_t)>\gamma}|I_t],$$

I_t is the information available at time t for the investor. This information set contained the past return values $\underline{y_t} = (y_t, y_{t-1} \ldots)$ and, possibly, other informations. The path dependence of ψ is not mentioned for notational convenience.

The European derivative asset, whose payoff $g(y)(= g(y_{t+1}))$ is written on the geometric return of the underlying asset, can be priced by means of a stochastic discount factor model [13]. The derivative price at date t is :

$$C(g) = E[Mg(y)], \tag{2.2}$$

[12]See also Esscher (1992), Buhlman et alii (1996), Gourieroux, Monfort (2001), Yao (2001).

[13]See Hansen, Richard (1987), Campbell, Lo, McKinlay, (1997) Chapter 8, Cochrane (2001), Gourieroux, Jasiak (2001), Chapter 13 .

where M denotes a stochastic discount factor. In an exponential-affine framework the stochastic discount factor is given by[14] :

$$M(y) = \exp[\alpha y + \beta]. \tag{2.3}$$

It is exponential-affine with respect to the geometric return $y(= y_{t+1})$[15]. Different motivations exist for the exponential affine restriction on the stochastic discount factor, which reduces the multiplicity of pricing formulas existing in this incomplete framework.

i) First the exponential-affine restriction underlies the usual approaches based on no arbitrage restrictions, or on equilibrium theory. For instance in a two period price exchange economy under preference restrictions [see e.g. Breeden, Litzenberger (1978), Huang, Litzenberger (1988)], the exponential-affine form of the stochastic discount factor corresponds to power utility functions.

ii) An exponential affine specification is obtained, when we look for the risk neutral distribution which is the closest to the historical one, with respect to entropy criterion [Stutzer (1996), Buchen, Kelly (1996)].

iii) The choice of an exponential affine sdf often leads to tractable computations, and provides results which are easy to compare with the standard Black-Scholes formula [see the examples given in Gerber, Shiu (1994)].

iv) Finally, it is seen below that it is appropriate to define spline historical and risk neutral densities, compatible with no arbitrage restrictions. Moreover it will be seen in Section 3 that this kind of property remains valid for a more general specification of the sdf, namely the exponential spline specification.

The arbitrage-free constraints are derived by applying the pricing formula to the zero-coupon bond with payoff 1 and to the risky asset with payoff $\exp y = S_{t+1}/S_t$. These constraints are :

[14] As above the time index is omitted for convenience. More explicit equations would be : $C_t(g) = E[M_{t,t+1}g(y_{t+1})|I_t]$, where : $M_{t,t+1} = \exp(\alpha_t y_{t+1} + \beta_t)$ is the stochastic discount factor for period $t, t+1$. The coefficients α_t, β_t and the derivative price $C_t(g)$ are I_t-measurable, whereas the stochastic discount factor $M_{t,t+1}$ is I_{t+1}-measurable.

[15] The stochastic discount factor is in general not exponential-affine with respect to the current and lagged values of the return ; indeed the lagged values can influence in a nonlinear way the change of probability through the sensitivity coefficients α and β [see the previous footnote].

$$\begin{cases} E[M \exp r] = 1, \\ \\ E[M \exp y] = 1. \end{cases}$$

They provide the values of the risk correcting factors α, β by solving the system below [16], which depends on the untruncated Laplace transform :

$$\begin{cases} \exp(\beta + r)\psi(\alpha, -\infty) = 1, \\ \\ \exp(\beta)\psi(\alpha + 1, -\infty) = 1. \end{cases} \tag{2.4}$$

Then the price of a European call with maturity one is easily deduced. Let us consider a call written on S_{t+1} with payoff $(S_{t+1} - kS_t)^+$, where k is the moneyness strike; the payoff can be written as : $S_t(\frac{S_{t+1}}{S_t} - k)^+$, or as : $S_t[\exp(y) - k]^+$. The price of a normalized payoff $[\exp(y) - k]^+$ is given by :

$$\begin{aligned} C(k) &= E[M(\exp y - k)^+] \\ &= E[\exp(\alpha y + \beta)[\exp y - k]\mathbb{1}_{y > \log k}], \\ C(k) &= \exp(\beta)[\psi(\alpha + 1, \log k) - k\psi(\alpha, \log k)], \end{aligned} \tag{2.5}$$

where α, β are the solutions of system (2.4).

2.2 Pricing formulas

Exponential-affine pricing can be applied to any return distribution [17]. In this section we consider the family of skewed Laplace distributions for several reasons.

i) Both historical and risk neutral distributions will belong to the Laplace family under no-arbitrage restrictions.

ii) The skewed Laplace distribution is compatible with the exponential fat tails observed on real data.

[16] When the time index is taken into account, the solutions α and β are generally path dependent, like function ψ.

[17] Under tail restrictions since the truncated Laplace transform has to exist in a neighbourhood of zero.

iii) Spline approximations of order 1 of the log-densities can be easily derived from this family.

Thus the skewed Laplace family will likely capture important stylized facts which are not captured in the Black-Scholes framework, while keeping the important property of staying in the same distribution family when moving from the historical to the risk neutral world.

This stability property is not specific to the skewed Laplace family since it is shared by the normal family underlying the Black-Scholes approach. However a nice property of the skewed Laplace family is that it is easily extended to the exponential spline family.

Let us consider a geometric return, whose conditional historical distribution is a skewed Laplace distribution denoted by $\mathcal{L}(b_0, b_1, c)$. The p.d.f is given by :

$$\begin{aligned} p(y) &= \frac{b_0 b_1}{b_0 + b_1} \exp[b_0(y-c)], \text{ if } y \leq c, \\ & \frac{b_0 b_1}{b_0 + b_1} \exp[-b_1(y-c)], \text{ if } y \geq c, \end{aligned}$$

where b_0 and b_1 are strictly positive and c is a location parameter. As already mentioned $p(y)$ must be seen as the conditional distribution of y_{t+1} given $\underline{y_t} = (y_t, y_{t-1}, \ldots)$ and, consequently, b_0 and b_1 and c are generally path dependent. For sake of notational simplicity this path dependency is not explicited. c is the mode of the (conditional) distribution, whereas b_0 and b_1 characterize the left and right exponential (conditional) tails, respectively. The mean of the distribution is :
$m = c + \frac{1}{b_1} - \frac{1}{b_o}$, and the variance is : $\sigma^2 = \frac{1}{b_0^2} + \frac{1}{b_1^2}$. This skewed Laplace distribution fits the conditional distribution of observed returns better than the Gaussian distribution. It admits fatter tails, which decrease at an exponential rate, and a sharp peak at the mode to balance the tail effect. By applying the general approach described in subsection 2.1, we get the pricing formulas below.

Proposition 1 : If the conditional historical distribution is a skewed Laplace distribution $\mathcal{L}(b_0, b_1, c)$ with $b_0 + b_1 > 1$, and if the stochastic discount factor is exponential-affine :

i) the conditional risk-neutral distribution is unique and corresponds to the skewed Laplace distribution $\mathcal{L}(b_0+\alpha, b_1-\alpha, c)$, with p.d.f. :

$$\begin{aligned}\pi(y) &= \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[(b_0+\alpha)(y-c)],\ \text{if } y \leq c,\\ & \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[-(b_1-\alpha)(y-c)],\ \text{if } y \geq c,\end{aligned}$$

where α is the solution of :

$$\exp(c-r)(b_0+\alpha)(b_1-\alpha) = (b_0+\alpha+1)(b_1-\alpha-1), \tag{2.6}$$

such that : $-b_0 < \alpha < b_1 - 1$.

ii) The price of the call written on $\exp y$ with payoff $(\exp y - k)^+$ is :

$$C(k) = C_1(k) = \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}\exp[-(b_1-\alpha-1)(\log k - c)],\ \text{if } \log k \geq c,$$

$$C(k) = C_2(k) = 1 - k\exp(-r) + \frac{b_1-\alpha-1}{(b_0+b_1)(b_0+\alpha)}\exp[(b_0+\alpha+1)(\log k - c)],\ \text{if } \log k \leq c.$$

iii) By the put-call parity relationship, the put prices are :
$P(k) = P_1(k) = -1 + k\exp(-r) + \dfrac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}\exp[-(b_1-\alpha-1)(\log k - c)]$, if $\log k \geq c$,

$$P(k) = P_2(k) = \frac{b_1-\alpha-1}{(b_0+b_1)(b_0+\alpha)}\exp[(b_0+\alpha+1)(\log k - c)],\ \text{if } \log k \leq c.$$

Proof : See Appendix 1.

The risk-neutral distribution is well defined whenever $-b_0 < \alpha < b_1$ and the stock price, corresponding to payoff $\exp(y)$, is finite whenever $-b_0 < \alpha < b_1 - 1$. These two inequalities are jointly satisfied only if $b_0 + b_1 > 1$ and under this condition is it is easily checked that equation (2.6) has a unique solution in α belonging to the interval $[-b_0, b_1 - 1]$.

The risk neutral distribution depends on b_0, b_1 through $b_0 + b_1$. Indeed $\pi(y)$ depends on b_0, b_1 through $b_0 + b_1, b_0 + \alpha$ and $b_1 + \alpha$, but $\delta_0 = b_0 + \alpha$ and $\delta_1 = b_1 - \alpha$ satisfy respectively the equations :

$$\exp(c-r)\delta_0(b_0+b_1-\delta_0) = (\delta_0+1)(b_0+b_1-\delta_0-1)$$

$$\exp(c-r)(b_0+b_1-\delta_1)\delta_1 = (b_0+b_1+1-\delta_1)(\delta_1-1)$$

which depend on b_0, b_1 only through b_0+b_1.

Since the risk neutral distribution depends on b_0, b_1 through b_0+b_1, the same is true for the call and put prices. More precisely, once α has been replaced by its value obtained from equation (2.6), these prices are functions of (b_0+b_1, C, k, r). Moreover it is easily seen that these prices are differentiable functions of k and c, although the two analytical expressions depend on the respective positions of $\log k$ and c. (see Appendix 1. iv).

The Laplace derivative pricing formulas have their own interest and can be easily compared with the standard Black-Scholes formulas, which assumes exponential-affine s.d.f, but Gaussian (conditional) return distribution. First the pricing formulas are simpler and in particular they avoid the use of the cdf of the standard normal distribution. Second they depend on two parameters c and b_o+b_1 instead of the single volatility in the Black-Scholes framework. This allows for more flexibility. Finally the risk correction involves the volatility parameter in the Black-Scholes framework, and a tail magnitude parameter in the Laplace framework. This special risk correction is easily understood. Indeed the payoff $\exp y$ of the underlying asset may be non integrable with respect to the conditional historical Laplace distribution [18]. More precisely if $b_1<1$, the payoff $\exp y$ is not integrable with respect to the conditional historical Laplace distribution, whereas it is integrable with respect to the conditional risk-neutral Laplace distribution, since $b_1-\alpha>1$. A consequence of the risk correction by α is to reduce the right tail in order to ensure this integrability and the existence of a finite stock price. Also note that the rebalancing of the tails has an impact on both the mean and the variance.

Finally the restrictive condition of homogeneous pricing formula of the Black-Scholes is not satisfied in the Laplace framework. More precisely the price of a European call written on S_{t+1} with strike K is given by : $C^*=S_tC(K/S_t)$. In the Laplace framework C^*/S_t is not, in general, an

[18] $\exp y$ is conditionally not integrable, if and only if the conditional expectation $E_t(S_{t+1})$ does not exist. In such a framework, the standard mean-variance portfolio management cannot be applied.

homogeneous function of K and S_t, since the coefficients b_0, b_1, c can be path dependent. This lack of homogeneity will allow for leverage effect and asymmetric smile observed on real data [Garcia, Renault (1998), Garcia, Luger, Renault (2000)].

2.3 Elasticity of the call price with respect to moneyness strike

Proposition 1 provides an explicit formula for the price of the call written on $\exp y$. It is easily checked that this price is a differentiable function of k, which decreases from 1 to 0, is convex and such that the elasticity of the call price [the put price, respectively] with respect to the moneyness strike is constant for $k \geq \exp c$ [$k \leq \exp c$, respectively].

In particular the call prices satisfy simple deterministic relationships. If k, k_1, k_2 are moneyness strikes larger than $\exp c$, we get :

$$\log C(k) = \log C(k_1) + \frac{\log k - \log k_1}{\log k_2 - \log k_1}[\log C(k_2) - \log C(k_1)]. \qquad (2.7)$$

When the parameters b_0, b_1, c are path independent, the elasticity of the call price C^* with respect to S_t is :

$$\begin{aligned}
\frac{\partial \log C^*}{\partial \log S_t} &= 1 + \frac{\partial \log C(K/S_t)}{\partial \log S_t} \\
&= 1 + \frac{\partial \log C}{\partial \log k}(K/S_t).\frac{\partial \log(K/S_t)}{\partial \log S_t} \\
&= 1 - \frac{\partial \log C}{\partial \log k}(K/S_t).
\end{aligned}$$

Therefore the condition of constant elasticity of C with respect to the moneyness strike for large k is equivalent to the condition of constant elasticity of C^* with respect to the current stock price.

Similarly constraint (2.7) is also valid when the derivatives are written on S_{t+1} instead of $\exp(y_{t+1})$. With obvious notations, relation (2.7) becomes :

$$\log C^*(K) = \log C^*(K_1) + \frac{\log K - logK_1}{\log K_2 - \log K_1}\{\log C^*(K_2) - \log C^*(K_1)\}.$$

2.4 Implied Black-Scholes Volatility

The pricing formula given in Proposition 1 can be numerically compared to the standard Black-Scholes formula. Since, for given values of r and k, the call price depends on two independent parameters, that are $b_0 + b_1$ and c, instead of only one parameter, namely σ, in the standard Black-Scholes, the Laplace pricing formula allows for implied location or tail effects. These features are observed on Figures 1 and 2, which provide the Black-Scholes implied volatilities for different sets of parameters b_0, b_1, c, and $r = 0$.[19] The Laplace model is appropriate for recovering the so-called smile, smirk and sneer effects observed in practice. Note that they can be recovered without introducing a time dependent volatility [as in Merton (1973), Dupire (1994)], but simply by suppressing the Gaussian assumption.

[Insert Figure 1 : Black-Scholes implied volatilities with c varying, $b_0 + b_1 = 10$ fixed, $k = 0.6$, top to bottom for c decreasing].

[Insert Figure 2 : Black-Scholes implied volatilities with $b_0 + b_1$ varying, $c = .06$ fixed, top to bottom for $b_0 + b_1$ increasing.]

2.5 Value of the call and historical parameters

The patterns of the call prices as functions of c and $b_0 + b_1$ are provided in Figures 3 and 4.

[Insert Figure 3 : Call price as a function of c, top to bottom for $b_0 + b_1$ increasing].

It is always difficult to understand how the call price depends on a location parameter, that are the mean in the standard Black-Scholes model and the mode c in the Laplace framework. This feature is clearly observed, when we consider the underlying stock with cash-flow $\exp y$. When the location parameter tends to $+\infty$ (resp. $-\infty$), the cash-flow tends to $+\infty$ (resp. 0), and the same is true for $(\exp(y) - k)^+$. Since the price of the underlying asset remains normalised to 1, it is natural that the price of the option

[19]These implied volatilities are simply obtained by solving, with respect to the volatility, the equation equating the Black Scholes formula to the exact price given by the formula of Proposition 1.

tends to 1. Contrary to the Black-Scholes case, in which the call price is independent of the mean, we observe a mean dependence in the Laplace framework. The symmetric pattern observed in Figure 3 is due to the special choice $k = 1, r = 0$, which implies $1 - k \exp -r = 0$ and identical call and put prices [20].

[Insert Figure 4 : Call price as a function of $b_0 + b_1$, top to bottom for c decreasing]

When $b_0 + b_1$ converges to 1, $b_0 + \alpha$ converges to 0, $b_1 - \alpha$ converges to 1 and the call price converges to one.

2.6 A special case

Let us consider the case $c = r$, where the mode of the (conditional) historical distribution corresponds to the riskfree return. The risk correcting factor α is the solution of :

$$(b_0 + \alpha)(b_1 - \alpha) = (b_0 + \alpha + 1)(b_1 - \alpha - 1)$$

$$\iff \quad \alpha = \frac{b_1 - b_0}{2} - \frac{1}{2}.$$

By replacing in the expression of the call-price, we get :

$$\begin{cases} C_1(k) &= \frac{1}{2\bar{b}} \exp[-(\bar{b} - 1/2)(\log k - r)], \text{ if } \log k \geq r, \\ C_2(k) &= 1 - k\exp(-r) + \frac{1}{2\bar{b}} \exp[(\bar{b} + 1/2)(\log k - r)], \text{ if } \log k \leq r. \end{cases}$$

As mentioned above, the pricing formula depends on the single parameter $\bar{b} = \frac{b_0 + b_1}{2}$, which measures the average tail magnitude. This parameter $\bar{b}$ has the same role as the volatility σ in the Black-Scholes model. When $\bar{b}$ increases, the average tail decreases. The derivatives of the call prices with respect to $\bar{b}$ are the analogues of the standard Black-Scholes vega. They are given by :

[20] It is easily checked that the correcting factor α satisfies : $\alpha = \alpha[b_0, b_1, \exp(r - c)] = b_1 - b_0 - 1 - \alpha[b_0, b_1, \exp(c - r)]$.

$$\frac{\partial C_1}{\partial \bar{b}}(k) = -\frac{1}{2\bar{b}^2} \exp[-(\bar{b} - 1/2)(\log k - r)][1 + \bar{b}(\log k - r)], \text{ if } \log k \geq r,$$

$$\frac{\partial C_2}{\partial \bar{b}}(k) = -\frac{1}{2\bar{b}^2} \exp[(\bar{b} + 1/2)(\log k - r)][1 - \bar{b}(\log k - r)], \text{ if } \log k \leq r.$$

These derivatives are negative, which implies a decreasing relationship between the average tail magnitude $\bar{b}$ and the call price. By inverting the pricing formula, we can define the implied (Laplace) tail magnitude associated with any observed call price. The surface of implied (Laplace) tail magnitude contains the same information as the call-price surface.

It is interesting to consider the admissible call prices when the historical variance $\sigma^2 = \frac{1}{b_0^2} + \frac{1}{b_1^2}$ is known. Since the price is a monotonous function of $\bar{b}$, we get an interval of admissible prices, whose bounds are obtained for the values of b_0, b_1 which optimize $b_0 + b_1$ submitted to $\sigma^2 = \frac{1}{b_0^2} + \frac{1}{b_1^2}$. This interval is easily deduced, for instance when $\log k \geq r$. We get :

$$C_1(k) \in [0, \frac{\sigma}{2\sqrt{2}} \exp(-[\frac{\sqrt{2}}{\sigma} - \frac{1}{2}](\log k - r))], \text{ if } \sigma < 2\sqrt{2},$$

$$C_1(k) \in [0, 1], \text{ if } \sigma \geq 2\sqrt{2}.$$

The interval increases with σ, and is equal to $[0, 1]$ in the limiting case $\sigma = 2\sqrt{2}$. The latter interval is the largest one compatible with the free arbitrage inequalities, since the constraints $0 \leq (\exp y - k)^+ \leq \exp y, \forall k$, imply $0 \leq C(k) \leq 1$.

2.7 The multiperiod case

Proposition 1 provides an explicit formula for the price of a European call (resp. European put) of maturity one. The time unit can be chosen arbitrarily; however if we want to price several European calls (resp. puts) at different maturities while satisfying the time consistency condition, we must choose the time unit such that all call maturities are multiple of this time unit. Proposition 1 still provides the various one step ahead risk neutral distribution. At times to maturity strictly larger than 1 there are no longer closed form expressions, but conditional risk neutral distributions and, therefore, call (or put) prices are easily obtained by simulations. If the parameters b_0, b_1, c are path dependent, the parameter α has to be computed at each period. More precisely the call price of maturity h at time t given by $e^{-rh} E_t^Q[\exp(y_{t+1} + \ldots + y_{t+h}) - k]^+$ can be evaluated by :

$$e^{-rh} \frac{1}{S} \sum_{s=1}^{S} [\exp(y_{t+1}^s + \ldots + y_{t+h}^s) - k]^+,$$

where $(y_{t+1}^s, \ldots, y_{t+h}^s)$ is a vector sequentially drawn in the various one step ahead risk neutral conditional distribution and S denotes the number of replications.

Note that simulation techniques are also required in the Gaussian framework whenever the conditional variances depend on the past, like in a GARCH framework, since, in this case, the sum $y_{t+1} + \ldots + y_{t+h}$ is no longer conditionally Gaussian.

3 Pricing with splines

The skewed Laplace transform can be considered as the mixture of two truncated exponential distributions. By increasing appropriately the number of element in this mixture it is possible to approximate any distribution. In this section we consider this extension, which is the basis for an arbitrage free nonparametric analysis of historical and risk neutral (conditional) distribution.

3.1 Exponential affine stochastic discount factor

The Laplace family distribution can be directly extended by increasing the number of exponential regimes for the density. Let us consider the p.d.f.

$$p(y) = \exp[a + yb_0 + \sum_{j=1}^{J} b_j (y - c_j)^+], \tag{3.1}$$

where a is fixed by the unit mass restriction, $c_1 < \ldots < c_J$ defines a partition of $I\!R, b_0 > 0, \sum_{j=0}^{J} b_j < 0$. This distribution is denoted by $\mathcal{S}(b_0, b_1, \ldots, b_J, c_1, \ldots, c_J)$. The specification corresponds to a spline approximation of the log-density by splines of degree 1. By increasing the number of nodes J and introducing finer partitions, we can approximate any p.d.f. [see e.g. De Boor (1978)]. When the time index is explicitly introduced the nonparametric approximation applies to the transition density, that is the conditional density of y_t given $\underline{y_{t-1}}$, and the different parameters can be path dependent.

This specification corresponds to a mixture of truncated exponential distributions. Indeed with the convention $c_0 = -\infty, c_{J+1} = +\infty$, the conditional p.d.f. can also be written as :

$$p(y) = \exp[a - B_j^* + B_j y], \text{ if } y \in (c_j, c_{j+1}) \text{ for } j = 0, \ldots, J, \tag{3.2}$$

where :

$$\begin{aligned} B_j^* &= \sum_{l=1}^{j} b_l c_l \text{ (with } B_0^* = 0), \\ B_j &= \sum_{l=0}^{j} b_l, \\ \exp a &= \left[\sum_{j=0}^{J} \frac{\exp(-B_j^*)}{B_j} \{\exp(B_j c_{j+1}) - \exp(B_j c_j)\} \right]^{-1}. \end{aligned} \tag{3.3}$$

Thus the conditional historical distribution is a mixture of truncated exponential distributions :

$$p_j(y) = B_j \frac{\exp(B_j y)}{\exp(B_j c_{j+1}) - \exp(B_j c_j)} 1\!\!1_{(c_j, c_{j+1})}(y), \tag{3.4}$$

with weights :

$$\begin{aligned} \pi_j &= \frac{\exp(-B_j^*)}{B_j} \{\exp(B_j c_{j+1}) - \exp(B_j c_j)\} \\ & [\Sigma_l \frac{\exp(-B_l^*)}{B_l} \{\exp(B_l c_{l+1}) - \exp(B_l c_l)\}]^{-1}. \end{aligned} \tag{3.5}$$

Proposition 2 : If the conditional historical distribution is specified as an exponential-affine spline, with $b_0 - B_J > 1$, and if the stochastic discount factor is exponential-affine :

i) the conditional risk neutral distribution is unique and is the exponential-affine spline distribution $\mathcal{S}(b_0 + \alpha, b_1, \ldots, b_J, c_1, \ldots, c_J)$ with p.d.f. :

$$\begin{aligned} q(y) &= \exp[a^q + y(b_0 + \alpha) + \sum_{j=1}^{J} b_j (y - c_j)^+] \\ &= \sum_{j=0}^{J} \{\exp[a^q - B_j^* + (B_j + \alpha) y] \mathbb{1}_{(c_j, c_{j+1})}(y)\}, \end{aligned}$$

where a^q is fixed by the unit mass restriction and α is solution of :

$$\exp r \sum_{l=0}^{J} \left\{ \frac{\exp(-B_l^*)}{B_l + \alpha} [\exp[(B_l + \alpha) c_{l+1}] - \exp[(B_l + \alpha) c_l]] \right\}$$

$$= \sum_{l=0}^{J} \left\{ \frac{\exp(-B_l^*)}{B_l + \alpha + 1} [\exp[(B_l + \alpha + 1) c_{l+1}] - \exp[(B_l + \alpha + 1) c_l]] \right\}.$$

and such that $-b_0 < \alpha < -\sum_{j=0}^{J} b_j - 1$, or $-b_0 < \alpha < -B_J - 1$.

(Note that when $J = 1$, the notation b_1 corresponds to the notation $b_0 - b_1$ o, section 2)

ii) The price of the call is given by :

$$
\begin{aligned}
C(k) &= C_j(k) \\
&= [\sum_{l=0}^{J} \frac{\exp(-B_l^*)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\}]^{-1} \\
&\quad [\frac{\exp((-B_j^*)}{B_j+\alpha+1}\{\exp[(B_j+\alpha+1)c_{j+1}] - \exp[(B_j+\alpha+1)\log k]\} \\
&- \frac{k\exp(-B_j^*)}{B_j+\alpha}\{\exp[(B_j+\alpha)c_{j+1}] - \exp[(B_j+\alpha)c_j]\} \\
&+ \sum_{l=j+1}^{J} \frac{\exp(-B_l^*)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\} \\
&- k\sum_{l=j+1}^{J} \frac{\exp(-B_l^*)}{B_l+\alpha}\{\exp[(B_l+\alpha)c_{l+1}] - \exp[(B_l+\alpha)c_l]\},
\end{aligned}
$$

for $\exp c_j \leq k \leq \exp c_{j+1}$.

Proof : See Appendix 2.

In statistical theory the approximations by splines are usually introduced to estimate regression functions. The result of Proposition 2 can be used in a similar way for nonparametric pricing. Indeed any conditional p.d.f. can be approximated as closely as wished by an exponential affine spline, when the partition is increased. Proposition 2 says that this approximation is appropriate for derivative pricing, since it provides approximations for both the historical and risk neutral distributions compatible with no arbitrage restriction. These approximations can be used for cross-sectional pricing, that is for pricing at a given date, a given maturity and any strike. The implementation is along the following lines :

i) Fix a partition $c_1, \ldots, c_J$. In the nonparametric approach it is natural to select a path independent partition.

ii) Estimate the (path dependent) parameters $b_j, j = 0, \ldots, J$ from historical data.

iii) Deduce the (path dependent) estimated risk correction α by solving the equation providing α after the replacement of $b_j, j = 0, \ldots, J$ by their estimates.

iv) Estimate the risk neutral (conditional) distribution by replacing $b_j, j = 0, \ldots, J$ and α by their estimates.

3.2 Exponential affine spline stochastic discount factor

By selecting a stochastic discount factor which is exponential affine in the return, we considered in Section 3.1 a two parameter specification of the s.d.f. These parameters have been fixed by the no-arbitrage restrictions leading to a unique admissible s.d.f. It is possible to increase the dimension of the parameter of the s.d.f. in a very simple way, by considering for instance :

$$M(y) = \exp[\beta + \alpha_0 y + \sum_{\ell=1}^{J} \alpha_\ell (y - c_\ell)^+],$$

The arbitrage restrictions imply two constraints on the $J + 2$ parameters $\beta, \alpha_0, \ldots, \alpha_J$. Thus J parameters are free and can be estimated from derivative prices. For instance when $J = 1$ and $c = 0$, we get different risk corrections for positive and negative returns, creating asymmetric smile effects.

Using the notations :

$$\begin{aligned} A_j &= \sum_{l=0}^{j} \alpha_l, \\ A_j^* &= \sum_{l=0}^{j} \alpha_l c_l \text{ (with } A_0^* = 0) \end{aligned}$$

the AOA conditions can be written :

$$\exp(r)EM(y) = \exp(r+\beta+a) \sum_{j=0}^{J} \exp(-B_j^* - A_j^*) \left\{ \frac{\exp[(B_j+A_j)c_{j+1}] - \exp[(B_j+A_j)c_j]}{B_j+A_j} \right\} = 1,$$

$$E[M(y)\exp(y)] = \exp(\beta+a) \sum_{j=0}^{J} \exp(-B_j^* - A_j^*) \left\{ \frac{\exp[(B_j+A_j+1)c_{j+1}] - \exp[(B_j+A_j+1)c_j]}{B_j+A_j+1} \right\} = 1.$$

Therefore the $\alpha_l, l = 0, \ldots, J$ satisfy the equation :

$$\exp(r) \sum_{l=0}^{J} \left\{ \frac{\exp(-B_l^* - A_l^*)}{B_l + A_l} [\exp[(B_l+A_l)c_{l+1}] - \exp[(B_l+A_l)c_l] \right\}$$

$$= \sum_{l=0}^{J} \left\{ \frac{\exp(-B_l^* - A_l^*)}{B_l + A_l + 1} [\exp[(B_l+A_l+1)c_{l+1})] - \exp[(B_l+A_l+1)c_l]] \right\}.$$

Thus Proposition 2 remains valid if B_l^* is replaced by $B_l^* + A_l^*$ and $B_l + \alpha$ is replaced by $B_l + A_l$. Finally note that some α_l may be equal or, equivalently, the partition appearing in the s.d.f. may be a subpartition of the partition appearing in the p.d.f; in this case the number of free parameters α_j is reduced.

4 Markov models

The aim of this section is to link the pricing formulas for different dates and various maturities in a simple way. The dynamics is introduced in the conditional Laplace distribution [resp. exponential-affine splines] by means of the different types of parameters, b_0, b_1 and c [resp. $b_j, j = 0, \ldots, J$ and c], which can be path dependent. In the subsections below, we introduce a simple dynamics, where the effect of the past is assumed to be well summarized by the regime indicator giving the interval $(-\infty, c)$ or (c, ∞), [resp. $(c_j, c_{j+1}), J = 1, \ldots, J$] which contains the lagged value. We first describe the extension in the special case of the conditional Laplace distribution considered in Section 2.2 before considering more general affine-exponential splines.

4.1 Dynamic Laplace Model

Let us consider the framework of Section 2.2 and introduce the dynamics. We assume a path independent location parameter c and define the regime indicator by :

$$z_t = \begin{cases} 1, & \text{if } y_t \geq c, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Moreover we assume that the conditional distribution of the geometric return y_{t+1} given the past $\underline{y_t} = (y_t, y_{t-1}, \ldots)$ is a skewed Laplace distribution, whose parameters depend on the past through the last regime only. Let us denote by $p(y_{t+1}|\underline{y_t})$ the conditional p.d.f of y_{t+1}, and $p(y; c, b_0, b_1)$ a Laplace p.d.f. with parameters c, b_0, b_1, we get :

$$\begin{aligned} p(y_{t+1}|\underline{y_t}) &= p(y_{t+1}; c, b_{00}, b_{10}), \text{ if } z_t = 0, \\ & p(y_{t+1}; c, b_{01}, b_{11}), \text{ if } z_t = 1. \end{aligned}$$

It is easily checked that the dichotomous qualitative process (z_t) defines a Markov chain with transition matrix :

$$\Pi = \begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix} = \begin{pmatrix} \dfrac{b_{10}}{b_{00}+b_{10}} & \dfrac{b_{00}}{b_{00}+b_{10}} \\ \dfrac{b_{11}}{b_{01}+b_{11}} & \dfrac{b_{01}}{b_{01}+b_{11}} \end{pmatrix}, \tag{4.2}$$

where : $\pi_{ij} = P[z_{t+1} = j | z_t = i]$.

Moreover the conditional historical distribution h steps ahead is :

$$p(y_{t+h}|\underline{y_t})$$

$$= p(y_{t+h}; c, b_{00}, b_{10})\pi_{00}^{(h-1)} + p(y_{t+h}; c, b_{01}, b_{11})\pi_{01}^{(h-1)}, \text{ if } z_t = 0,$$

$$p(y_{t+h}|\underline{y_t})$$

$$= p(y_{t+h}; c, b_{00}, b_{10})\pi_{10}^{(h-1)} + p(y_{t+h}; c, b_{01}, b_{11})\pi_{11}^{(h-1)}, \text{ if } z_t = 1,$$

where $\pi_{i,j}^{(h-1)}$ is the element (i,j) of the matrix Π^{h-1}.

The exponential-affine stochastic discount factor for the period $t, t+1$ is : $M_{t,t+1} = \exp(\beta_t + \alpha_t y_{t+1})$, where α_t, β_t depend on the regime prevailing at date t. Thus we get different corrections (α_0, β_0) and (α_1, β_1) according to the regime. The conditional risk neutral distribution h steps ahead is :

$$q(y_{t+h}|\underline{y_t})$$

$$= p(y_{t+h}; c, b_{00} + \alpha_0, b_{10} - \alpha_0)\pi_{00}^{q^{(h-1)}} + p(y_{t+h}; c, b_{01} + \alpha_1, b_{11} - \alpha_1)\pi_{01}^{q^{(h-1)}}, \text{if } z_t = 0,$$

$$= p(y_{t+h}; c, b_{00} + \alpha_0, b_{10} - \alpha_0)\pi_{10}^{q^{(h-1)}} + p(y_{t+h}; c, b_{01} + \alpha_1, b_{11} - \alpha_1)\pi_{11}^{q^{(h-1)}}, \text{if } z_t = 1,$$

where :

$$\Pi^q = \begin{pmatrix} \dfrac{b_{10} - \alpha_0}{b_{00} + b_{10}} & \dfrac{b_{00} - \alpha_0}{b_{00} + b_{10}} \\ \\ \dfrac{b_{11} - \alpha_1}{b_{01} + b_{11}} & \dfrac{b_{01} - \alpha_1}{b_{01} + b_{11}} \end{pmatrix}.$$

We immediately deduce the price at t of a European call with moneyness strike k and time to maturity h. This price is given by :

$$C_t(k, h) = \exp[-r(h-1)][C(k; c, b_{00} + \alpha_0, b_{10} - \alpha_0)\pi_{00}^{h^{(h-1)}} + C(k; c, b_{01} + \alpha_1, b_{11} - \alpha_1)\pi_{01}^{q^{(h-1)}}], \text{ if } z_t = 0,$$

$$C_t(k, h) = \exp[-r(h-1)][C(k, c, b_{00} + \alpha_0, b_{10} - \alpha_0)\pi_{10}^{h^{(h-1)}} + C(k; c, b_{01} + \alpha_1, b_{11} - \alpha_1)\pi_{11}^{q^{(h-1)}}], \text{ if } z_t = 1.$$

where $C(k; c, b_0, b_1)$ is the call price at maturity one associated with the Laplace distribution $p(y; c, b_0, b_1)$.

As an illustration, let us consider the special case $c = r$ and denote : $\bar{b}_0 = \frac{b_{00}+b_{10}}{2}, \bar{b}_1 = \frac{b_{01}+b_{11}}{2}$. We get :

$$\Pi^q = \begin{pmatrix} \frac{1}{2}+\frac{1}{4\bar{b}_0} & \frac{1}{2}-\frac{1}{4\bar{b}_0} \\ \frac{1}{2}+\frac{1}{4\bar{b}_1} & \frac{1}{2}-\frac{1}{4\bar{b}_1} \end{pmatrix}$$

and, for $\log, k \geq r, y_t \geq r$ for instance :

$$\begin{aligned} C_t(k,h) &= \exp[-r(h-1))]\frac{1}{2\bar{b}_0}\exp[-\bar{b}_0 - 1/2)(\log k - r)]\pi_{00}^{q^{(h-1)}} \\ &+ \frac{1}{2\bar{b}_1}\exp[-(\bar{b}_1 - 1/2)(\log k - r)]\pi_{01}^{q^{(h-1)}}]. \end{aligned}$$

This example shows that this special dynamic model may capture different tail magnitudes in the different regimes, which is the analogue of stochastic volatility models.

In the more general case where the payoff at $t+h$ is given by $[\exp(y_{t+1} + \ldots + y_{t+h}) - k]^+$, a simulation method, analogous to that described in Section 2.7, is required. Note however that only two values of α are necessary and, therefore, the computations are simple.

4.2 Dynamic exponential-affine splines

The approach extends the dynamic Laplace model by introducing a larger number of regimes. The regimes are defined by means of a partition $c_j, j = 0, \ldots, J$, which is assumed path independent. Then the multiregime indicator at date t is :

$$z_t = j, \text{ if } y_t \in [c_j, c_{j+1}], j = 0, \ldots, J. \tag{4.3}$$

If $c = (c_1, \ldots, c_J), b = (b_0, \ldots, b_J)'$ denote the two types of parameters, we assume that the conditional distribution of the geometric return is an exponential spline affine distribution, which depends on the past by means of the most recent regime :

$$p(y_{t+1}|\underline{y_t}) = p(y_{t+1}|z_t). \tag{4.4}$$

These conditional distributions differ by the value of parameter b which is not the same in each regime. b^j denotes the value of b, when $Z = j$:

$$p(y_{t+1}|z_t) = p(y_{t+1}; c, b^j), \text{if } z_t = j. \tag{4.5}$$

As in the previous subsection, the polytomous qualitative process (z_t) defines a Markov chain, with a transition matrix Π, with elements $\pi_{ij} = P[z_{t+1} = j|z_t = i]$ functions of the basic parameters $c, b^j, j = 0, \dots, J$.

Then the conditional historical distribution h steps ahead is :

$$p(y_{t+h}|y_t) = \sum_{j=0}^{J} p(y_{t+h}; c, b^j)\pi_{i,j}^{(h-1)}, \text{ if } z_t = i, \tag{4.6}$$

When the s.d.f. is exponential affine the conditional risk-neutral density is :

$$q(y_{t+h}|y_t) = \sum_{j=0}^{J} p(y_{t+h}; c, \tilde{b}^j)\tilde{\pi}_{ij}^{(h-1)} \text{ if } z_t = i, \tag{4.7}$$

where : $\tilde{b}_0^j = b_0 + \alpha^j, \tilde{b}_l^j = b_l^j$, if $l = 1, \dots, J$, and $\tilde{\Pi}$ is deduced from Π after replacement of b^j by $\tilde{b}^j$.

A further extension is obtained straightforwardly if the sdf is exponential spline.

This dynamic approach is the basis for dynamic semiparametric pricing under the assumption of a Markov process for geometric returns. Indeed when the partition $c_j, j = 0, \dots, J$ increases, the exponential-affine spline approximation with multiregime will tend to the conditonal p.d.f. $p(y_{t+1}|y_t)$ itself. It provides numerical approximations for call-prices in the Markov famework, which mixes a standard multinomial tree with a spline smoothing. The numerical advantage of the additional smoothing is to diminish the erratic evolutions of the approximated derivative prices, which are usually observed when the number of nodes in the tree increases.

5 Concluding remarks

The success of the Black-Scholes approach is due to a simple analytical formula for European call prices. However this formula is based on restrictive assumptions and may induce various mispricing. For instance the implied volatility has to be constant with the moneyness strike, whereas smile effects are often observed. The implied volatility has to be independent of the time to maturity, whereas an increasing dependence may be observed. The aim of this paper was to introduce alternative derivative pricing methods. We first derived a pricing formula for the skewed conditional Laplace distribution, before extending the analysis to exponential-affine splines. Finally, we introduced underlying Markov regimes in order to link the derivative prices for different dates and residual maturities.

Appendix 1 :
Pricing with Laplace distribution

i) The truncated Laplace transform

Let us assume $\gamma > c$ and $u < b_1$; we get :

$$\begin{aligned}
\psi(u,\gamma) &= E[\exp(uy)\mathbb{1}_{y>\gamma}] \\
&= \exp(uc)E\{\exp[u(y-c)]\mathbb{1}_{y>\gamma}\} \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_\gamma^\infty \exp[-(b_1-u)(y-c)]dy \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-u)(\gamma-c)]}{b_1-u}.
\end{aligned}$$

If $\gamma < c$, we get :

$$\begin{aligned}
\psi(u,\gamma) &= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_c^\infty \exp[-(b_1-u)(y-c)]dy \\
&+ \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_\gamma^c \exp[(b_0+u)(y-c)]dy \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_1-u} \\
&+ \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_0+u}\{1-\exp[(b_0+u)(\gamma-c)]\}.
\end{aligned}$$

Note that the truncated Laplace transform is defined for $u \in (-b_0, b_1)$.

ii) The arbitrage free conditions

If $-b_0 < u < b_1$ the Laplace transform is given by :

$$\begin{aligned}\psi(u,-\infty) &= exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_1-u}+\exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_0+u}\\ &= \exp(uc)\frac{b_0 b_1}{(b_0+u)(b_1-u)}.\end{aligned}$$

Thus the arbitrage free conditions become :

$$\begin{cases}\exp(\beta+r)\psi(\alpha,-\infty)=1,\\ \exp(\beta)\psi(\alpha+1,-\infty)=1,\end{cases}$$

$$\iff \begin{cases}\exp(\beta+r+\alpha c)\dfrac{b_0 b_1}{(b_0+\alpha)(b_1-\alpha)}=1,\\ \\ \exp[\beta+(\alpha+1)c]\dfrac{b_0 b_1}{(b_0+\alpha+1)(b_1-\alpha-1)}=1.\end{cases}$$

In particular the risk correcting factor is the solution of the second degree equation, satisfying $-b_0 < \alpha < b_1 - 1$:

$$\exp(c-r)(b_0+\alpha)(b_1-\alpha)=(b_0+\alpha+1)(b_1-\alpha-1).$$

It is easily checked that this equation has a unique solution in the interval $(-b_0, b_1-1)$, where the Laplace transforms $\psi(\alpha,-\infty)$ and $\psi(\alpha+1,-\infty)$ are both defined.

iii) **The price of the call**.

For $logk > c$, we get :

$$
\begin{aligned}
C(k) &= \exp\beta[\psi(\alpha+1, logk) - k\psi(\alpha, logk)] \\
&= \exp\beta\left\{\exp[(\alpha+1)c]\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-\alpha-1)(logk-c)]}{b_1-\alpha-1}\right. \\
&\left.-k\exp(\alpha c)\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-\alpha)(logk-c)]}{b_1-\alpha}\right\} \\
&= \exp\beta\exp[(\alpha+1)c]\exp[-(b_1-\alpha-1)(logk-c)]\frac{b_0 b_1}{b_0+b_1}\frac{1}{(b_1-\alpha)(b_1-\alpha-1)} \\
&= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}\exp[-(b_1-\alpha-1)(\log k - c)],
\end{aligned}
$$

by the arbitrage free condition.

The computation is similar for $logk < c$ and provides :

$$
C(k) = 1 - k\exp(-r) + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_o+\alpha}\exp[(b_0+\alpha+1)(logk-c)].
$$

iv) **Continuity of the pricing function**.

The value of the call is a continuous function of k. Indeed we get :

$$
\begin{aligned}
C_1(\exp c) &= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}, \\
C_2(\exp c) &= 1 - \exp(c-r) + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_0+\alpha} \\
&= 1 - \frac{(b_0+\alpha+1)(b_1-\alpha-1)}{(b_0+\alpha)(b_1-\alpha)} + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_0+\alpha} \\
&= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}
\end{aligned}
$$

The continuity property is still satisfied for the derivative of the value of the call with respect to k. Indeed the first order derivative of the pricing function is :

$$
\begin{aligned}
\frac{dC_1(k)}{dk} &= -\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)}\frac{1}{k}\exp[-(b_1-\alpha)(\log k - c)],\\
\frac{dC_2(k)}{dk} &= -\exp(-r)+\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)}\frac{1}{k}\exp[(b_0+\alpha)(\log k - c)].
\end{aligned}
$$

At the limiting point $k = \exp c$, we get :

$$
\begin{aligned}
\frac{dC_1(\exp c)}{dk} &= -\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)}\exp(-c),\\
\frac{dC_2(\exp c)}{dk} &= -\exp(-r)+\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)}\exp(-c).
\end{aligned}
$$

and :

$$
\begin{aligned}
\frac{dC_1(\exp c)}{dk} &= \frac{dC_2(\exp c)}{dk}\\
\Longleftrightarrow \quad \exp(c-r) &= \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)}+\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)}\\
&= \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_1-\alpha)(b_0+\alpha)},
\end{aligned}
$$

which is exactly the equation defining α

v) **Risk neutral distribution**

The p.d.f. of the risk neutral distribution is still a Laplace distribution. Indeed this p.d.f. is given by :

$$
\begin{aligned}
q(y) &= \exp(r)\frac{b_0b_1}{b_0+b_1}\exp(\beta+\alpha c)\exp[(b_0+\alpha)(y-c)],\ \text{if } y\le c,\\
& \quad \exp(r)\frac{b_0b_1}{b_0+b_1}\exp(\beta+\alpha c)\exp[-(b_1-\alpha)(y-c)],\ \text{if } y\ge c.
\end{aligned}
$$

By using the arbitrage free condition, we get :

$$q(y) = \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[(b_0+\alpha)(y-c)], \text{ if } y \leq c,$$

$$\frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[-(b_1-\alpha)(y-c)], \text{ if } y > c.$$

Finally it is easily checked that the risk neutral distribution depends on b_0, b_1 through b_0+b_1 and c only. This property is satisfied if both $\alpha_0 = b_0+\alpha$ and $\alpha_1 = b_1-\alpha$ depend on b_0+b_1 and c only. It is easily seen that α_0 and α_1 are solutions of the equations :

$$\exp(c-r)\alpha_0(b_0+b_1-\alpha_0) = (\alpha_0+1)(b_0+b_1-\alpha_0-1),$$

$$\exp(c-r)\alpha_1(b_0+b_1-\alpha_1) = (\alpha_1-1)(b_0+b_1-\alpha_1+1),$$

and the result follows.

Appendix 2 :
Pricing with exponential-affine splines

i) **The historical distribution**

The distribution is given by :

$$p(y) = \exp[a + b_0 y + \sum_{j=1}^{J} b_j (y - c_j)^+],$$

where the constant a is fixed by the unit mass restriction. This p.d.f. can also be written as :

$$p(y) = \exp(a - B_j^* + B_j y), \text{ if } y \in (c_j, c_{j+1}),$$

where :

$$\begin{aligned} B_j^* &= \sum_{l=1}^{j} b_l c_l \text{ (with } B_0^* = 0), \\ B_j &= \sum_{l=0}^{j} b_l. \end{aligned}$$

Then the integral of the p.d.f. is :

$$\begin{aligned} \int_{-\infty}^{+\infty} p(y)dy &= \sum_{j=0}^{J} \int_{c_j}^{c_{j+1}} \exp(a - B_j^* + B_j y) dy \\ &= \exp(a) \sum_{j=0}^{J} (\exp(-B_j^*) \frac{\exp B_j y}{B_j}]_{c_j}^{c_{j+1}}) \\ &= \exp(a) \sum_{j=0}^{J} \frac{\exp -B_j^*}{B_j} [\exp B_j c_{j+1} - \exp B_j c_j]. \end{aligned}$$

We deduce the expression of the p.d.f :

$$
\begin{aligned}
p(y) &= \sum_{j=0}^{J} \{\exp[-B_j^* + B_j y]\mathbf{1}_{(c_j,c_{j+1})}(y)\} \left\{ \sum_{j=0}^{J} \frac{\exp -B_j^*}{B_j} (\exp B_j c_{j+1} - \exp B_j c_j) \right\}^{-1} \\
&= \left\{ \sum_{j=0}^{J} \frac{\exp(-B_j^*)}{B_j} (\exp B_j c_{j+1} - \exp B_j c_j) \frac{B_j \exp B_j y}{\exp B_j c_{j+1} - \exp B_j c_j} \mathbf{1}_{(c_j,c_{j+1})}(y) \right\} \\
& \left\{ \sum_{j=0}^{J} \frac{\exp -(B_j^*)}{B_j} (\exp B_j c_{j+1} - \exp B_j c_j) \right\}^{-1} .
\end{aligned}
$$

ii) **The truncated Laplace transform :**

Let us assume $\gamma \in (c_j, c_{j+1})$; we get :

$$
\begin{aligned}
\psi(u,\gamma) &= E[\exp(uy)\mathbb{1}_{y>\gamma}] \\
&= \int_{\gamma}^{c_{j+1}} \exp(a - B_j^* + B_j y + uy) dy \\
&+ \sum_{l=j+1}^{J} \int_{c_l}^{c_{l+1}} \exp(a - B_l^* + B_l y + uy) dy \\
&= \frac{\exp(a - B_j^*)}{B_j + u} \{\exp[(B_j + u)c_{j+1}] - \exp[(B_j + u)\gamma]\} \\
&+ \sum_{l=j+1}^{J} \frac{\exp(a - B_l^*)}{B_l + u} \{\exp[(B_l + u)c_{l+1}] - \exp[(B_l + u)c_l]\}.
\end{aligned}
$$

iii) **The arbitrage free conditions**

The (untruncated) Laplace transform is given by :

$$
\psi(u,-\infty) = \sum_{l=0}^{J} \frac{\exp(a - B_l^*)}{B_l + u} \{\exp[(B_l + u)c_{l+1}] - \exp[(B_l + u)c_l]\},
$$

and the correcting factor α is solution of the equation :

$$
\exp(r)\psi(\alpha,-\infty) = \psi(\alpha+1,-\infty)
$$

or equivalently :

$$\exp(r)\sum_{l=0}^{J}\left\{\frac{\exp(-B_l^*)}{B_l+\alpha}(\exp[(B_l+\alpha)c_{l+1}]-\exp[(B_l+\alpha)c_l])\right\}$$

$$=\sum_{l=0}^{J}\left\{\frac{\exp(-B_l^*)}{B_l+\alpha+1}(\exp[(B_l+\alpha+1)c_{l+1}]-\exp[(B_l+\alpha+1)c_l])\right\}.$$

iv) **The risk-neutral distribution**

By multiplying the historical p.d.f by $\exp(\alpha y+\beta+r)$, we get a risk-neutral density with an exponential-affine spline representation. The limiting points of the partition $c_j, j=1,\dots,J$ are unchanged, whereas the parameters of the truncated exponential distributions become : $B_j^q=B_j+\alpha$. Since : $B_j^q=\sum_{l=0}^{j} b_l^q$, we immediately deduce that :

$$b_0^q=b_0+\alpha, b_j^q=b_j, j=1,\dots,J,$$

$$B_j^{*q}=B_j^*, j=0,\dots,J.$$

Thus the risk-neutral p.d.f. is :

$$\begin{aligned} q(y) &= \exp[a^q+y(b_0+\alpha)+\sum_{j=1}^{J} b_j(y-c_j)^+] \\ &= \sum_{j=0}^{J}[\exp[a^q-B_j^*+(B_j+\alpha)y]\mathbb{1}_{(c_j,c_{j+1})}(y)]. \end{aligned}$$

v) **The price of a call**

Let us assume $\log k\in(c_j,c_{j+1})$; the price of a call is given by :

$$
\begin{aligned}
C(k) &= \frac{1}{\psi(\alpha+1,-\infty)}[\psi(\alpha+1,\log k) - k\psi(\alpha,\log k)] \\
&= [\sum_{l=0}^{J}\frac{\exp(-B_l^*)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\}]^{-1} \\
&\quad [\frac{\exp(-B_j^*)}{B_j+\alpha+1}\{\exp[(B_j+\alpha+1)c_{j+1}] - \exp[(B_j+\alpha+1)\log k]\} \\
&- k\frac{\exp(-B_j^*)}{B_j+\alpha}\{\exp[(B_j+\alpha)c_{j+1}] - \exp[(B_j+\alpha)\log k]\} \\
&+ \sum_{l=j+1}^{J}\frac{\exp(-B_l^*)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\} \\
&- k\sum_{l=j+1}^{J}\frac{\exp(-B_l^*)}{B_l+\alpha}\{\exp[(B_l+\alpha)c_{l+1}] - \exp[(B_l+\alpha)c_l]\}]
\end{aligned}
$$

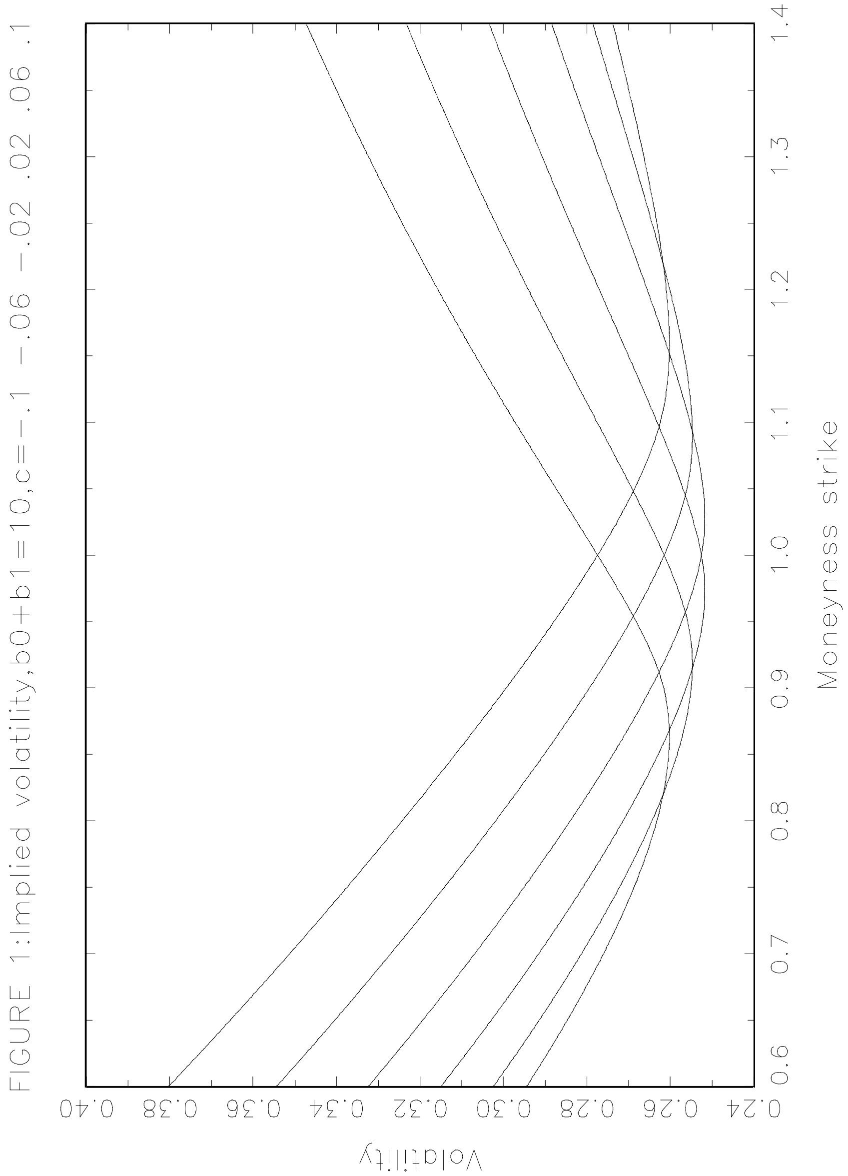


FIGURE 1:Implied volatility,b0+b1=10,c=-.1 -.06 -.02 .02 .06 .1
Volatility
0.24 0.26 0.28 0.30 0.32 0.34 0.36 0.38 0.40
Moneyness strike
0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4

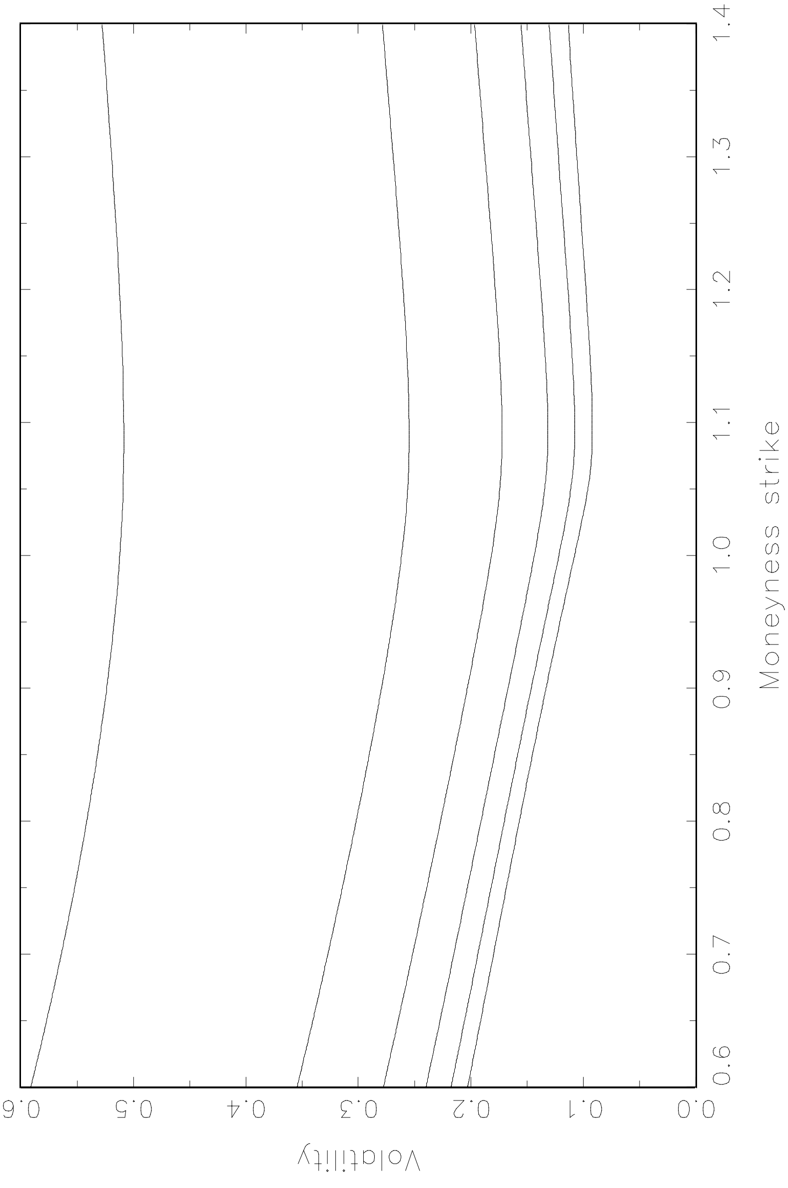
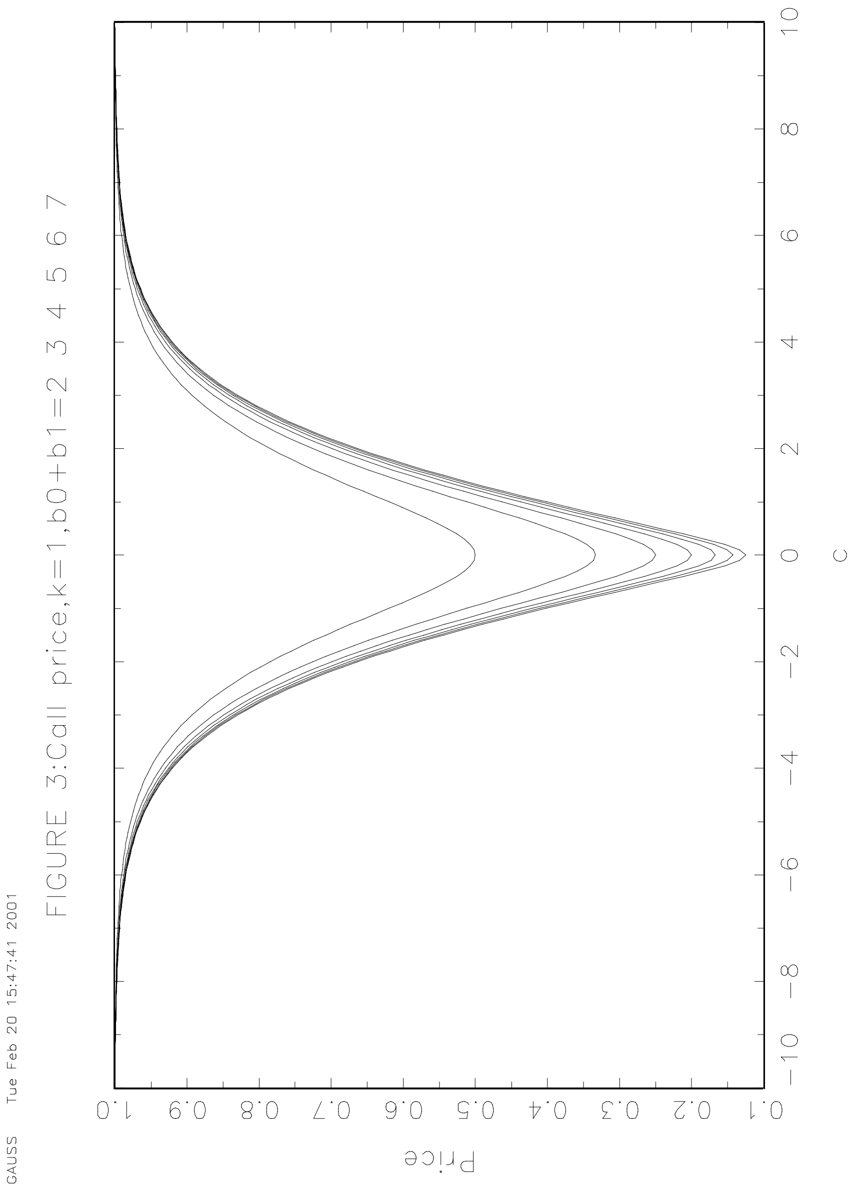
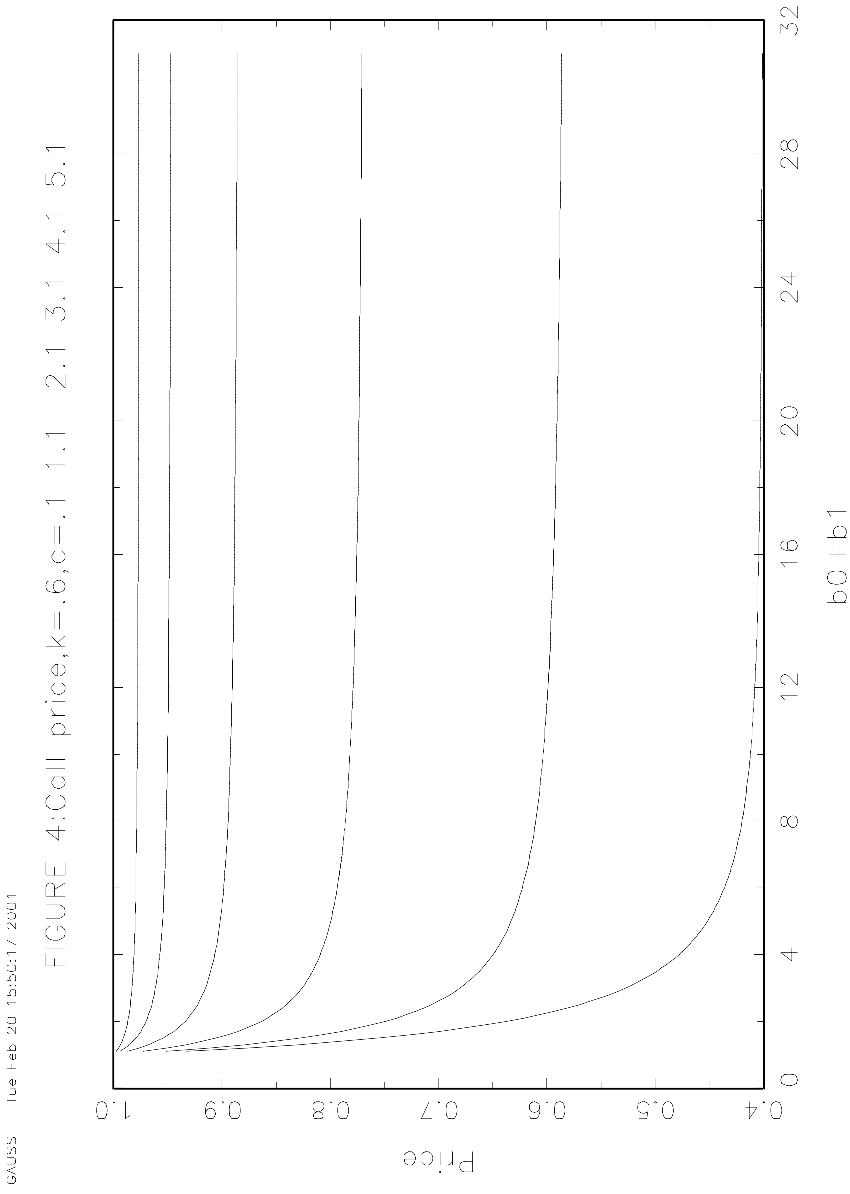


FIGURE 2:Implied volatility,c=.06 b0+b1=5 10 15 20 25 30


GAUSS Tue Feb 20 15:47:41 2001
FIGURE 3:Call price,k=1,b0+b1=2 3 4 5 6 7
Price
c


GAUSS Tue Feb 20 15:50:17 2001
FIGURE 4:Call price,k=.6,c=.1 1.1 2.1 3.1 4.1 5.1
Price
1.0
0.9
0.8
0.7
0.6
0.5
0.4
b0+b1
0
4
8
12
16
20
24
28
32

R E F E R E N C E S

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