Econometric Asset Pricing Modelling

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PURPOSE OF THE PAPER :

- to **build** a bridge between **Econometric Modelling** and **Asset Pricing**:
  $\hookrightarrow$ we propose a **general framework** able to produce models **dealing** at the same time **with usual econometric problems** (historical analysis, prediction ...) and **pricing of derivative assets**.

This general econometric approach is

BASED ON THREE MAIN INGREDIENTS:

i) the discrete-time **historical dynamics of the factor** representing the economy

ii) the **Stochastic Discount Factor (SDF)**

iii) the discrete-time **risk-neutral (R.N.) dynamics.**

- The **conditional Laplace transform** is the central mathematical tool used for the specification of the historical and R.N. dynamics

- The **SDF** is assumed to be **exponential-affine** [see Gourieroux and Monfort (2007)],

$\hookrightarrow$ its specification is **equivalent to the specification** of a

factor loading vector and of the short rate,

if the latter is neither exogenous nor a known function of the factor.

The **three elements can not be defined independently**
and **we distinguish three modelling strategies**

a) the **Direct Modelling strategy**:
we specify **the historical dynamics and the SDF**
($\hookrightarrow$ the factor loading vector and, possibly, the short rate)
$\longrightarrow$ the R.N. dynamics is obtained as a by-product.

b) the **Risk-Neutral Constrained Direct Modelling strategy**:
we specify **the historical dynamics and we constrain the R.N. dynamics**
to belong to a given family $\longrightarrow$ typically the family of Car processes
[Darolles, Gourieroux and Jasiak (2006)];
$\longrightarrow$ the factor loading vector characterizing the SDF is obtained as a by-product.

c) the **Back Modelling strategy**:
we specify **the R.N. dynamics (and, possibly, the short rate)**
as well as **the factor loading (SDF)**
$\longrightarrow$ the historical dynamics is obtained as by-product.

$\Rightarrow$ we get **three kinds of Econometric Asset Pricing Models (EAPM)**.

In the Back Modelling approach, and in the Risk-Neutral Constrained Direct Modelling approach

- The **factor describing the R.N. dynamics** is assumed to be, in general, a **Car process** in order to facilitate pricing implementation and econometric analysis.

- However, **we are able to derive asset pricing models where**:

- **even if the historical and R.N. dynamics** of the factor **IS NOT Car**,

- **the introduction of a new variable** (function of the initial factor) **defines a new (extended) factor which turns out to be Car** at least under the R.N. probability
  $\Rightarrow$ **explicit or quasi explicit pricing formulas can be obtained**.

- This extended factor process will be called (**historical or risk-neutral**) **Extended Car process**.

- For all the strategies, **we discuss basic problems of econometric modelling**
  - **parameterization**,
  - **identification** and
  - **internal consistency** with the AAO assumption
- We also propose **interpretations of the factor loading vector** in terms of **market price of risk**.
- The **general modelling strategies are applied to two important cases**:
  1. **Econometric Security Market Modelling**
     [Switching Regime Models, SV Models, (Regime-Switching) GARCH Models]
  2. **Term Structure of Interest Rates Modelling**
     [VAR Models, Switching VAR Models, Wishart Models]

  $\hookrightarrow$ **Example of joint modelling** of geometric returns, dividends and short rates

- **We stress the usefulness of the**
  Risk-Neutral Constrained Modelling approach
  and of the **Back Modelling** approach,
  - both allowing to **conciliate a flexible historical dynamics**
  - and a **Car risk-neutral dynamics**

  ⇒ **explicit or quasi explicit** pricing formulas
  for various derivative products.

- Moreover, **we highlight the possibility to specify asset pricing models**
  - able to **accommodate non-affine historical and R.N. factor dynamics**
  - with **tractable pricing formulas**.

Outline :

1. **Historical and Risk-Neutral Dynamics**
2. **Risk Premia and Market Price of Risk**
3. **Econometric Asset Pricing Model (EAPM)**
4. **Applications to Econometric Security Market Modelling**
5. **Applications to Econometric Term Structure Modelling**
6. **An Example of Econometric Security Market Model with Stochastic Dividends and Short Rates**

HISTORICAL AND RISK-NEUTRAL DYNAMICS :

- **Information** : we have an economy over the period $\{0, \ldots, T\}$
  the new information in the economy at date t : w_t
  the overall information at date t : $\underline{w_t} = (w_t, w_{t-1}, ..., w_0)$
  $w_t = K$-dim **factor (or state vector)**, and may be
  observable, partially observable or unobservable to the econometrician

- **Historical Dynamics**: defined by the joint distribution of $\underline{w_T}$ denoted $\mathbb{P}$
  or the conditional p.d.f. $f_t(w_{t+1}|\underline{w_t})$,
  the conditional Laplace Transform $\varphi_t(u) = \varphi_t(u|\underline{w_t}) = E[\exp(u'w_{t+1})|\underline{w_t}]$
  the conditional Log-Laplace transform $\psi_t(u) = \psi_t(u|\underline{w_t}) = \text{Log}[\varphi_t(u|\underline{w_t})]$

- **The Stochastic Discount Factor (SDF)**: Hansen and Richard (1987)
  Existence and uniqueness of a price + Linearity and continuity + AAO
  $\Rightarrow$ there exists, and unique, a positive SDF $M_{t,t+1}(\underline{w_{t+1}}) \in L_{2,t+1}$, such that:
  the price at date t of the payoff $g(\underline{w_s}) \in L_{2,s}$ delivered at $s > t$ is given by :

$$p_t\left[g(\underline{w_s})\right] = E_t\left[M_{t,t+1} ... M_{s-1,s}\, g(\underline{w_s})\right] .$$

- **Exponential-affine SDF**: $M_{t,t+1}(\underline{w}_{t+1})$ has an exponential-affine form :

$$M_{t,t+1} = \exp\left[\alpha_t(\underline{w}_t)' w_{t+1} + \beta_t(\underline{w}_t)\right] ,$$

α_t is the "factor loading" or "sensitivity" vector.

→ **justification of this exponential-affine specification**:

a) **this form naturally appears in equilibrium models** like CCAPM, or Consumption-based asset pricing models

b) in general **continuous time security models** the **discretized version of the SDF is exponential-affine**

c) exponential-affine specification is particularly well adapted to the **Laplace Transform** which is a central tool in discrete-time asset pricing theory.

- Since $\exp(-r_{t+1}) = E_t(M_{t,t+1}) = \exp\left[\psi_t(\alpha_t \mid \underline{w}_t) + \beta_t\right]$, the SDF can also be written :

$$M_{t,t+1} = \exp\left[-r_{t+1}(\underline{w}_t) + \alpha_t'(\underline{w}_t) w_{t+1} - \psi_t(\alpha_t | \underline{w}_t)\right] .$$

- **Risk-Neutral Dynamics**: it is **another joint distribution of $\underline{w}_T$ denoted $\mathbb{Q}$**

- **it is defined by the conditional p.d.f.** $f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = f_t(w_{t+1}|\underline{w}_t) d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t)$ with $d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = \exp(r_{t+1}) M_{t,t+1}(\underline{w}_{t+1}) = \exp\left[\alpha_t' w_{t+1} - \psi_t(\alpha_t)\right]$

- **the joint p.d.f. of $\mathbb{Q}$ whit respect to $\mathbb{P}$ is** :
  $\xi_T = \frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{t=0}^{T-1} d_t^{\mathbb{Q}}(w_{t+1}/\underline{w}_t) = \prod_{t=0}^{T-1} \exp(r_{t+1}) M_{t,t+1}$

- and **the basic pricing formula can be rewritten**

$$p_t\left[g(\underline{w}_s)\right] = E_t^{\mathbb{Q}}\left[\exp(-r_{t+1} - \ldots - r_s) g(\underline{w}_s)\right] \; ;$$

- the **conditional R.N. Laplace**, and **Log-Laplace**, transforms of w_{t+1}, given $\underline{w}_t$, are

$$\varphi_t^{\mathbb{Q}}(u|\underline{w}_t) = \frac{\varphi_t(u+\alpha_t)}{\varphi_t(\alpha_t)} \, , \quad \psi_t^{\mathbb{Q}}(u) = \psi_t(u+\alpha_t) - \psi_t(\alpha_t)$$

- **conversely, we get:**

$d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \exp\left[-\alpha_t' w_{t+1} + \psi_t(\alpha_t)\right]$ and $\psi_t(u) = \psi_t^{\mathbb{Q}}(u-\alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t)$

$d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \exp\left[-\alpha_t' w_{t+1} - \psi_t^{\mathbb{Q}}(-\alpha_t)\right]$ and $d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = \exp\left[\alpha_t' w_{t+1} + \psi_t^{\mathbb{Q}}(-\alpha_t)\right]$

RISK PREMIA AND MARKET PRICE OF RISK :

- **Geometric and Arithmetic Risk Premia** : p_t is the asset price at t geometric and arithmetic return between t and $t+1$:

$$\begin{aligned} \rho_{G,t+1} &= \text{Log}\left(\frac{p_{t+1}}{p_t}\right) \\ \rho_{A,t+1} &= \frac{p_{t+1}}{p_t} - 1 = \exp(\rho_{G,t+1}) - 1 \end{aligned}$$

in the case of the risk-free asset :

$$\begin{aligned} \rho^f_{G,t+1} &= r_{t+1}\,, \\ \rho^f_{A,t+1} &= \exp(r_{t+1}) - 1 = r_{A,t+1}\,. \end{aligned}$$

So, we can define two risk premia of the given asset :

$$\begin{aligned} \pi_{Gt} &= E_t(\rho_{G,t+1}) - r_{t+1}\,, \\ \pi_{At} &= E_t(\rho_{A,t+1}) - r_{A,t+1} = E_t\left[\exp(\rho_{G,t+1})\right] - \exp(r_{t+1})\,. \end{aligned}$$

- **Let us now consider two important particular cases**
  ↪ **more explicit forms of these risk premia**
  ↪ **intuitive interpretations of the factor loading vector α_t.**

1. **The K-factor w_{t+1} is a vector of geometric returns** :

for each $i \in \{1, ..., K\}$, $w_{t+1,i} = Log(p_{t+1,i}/p_{t,i})$ and the risk premia can be written :

$$\begin{aligned} \pi_{Gt,i} &= e_i' \psi_t^{(1)}(0) - \psi_t(\alpha_t + e_i) + \psi_t(\alpha_t) \\ \pi_{At,i} &= \varphi_t(e_i) - \frac{\varphi_t(\alpha_t + e_i)}{\varphi_t(\alpha_t)}. \end{aligned}$$

- Moreover, considering first order expansions around $\alpha_t = 0$ and neglecting conditional cumulants of order strictly larger than 2 we get:

$$\begin{aligned} \pi_{Gt} &\simeq -\frac{1}{2} vdiag(\Sigma_t) - \Sigma_t \alpha_t \\ \pi_{At} &\simeq -diag\left[\varphi_t(e_i)\right] \Sigma_t \alpha_i \,. \end{aligned}$$

where $[\varphi_t(e_i)] := (\varphi_t(e_1), \ldots, \varphi_t(e_K))'$,

$\Sigma_t =$ conditional variance-covariance matrix of $w_{t+1} \,|\, \underline{w_t}$.

$\Rightarrow$ α_t can be viewed as the **opposite of a Market Price of Risk vector**.

2. **The K-factor w_{t+1} is a vector of yields** :

for each $i \in \{1, ..., K\}$, $w_{t+1,i} = h_i r(t+1, h_i)$ and the risk premia can be written :

$$
\begin{aligned}
\pi_{Gt,i} &= -e_i \psi_t^{(1)}(0) - \psi_t(\alpha_t - e_i) + \psi_t(\alpha_t) \\
\pi_{At,i} &= \exp\left[(h_i+1) r(t, h_i+1)\right] \left[\varphi_t(-e_i) - \frac{\varphi_t(\alpha_t - e_i)}{\varphi_t(\alpha_i)}\right].
\end{aligned}
$$

- Expanding around $\alpha_t = 0$, and neglecting conditional cumulants of order strictly larger than 2, we get :

$$
\begin{aligned}
\pi_{Gt} &\simeq -\frac{1}{2} vdiag\,(\Sigma_t) + \Sigma_t \alpha_t \\
\pi_{At} &\simeq diag\left[\varphi_t(-e_i) \exp\left((h_i+1) r(t, h_i+1)\right)\right] \Sigma_t \alpha_t .
\end{aligned}
$$

$\Rightarrow$ α_t can be viewed as a **Market Price of Risk vector**.

- In both cases the formula for π_{Gt} is exact when the conditionally Gaussian setting is considered.

ECONOMETRIC ASSET PRICING MODEL (EAPM) :

The true values of $\psi_t, M_{t,t+1}$ or $\psi_t^{\mathbb{Q}}$, are unknown by the econometrician

$\longrightarrow$ **they have to be specified and parameterized**

$\hookrightarrow$ **we have to specify an Econometric Asset Pricing Model (EAPM)**.

We are going to present three ways of specifying an EAPM:

a. **the Direct Modelling**

b. **the R.N. Constrained Direct Modelling**

c. **the Back Modelling**.

In all approaches, **we first need to make more precise**

the **status of the short rate** r_{t+1}.

- **The status of the short rate**:

  the short rate r_{t+1} is a function of $\underline{w}_t$

  this function may be known or unknown by the econometrician

  A) **It is known in two main cases :**

  i) r_{t+1} is exogenous $\Rightarrow r_{t+1}(\underline{w}_t)$ does not depend on $\underline{w}_t$,
  $\hookrightarrow r_{t+1}(.)$ is a known constant function of $\underline{w}_t$;

  ii) r_{t+1} is an endogenous factor $\Rightarrow r_{t+1}$ is a component of w_t.

  B) **If the function $r_{t+1}(\underline{w}_t)$ is unknown**
  $\hookrightarrow$ it has to be specified parametrically:

$$\left\{ r_{t+1}(\underline{w}_t, \tilde{\theta}), \tilde{\theta} \in \tilde{\Theta} \right\} ,$$

  where $r_{t+1}(.,.)$ is a known function.

- **The Direct Modelling**: **it is based on the following steps**

1^{st} - **we specify the historical dynamics**, i.e. we choose a parametric family for the conditional Log-Laplace transform $\psi_t(u \,|\, \underline{w_t})$:

$$\{\psi(u \,|\, \underline{w_t}; \theta_1), \theta_1 \in \Theta_1\} \ .$$

2^{nd} - **we specify the SDF :**

$$M_{t,t+1} \ = \ \exp\left[-r_{t+1}(\underline{w_t}) + \alpha_t'(\underline{w_t})w_{t+1} - \psi_t(\alpha_t|\underline{w_t})\right] \ ,$$

once r_{t+1} has been specified, the remaining function to be specified is $\alpha_t(\underline{w_t})$: **we assume that $\alpha_t(\underline{w_t})$** belongs to a parametric family :

$$\{\alpha_t(\underline{w_t}; \theta_2), \theta_2 \in \Theta_2\} \ .$$

$\Rightarrow$ the SDF takes the following parametric specification:

$$M_{t,t+1}(\underline{w_{t+1}}, \theta) \ = \ \exp\left\{-r_{t+1}(\underline{w_t}, \tilde{\theta}) + \alpha_t'(\underline{w_t}, \theta_2)w_{t+1} - \psi_t\left[\alpha_t(\underline{w_t}, \theta_2)|\underline{w_t}; \theta_1\right]\right\} \ ,$$

where $\theta = (\tilde{\theta}', \theta_1', \theta_2')' \in \tilde{\Theta} \times \Theta_1 \times \Theta_2 = \Theta$;
$\tilde{\Theta}$ may be reduced to one point.

3^{rd} - **We have to satisfy internal consistency conditions (AAO).**

For any payoff $g(\underline{w}_s)$ delivered at $s > t$, with price $p(\underline{w}_t)$ at t which is a known function of $\underline{w}_t$, we must have :

$$p(\underline{w}_t) = E\left\{M_{t,t+1}(\theta)...M_{s-1,s}(\theta)\, g(\underline{w}_s) \,|\, \underline{w}_t, \theta_1\right\} \quad \forall\, \underline{w}_t, \theta\,.$$

⇒ these AAO pricing conditions may imply strong constraints on the parameter θ,
↪ for instance when components of w_t are returns
of some assets or interest rates with various maturities
[see following examples]

- **the previous steps imply** the specification of the R.N. dynamics :

$$\psi^{\mathbb{Q}}(u|\underline{w}_t, \theta_1, \theta_2) = \psi_t\left[u + \alpha_t(\underline{w}_t, \theta_2)|\underline{w}_t, \theta_1\right] - \psi_t\left[\alpha_t(\underline{w}_t, \theta_2)|\underline{w}_t, \theta_1\right]\,.$$

- **The R.N. Constrained Direct Modelling - 1**: **Motivation**

a. **It may be important to control the family of R.N. dynamics** and, possibly, the specification of the short rate, if we want to have
$\Rightarrow$ **explicit or quasi-explicit formulas** for the price of some derivatives.

$\hookrightarrow$ **for instance**, it is often convenient to impose that the R.N. dynamics be described by a **Car (Compound Autoregressive) process** [see Darolles, Gourieroux, Jasiak (2006) for details]

$\hookrightarrow$ indeed, a Car process is characterized by
an **exponential-affine multi-horizon (complex) Laplace transform**
$\Rightarrow$ multi-horizon pricing formula for derivative products
are easily derived and implemented [Transform Analysis]
[Duffie, Pan and Singleton (2000), and
Gourieroux, Monfort and Polimenis (2003)]

b. **At the same time, if we want to control the historical dynamics**
$\hookrightarrow$ for instance to have good fitting when w_t is observable,
$\Rightarrow$ the **by-product of the modelling becomes the factor loading vector** $\alpha_t(\underline{w_t})$.

$\Rightarrow$ this is the R.N. Constrained Direct Modelling approach.

- **The R.N. Constrained Direct Modelling - 2**: more precisely

  i) **we select a family** $\{\psi_t(u|\underline{w_t}, \theta_1), \theta_1 \in \Theta_1\}$
  [$\hookrightarrow$ to explain, for instance, some stylized fact about w_t]

  ii) **and a family** $\{\psi_t^{\mathbb{Q}}(u|\underline{w_t}, \theta^*), \theta^* \in \Theta^*\}$
  [$\hookrightarrow$ to have explicit or quasi explicit pricing formula]

  iii) **such that, for any pair** $(\psi_t^{\mathbb{Q}}, \psi_t)$ **belonging to these families**, **there exists a unique function** $\alpha_t(\underline{w_t})$ denoted by $\alpha_t(w_t; \theta_1, \theta^*)$ satisfying :

$$\psi_t^{\mathbb{Q}}(u|\underline{w_t}) = \psi_t\left[u + \alpha_t(w_t)|\underline{w_t}\right] - \psi_t\left[\alpha_t(\underline{w_t})|\underline{w_t}\right] .$$

  – once the parameterization $(\tilde{\theta}, \theta_1, \theta^*) \in \tilde{\Theta} \times \Theta_1^*$ is defined, **internal consistency conditions may be imposed**.

- **The Back Modelling**: **it is based on the following steps**

1st - **we specify the R.N. dynamics** [and possibly $r_{t+1}(\underline{w_t})$] : $\psi_t^{\mathbb{Q}}(u|\underline{w_t};\theta_1^*)$

2nd - **we take into account**, if relevant, **internal consistency conditions** of the form :

$$p(\underline{w_t}) = E_t^{\mathbb{Q}}\left[\exp(-r_{t+1}(\underline{w_t},\tilde{\theta}) - \dots - r_s(\underline{w_s},\tilde{\theta}))g(\underline{w_s})|\underline{w_t},\theta_1^*\right] , \forall \underline{w_t}\,,\tilde{\theta}\,,\theta_1^*.$$

3rd - once this it is done, **the specification of $\alpha_t(\underline{w_t})$ is chosen**, without any constraint, **providing the family** $\{\alpha_t(\underline{w_t},\theta_2^*),\theta_2^*\in\Theta_2^*\}$.

- **the historical dynamics is a by-product** :

$$\psi_t(u|\underline{w_t};\theta_1^*,\theta_2^*) = \psi_t^{\mathbb{Q}}[u-\alpha_t(\underline{w_t},\theta_2^*)|\underline{w_t};\theta_1^*] - \psi_t^{\mathbb{Q}}[-\alpha_t(\underline{w_t},\theta_2^*)|\underline{w_t},\theta_1^*] \ .$$

- **Inference in an Econometric Asset Pricing Model**:

i) **In order to estimate an EAPM, we assume that**
the econometrician observes, at dates $t \in \{0, \ldots, T\}$,
a set of prices corresponding to payoffs $g_i(\underline{w}_s), i \in \{1, ..., J_t\}$,
given by (using the parameter notations of Direct Modelling) :

$$q_{ti}(\underline{w}_t, \theta) = E\left[g_i(\underline{w}_s) M_{t,s}(\underline{w}_s, \theta_2, \tilde{\theta}))|\underline{w}_t, \theta_1\right] , \; i \in \{1, ..., J_t\} .$$

$\Rightarrow$ we have two kinds of equations,
"transition equations" and "measurement equations", that is:

$$\begin{aligned} w_t &= \tilde{q}_t(\underline{w}_{t-1}, \varepsilon_{1t}, \theta_1) , && \textbf{the historical dynamics of the factors} \\ x_t &= q_t(\underline{w}_t; \theta) , && \textbf{the observations} \end{aligned}$$

($\varepsilon_{1,t}$ is a white noise which can be chosen Gaussian without loss of generality).

ii) if r_{t+1} **is not a known function of** $\underline{w}_t$,
$\Rightarrow$ **we must have** $r_{t+1} = r_{t+1}(\underline{w}_t, \tilde{\theta})$ **among "measurement equations"**,

iii) **if some components of w_t are observed**

$\Rightarrow$ they should appear **in the "measurement equations" without parameters**.

- **It is a nonlinear state space model** $\rightarrow$ appropriate econometric methods may be used for inference in this system

  [Maximum Likelihood methods based on Kalman filter, Kitagawa-Hamilton filter, Simulations based methods or Indirect Inference.]

iv) For given x_t's, **"measurement equations" may have no solutions in w_t's**

$\Rightarrow$ in this case, an additional noise in often introduced leading to

$x_t = q_t(\underline{w}_t; \theta) + \varepsilon_{2t}$, "modified measurement equations".

v) **when w_t is (partially) observable**, **θ_1 may be identifiable from "transition equations"** $\Rightarrow$ **in this case a two step method is available** :

a) **ML** estimation of θ_1 from "transition equations";

b) estimation of θ_2, and possibly $\tilde{\theta}$, by **Nonlinear Least Square** using "modified measurement equations" in which θ_1 is replaced by its ML estimator

(and, possibly, unobserved components of w_t by their smoothed value).

APPLICATIONS TO ECONOMETRIC SECURITY MARKET MODELLING

- **General Setting**:

  a) we assume that the short rate r_{t+1} **is exogenous**

  b) $w_t = (y_t, z_t')'$, $\dim(y_t) = K_1$, $\dim(w_t) = K_2 = K - K_1$, and
  $y_t =$ vector observable geometric returns,
  $z_t =$ factors not observed by the econometrician

  c) **since** $1 = E_t[M_{t,t+1} \exp(y_{j,t+1})]$ **for each** $j \in \{1, ..., K_1\}$,
  we have prices at t which are known functions of $\underline{w}_t$, namely 1,
  $\Rightarrow$ we have **to guarantee internal consistency conditions**.

  $\rightarrow$ In the **Direct Modelling** approach, and in the
  R.N. Constrained Direct Modelling one,
  these conditions are [using Direct Modelling notation] :

$$r_{t+1} = \psi_t \left[\alpha_t(\underline{w}_t; \theta_2) + e_j | \underline{w}_t, \theta_1 \right] - \psi_t \left[\alpha_t(w_t; \theta_2) | \underline{w}_t, \theta_1 \right] \ ,$$

$$\forall \ \underline{w}_t, \theta_1, \theta_2 \, ; \ j \in \{1, ..., K_1\} \, .$$

→ In the **Back Modelling** approach, these conditions are :

$$r_{t+1} = \psi_t^{\mathbb{Q}}(e_j | \underline{w}_t; \theta_1^*)\,, \ \ \forall \underline{w}_t, \theta_1^*\,, \,; j \in \{1, ..., K_1\}\,.$$

→ if $w_t \sim$ R.N. Car(1) process with conditional R.N. Log-Laplace transform $\psi_t^{\mathbb{Q}}(u \,|\, \underline{w}_t) = a^{\mathbb{Q}}(u)' w_t + b_t^{\mathbb{Q}}(u)$, the previous **internal consistency conditions** are given by (using the Back Modelling notation):

$$\begin{cases} a^{\mathbb{Q}}(e_j, \theta_1^*) & = & 0\,, \\ b_t^{\mathbb{Q}}(e_j, \theta_1^*) & = & r_{t+1}\,, \ \ \forall \theta_1^*\,; \, j \in \{1, ..., K_1\}\,. \end{cases}$$

- **Conditionally Gaussian Models**: $w_t = y_t$

- **if we follow the Direct Modelling approach :**

  i) **we specify the historical dynamics :**

$$y_{t+1} = m_t(\underline{y}_t, \theta_1) + \Sigma_t^{1/2}(\underline{y}_t, \theta_1)\varepsilon_{t+1}\,, \;\; \varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0, I_K)$$

  ii) and, for a given function $\alpha_t(\underline{w}_t)$, the **internal consistency conditions are** :

$$r_{t+1} = e_j' m_t(\underline{y}_t; \theta_1) + \frac{1}{2}(\alpha_t + e_j)'\Sigma_t(\underline{y}_t; \theta_1)(\alpha_t + e_j) - \frac{1}{2}\alpha_t'\Sigma_t(y_t; \theta_1)\alpha_t$$

giving : $\alpha_t = \Sigma_t^{-1}(\underline{y}_t; \theta_1)\left[r_{t+1}e - m_t(y_t; \theta_1) - \frac{1}{2}\, vdiag\; \Sigma_t(\underline{y}_t; \theta_1)\right]$

$\Rightarrow$ α_t is uniquely defined (no additional parameterization is needed)

and $m_t - r_{t+1}e = -\frac{1}{2} vdiag\, \Sigma_t - \Sigma_t\alpha_t = \pi_{Gt}$

iii) thus, **the conditional R.N. distribution is :**

$$y_{t+1} = r_{t+1}e - \frac{1}{2}\, vdiag\; \Sigma_t(y_t; \theta_1) + \Sigma_t^{1/2}(\underline{y}_t; \theta_1)\xi_{t+1}$$

where $\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0, I_K)$, and we get $\xi_{t+1} = \Sigma_t^{-1/2}(m_t - r_{t+1}e + \frac{1}{2}\, vdiag\; \Sigma_t) + \varepsilon_{t+1}$.

- **if we follow the Back Modelling approach :**

  i) **we start with the R.N. dynamics :**

$$\psi_t^{\mathbb{Q}}(u|y_t,\theta_1^*) = u' m_t^{\mathbb{Q}}(y_t,\theta_1^*) + \frac{1}{2} u' \Sigma_t^{\mathbb{Q}}(y_t,\theta_1^*) u$$

  ii) and **we impose the internal consistency conditions:**

$$r_{t+1} = m_{jt}^{\mathbb{Q}}(y_t,\theta_1^*) + \frac{1}{2}\Sigma_{jj,t}^{\mathbb{Q}}\,,\ j \in \{1,\ldots,K\}\,,$$

  $\Rightarrow$ the **R.N. dynamics compatible with the A.A.O. is**:

$$\psi_t^{\mathbb{Q}}(u|y_t,\theta_1^*) = u' e r_{t+1} - \frac{1}{2} u'\ vdiag\ \Sigma_t^{\mathbb{Q}} + \frac{1}{2} u' \Sigma_t^{\mathbb{Q}} u\,.$$

  iii) choosing any $\alpha_t(\underline{w}_t;\theta_2^*)$, we **deduce the historical dynamics :**

$$\begin{aligned} \psi_t(u|y_t;\theta_1^*,\theta_2^*) \quad = \quad & u' \Big[e r_{t+1} - \tfrac{1}{2}\ vdiag\ \Sigma_t^{\mathbb{Q}}(\underline{y}_t,\theta_1^*) \\ & - \Sigma_t^{\mathbb{Q}}(\underline{y}_t,\theta_1^*)\alpha_t(\underline{w}_t,\theta_2^*) \Big] + \tfrac{1}{2} u' \Sigma_t^{\mathbb{Q}}(\underline{y}_t,\theta_1^*) u\,, \end{aligned}$$

  and θ_1^* and θ_2^* can be identified from the dynamics of y_t only.

- **Direct Modelling of Switching Regime Models**: $w_t = (y_t, z_t')'$
  - y_t = observable geometric return
    z_t = J-state homogeneous Markov chain, valued in $(e_1, ..., e_J)$, and unobservable by the econometrician.
  - we **first define the historical dynamics of** w_{t+1}:

$$y_{t+1} = \mu_t(\underline{y_t}, \underline{z_t}, z_{t+1}; \theta_{11}) + \sigma_t(\underline{y_t}, \underline{z_t}, z_{t+1}; \theta_{11})\varepsilon_{t+1} \,, \quad \varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,1)$$

$$\mathbb{P}(z_{t+1} = e_j | \underline{y_t}, \underline{z_{t-1}}, z_t = e_i) = \mathbb{P}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij} \,,$$

that is

$$\begin{aligned} \varphi_t(u,v) &= E_t \exp(u y_{t+1} + v' z_{t+1}) \\ &= \Lambda(\underline{y_t}, z_t, \theta_1)' z_t = \textstyle\sum_{i=1}^{J} \Lambda_i(\underline{y_t}, e_i, \theta_1) \mathbb{1}_{e_i}(z_t) \,, \end{aligned}$$

where

$$\Lambda_i(\underline{y_t}, e_i, \theta_1) = \textstyle\sum_{j=1}^{J} \pi_{ij} \exp\left[v' e_j + u \mu_t(\underline{y_t}, e_i, e_j; \theta_{11}) + \frac{1}{2} u^2 \sigma_t^2(\underline{y_t}, e_i, e_j; \theta_{11})\right] \,,$$

  - then, **the SDF is specified as :**

$$M_{t,t+1} = \exp\left[-r_{t+1} + \gamma_t(\underline{w_t}, \theta_2)\, y_{t+1} + \delta_t(\underline{w_t}, \theta_2)' z_{t+1} - \psi_t(\gamma_t, \delta_t)\right] \,,$$

with $\delta_{Jt} = 0$ for identification reasons.

- the **internal consistency condition is:**

$$\varphi_t(\gamma_t + 1, \delta_t) = \exp(r_{t+1})\varphi_t(\gamma_t, \delta_t)\,.$$

for a given δ_t, this equation has a unique solution in γ_t
$\Rightarrow \gamma_t(\underline{w}_t, \theta_2)$ can be written $\gamma_t\left[\delta_t(\underline{w}_t, \theta_2)\right]$
and we only have to specify $\delta_{1t}(\underline{w}_t, \theta_2), ..., \delta_{J-1}(\underline{w}_t, \theta_2)$.

- **the associated R.N. dynamics is defined by**

$$\varphi_t^{\mathbb{Q}}(u, v) = \Lambda^{\mathbb{Q}}(\underline{y}_t, z_t, \theta_1, \theta_2)' z_t$$

where

$$\begin{aligned}\Lambda_i^{\mathbb{Q}}(\underline{y}_t, e_i, \theta_1, \theta_2) \quad &= \quad \textstyle\sum_{j=1}^{J} \pi_{ij,t}^{*} \exp\left\{v' e_j + u\left[\mu_t(\underline{y}_t, e_i, e_j; \theta_{11}) + \gamma_t \sigma_t^2(\underline{y}_t, e_i, e_j; \theta_{11})\right]\right. \\ &\qquad \left. + \tfrac{1}{2} u^2 \sigma_t^2(\underline{y}_t, e_i, e_j; \theta_{11})\right\}\,,\end{aligned}$$

and with

$$\pi_{ij,t}^{*} = \frac{\pi_{ij} \exp\left[\delta_t' e_j + \gamma_t \mu_t(\underline{y}_t, e_i, e_j; \theta_{11}) + \frac{1}{2}\gamma_t^2 \sigma_t^2(\underline{y}_t, e_i, e_j; \theta_{11})\right]}{\sum_{j=1}^{J} \pi_{ij} \exp\left[\delta_t' e_j + \gamma_t \mu_t(\underline{y}_t, e_i, e_j; \theta_{11}) + \frac{1}{2}\gamma_t^2 \sigma_t^2(\underline{y}_t, e_i, e_j; \theta_{11})\right]}\,.$$

- **in other words**, the R.N. dynamics is defined by :

$$y_{t+1} = \mu_t(\underline{y}_t, \underline{z}_t, z_{t+1}; \theta_{11}) + \gamma_t \left[\delta_t(\underline{w}_t, \theta_2)\right] \sigma_t^2(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_2)$$
$$+\sigma_t(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_{11})\xi_{t+1}$$

where

$$\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)\,,$$
$$\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \pi^*_{ij,t}\,.$$

- in particular, we get :

$$\varepsilon_{t+1} = \gamma_t \left[\delta_2(\underline{w}_t, \theta_2)\right] \sigma_t(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_{11}) + \xi_{t+1}\,.$$

- **Back Modelling of Switching Regime Models**: $w_t = (y_t, z_t')'$

  the Direct Modelling has two main drawbacks

  a. **internal consistency condition (ICC):** solved numerically in γ_t, for any t

  b. **the R.N. dynamics is not Car**, in general,
  $\longrightarrow$ the pricing of derivatives need simulations which, in turn,
  imply to solve the ICC for any t and any path.

  $\Rightarrow$ **we solve these problems using the Back Modelling approach**

- we **start from a Car R.N. dynamics:**

$$y_{t+1} = \nu_t + \rho y_t + \nu_1' z_t + \nu_2' z_{t+1} + (\nu_3' z_{t+1})\xi_{t+1}\,, \quad \xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$$

$$\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j \,|\, z_t = e_i) = \pi^*_{ij}\,.$$

indeed, the associated conditional R.N. Log-Laplace transform is

$$\psi_t^{\mathbb{Q}}(u,v) \quad = \quad u(\nu_t + \rho y_t + \nu_1' z_t) + \Lambda'(u, v, \nu_2, \nu_3, \pi^*) z_t\,,$$

where

$$\Lambda_i(u, v, \nu_2, \nu_3, \pi^*) = \text{ Log } \sum_{j=1}^{J} \pi^*_{ij} \exp\left(u\nu_{2j} + \frac{1}{2}u^2\nu_{3j}^2 + v_j\right)\,.$$

- **the ICC** is $\psi_t^{\mathbb{Q}}(1,0) = r_{t+1}$, that is

$$-r_{t+1} + \nu_t + \rho y_t + \nu_1' z_t + \lambda'(\nu_2, \nu_3, \pi^*) z_t = 0$$

$$\forall\, y_t,\ z_t,\ \nu_t,\ \nu_1,\ \nu_2,\ \nu_3 \text{ and } \pi^*$$

with

$$\lambda_i(\nu_2, \nu_3, \pi^*) = \text{ Log } \textstyle\sum_{j=1}^{J} \pi_{ij}^* \exp\left(\nu_{2j} + \frac{1}{2}\nu_{3j}^2\right) .$$

$\Rightarrow$ **the R.N. dynamics compatible with the AAO condition (ICC) is :**

$$y_{t+1} = r_{t+1} - \lambda'(\nu_2, \nu_3, \pi^*) z_t + \nu_2' z_{t+1} + (\nu_3 z_{t+1}) \xi_{t+1} ,$$

where

$$\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$$

$$\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j \,|\, z_t = e_i) = \pi_{ij}^* .$$

- for identification reasons we assume $\nu_{2J} = 0$.

- **the historical dynamics can then be deduced** by specifying $\gamma_t(\underline{w}_t, \theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$ without any constraints in

$$M_{t,t+1} = \exp[-r_{t+1} + \gamma_t(\underline{w}_t, \theta_2^*) y_{t+1} + \delta_t(\underline{w}_t, \theta_2^*)' z_{t+1} - \psi_t(\gamma_t, \delta_t | \underline{w}_t, \theta_1^*)]$$

(and assuming, for instance, $\delta_{Jt} = 0$) giving the Log-Laplace transform :

$$\psi_t(u, v) = u(r_{t+1} - \lambda' z_t) + [\Lambda(u - \gamma_t, v - \delta_t) - \Lambda(-\gamma_t, -\delta_t)]' \, z_t$$

where

$$\Lambda_i(u - \gamma_t, v - \delta_t) - \Lambda_i(-\gamma_t, -\delta_t) = \text{ Log } \sum_{j=1}^{J} \pi_{ij,t} \exp\left[u(\nu_{2j} - \gamma_t \nu_{3j}) + \frac{1}{2} u^2 \nu_{3j}^2 + v_j\right]$$

with

$$\pi_{ij,t} = \frac{\pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_t^2 \nu_{3j}^2 - \delta_{jt}\right)}{\sum_{j=1}^{J} \pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_t^2 \nu_{3j}^2 - \delta_{jt}\right)}.$$

- in other words, we have

$$y_{t+1} = r_{t+1} - \lambda'(\nu_2, \nu_3, \pi^*) z_t + (\nu_2 - \gamma_t \nu_3^2)' z_{t+1} + (\nu_3' z_{t+1}) \varepsilon_{t+1} \,,$$

$$\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,1)\,, \;\; \mathbb{P}(z_{t+1} = e_j \,|\, \underline{y_t}, \underline{z_{t-1}}, z_t = e_i) = \pi_{ij,t}$$

$$\lambda_i(\nu_2, \nu_3, \pi^*) = \text{Log} \sum_{j=1}^{J} \pi_{ij}^* \exp\left(\nu_{2j} + \frac{1}{2}\nu_{3j}^2\right) \,,$$

and

$$\varepsilon_{t+1} = \xi_{t+1} + \gamma_t \, (\nu_3' z_{t+1}) \,.$$

- since γ_t and δ_t are arbitrary functions of $\underline{w_t}$ (assuming, for instance, $\delta_{Jt} = 0$), we obtain **a large class of historical switching regime dynamics** which can be **matched with a Car switching regime R.N. dynamics**.

- **identification problem** when γ and δ are constant:
identifiable: π_{ij}, ν_3, $(\nu_2 - \gamma \nu_3^2)$ and $\lambda'(\nu_2, \nu_3, \pi^*)$ $\rightarrow$ $J(J+2)-1$ parameters
to be estimated: π_{ij}^*, ν_2, ν_3, γ, δ $\rightarrow$ $J(J+2)-1$ parameters also.
$\Rightarrow$ **all the parameters might be estimated from the observations** of the $y_t's$.

- **Back Modelling of Stochastic Volatility Models**: $w_t = (y_t, \sigma_t^2)'$

- $y_t =$ observable geometric return
  $\sigma_t^2 =$ unobservable stochastic variance,

- **we start from a Car R.N. dynamics** for w_t:

$$y_{t+1} = \lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2 + (\lambda_3 \sigma_t)\xi_{t+1}\,, \quad \xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$$

$$\sigma_{t+1}^2 \,|\, \underline{y_t},\, \underline{\sigma_t^2} \overset{\mathbb{Q}}{\sim} ARG(1, \nu, \varrho)$$

  that is:

$$\psi_t^{\mathbb{Q}}(u,v) = (\lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2)u + \tfrac{1}{2}\lambda_3^2 \sigma_t^2 u^2 + a^{\mathbb{Q}}(v)\sigma_t^2 + b^{\mathbb{Q}}(v)$$

$$a^{\mathbb{Q}}(v) = \tfrac{\rho v}{1-v}\,, \quad b^{\mathbb{Q}}(v) = -\nu \mathrm{Log}(1-v)\,, \;\; v < 1\,, \rho > 0\,, \nu > 0.$$

- **the ICC** is $\psi_t^{\mathbb{Q}}(1,0) = r_{t+1}$, which implies $\lambda_t = r_{t+1},\ \lambda_1 = 0,\ \lambda_2 = -\frac{1}{2}\lambda_3^2$,
  $\Rightarrow$ the **R.N. dynamics compatible with the AAO is**

$$\psi_t^{\mathbb{Q}}(u,v) = \left(r_{t+1} - \tfrac{1}{2}\lambda_3^2 \sigma_1^2\right) u + \tfrac{1}{2}\lambda_3^2 \sigma_t^2 u^2 + a(v)^{\mathbb{Q}}\sigma_t^2 + b(v)^{\mathbb{Q}}\,.$$

- **the historical dynamics** is defined by specifying $\gamma_t(\underline{w}_t\theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$, and we get :

$$\psi_t(u,v) \;=\; \left(r_{t+1} - \tfrac{1}{2}\lambda_3^2\sigma_t^2 - \lambda_3^2\sigma_t^2\gamma_t\right) u + \tfrac{1}{2}\lambda_3^2\sigma_t^2 u^2 + a_t(v)\sigma_t^2 + b_t(v)\,,$$

with

$$a_t(v) \;=\; \frac{\rho_t v}{1 - v\mu_t}\,,\; b_t(v) = -\nu \mathrm{Log}(1 - v\mu_t)\,,$$

$$\rho_t \;=\; \frac{\rho}{(1+\delta_t)^2}\,,\; \mu_t = \frac{1}{1+\delta_t}\,.$$

- The **historical dynamics can be written:**

$$y_{t+1} = r_{t+1} - \frac{1}{2}\lambda_3^2\sigma_t^2 - \lambda_3^2\sigma_t^2\gamma_t + \lambda_3\sigma_t\varepsilon_{t+1}$$

where

$$\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,1)\,,\;\; \sigma_{t+1}^2 \,|\, \underline{y}_t, \underline{\sigma_t^2} \overset{\mathbb{P}}{\sim} ARG(\mu_t, \nu, \rho_t)\,;$$

the conditional historical distribution of $\sigma_{t+1}^2 \,|\, (\underline{y}_t, \underline{\sigma_t^2})$, **is not affine** in σ_t^2, except in the case where δ_t is constant (or a deterministic function of t).

- **Moreover we have** :

$$\varepsilon_{t+1} = \xi_{t+1} + (\lambda_3 \sigma_t)\gamma_t \,.$$

- **Identification problem** when γ and δ are constant:

i) the identifiable parameters are λ_3, γ, ρ and μ ($\nu = 1$ for identification reasons)
$\rightarrow$ we have four identifiable parameters.

ii) the parameters to be estimated are λ_3, ν, ρ, γ, δ, i.e. five parameters
$\Rightarrow$ so these parameters are not identifiable from the dynamics of the $y_t' s$
$\rightarrow$ **observations of derivative prices must be added.**

- **Back Modelling of Switching GARCH Model with leverage effect: a first application of Extended Car Processes**

  $w_t = (y_t, z_t')'$,

  $y_t =$ observable geometric return

  $z_t = J$-state process, valued in $\{e_1, ..., e_J\}$,

  and unobservable by the econometrician.

- in this example following a Back Modelling approach, we consider specifications generalizing those proposed by Heston and Nandi (2000) [see also Elliot, Siu and Chan (2006)].

  $\hookrightarrow$ **Switching Regime Model** + **GARCH effect (with leverage)**

- **the R.N. dynamics is given by:**

$$y_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \nu_3' z_{t+1} + \nu_4 \sigma_{t+1}^2 + \sigma_{t+1}\xi_{t+1}\,, \quad \xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$$

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma_t^2$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*\,.$$

- **note that:**

  a) σ^2_{t+1} is a deterministic function of $(\underline{\xi_t},\ \underline{z_t})$, and therefore of $\underline{w_t} = (\underline{y_t}, \underline{z_t})$;

  b) in this switching GARCH(1,1) model, ξ_t replaces the usual term $\sigma_t\ \xi_t$;

  c) the term $\alpha_2\sigma_t$ captures an asymmetric or "leverage" effect;

- **the ICC** $\psi_t^{\mathbb{Q}}(1,0) = r_{t+1}\,,\ \forall \underline{w_t}$, implies :

$$\begin{cases} \nu_1 = 0 \\ \nu_t + \nu_2' z_t + \nu_4 \sigma^2_{t+1} = r_{t+1} - \lambda'(\nu_3, \pi^*) z_t - \frac{1}{2}\sigma^2_{t+1}\,, \end{cases}$$

  with $\lambda_i(\nu_3, \pi^*) = \text{Log} \sum_{j=1}^{j} \pi^*_{ij} \exp(\nu_{3j})$, $i \in \{1, \ldots, J\}$, and, therefore,

  the arbitrage-free R.N. dynamics is:

$$y_{t+1} = r_{t+1} - \lambda'(\nu_3, \pi^*) z_t - \frac{1}{2}\sigma^2_{t+1} + \nu_3' z_{t+1} + \sigma_{t+1}\xi_{t+1}\,,\ \ \xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$$

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma^2_t$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y_t}, \underline{z_{t-1}}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi^*_{ij}\,,$$

- **the historical dynamics is obtained** by specifying $\gamma_t(\underline{w_t}, \theta_2^*)$ and $\delta_t(\underline{w_t}, \theta_2^*)$ [with, for instance, $\delta_{Jt} = 0$] and in particular we have :

$$y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t - \frac{1}{2}\sigma_{t+1}^2 - \gamma_t(\underline{w_t}, \theta_2^*)\sigma_{t+1}^2 + \nu_3' z_{t+1} + \sigma_{t+1}\varepsilon_{t+1}\,,$$

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma_t^2$$

$$\mathbb{P}(z_{t+1} = e_j | \underline{y_t}, \underline{z_{t-1}}, z_t = e_i) = \pi_{ij,t}\,, \quad \pi_{ij,t} = \frac{\pi_{ij}^* \exp(-\gamma_t\nu_{3j} - \delta_{jt})}{\sum_{j=1}^J \pi_{ij}^* \exp(-\gamma_t\nu_{3j} - \delta_{jt})}\,.$$

- Moreover, we get $\xi_{t+1} = \varepsilon_{t+1} - \gamma_t\sigma_{t+1}$ and, therefore, the equation giving σ_{t+1}^2 can be rewritten :

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1 \left[\varepsilon_t - (\alpha_2 + \gamma_t)\sigma_t\right]^2 + \alpha_3\sigma_t^2\,.$$

- Observe that $w_t = (y_t, z_t')$ **does not have a Car R.N. dynamics**
  $\Rightarrow$ so the pricing seems a priori difficult.
  BUT, the augmented factor $w_t^a := (y_t, \sigma_{t+1}^2, z_t')'$ is **R.N. Car**
  $\Rightarrow$ and therefore **the pricing methods, based on Car dynamics, apply.**

- In particular, **the R.N. conditional Log-Laplace transform** of w_{t+1}^a, given $\underline{w_t^a}$, is :

$$\psi_t^{\mathbb{Q}}(u,v,\tilde{v}) = a_1^{\mathbb{Q}}(u,v,\tilde{v})' z_t + a_2^{\mathbb{Q}}(u,\tilde{v})\sigma_{t+1}^2 + b_t^{\mathbb{Q}}(u,\tilde{v})\,,$$

where

$$a_1^{\mathbb{Q}}(u,v,\tilde{v}) = \tilde{\Lambda}(u,v,\tilde{v},\nu_3,\omega,\pi^*) - \lambda(\nu_3,\pi^*)\,u$$

$$\text{with} \quad \tilde{\Lambda}_i(u,v,\tilde{v},\nu_3,\omega,\pi^*) = \text{Log}\sum_{j=1}^{J}\pi_{ij}^*\,\text{exp}(u\nu_{3j} + v_j + \tilde{v}\omega_j)\,,\ i \in \{1,\ldots,J\}\,,$$

$$a_2^{\mathbb{Q}}(u,\tilde{v}) = -\frac{1}{2}u + \tilde{v}(\alpha_1\alpha_2^2 + \alpha_3) + \frac{(u - 2\alpha_1\alpha_2\tilde{v})^2}{2(1-2\alpha_1\tilde{v})}$$

$$b_t^{\mathbb{Q}}(u,\tilde{v}) = ur_{t+1} - \frac{1}{2}\text{Log}(1-2\alpha_1\tilde{v})\,,$$

which is affine in $(z_t, \sigma_{t+1}^2)'$, with an intercept deterministic function of time.

- **Identification problem** when γ and δ are constant:

identifiable: ν_3, λ, ω, α_1, $(\alpha_2+\gamma)$, α_3, and π_{ij}, i.e. $J(J+2)+3$ parameters

to be estimated: ν_3, $\omega, \alpha_1, \alpha_2, \alpha_3$, π_{ij}^*, γ, δ, that is, $J(J+2)+2$ parameters.

$\Rightarrow$ the **historical model is over identified.**

- **Back Modelling of Switching IG GARCH Models**: **a second application of Extended Car Processes**

  $w_t = (y_t, z_t')'$

  $y_t =$ observable geometric return

  $z_t = J$-state process, valued in $\{e_1, ..., e_J\}$,

  and unobservable by the econometrician.

- in this case, following a Back Modelling approach, we introduce **several generalizations of the Inverse Gamma (IG) GARCH model** proposed by Christoffersen, Heston and Jacobs (2006):

  $i)$ we consider **switching regimes** in the (historical and risk-neutral) dynamics of y_t and in the GARCH variance σ^2_{t+1};

  $ii)$ we **price not only the factor risk** but also the **regime-shift risk**;

  $iii)$ **risk correction coefficients are in general time-varying**;

- **the R.N. dynamics is assumed to be:**

$$y_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \nu_3' z_{t+1} + \nu_4 \sigma_{t+1}^2 + \eta \xi_{t+1}$$

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} IG\left(\frac{\sigma_{t+1}^2}{\eta^2}\right)$$

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1 \sigma_t^2 + \alpha_2 \xi_t + \alpha_3 \frac{\sigma_t^4}{\xi_t}$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*$$

- **the ICC** $\psi_t^{\mathbb{Q}}(1,0) = r_{t+1}\,,\ \forall \underline{w}_t$, implies :

$$r_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \lambda'(\nu_3, \pi^*) z_t + \sigma_{t+1}^2 \left(\nu_4 + \frac{1}{\eta^2} \left[1 - (1 - 2\eta)^{1/2} \right] \right),$$

with $\lambda_i(\nu_3, \pi^*) = \text{Log} \sum_{j=1}^{j} \pi_{ij}^* \exp(\nu_{3j}),\ i \in \{1, \dots, J\}$, and, therefore,

the arbitrage-free R.N. dynamics is:

$$y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t - \frac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\sigma^2_{t+1} + \nu_3' z_{t+1} + \eta\xi_{t+1}$$

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1 \sigma^2_t + \alpha_2 \xi_t + \alpha_3 \frac{\sigma^4_t}{\xi_t}$$

$$\xi_{t+1} \,|\, \underline{\xi_t}, \underline{z_{t+1}} \overset{\mathbb{Q}}{\sim} IG\left(\frac{\sigma^2_{t+1}}{\eta^2}\right)$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y_t}, \underline{z_{t-1}}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi^*_{ij}\,,$$

- given the specification of $\gamma_t(\underline{w_t}, \theta_2^*)$ and $\delta_t(\underline{w_t}, \theta_2^*)$ [with, for instance, $\delta_{Jt} = 0$], the conditional historical Log-Laplace transform of the factor is given by:

$$\psi_t(u,v) = \left(r_{t+1} - \lambda' z_t - \tilde{\eta}_t^{-3/2}\eta^{-1/2}\left[1-(1-2\eta)^{1/2}\right]\tilde{\sigma}^2_{t+1}\right)u$$

$$+ \left[\Lambda(u-\gamma_t, v-\delta_t, \nu_3, \pi^*) - \Lambda(-\gamma_t, -\delta_t, \nu_3, \pi^*)\right]' z_t$$

$$+ \frac{\tilde{\sigma}^2_{t+1}}{\tilde{\eta}^2_t}\left[1-(1-2u\tilde{\eta}_t)^{1/2}\right],$$

where $\tilde{\eta}_t = \frac{\eta}{1+2\gamma_t\eta}$ and $\tilde{\sigma}^2_{t+1} = \sigma^2_{t+1}\left(\frac{\tilde{\eta}_t}{\eta}\right)^{3/2}$.

- **so, the non-affine historical dynamics** is given by :

$$y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t + \nu_3' z_{t+1} - \tilde{\eta}_t^{-3/2} \eta^{-1/2} \left[1 - (1-2\eta)^{1/2}\right] \tilde{\sigma}^2_{t+1} + \tilde{\eta}_t \varepsilon_{t+1}$$

$$\varepsilon_{t+1} \,|\, \underline{\varepsilon_t}, \underline{z_{t+1}} \sim IG\left(\frac{\tilde{\sigma}^2_{t+1}}{\tilde{\eta}_t^2}\right)$$

with $\eta \xi_{t+1} = \tilde{\eta}_t \varepsilon_{t+1}$ and

$$\tilde{\sigma}^2_{t+1} = \tilde{\omega}_t' z_t + \tilde{\alpha}_{1,t} \tilde{\sigma}_t^2 + \tilde{\alpha}_{2,t} \varepsilon_t + \tilde{\alpha}_{3,t} \frac{\tilde{\sigma}_t^4}{\varepsilon_t}$$

$$\mathbb{P}(z_{t+1} = e_j | \underline{y_t}, \underline{z_{t-1}}, z_t = e_i) = \pi_{ij,t} \,,$$

where $\tilde{\omega}_t = \omega(\tilde{\eta}_t/\eta)^{3/2}$, $\tilde{\alpha}_{1,t} = \alpha_1(\tilde{\eta}_t/\tilde{\eta}_{t-1})^{3/2}$,

$\tilde{\alpha}_{2,t} = \alpha_2(\tilde{\eta}_t^{3/2} \tilde{\eta}_{t-1}/\eta^{5/2})$ and $\tilde{\alpha}_{3,t} = \alpha_3 \tilde{\eta}_t^{3/2}/(\tilde{\eta}^4_{t-1} \eta^{-5/2})$.

- Observe that $w_t = (y_t, z_t')'$ **does not have a Car R.N. dynamics**

$\Rightarrow$ so the pricing seems a priori difficult.

BUT, the augmented factor $w_t^a := (y_t, \sigma^2_{t+1}, z_t')'$ is **R.N. Car**

- Indeed, **the R.N. conditional Log-Laplace transform** of w^a_{t+1}, given $\underline{w}^a_t$, is :

$$\psi^{\mathbb{Q}}_t(u,v,\tilde{v}) = a^{\mathbb{Q}}_1(u,v,\tilde{v})' z_t + a^{\mathbb{Q}}_2(u,\tilde{v})\sigma^2_{t+1} + b^{\mathbb{Q}}_t(u,\tilde{v})\,,$$

where

$$\begin{aligned}
a^{\mathbb{Q}}_1(u,v,\tilde{v}) &= \tilde{\Lambda}(u,v,\tilde{v},\nu_3,\omega,\pi^*) - \lambda(\nu_3,\pi^*)\,u \\
\text{with} \quad & \tilde{\Lambda}_i(u,v,\tilde{v},\nu_3,\omega,\pi^*) = \text{Log}\sum_{j=1}^{J}\pi^*_{ij}\,\text{exp}(u\nu_{3j}+v_j+\tilde{v}\omega_j)\,,\ i\in\{1,\ldots,J\}\,, \\
a^{\mathbb{Q}}_2(u,\tilde{v}) &= \tilde{v}\alpha_1 - \frac{1}{\eta^2}\left(u\left[1-(1-2\eta)^{1/2}\right]+1-\sqrt{\left(1-2\tilde{v}\alpha_3\eta^4\right)(1-2(u\eta+\tilde{v}\alpha_2))}\right)\,, \\
b^{\mathbb{Q}}_t(u,\tilde{v}) &= ur_{t+1} - \frac{1}{2}\text{Log}(1-2\tilde{v}\alpha_3\eta^4)\,,
\end{aligned}$$

which is affine in $(z_t,\sigma^2_{t+1})'$, with an intercept deterministic function of time.

- **Identification problem** when γ and δ are constant:
identifiable from the historical dynamics: $3J+J(J-1)+4$ parameters to be estimated: ν_3 (with $\nu_{3J}=0$), $\omega,\alpha_1,\alpha_2,\alpha_3$, π^*_{ij}, γ, δ (with $\delta_J=0$), and η, that is, $2(J-1)+J+5+J(J-1) = 3J+J(J-1)+3$ parameters.
$\Rightarrow$ the **historical model is over identified.**

- **If the R.N. dynamics** of w_t is Car **and if** r_{t+1} is **an affine function** of w_t,
  $\Rightarrow$ the term structure of interest rates $[r(t,h), h \in \{1,...,H\}]$
  is easily determined recursively and is affine in w_t
  [see Gouriéroux, Monfort and Polimenis (2003)]

- indeed, if $\psi_t^{\mathbb{Q}}(u|w_t;\theta_1^*) = a^{\mathbb{Q}}(u,\theta_1^*)'w_t + b^{\mathbb{Q}}(u,\theta_1^*)$ and $r_{t+1} = \tilde{\theta}_1 + \tilde{\theta}_2' w_t$,
  then $r(t,h) = -\dfrac{c_h'}{h} w_t - \dfrac{d_h}{h}$,
  where $c_h = -\tilde{\theta}_2 + a^{\mathbb{Q}}(c_{h-1})$, $d_h = d_{h-1} - \tilde{\theta}_1 + b^{\mathbb{Q}}(c_{h-1})$, with $c_0 = 0$, $d_0 = 0$.

- **Moreover, applying the transform analysis,** various interest rates derivatives have quasi explicit pricing formulas.

- **Note that if** the i^{th} component of w_t is a rate $r(t,h_i)$, $i \in \{1,...,K_1\}$,
  we must satisfy the internal consistency conditions:

$$c_{h_i} = -h_i e_i, \; d_{h_i} = 0, \;\; i \in \{1,...,K_1\} \, .$$

- **Direct Modelling of the VAR(p) Factor-Based Term Structure Model**:

- we consider the **one factor case**, but the results can be extended to the multivariate case [see Monfort and Pegoraro (2006)]

- the factor w_t, which **may be observable or unobservable**, has a **historical dynamics** given by a Gaussian AR(p) model:

$$w_{t+1} = \nu + \varphi' W_t + \sigma \varepsilon_{t+1}$$

where $\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,1)$, $\varphi = (\varphi_1, ..., \varphi_p)'$ and $W_t = (w_t, ..., w_{t+1-p})'$.

- the **SDF is given by**

$$M_{t,t+1} = \exp\left[-r_{t+1} + \alpha_t w_{t+1} - \psi_t(\alpha_t)\right]$$

with $\psi_t(u) = (\nu + \varphi' W_t)u + \frac{1}{2}\sigma^2 u^2$

and $\alpha_t = \alpha_0 + \alpha' W_t$,

and **the short rate is given by** : $r_{t+1} = \tilde{\theta}_1 + \tilde{\theta}_2' W_t$

- The associated conditional R.N. Log-Laplace transform is given by:

$$\begin{aligned}\psi_t^{\mathbb{Q}}(u) &= \psi_t(u+\alpha_t) - \psi_t(\alpha_t)\\ &= [(\nu+\sigma^2\alpha_0)+(\varphi+\sigma^2\alpha)'W_t]u + \tfrac{1}{2}\sigma^2u^2\,.\end{aligned}$$

$\Rightarrow$ the **R.N. dynamics of the factor** is given by :

$$w_{t+1} = \nu + \sigma^2\alpha_0 + (\varphi+\sigma^2\alpha)'W_t + \sigma\xi_{t+1}$$

where $\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$. Moreover, we have that $\varepsilon_{t+1} = \xi_{t+1} + \sigma(\alpha_0 + \alpha' W_t)$.

- **The yield-to-maturity formula** at date t is given by :

$$r(t,h) = -\frac{c_h'}{h}W_t - \frac{d_h}{h}\,, \quad h \geq 1\,,$$

with

$$\begin{cases} c_h = -\tilde{\theta}_2 + \Phi' c_{h-1} + c_{1,h}\sigma^2\alpha \\ d_h = -\tilde{\theta}_1 + c_{1,h-1}(\nu+\sigma^2\alpha_0) + \frac{1}{2}c_{1,h-1}^2\sigma^2 + d_{h-1} \\ c_0 = 0 \;\; d_0 = 0\,. \end{cases}$$

- **R.N. Constrained Direct Modelling of the Switching VAR(p) Factor-Based Term Structure Model**: $w_t = (x_t, z_t')'$

  x_t is observable or unobservable

  z_t is a J-state non-homogeneous Markov chain valued in $\{e_1, ..., e_J\}$ and unobservable by the econometrician

- **the historical dynamics** is given by :

$$x_{t+1} = \nu(Z_t) + \varphi_1(Z_t)x_t + ... + \varphi_p(Z_t)x_{t+1-p} + \sigma(Z_t)\varepsilon_{t+1}$$

$$\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,1)\,, \;\; \mathbb{P}(z_{t+1} = e_j | \underline{x_t}\; \underline{z_{t-1}}, z_t = e_i) = \pi(e_i, e_j; X_t)$$

$$Z_t = (z_t', ..., z_{t-p}')\,, \;\; X_t = (x_t, ..., x_{t+1-p})'\,.$$

$\longrightarrow$ observe that, the joint dynamics of $(x_t, z_t')'$ is not Car.

- **we specify the SDF** in the following way :

$$M_{t,t+1} = \exp\left[-r_{t+1} + \Gamma(Z_t, X_t)\varepsilon_{t+1} - \frac{1}{2}\Gamma(Z_t, X_t)^2 - \delta(Z_t, X_t)' z_{t+1}\right],$$

with $\Gamma(Z_t, X_t) = \gamma(Z_t) + \tilde{\gamma}(Z_t)' X_t$
and $r_{t+1} = \tilde{\theta}_1' X_t + \tilde{\theta}_2' Z_t$.

- In order to ensure that $E_t M_{t,t+1} = \exp(-r_{t+1})$, we add the normalization condition :

$$\sum_{j=1}^{J} \pi(e_i, e_j, X_i) \exp[-\delta(Z_t, X_t)' e_j] = 1, \;\; \forall Z_t, X_t\,.$$

- It is easily seen that **the R.N. dynamics is given by :**

$$x_{t+1} = \nu(Z_t) + \gamma(Z_t)\sigma(Z_t) + [\varphi(Z_t) + \tilde{\gamma}(Z_t)\sigma(Z_t)]' X_t + \sigma(Z_t)\xi_{t+1}$$

$$\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{x}_t, \underline{z}_{t-1}, z_t = e_i) = \pi(e_i, e_j, ; X_t) \exp[-\delta(Z_t, X_t)' e_j] \, .$$

$\longrightarrow$ and it is not Car.

- **BUT we want the R.N. dynamics** of (w_t) to be Car. WE IMPOSE:

i) $\sigma(Z_t) = \sigma^{*'} Z_t$ (linearity in $z_t, ..., z_{t-p}$)

ii) $\gamma(Z_t) = \frac{\nu^{*'} Z_t - \nu(Z_t)}{\sigma^{*'} Z_t}$

iii) $\tilde{\gamma}(Z_t) = \frac{\varphi^{*} - \varphi(Z_t)}{\sigma^{*'} Z_t}$

iv) $\delta_j(Z_t, X_t) = \text{Log} \left[\frac{\pi(z_t, e_j, X_t)}{\pi^{*}(z_t, e_j)} \right]$,

$\Rightarrow$ the **associated R.N. (Constrained) Car dynamics is:**

$$X_{t+1} = \Phi^* X_t + [\nu^{*'} Z_t + (\sigma^{*'} Z_t)\xi_{t+1}] e_1 \,,$$

$$\Phi^* = \begin{bmatrix} \varphi_1^* & \dots & \dots & \varphi_{p-1}^* & \varphi_p^* \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} \text{ is a } (p \times p) \text{ matrix} \,,$$

$$\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1) \,,$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{x_t}, \underline{z_{t-1}}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*$$

and the affine (in X_t and Z_t) term structure of interest rates is easily derived [see Monfort and Pegoraro (2006) for the proof].

- **Back Modelling of the VAR(p) Factor-Based Term Structure Model**

  we consider the (bivariate) case where $w_t = [r(t,1), r(t,2)]'$

- **we want to impose the following VAR(1) R.N. dynamics:**

$$w_{t+1} = \nu + \Phi w_t + \varepsilon_{t+1}\,,$$

  where $\varepsilon_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,\Sigma)$.

- In this case the **internal consistency conditions are satisfied if** we impose

$$\begin{cases} -2e_2 = a^{\mathbb{Q}}\begin{pmatrix}-1\\0\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\,, \\ 0 = b^{\mathbb{Q}}\begin{pmatrix}-1\\0\end{pmatrix}\,, \end{cases}$$

  where $a^{\mathbb{Q}}(u) = \Phi' u$, $b^{\mathbb{Q}}(u) = u'\nu + \frac{1}{2}u'\Sigma u$.

  and the **arbitrage-free R.N. dynamics** of w_t is

$$\begin{cases} r(t+1,1) = \frac{1}{2}\sigma_1^2 - r(t,1) + 2r(t,2) + \varepsilon_{1,t+1} \\ r(t+1,2) = \nu_2 + \varphi_{21} r(t,1) + \varphi_{22} r(t,2) + \varepsilon_{2,t+1}\,, \end{cases}$$

  with $\varepsilon_t \overset{\mathbb{Q}}{\sim} IIN(0,\Sigma)$.

- Now, if **we move back to the historical conditional Log-Laplace transform**, we get:

$$\psi_t^{\mathbb{P}}(u) = u' \left[\begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} w_t \right] - u'\Sigma\alpha_t + \tfrac{1}{2}u'\Sigma u \,.$$

- If we assume $\alpha_t = \gamma + \Gamma w_t$, we get:

$$\psi_t^{\mathbb{P}}(u) = u' \left\{ \begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} - \Sigma\gamma + \left[\begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} - \Sigma\Gamma \right] w_t \right\} + \frac{1}{2}u'\Sigma u \,,$$

  or, equivalently

$$w_{t+1} = \begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} - \Sigma\gamma + \left[\begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} - \Sigma\Gamma \right] w_t + \xi_{t+1} \,,$$

  where $\xi_{t+1} \overset{\mathbb{P}}{\sim} IIN(0, \Sigma)$ and $\xi_t = \varepsilon_t + \Sigma(\gamma + \Gamma w_t)$.

- If $\Gamma = 0$, the historical dynamics of w_t is **constrained**,

  the parameters Σ,φ_{12} and φ_{22} are identifiable from the observations on w_t,

  whereas γ and ν_2 are not.

  If $\Gamma \neq 0$, the historical dynamics of w_t is **not constrained**

  and only Σ is identifiable from the observations on w_t.

- **Direct Modelling of Wishart Term Structure Models**:

  a third application of Extended Car Processes

  We consider an unobservable factor W_t which follows a Wishart autoregressive (WAR) process, that is, a processes valued in the space of $(n \times n)$ symmetric positive definite matrices.

- The **conditional historical Log-Laplace transform** of W_{t+1} given W_t is:

$$\begin{aligned} \psi_t^{\mathbb{P}}(\Gamma) &= \text{Log}\{E_t \exp(Tr\Gamma W_{t+1})\} \\ &= Tr\left[M'\Gamma(I_n - 2\Sigma\Gamma)^{-1}MW_t\right] - \frac{K}{2}\text{Log det}[(I_n - 2\Sigma\Gamma)], \end{aligned}$$

  where Γ is a $(n \times n)$ matrix of coefficients, which can be chosen symmetric [since, with obvious notations, $Tr(\Gamma W_{t+1}) = \sum_{i\ j} \Gamma_{ij} W_{ij,t+1} = \sum_{i \leq j}(\Gamma_{ij} + \Gamma_{ji}) W_{ij,t+1}$].

- **This dynamics is Car(1)** and, if K is integer, it can be defined as:

$$\begin{aligned} &W_t = \textstyle\sum_{k=1}^{K} x_{k,t} x'_{k,t}\,, \quad (K \geq n) \\ &x_{k,t+1} = M x_{k,t} + \varepsilon_{k,t+1}\,, \ k \in \{1, \ldots, K\} \\ &\varepsilon_{k,t+1} \overset{\mathbb{P}}{\sim} IIN(0, \Sigma)\,, \ k \in \{1, \ldots, K\}\,, \text{ independents.} \end{aligned}$$

  Since W_t is not observed, it can be normalized by $\Sigma = I_n$.

- **The SDF is defined by:**

$$M_{t,t+1} = \exp\left[Tr(CW_{t+1}) + d\right] \ ,$$

where C is a $(n \times n)$ symmetric matrix and d is a scalar.

- **The associated R.N. dynamics is also Car(1):**

$$\begin{aligned} \psi_t^{\mathbb{Q}}(\Gamma) &= Tr\left[M'\left\{(C+\Gamma)[I_n - 2(C+\Gamma)]^{-1} - C(I_n - 2C)^{-1}\right\} MW_t\right] \\ &\quad -\frac{K}{2}\text{Log}\det[(I_n - 2(I_n - 2C)^{-1}\Gamma)] \,; \end{aligned}$$

- **The term structure of interest rates** at date t is affine in W_t and given by:

$$\begin{aligned} r(t,h) &= -\frac{1}{h}Tr[A(h)W_t] - \frac{1}{h}b(h) \,, \ h \geq 1 \\ A(h) &= M'[C + A(h-1)]\left\{I_n - 2[C + A(h-1)]\right\}^{-1} M \\ b(h) &= d + b(h-1) - \frac{K}{2}\text{Log}\det[I_n - 2(C + A(h-1))] \\ A(0) &= 0 \,, \ b(0) = 0 \,. \end{aligned}$$

- **In particular**, if K is integer, we get:

$$\begin{aligned} r(t,h) &= -\frac{1}{h}Tr[\sum_{k=1}^{K} A(h)x_{k,t}x'_{k,t}] - \frac{1}{h}b(h)\,,\ h \geq 1 \\ &= -\frac{1}{h}\sum_{k=1}^{K} x'_{k,t}A(h)x_{k,t} - \frac{1}{h}b(h)\,,\ h \geq 1\,, \end{aligned}$$

which is a sum of quadratic forms in $x_{k,t}$.

- If $K = 1$, we get the standard **Quadratic Term Structure Model**
$\hookrightarrow$ therefore, a special affine model.

- We can also define a **Quadratic Term Structure Model with a Linear Term: the historical dynamics** of x_{t+1} is given by:

$$x_{t+1} = m + Mx_t + \varepsilon_{t+1}\,,$$

$$\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0, \Sigma)\,;$$

- **the factor** $w_t = [x_t', vech(x_t x_t')]'$ can be shown to be Car(1), **and choosing:**

$$\begin{aligned} M_{t,t+1} &= \exp\left[C' x_{t+1} + Tr(C x_{t+1} x_{t+1}') + d\right] \\ &= \exp(C' x_{t+1} + x_{t+1}' C x_{t+1} + d)\,,\ (C \text{ is a symmetric } (n \times n) \text{ matrix})\,, \end{aligned}$$

→ **the term structure is found to be affine** in w_t, that is, of the form:

$$r(t,h) = x_t' \Lambda(h)\, x_t + \mu(h)' x_t + \nu(h)\,, h \geq 1\,,$$

where $\Lambda(h), \mu(h)$ and $\nu(h)$ follow recursive equations.

- **An Example of Back Modelling for a Security Market Model with Stochastic Dividends and Short Rates**

  Here we consider an Econometric Security Market Model where

  $\hookrightarrow$ **the risky assets are dividend-paying assets** and the **short rate is endogenous.**

  More precisely, the factor is given by $w_t = (y_t, \delta_t, r_{t+1})'$, where:

  – $y_t = (y_{1,t}, \ldots, y_{K_1,t})'$ denotes, for each date t, the K_1-dimensional vector of geometric returns associated to cum dividend prices $S_{j,t},\ j \in \{1, \ldots, K_1\}$;

  – $\delta_t = (\delta_{1,t}, \ldots, \delta_{K_1,t})$ is the associated K_1-dimensional vector of (geometric) dividend yields and, denoting $\tilde{S}_{j,t}$ as the ex dividend price of the j^{th} risky asset, we have $S_{j,t} = \tilde{S}_{j,t} \exp(\delta_{j,t})$;

  – r_{t+1} denotes the (predetermined) stochastic short rate for the period $[t, t+1]$;

- **the R.N. dynamics** of w_t is assumed to be Gaussian VAR(1):

$$w_{t+1} = A_0 + A_1\, w_t + \varepsilon_{t+1}\,,\ \ \varepsilon_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0, \Sigma)\,.$$

- **AAO restrictions**, applied to the K_1-dimensional vector y_{t+1}, are given by :

$$E_t^{\mathbb{Q}}\left[\exp(y_{j,t+1})\right] = \exp(r_{t+1} - \delta_{j,t})\,, \;\; j \in \{1,\dots,K_1\}\,,$$

$$\Longleftrightarrow \quad \begin{cases} a^{\mathbb{Q}}(e_j) & = & A_1{}' e_j & = & e_K - e_{j+K_1}\,, \;\; j \in \{1,\dots,K_1\}\,, \\ b^{\mathbb{Q}}(e_j) & = & A_0{}' e_j + \frac{1}{2}e_j'\Sigma e_j \, = \, 0\,, \;\; j \in \{1,\dots,K_1\}. \end{cases}$$

- in other words, the K_1 first equations of R.N. dynamics are:

$$y_{j,t+1} = -\frac{1}{2}\sigma_j^2 + r_{t+1} - \delta_{j,t} + \varepsilon_{j,t+1}\,, \;\; j \in \{1,\dots,K_1\}\,.$$

- then, **coming back to the historical dynamics** of w_t, we get :

$$\psi_t(u) = \psi_t^{\mathbb{Q}}(u - \alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t) = u'\left(A_0 + A_1 w_t\right) + \frac{1}{2}u'\Sigma u - \alpha_t'\Sigma u\,.$$

and, **if we impose** $\alpha_t = (\alpha_0 + \alpha\, w_t)$, we have

$$w_{t+1} = A_0 - \Sigma\alpha_0 + (A_1 - \Sigma\alpha)w_t + \xi_{t+1}\,, \;\; \xi_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,\Sigma)\,,$$

$$\text{and } \xi_{t+1} = \varepsilon_{t+1} + \Sigma(\alpha_0 + \alpha\, w_t)\,.$$

$i)$ **under the historical probability** : any VAR(1) distribution can be reached but only Σ is identifiable

$ii)$ **if we add the constraint** $\alpha = 0$, then the historical dynamics of w_t is constrained, and A_0 and α_0 are not identifiable.

- **Conclusions**:

- **In this paper we have proposed a general econometric approach to asset pricing modelling** based on three main elements :
  (i) the **historical discrete-time dynamics of the factor** representing the information;
  (ii) the **Stochastic Discount Factor (SDF);**
  (iii) the **risk-neutral (R.N.) dynamics of the factor.**

- **We have presented three modelling strategies :**
  a) the **Direct Modelling**,
  b) the **R.N. Constrained Direct Modelling** and
  c) the **Back Modelling**.

- In all the approaches **we have studied the internal consistency conditions**, induced by the AAO restrictions, **and the identification problem.**

- **Important role played by** the **R.N. Constrained Direct Modelling** and the **Back Modelling** strategies in determining AT THE SAME TIME :

  1 - **flexible Historical dynamics**

  $\longrightarrow$ to explain stylized facts about the observable factor

  2 - and a **Car R.N. dynamics**

  $\longrightarrow$ explicit or quasi explicit pricing formula for various contingent claims

- **Possibility to derive asset pricing models** able to accommodate **non-affine historical** and **risk-neutral factor dynamics** with **tractable pricing formulas**.

  $\longrightarrow$ this result is **achieved when the starting R.N. non-affine factor** can be **modified to be a R.N. Extended Car process**.

- **These strategies, already implicitly been adopted in several studies,**

  $\longrightarrow$ clearly could be **the basis for the specification of new asset pricing** models leading to promising empirical analysis.