Age and Term Structures in Duration Models

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Abstract

We consider duration models with an intensity depending on observable factors. At any date t, the current and lagged values of the factors are known, but not its future values ; then the future intensities are stochastic. This framework is convenient for studying how the observable intensities or the links between durations depend on the date (age structure) and on the horizon (term structure). We analyse how the factor dynamics affects both types of structures and give financial interpretations of the results for credit risk.

Keywords :Duration, Laplace Transform, Heterogeneity, Stochastic Intensity, Term Structure, Credit Risk, Copulas.

JEL number :

Structure par terme et âge dans les modèles de durée
Résumé

Nous considérons des modèles de durée dont l'intensité peut dépendre de facteurs observables. A toute date t, les valeurs présentes et passées des facteurs sont connues, mais pas leurs valeurs futures, de sorte que les intensités futures apparaissent stochastiques. Ce contexte est adapté pour étudier comment les intensités observables ou les mesures de liaison entre durées dépendent de la date (structure par age) et de l'horizon (structure par terme). Nous analysons comment la dynamique du facteur influe sur ces structures et donnons des interprétations financières des résultats pour le risque de crédit.

Mots clés : Durée, transformée de Laplace, hétérogénéité, intensité stochastique, structure par terme, risque de crédit, copule.

1 INTRODUCTION

Omitted heterogeneity in duration models is well-documented in the statistical and econometric literature. Typically it induces negative duration dependence at the marginal level, due to the so-called mover-stayer phenomenon, and creates spurious dependence when competing risk are considered. However this analysis generally assumes an individual time invariant heterogeneity. The aim of this survey is to analyse the effects of time varying heterogeneity. Loosely speaking the duration model is defined by means of an intensity which depends on underlying factors (or heterogeneity components). When the factors are observable, their current and lagged values are known at any date t, but not their future values. Then the future underlying intensities are stochastic. We explain how to compute the duration distribution at any date t, that is when the information set includes the past individual histories and the current and lagged factor values. We study especially how the observable intensity, the link between competing durations, or the factor distribution depend on the date (age structure) and on the horizon (term structure). In particular we discuss the influence of the factor dynamics on the patterns of age and term structures. The theoretical results are illustrated by credit risk analysis.

In Section 2 we consider the case of two duration variables and define the various age and term structures. They concern survivor functions, intensities, but also functional dependence measures between both variables. By considering the application to corporate default, interpretations of survivor functions [resp. intensities] are given in terms of difference between the prices of corporate bonds and T-bonds [resp. forward spread of interest rates]. Then we exhibit the pattern of the term structure and its time evolution in the standard model with unobserved individual heterogeneity and discuss several nonparametric constraints, which can be introduced by means of the age or term structures. The aim of Section 3 is to extend the results, when the underlying intensities depend on time dependent factors observable up to the current date. This situation is important for financial applications, since it underlies the analysis of the term structure of interest rate [Lando (1998), Duffie, Pan, Singleton (2000), Gourieroux, Monfort, Polimenis (2002), (2003)]. It is also useful for microeconomic applications to account for moral hazard phenomenon. In this case the factor variable can be interpreted as the time varying effort of the individual [Gourieroux, Jasiak (2002)]. This framework corresponds to the stochastic intensity model initially introduced by Cox []. We emphasize the distinction between spot and forward intensities and completely study the special case of exponential affine intensity models [Duffie, Filipovic, Schachermayer (2001), Gourieroux, Monfort, Polimenis (2003)]. Section 4 deals with the age and term structures corresponding to the minimum and maximum of duration variables. Indeed it is useful to consider these variables both for financial purpose, in particular for pricing credit derivatives such as a first-to-default basket, and for microeconomic analysis of competing risks.

2 Age and term structures

Let us consider two duration variables X and Y, with either continuous or discrete values. In the first case their range is $\mathbb{R}^+$, whereas it is $\mathbb{N}/0$ in the second one. Their joint survivor function is denoted by :

$$S(x, y) = P(X > x, Y > y), \tag{2.1}$$

where x, y belong to $\mathbb{R}^+$ in the continuous case [resp. to $\mathbb{N}$ in the discrete case]. For illustration the duration variables can represent two competing risks faced by an individual. They can also concern two similar risks faced by two different individuals. This second situation is typical of credit risk analysis, in which a portfolio including several credits is followed up. The price of this portfolio and the underlying risk depend on the times to default on these credits. Default analysis has to take into account the individual rates of default, but also the dependence between defaults, called default correlation. Moreover the risk analysis has to be performed during the whole life of the portfolio[3].

2.1 Definitions

It is well-known that the distribution of the durations (or of the residual durations) depend on the age of the individual [resp. of the credits]. An age [resp. term] structure defines the way characteristics of the distribution depend on the age [resp. on the horizon].

[3]Application of duration models to credit risk can be found in e.g. Lando (1998), Schonbucher (2000), Gourieroux, Monfort (2002), Gourieroux, Monfort, Polimenis (2002)b.

i) Age and term structures of one-dimensional survivor functions

Let us first focus on the age and term structures corresponding to the survivor function of the first variable X. Different survivor functions taking age and term into account can be introduced. The definitions below are valid for x, h, k in $I\!R^+$ (resp. $I\!N$) in the continuous case [resp. discrete case].

The marginal term structure of the survivor function of X at age h associates with any residual age x (or term, or maturity) the quantity :

$$S_h^m(x) = P[X > x + h | X > h] = \frac{S(x+h, 0)}{S(h, 0)}. \tag{2.2}$$

This is the survivor function of the residual duration $X - h$, when the individual [resp. credit] is still alive at h. This term structure depends on age h. When x is fixed and h varies, we get the age structure for given horizon x. The analysis of this survivor function is standard in duration analysis [see e.g. Kalbfleisch, Prentice (1980), Lancaster (1990)]. However the definition neglects the information available at age h on the second duration variable, since it implicitely assumes the independence between both variables. When X and Y are linked, other survivor functions can be considered according to the situation of Y.

If at date h the second event has not yet occurred, we can consider a (spot) term structure of the survivor function of X. It associates to any term x the quantity :

$$S_h^s(x) = P[X > x + h | X > h, Y > h] = \frac{S(x+h, h)}{S(h, h)}. \tag{2.3}$$

If at date h the second event has already occurred at a previous date $k \leq h$, we get another pattern of the term structure :

$$S_h^s(x|k) = P[X > x + h | X > h, Y = k], \forall h \geq k. \tag{2.4}$$

In the continuous case this quantity is given by :

$$S_h^s(x|k) = \frac{\frac{\partial}{\partial k} P[X > x + h, Y > k]}{\frac{\partial}{\partial k} P[X > h, Y > k]} = \frac{\frac{\partial S}{\partial y}(x+h, k)}{\frac{\partial S}{\partial y}(h, k)}, \forall h \geq k. \tag{2.5}$$

Similarly in the discrete case, we get :

$$S_h^s(x|k) = \frac{S(x+h,k) - S(x+h,k-1)}{S(h,k) - S(h,k-1)}, \forall h \geq k. \tag{2.6}$$

ii) Term structure of one-dimensional intensities.

In a continuous time framework it is usual to summarize the duration distribution by means of the hazard function. For a single duration variable X with survivor function $S(.)$, the hazard function measures the instantaneous rate of occurrence of the event :

$$\lambda(x) = \lim_{dx \to 0} \frac{1}{dx} P[X < x + dx | X > x] = -\frac{d \log S(x)}{dx}. \tag{2.7}$$

Thus the marginal term structure of hazard intensities can be defined as :

$$\lambda_h^m(x) = -\frac{\partial \log S_h^m(x)}{\partial x}, \tag{2.8}$$

and the (forward) term structures of hazard intensities as :

$$\lambda_h^f(x) = -\frac{\partial \log S_h^s(x)}{\partial x}, \tag{2.9}$$

$$\lambda_h^f(x|k) = -\frac{\partial \log S_h^s(x|k)}{\partial x}, \tag{2.10}$$

respectively. As above these term structures depend on age h.

In discrete time the hazard intensity is replaced by $1 - \pi(x)$, where $\pi(x)$ is the survivor intensity defined by :

$$\pi(x) = P[X > x | X > x - 1] = \frac{S(x)}{S(x-1)}. \tag{2.11}$$

Thus in discrete time we can also introduce the marginal and forward term structures of survivor intensities, denoted by $\pi_h^m, \pi_h^f, \pi_h^f(.|k)$, respectively, for age h.

iii) Age structure of copula

The term structures defined above are characteristics of the one-dimensional distribution of the first variable (or of the second variable if similar computations are performed for variable Y). However in a situation of competing risks we are also interested in the dependence between X and Y at any age h. This question matters only if no event occurs before date h.

The (spot) age structure of the joint survivor function associates with any age h the quantities :

$$S_h^s(x,y) = P[X > x+h, Y > y+h | X > h, Y > h] = \frac{S(x+h, y+h)}{S(h,h)}. \quad (2.12)$$

To distinguish the age structure of the one-dimensional distributions and the age structure of the dependence, we can introduce the associated survivor copula [see Sklar (1959), Joe (1997), Nelsen (1999)]. In continuous time it is defined by :

$$C_h^s(u,v) = S_h^s[(S_{h,x}^s)^{-1}(u), (S_{h,y}^s)^{-1}(v)], \quad (2.13)$$

where $S_{h,x}^s(x) = \frac{S(x+h,h)}{S(h,h)}, S_{h,y}^s(y) = \frac{S(h,y+h)}{S(h,h)}$.

This copula defines a cdf on $[0,1]^2$, which summarizes all dependencies between residual durations $X-h, Y-h$, which are invariant by increasing transformations. [4]

iv) Joint intensities

Finally we can also consider term structure of intensities taking into account the possibility of dependence.

Let us consider the continuous time framework. A continuous bivariate distribution can be characterized by means of the following intensity functions :

[4] In particular by time deformation after h.

$$\lim_{dx\to 0} P[X < y + x + dx | X > y, Y = y] = \gamma_1(x,y),$$

$$\lim_{dy\to 0} P[Y < x + y + dy | X = x, Y > x + y] = \gamma_2(x,y),$$

$$\lim_{dx\to 0} P[X < x + dx, Y > x + dx | X > x, Y > x] = \lambda_1(x),$$

$$\lim_{dy\to 0} P[X > y + dy, Y < y + dy | X > y, Y > y] = \lambda_2(y).$$

It can be checked that these intensities can be chosen arbitrarily.[5] The corresponding expression of the joint density function is :

$$\begin{aligned} f(x,y) &= \lambda_2(y)\exp -[\wedge_1(y) + \wedge_2(y)]\gamma_1(x-y,y)\exp -\Gamma_1(x-y,y), \text{if } x > y, \\ &= \lambda_1(x)\exp -[\wedge_1(x) + \wedge_2(x)]\gamma_2(x,y-x)\exp -\Gamma_2(x,y-x), \text{if } y > x, \\ \text{where } \wedge_i(x) &= \int_0^x \lambda_i(u)du, \Gamma_1(x,y) = \int_0^x \gamma_1(u,y)du, \\ \Gamma_2(x,y) &= \int_0^y \gamma_2(x,v)dv. \end{aligned}$$

2.2 Pricing interpretation

At this stage it is necessary to justify the terminology "spot", "forward" and "term structure" used in Section 2.1, and for this purpose to provide price interpretations of the computations above.

Let us assume two firms of a same industrial sector, created at the same date. If we are interested in default (failure) analysis the variables X and Y measure their lifetimes. These variables are generally not independent. First there can exist some (unobservable) industrial sector effect increasing default intensity of all firms of the sector, which will create a positive dependence. Second if the number of firms in the sector is rather small, the failure of one firm can increase the monopolistic power of the other ones and improve their situation, which induces a negative dependence effect.

[5] We assume that both events cannot occur jointly during a small time interval. Otherwise the joint distribution includes a degenerate component on the 45 % line (see example 3).

Let us now assume that both firms are issuing bonds. The corporate bonds are traded and priced on a bond market, and their prices are naturally compared with the prices of the Treasury bonds, which are considered as riskfree. These bonds are both traded on spot and forward markets.

Let us consider a zero-coupon T-bond paying 1$ at date T. Its (spot) price at date t is denoted by $B(t,T)$. On the forward market, its forward price at t for a payment at an intermediate date $\tau, t \leq \tau \leq T$ is denoted by $B^f(t,\tau,T)$. By the no-arbitrage condition, it is known that :

$$B^f(t,\tau,T) = \frac{B(t,T)}{B(t,\tau)}. \tag{2.14}$$

Similar notations can be introduced for the spot and forward prices of the zero-coupon bonds issued by both corporates. These prices are $B_1(t,T), B_2(t,T)$, $B_1^f(t,\tau,T), B_2^f(t,\tau,T)$. By the no-arbitrage condition the prices of the corporate bonds and of the T-bonds can be related. Under the assumption of independence between default and riskfree rates, we get :

$$B_1(t,T) = B(t,T) \overset{Q}{E_t} [\mathbb{1}_{X>T}], \tag{2.15}$$

$$B_2(t,T) = B(t,T) \overset{Q}{E_t} [\mathbb{1}_{Y>T}], \tag{2.16}$$

where Q denotes a risk neutral probability and $\overset{Q}{E_t}$ the conditional expectation with respect to the information available at time t.

Thus at date t the term structures of T-bonds and corporate bonds :

$x \to B(t,t+x), x \to B_1(t,t+x)$

are related through the term structure of the conditional survivor function :

$x \to \overset{Q}{E_t} [\mathbb{1}_{X>t+x}] = Q[X > t+x | I_t]$,

where Q is the risk neutral probability and I_t the information available at time t.

In the term structures introduced in Section 2.1 for the survivor functions, the conditional information consists in observations on default for both X and Y[6]. The corresponding values provide the term structure of the relative

[6]But it does not include the information on past prices. This extension is discussed in Section 3.

discrepancy between the corporate bond and T-bond prices, and admit a spot interpretation.

Let us now consider the pricing interpretation of the (risk neutral) survivor intensity. We get :

$$\begin{aligned} B_1^f(t,t+x,t+x+1) &= B^f(t,t+x,t+x+1)\frac{Q[X>t+x+1|I_t]}{Q[X>t+x|I_t]} \\ &= B^f(t,t+x,t+x+1)\pi_t^f(x). \end{aligned} \tag{2.17}$$

Thus the survivor intensity admits a forward interpretation. In fact if we introduce the geometric forward short term interest rates $r^f(t,t+x,t+x+1)$, we get :

$$\begin{aligned} &\exp -r_x^f(t,t+x,t+x+1) = \exp -r^f(t,t+x,t+x+1)\pi_x^f(x) \\ &\Leftrightarrow -\log \pi_t^f(x) = r_x^f(t,t+x,t+x+1) - r^f(t,t+x,t+x+1). \end{aligned} \tag{2.18}$$

Thus the term structure of survivor intensities is directly related to the term structure of the spread of the forward interest rates. Finally note that the first survivor intensity $\pi_t^f(0)$ can be interpreted as a spot intensity, directly related with the spread between the short term spot interest rates.

2.3 Model with unobserved individual heterogeneity

In duration models the hazard functions and the dependence between duration variables are very sensitive to unobserved individual heterogeneity. Typically if X and Y are independent with constant hazard function conditionally to a time invariant unobserved heterogeneity Z, we observe decreasing hazard function, due to the so-called mover-stayer phenomenon, and instantaneous dependence between the durations. The aim of this section is to review and complete these well-known results, especially concerning the age and term structure effects.

2.3.1 The model

We assume that :

Assumption A1 : X and Y are independent conditionally to a nonnegative factor Z;

Assumption A2 : X and Y have the same conditional distribution with survivor function :

$$P[X > x|Z] = \exp(-xZ), P[Y > y|Z] = \exp(-yZ).$$

Up to appropriate time deformations [see Appendix 1], this specification is often called Multivariate Mixed Proportional Hazard Model (MMPH) [see e.g. Van den Berg (1997), (2001)].

In continuous time, the hazard function is constant equal to Z, and the distribution of X (resp. Y) conditionally to Z is exponential with parameter Z. In discrete time the survivor intensity is constant equal to $\exp -Z$ and the conditional distribution of X (resp. Y) is geometric with parameter $\exp -Z$.

In this simple framework all term structures of interest depend on the distribution of Z only. It is important to note that an appropriate characterization of this distribution is :

in continuous time, the real Laplace transform defined by :

$$\psi(u) = E(\exp -uZ), u \in \mathbb{R}^+; \tag{2.19}$$

in discrete time the moment generating function defined by :

$$\psi(h) = E(\exp -hZ), h \in \mathbb{N}. \tag{2.20}$$

In any case, since Z is a nonnegative variable, the moment generating function (and then the real Laplace transform) characterizes the Z-distribution (see Feller (1971)). Moreover a real Laplace transform or moment generating function is characterized by conditions of alternating sign:

$$(-1)^n \psi^{(n)}(u) \geq 0, \forall u \in \mathbb{R}^+, \forall n \in \mathbb{N}, \tag{2.21}$$

for continuous argument [Bernstein theorem],

$$(-1)^n \Delta^n \psi(h) \geq 0, \forall h \geq n, h, n \in \mathbb{N}, \tag{2.22}$$

for discrete argument, where Δ denotes the differencing operator [Hausdorff theorem].

When Z is unobservable the joint survivor function of (X, Y) becomes :

$$\begin{aligned} S(x,y) &= P[X > x, Y > y] \\ &= EP[X > x, Y > y|Z] \\ &= E(\exp -(x+y)Z) \\ &= \psi(x+y). \end{aligned} \tag{2.23}$$

The importance of the Laplace transform clearly appears in the expression of the survivor function. Direct reasoning based on ψ will avoid the use of integral formulations of the pdf (see Van den Berg (1997), (2001) for such presentation).

2.3.2 Age structure of the heterogeneity distribution

Before deriving the relevant age and term structures, it is useful to consider how the heterogeneity distribution varies with the age h. The analysis depends on the information available at h, and is easy to develop (in continuous time) using of Laplace transform.

i) Let us first consider the conditioning set $X > h$. We get :

$$\begin{aligned} & E[\exp(-uZ)|X > h] \\ = \; & \frac{E[\exp(-uZ)\mathbb{1}_{X>h}]}{P(X > h)} \\ = \; & \frac{E(\exp(-uZ)P[X > h|Z])}{P[X > h]} \\ = \; & \frac{E[\exp -(u+h)Z]}{P[X > h]}. \end{aligned}$$

Therefore :

$$\tilde{\psi}_h(u) = E(\exp -uZ|X > h) = \frac{\psi(u+h)}{\psi(h)}. \tag{2.24}$$

ii) When the conditioning set is $X > h, Y > h$, a similar computation provides :

$$\psi_h(u) = E(\exp -uZ | X > h, Y > h) = \frac{\psi(u+2h)}{\psi(2h)}. \quad (2.25)$$

Thus the Laplace transforms are modified by an appropriate shift and a standardization to ensure the unitary value for $u = 0$.

For any nonnegative values u, v, h, we have :

$$\psi_h(u+v) - \psi_h(u)\psi_h(v) = \text{ Cov } (\exp -uZ, \exp -vZ | X > h, Y > h) \geq 0,$$

since $\exp -uZ$ and $\exp -vZ$ are both decreasing in Z. We deduce the Proposition below.

Proposition 1 : The age structure of the heterogeneity distribution is such that the functions $h \to \psi_h$ and $h \to \tilde{\psi}_h$ are both increasing.

Since they are upper bounded by 1, they converge asymptotically, to some limiting Laplace transforms $\tilde{\psi}_\infty$ and ψ_∞, respectively.

This is the mathematical explanation of the mover-stayer phenomenon. Since the Laplace transform ψ_h increases, the heterogeneity decreases with the age. At the limit the model becomes homogenous if $\tilde{\psi}_\infty$ and ψ_∞ correspond to an exponential function $\exp -uz_o$, say.

Finally, for the conditioning set $X > h, Y = k$, where $k \leq h$, we get :

$$\psi_{h|k}(u) = E[\exp(-uZ) | X > h, Y = k] = \frac{\psi'(u+h+k)}{\psi'(h+k)}. \quad (2.26)$$

Example 1 : For an heterogeneity factor with gamma distribution and parameter ν, the Laplace transform is : $\psi(u) = (1+u)^{-\nu}$. Then :

$$\tilde{\psi}_h(u) = \frac{(1+h)^\nu}{(1+h+u)^\nu}, \psi_h(u) = \frac{(1+2h)^\nu}{(1+2h+u)^\nu}, \psi_{h|k}(u) = \frac{(1+h+k)^{\nu+1}}{(1+h+k+u)^{\nu+1}}.$$

Thus, according to the conditioning set, the heterogeneity factor at age h is such that :

$$(1+h)Z \sim \gamma(\nu), (1+2h)Z \sim \gamma(\nu), (1+h+k)Z \sim \gamma(\nu+1),$$

respectively. We get a multiplicative effect of the age on the heterogeneity factor and possibly a change of degree of freedom. In the long run $h \to \infty$, the different Laplace transforms tend to $\tilde{\psi}_\infty = \psi_\infty = \psi_{\infty|k} = 1$, corresponding to the point mass at zero. Asymptotically the model becomes homogenous.

Example 2 : When the heterogeneity factor follows a stable distribution on $I\!R^+$ with stability parameter θ, the Laplace transform is :

$$\begin{aligned}
\psi(u) &= \exp -(u)^{1/\theta}, \theta > 1, \\
\text{Then : } \psi(u) &= \exp -((u+h)^{1/\theta} - (h)^{1/\theta}), \\
\psi_h(u) &= \exp -((u+2h)^{1/\theta} - (2h)^{1/\theta}), \\
\psi_{h|k}(u) &= \frac{(u+h+k)^{1/\theta-1} \exp -(u+h+k)^{1/\theta}}{(h+k)^{1/\theta-1} \exp -(h+k)^{1/\theta}}, u \geq 0.
\end{aligned}$$

2.3.3 Term structure of one-dimensional survivor functions.

The survivor functions are directly related to the Laplace transforms of the heterogeneity distributions.

$$S_h^m(x) = \frac{\psi(x+h)}{\psi(h)} = \tilde{\psi}_h(x), \quad (2.27)$$

$$S_h^s(x) = \frac{\psi(x+2h)}{\psi(2h)} = \psi_h(x), \quad (2.28)$$

$$S_h^s(x|k) = \psi'(x+h+k)/\psi'(h+k) = \psi_{h|k}(x). \quad (2.29)$$

Let us consider the continuous framework for instance. The associated hazard functions are given by :

$$\lambda_h^m(x) = -\frac{\partial \log S_h^m(x)}{\partial x} = -\frac{\psi'(x+h)}{\psi(x+h)}, \quad (2.30)$$

$$\lambda_h^f(x) = -\frac{\partial \log S_h^s(x)}{\partial x} = -\frac{\psi'(x+2h)}{\psi(x+2h)}, \quad (2.31)$$

$$\lambda_h^f(x|k) = -\frac{\partial \log S_h^s(x|k)}{\partial x} = -\frac{\psi"(x+h+k)}{\psi'(x+h+k)}. \quad (2.32)$$

Example 1 (continued) : For a gamma heterogeneity factor the survivor functions at age h are :

$$S_h^m(x) = \frac{(1+h)^\nu}{(1+h+x)^\nu}, S_h^s(x) = \frac{(1+2h)^\nu}{(1+2h+x)^\nu}, S_h^s(x|k) = \frac{(1+h+k)^{\nu+1}}{(1+h+k+x)^{\nu+1}}.$$

They correspond to homothetic Pareto distributions (see e.g. Kalbfleisch, Prentice (1980), Lancaster (1990)) with parameters ν and $\nu+1$, respectively.

Example 2 (continued) : For a stable heterogeneity factor the survivor functions at age h are :

$$S_h^m(x) = \exp[-(x+h)^{1/\theta} - h^{1/\theta}],$$

$$S_h^s(x) = \exp[-(x+2h)^{1/\theta} - (2h)^{1/\theta}].$$

They correspond to truncated Weibull distribution.

2.3.4 Age structure of the copula

At age h the joint survivor function of residual durations is :

$$S_h^s(x,y) = \frac{\psi(x+y+2h)}{\psi(2h)} = \psi_h(x+y). \quad (2.33)$$

We immediately deduce the age structure of the associated copulas :

$$C_h^s(u,v) = \psi_h[\psi_h^{-1}(u) + \psi_h^{-1}(v)] \quad (2.34)$$

$$= \frac{\psi[\psi^{-1}(u\psi(2h)) + \psi^{-1}(v\psi(2h)) - 4h]}{\psi(2h)}. \quad (2.35)$$

As expected we get an Archimedean copula with generator ψ_h [Genest, Mc Kay (1986) a,b]. As a direct consequence the scalar measures of nonlinear dependence, such as the Kendall tau, will depend on the age. If ψ is selected in a parametric family, we get lower and upper bounds for the Kendall tau and these bounds vary with the age [see Van den Berg (1997) for bounds computed at age 0, only]. The pattern of the dependence as a function of the age is an important characteristic of joint duration models.

Example 1 (continued) : For a gamma heterogeneity factor the copula at age h is given by :

$$\begin{aligned} C_h^s(u,v) &= \frac{(1+2h)^\nu}{[1+2h+(1+2h)\dfrac{1-u^{1/\nu}}{u^{1/\nu}}+(1+2h)\dfrac{1-v^{1/\nu}}{v^{1/\nu}}]^\nu} \quad (2.36) \\ &= [u^{-1/\nu}+v^{-1/\nu}-1]^{-\nu}. \quad (2.37) \end{aligned}$$

Since the heterogeneity disappears for large h, we would have expected asymptotically some independence between X and Y. However we get a flat age structure of copulas, corresponding to a Clayton Copula [Kimeldorf, Sampson (1975), Clayton (1978)]. In fact it is easily checked that two functions ψ and $\tilde{\psi}$ with proportional inverses generate the same Archimedean copula :

$$\psi[\psi^{-1}(u)+\psi^{-1}(v)] = \tilde{\psi}[\tilde{\psi}^{-1}(u)+\tilde{\psi}^{-1}(v)].$$

In the gamma framework, we get :

$$\psi_h^{-1} = (1+2h)\psi_0^{-1}, \forall h,$$

and the result follows.

Example 2 : (continued) : For a stable heterogeneity factor the copula at age h is :

$$C_h^s(u,v) = \exp - \left\{[(2h)^{1/\theta} - \log u)^\theta + ((2h)^{1/\theta} - \log v)^\theta - 4h]^{1/\theta} - (2h)^{1/\theta}\right\}.$$

In particular at the initial date $h=0$, the copula is a Gumbel copula [Gumbel (1960)] :

$$C_0^s(u,v) = \exp -[(-\log u)^\theta + (-\log v)^\theta]^{1/\theta}.$$

2.4 Constrained model

The introduction of an individual heterogeneity factor is an indirect way for constraining the joint distribution of X, Y. Indeed under the assumptions of Appendix 1 this joint distribution is characterized by three one dimensional functional parameters, that are the two marginal cdf of X and Y and the Laplace transform of the heterogeneity factor (which characterizes the copula). The aim of this section is to discuss other nonparametric constraints easy to interprete [7].

2.4.1 Age independent joint survivor function

It is well-known that, in the continuous case, the exponential distribution is the one-dimensional distribution which features the lack of memory property. It is such that the conditional distribution of $X - h$ given $X > h$ is independent of age h. Let us now consider the same question in a bivariate framework and look for distributions such that :

$$S_h^s(x, y) = P[X > x + h, Y > y + h | X > h, Y > h],$$

is independent of h. The property below is proved in Appendix 2.

Proposition 2 : Let us assume continuous marginal distributions with densities tending to zero at infinity. The joint survivor function is age independent if and only if it can be written as :

$$\begin{aligned} S(x, y) &= \exp(-\lambda y) S_x(x - y), \text{ if } x > y, \\ &= \exp(-\lambda x) S_y(y - x), \text{ if } y > x, \end{aligned}$$

where λ is a nonnegative scalar and S_x, S_y are the marginal survivor functions of X and Y, respectively. The pdf f_x and f_y of the marginal distributions have to satisfy :

$$-f_x'/f_x \leq \lambda, -f_y'/f_y \leq \lambda \leq f_x(0) + f_y(0).$$

The joint distribution can be decomposed into a continuous component and a degenerate component associated with a one-dimensional continuous distribution restricted to the 45% line :

[7] See Gagliardini, Gourieroux (2002) for a general presentation of constrained nonparametric bivariate distributions.

$$S(x,y) = S^c(x,y) + S^d(x,y), \text{ say},$$

where the degenerate component is :

$$S^d(x,y) = \frac{f_x(0) + f_y(0) - \lambda}{\lambda} \exp[-\lambda \max(x,y)].$$

In the general case the constrained joint distributions are characterized by the two marginal distributions and a scalar dependence parameter λ. It is easily checked that the minimum of both durations $\min(X,Y)$ follows an exponential distribution with parameter λ. The distribution of $X - h$ given $X > h$ generally depends of the age, since it is not necessarily exponential, whereas it is independent of the age when the conditioning set is $X > h, Y > h$.

The associated survivor copula C_h^s is also independent of the age. Let us provide its expression when $S_x = S_y$. We get :

$$\begin{aligned} C^s(u,v) &= \exp[-\lambda S_x^{-1}(v)] S_x[S_x^{-1}(u) - S_x^{-1}(v)], \text{ if } u < v, \\ &= \exp[-\lambda S_x^{-1}(u)] S_x[S_x^{-1}(v) - S_x^{-1}(u)], \text{ if } v < u. \end{aligned}$$

Thus :

$$C^s(u,v) = \exp[-\lambda \min(S_x^{-1}(u), S_x^{-1}(v))] S_x(|S_x^{-1}(v) - S_x^{-1}(u)|), \forall u, v \in (0,1).$$

Example 3 : Let us consider the special case of marginal exponential distributions with parameter γ_x and γ_y, respectively. The joint survivor function is given by :

$$\begin{aligned} S(x,y) &= \exp[-(\lambda - \gamma_x)y] \exp(-\gamma_x x), \text{ if } x > y, \\ &= \exp[-\gamma_y y] \exp[-(\lambda - \gamma_y)x], \text{ if } y > x. \end{aligned}$$

The condition of Proposition 2 becomes :

$$\lambda \geq \gamma_x, \lambda \geq \gamma_y, \lambda \leq \gamma_x + \gamma_y.$$

The form of the joint distribution can be interpreted in terms of default intensities. Indeed it can be checked that :

$$\lim_{dh\to 0} \frac{1}{dh} P[X < h + dh, Y < h + dh | X > h, Y > h] = \gamma_x + \gamma_y - \lambda,$$

$$\lim_{dh\to 0} \frac{1}{dh} P[X < h + dh, Y > h + dh | X > h, Y > h] = \lambda - \gamma_x,$$

$$\lim_{dh\to 0} \frac{1}{dh} P[X > h + dh, Y < h + dh | X > h, Y > h] = \lambda - \gamma_y,$$

$$\lim_{dh\to 0} \frac{1}{dh} P[X > h + k + dh | X > h + k, Y = k] = \gamma_x,$$

$$\lim_{dh\to 0} \frac{1}{dh} P[Y > h + k + dh | X = k, Y > h + k] = \gamma_y.$$

Thus the five default intensities of interest are time and path independents. Moreover they depend on three parameters only to ensure jointly the marginal and global lack of memory property.

2.4.2 Age independent copulas

We have noted that in the model with unobserved individual gamma heterogeneity or in the constrained model of Section 2.4.1 the copula does not depend on the age h. Are they the only cases ?

Since the age structure of copula is invariant with respect to time deformation on durations X and Y, the restriction involves the initial copula only. The proposition below is proved in Appendix 3.

Proposition 3 : Let us consider two homogenous risks, that is a symmetric copula and also assume that the copula is differentiable. The duration model features age independent copula if and only if :

$$C(u, v) = C_1(u, v)\{u - \frac{1}{2}C_1(1, u)\} + C_1(v, u)\{v - \frac{1}{2}C_1(1, v)\}, \forall u, v,$$

where $C_1(u, v)$ denotes the partial derivative with respect to the first argument.

Since the copula is symmetric the condition can also be written as :

$$1 = \frac{\partial \log C(u,v)}{\partial u}\{u - \frac{1}{2}C_1(1,u)\} + \frac{\partial \log C(u,v)}{\partial v}\{v - \frac{1}{2}C_1(1,v)\}, \quad (2.38)$$

and explains how to derive the copula from the one-dimensional function $C_1(1,u)$. This function characterizes the conditional distribution of V when $u = 1$. It is easily checked that the knowledge of this function is equivalent to the knowledge of the distribution of the maximal standardized duration $\max(U,V)$. Indeed we get :

$P[Z = \max(U,V) < z] = C(z,z)$,

and the associated density of Z is $2C_1(z,z)$. By taking $u = v = z$ in equation (2.38), we deduce :

$$2C_1(z,z)/C(z,z) = [z - 1/2C_1(1,z)]^{-1}, \quad (2.39)$$

and the one-to-one relationship between both distributions.

2.4.3 Term structures independent of the date of occurrence of the other event.

Finally we can look for situations in which the conditional survivor functions : $P[X > x+h|X > h, Y = k]$ and $P[Y > y+h|Y > h, X = k]$ are independent of the date k of occurrence of the other event. Proposition 4 below is proved in Appendix 4.

Proposition 4 : The conditional survivor functions $P[X > x+h|X > h, Y = k]$ and $P[Y > y+h|Y > h, X = k]$ are independent of k for any $k \leq h$ and any x, y, if and only if the joint pdf of (X,Y) can be written as :

$$\begin{aligned} f(x,y) &= \tilde{a}_1(x)\tilde{b}_1(y), \text{ for } x > y, \\ &= \tilde{a}_2(x)\tilde{b}_2(y), \text{ for } x < y. \end{aligned}$$

The function $\tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2$ are not uniquely defined. They can be standardized as :

$$
\begin{aligned}
f(x,y) &= \mu_1 a_1(x) b_1(y), \text{ for } x > y, \\
&= \mu_2 a_2(x) b_2(y), \text{ for } y > x,
\end{aligned}
\tag{2.40}
$$

where a_1, b_1, a_2, b_2 are probability densities. Then a_1 (resp. a_2) provides the pdf of X given $Y = 0$ (resp. $Y = \infty$), b_1 (resp. b_2) provides the pdf of Y given $Y = \infty$, (resp. $X = 0$).

3 Factor Duration Models

A drawback of the standard model with unobserved heterogeneity considered in the literature [see Section 2.3] is to assume a time invariant factor Z. As a consequence the pattern of cross-sectional dependence and its age structure are closely related by means of the one-dimensional Laplace transform ψ. The aim of this section is to break down this link by introducing as many factors as dates. To simplify the presentation, we focus on the discrete time framework. Then the time invariant heterogeneity factor will be replaced by a factor process $Z = (Z_t)$.

The factor duration model is presented in Section 3.1. We discuss the information set and the associated age and term structures are derived when the factor process is compound autoregressive (CAR). In particular we consider the case of an autoregressive gamma process, which extends directly the example of gamma heterogeneity discussed in Section 2.

3.1 The specification

The model is specified in a discrete time framework. It is based on the following assumptions.

Assumption A.1* : There exists an underlying markovian factor process $Z = (Z_t)$.

Assumption A.2* : The durations X and Y are independent conditionally to past, current and future values of process (Z).

Assumption A.3* : The conditional distributions of X and Y given the factor process admit the survivor intensities :

$$\pi_x(h, Z_{h+1}) = P[X > h+1 | X > h, (Z)] = \exp[-\alpha_x(h+1)' Z_{h+1} + \beta_x(h+1)],$$

$$\pi_y(h, Z_{h+1}) = P[Y > h+1 | Y > h, (Z)] = \exp[-\alpha_y(h+1)' Z_{h+1} + \beta_y(h+1)].$$

The exponential affine factor specification of the survivor intensity is standard in duration literature [see e.g. Flinn, Heckman (1982), Heckman, Walker (1990), or Van den Berg (1997), (2001)]. It corresponds in continuous time to the affine specification of the hazard function [see e.g. Lando (1994), (1998), Duffie, Singleton (1999)].

In this framework it is possible to define as in Sections 2.1 - 2.2 the age and term structures of survivor functions, intensities... However we have to highlight the role of the information, which is assumed available at date h. Let us consider the survivor functions for the discussion .

i) Unobservable factor

If the factor process is not observable, the different survivor functions are defined as in Section 2.1 :

$$\begin{aligned} S^m_{x,h}(x) &= P[X > x+h | X > h], \\ S^s_{x,h}(x) &= P[X > x+h | X > h, Y > h], \\ S^s_{x,h}(x|k) &= P[X > x+h | X > h, Y = k]. \end{aligned}$$

Their expressions are obtained by integrating out all factor values. For instance, we get :

$$\begin{aligned} S^m_{x,h}(x) &= \frac{P[X > x+h]}{P[X > h]} \\ &= \frac{EP[X > x+h | (Z)]}{EP[X > h | (Z)]} \\ &= \frac{E\left[\prod_{k=1}^{x+h} \pi_x(k, Z_k)\right]}{E\left[\prod_{k=1}^{h} \pi_x(k, Z_k)\right]}. \end{aligned}$$

ii) Observable factor

If the factor is observable, the current and lagged values of the factor are known at date h, but the future ones are not. The survivor functions become :

$$\begin{aligned} S^m_{x,h}(x;\underline{Z_h}) &= P[X > x+h | X > h; \underline{Z_h}], \\ S^s_{x,h}(x;\underline{Z_h}) &= P[X > x+h | X > h, Y > h; \underline{Z_h}], \\ S^s_{x,h}(x|k,\underline{Z_h}] &= P[X > x+h | X > h, Y = k; \underline{Z_h}]. \end{aligned}$$

where $\underline{Z_h} = (Z_h, Z_{h-1}, Z_{h-2}\ldots)$.

Their expressions are obtained by integrating out all future factor values. For instance, we get :

$$\begin{aligned} S^m_{x,h}(x,\underline{Z_h}) &= \frac{P[X > x+h|\underline{Z_h}]}{P[X > h|\underline{Z_h}]} \\ &= E\left[\prod_{k=h+1}^{x+h} \pi_x(k,Z_k)|\underline{Z_h}\right] \\ &= E\left[\prod_{k=h+1}^{x+h} \pi_x(k,Z_k)|Z_h\right], \text{ by the Markov Assumption } A.1^*. \end{aligned}$$

3.2 Exponential affine duration model

For observable, or unobservable factor, the survivor functions involve predictions of product of survivor intensities. For instance :

$$\begin{aligned} S^m_{x,h}(x,\underline{Z}_h) &= E\left[\prod_{k=h+1}^{x+h} \pi_x(k,Z_k)|Z_h\right] \\ &= E\left[\exp \sum_{k=h+1}^{x+h} [-\alpha_x(k)'Z_k + \beta_x(k)]|Z_h\right]. \end{aligned} \tag{3.1}$$

They admit tractable expressions if the distribution of the factor process has a simple conditional multivariate Laplace transform. This explains the importance of the compound autoregressive process [CAR] introduced by Darolles et alii (2002), Polimenis (2001) [see also Duffie, Filipovic, Schachermayer (2001) for its analogue in continuous time, called affine process].

The exponential affine duration model is obtained under the additional assumption :

Assumption A4* : The factor process $Z = (Z_t)$ is compound autoregressive. Its conditional Laplace transform is such that :

$$E[\exp u' Z_{t+1} | Z_t] = \exp[a(u)' Z_t + b(u)], \text{ for any } u.$$

Thus the conditional Laplace transform is an exponential affine function of the current value Z_t.

The tractability of the CAR process is due to the Lemma below which explains how to compute recursively any multivariate conditional Laplace transform of the factor process [see Darolles et alii (2002), or Gourieroux, Monfort, Polimenis (2002)b for a proof].

Lemma 1 : For any deterministic sequence $[u]$ of vectors $(u_s, s = 1, \ldots)$. we have :

$$\begin{aligned} & E[\exp(u'_{t+1} Z_{t+1} + \ldots + u'_{t+K} Z_{t+K}) | Z_t] \\ = \; & \exp[A^{[u]}(t, t+K)' Z_t + B^{[u]}(t, t+K)], \end{aligned}$$

where the operators $A^{[u]}, B^{[u]}$ depend on functions $a(.)$ and $b(.)$ as well as on sequence $[u]$ and satisfy the backward recursion :

$$A^{[u]}(t, t+K) = a[u_{t+1} + A^{[u]}(t+1, t+K)],$$

$$B^{[u]}(t, t+K) = b[u_{t+1} + A^{[u]}(t+1, t+K)] + B^{[u]}(t+1, t+K), \text{ for } K > 0,$$

with terminal conditions :

$$A^{[u]}(t, t) = 0, B^{[u]}(t, t) = 0, \forall t.$$

The Lemma can be directly used to derive the expressions of the survivor functions. For instance we deduce from (3.1) the following corollary.

Corollary 1 : In the exponential affine duration model the survivor function $S^m_{x,h}$ is given by :

$$S^m_{x,h}(x;\underline{Z}_h) = \exp\left\{A^{-[\alpha_x]}(h,x+h)Z_h + B^{-[\alpha_x]}(h,x+h) + \sum_{k=h+1}^{x+h}\beta_x(k)\right\}.$$

The other survivor functions, intensities, copulas are derived in the same way for observable factors.

When the factor process is not observable the expression of the survivor function is modified (see 3.1.i)) and the initial factor value has to be reintegrated out. We get :

$$\begin{aligned} S^m_{x,h}(x) &= \frac{EP[X > x+h|Z_o]}{EP[X > h|Z_o]} \\ &= \frac{E\left\{\exp[A^{-[\alpha_x]}(0,x+h)'Z_o + B^{-[\alpha_x]}(0,x+h) + \sum_{k=1}^{x+h}\beta_x(k)\right\}}{E\left\{\exp[A^{-[\alpha_x]}(0,h')Z_o + B^{-[\alpha_x]}(0,h) + \sum_{k=1}^{h}\beta_x(k)]\right\}} \end{aligned} \tag{3.2}$$

The expression of the survivor function involves the marginal Laplace transform of Z_o. It is proved in Darolles et alii (2002) that the CAR process is stationary if $\lim_{n\to\infty} a^{oK}(u) = 0$, where a^{oK} denotes function a compounded h times with itself. Under the stationarity condition, we get :

$$\begin{aligned} E[\exp u'Z_{t+1}] &= EE(\exp u'Z_{t+1}|Z_t) \\ &= E\exp[a(u)'Z_t + b(u)], \\ \text{or } \exp c(u) &= \exp[c[a(u)] + b(u)], \end{aligned} \tag{3.3}$$

where $\exp c(u) = E(\exp u'Z_t)$ is the Laplace transform of the stationary distribution. Thus we get :

$$E\exp(u'Z_{t+1}) = \exp c(u),$$

where c is the solution of : $c(u) - c[a(u)] = b(u)$.

We deduce :

$$
\begin{aligned}
S^m_{x,h}(x) &= \exp\Big[c[A^{-[\alpha_x]}(0,x+h) - c[A^{-[\alpha_x]}(0,h)] \\
&+ B^{-[\alpha_x]}(0,x+h) - B^{-[\alpha_x]}(0,h) + \sum_{k=h+1}^{x+h} \beta_x(k)\Big]. \qquad (3.4)
\end{aligned}
$$

3.2.1 Geometric duration with gamma autoregressive heterogeneity

i) The model

As an illustration we will consider the direct extension of the MMPH model with gamma heterogeneity. The survivor intensities are given by :

Assumption A.3 :**

$$\pi_x(h, Z_{h+1}) = \exp(-\alpha Z_{h+1}),$$

$$\pi_y(h, Z_{h+1}) = \exp(-\alpha Z_{h+1}).$$

Thus the duration dependence is only introduced by means of the factor, since the sensitivity coefficients are term independent. Moreover the sensitivity coefficients are identical for both durations. As a consequence the distribution of (X, Y) is symmetric. This equidependence assumption is important for the analysis of homogenous class of individual credits [see e.g. Gourieroux, Monfort (2002)].

If $Z_h = Z, \forall h$, we get the standard model with omitted individual heterogeneity. If the components Z_h are i.i.d. the observable survivor intensities are constant equal to $E(\exp -\alpha Z_h)$, identical for both duration variables. X and Y are independent, with the same geometric distribution.

Let us now introduce a factor dynamics to fulfill the gap between these extreme situations. It is natural to consider a process with a distribution extending the standard one-dimensional gamma distribution. Such a process is the autoregressive gamma process (ARG) [Gourieroux, Jasiak (2000)], which is the discretized version of the Cox-Ingersoll-Ross diffusion process [Cox, Ingersoll, Ross (1985)]. The transition distribution of an ARG process is defined as follows.

Assumption A.4 :** $Z = (Z_t)$ is an ARG process. Conditionally to Z_t, an intermediate discrete variable N_{t+1} is drawn in a Poisson distribution $\mathcal{P}[\lambda Z_t]$. Then Z_{t+1} is drawn in the gamma distribution $\gamma(\nu + N_{t+1})$.

The ARG process is compound autoregressive. Indeed we get :

$$\begin{aligned}
& E[\exp uZ_{t+1}|Z_t) \\
= \; & E[E(\exp uZ_{t+1}|N_{t+1})|Z_t] \\
= \; & E\left(\frac{1}{(1-u)^{N_{t+1}}}|Z_t\right)\frac{1}{(1-u)^{\nu}} \\
= \; & \exp\left[\frac{\lambda u Z_t}{1-u} - \nu \log(1-u)\right].
\end{aligned}$$

Thus $a(u) = \dfrac{\lambda u}{1-u}$, $b(u) = \nu \log(1-u)$.

The stationarity condition is $|\lambda| \leq 1$. The function a^{oK} is given by :

$$a^{oK}(u) = \frac{\lambda^K u}{1 - u\dfrac{1-\lambda^K}{1-h}}, \text{ if } \lambda < 1,$$

$$a^{oK}(u) = \frac{u}{1-Ku}, \text{ if } \lambda = 1.$$

It tends to zero at a geometric rate if $\lambda < 1$. This rate is hyperbolic in the limiting case $\lambda = 1$, and the process features long memory.

ii) Marginal survivor function

Let us apply Corollary 1. The function $A^{[-\alpha]}$, satisfies the recursion.

$$\begin{aligned}
A^{-[\alpha]}(t, t+K) &= a[-\alpha + A^{-[\alpha]}(t+1, t+K)] \\
&= a_{-\alpha}[A^{[-\alpha]}(t+1, t+K)], \text{ (say)},
\end{aligned}$$

with terminal condition $A^{[-\alpha]}(t+K, t+K) = 0$. We deduce :

$$A^{[-\alpha]}(t, t+K) = a_{-\alpha}^{oK}(0), \tag{3.5}$$

where : $a_{-\alpha}(u) = a(-\alpha + u)$.

Similarly we get :

$$B^{[-\alpha]}(t, t+h) = \sum_{k=1}^{h-1} b_{-\alpha}[a_{-\alpha}^{oh}(0)]. \tag{3.6}$$

We deduce from Corollary 1 the expression of the survivor function

Corollary 2 : Under Assumption A3*, the survivor function is given by :

$$S_{x,h}^{m}(x, \underline{Z}_h) = \exp[(a_{-\alpha})^{ox}(0)Z_h + \sum_{k=1}^{x-1} b_{-\alpha}[a_{-\alpha}^{ok}(0)]].$$

Then we could particularize the formula for the ARG process, where we have :

$$a_{-\alpha}(u) = a(u-\alpha) = \frac{\lambda(u-\alpha)}{1-u+\alpha}.$$

4 Minimum and maximum of duration variables

Let us now discuss how the nonlinear dependence between duration variables influence the joint distribution of the associated minimum and maximum durations (that is the order statistics). This analysis is required when we are concerned by correlated competing risks. Let us provide two examples in finance and insurance.

i) Risk derivatives are financial assets backed on a pool of individual credits. For instance a first-to-default basket pays 1$ at a contractual maturity H, if all credits in the pool have not defaulted before H. The pricing of the first-to-default basket requires the distribution of the minimum of individual lifetimes. Risk derivatives are also traded on the markets for the second-, third-to-default basket...

ii) Long term care insurance is interested in the future needs of the elderly. Different competing inabilities are considered for eating, washing him/herself, walking or concerning incontinence [see e.g. Rickayzen, Walsh (2002)]. The insurance contract can provide an annuity if three disabilities are observed, 50 % of the annuity if only two are observed.

4.1 Marginal distributions

They are easily deduced from the joint duration distribution :

$$S(x,y) = C[S_x(x), S_y(y)],$$

where C denotes the survivor copula at time origin.

The survivor function of the minimal duration is :

$$\begin{aligned} S_m(z) = P[\min(X,Y) > z] &= P[X > z, Y > z] \\ &= C[S_x(z), S_y(z)]. \end{aligned} \tag{4.1}$$

The survivor function of the maximal duration is :

$$\begin{aligned} S_M(z) = P[\max(X,Y) > z] &= P[X > z] + P[Y > z] - P[X > z, Y > z] \\ &= S_x(z) + S_y(z) - C[S_x(z), S_y(z)]. \end{aligned} \tag{4.2}$$

For given marginal duration distributions S_x, S_y, the survivor function S_m of the minimal duration [resp. S_M of the maximal duration] is an increasing [resp. decreasing] function of the copula. Let us consider two survivor copulas C_1, C_2 such that $C_1(u,v) \leq C_2(u,v), \forall u, v$. Then the two durations X, Y are more positive quadrant dependent with C_1 than with C_2 [8]. An increase of positive quadrant dependence implies a larger survivor function S_m and a smaller survivor function S_M. Thus the minimum is larger and the maximum is smaller for first order stochastic dominance. The dependence between the durations has a direct effect on the locations of the minimal and maximal values.

[8] see Dhaene, Goovaerts (1996), Tchen (1980), Muller, Scarsini (2000) for properties of the positive quadrant dependence ordering.

The limiting situations are obtained when X and Y are in an increasing deterministic relationship and in a decreasing one, respectively. The limiting copulas are $C_1(u,v) = \min(u,v)$ and $C_2(u,v) = \max(u+v-1,0)$, respectively.

For the deterministic positive dependence, we get :

$$S_m^1 = \min(S_x, S_y), S_M^1 = S_x + S_y - \min(S_x, S_y).$$

For the deterministic negative dependence, we get :

$$S_m^2 = \max(S_x + S_y - 1, 0), S_M^2 = S_x + S_y - \max(S_x + S_y - 1, 0).$$

Example 4 : When the durations have the same marginal distribution S_x, we get :

$$S_m^2 = \max(2S_x - 1, 0) \leq S_m \leq S_m^1 = S_x = S_M^1 \leq S_M \leq S_M^2 = 2S_x - \max(2S_x - 1, 0).$$

4.2 Model with unobserved individual heterogeneity

A more complete analysis of the joint distribution of the minimal and maximal durations can be done in the framework of Section 2.3. The duration variables follow independent, exponential distributions with a stochastic intensity Z : $S(x,y) = \psi(x+y)$, where ψ is the real Laplace transform of Z.

To avoid the constraint $\min(X,Y) \leq \max(X,Y)$ between the two statistics of interest, let us focus on the joint distribution of :

$$D_1 = \min(X,Y), D_2 = \max(X,Y) - \min(X,Y).$$

When Z is known, it is easily checked that the variables D_1, D_2 are independent, with exponential distributions. The intensities are $2Z$ and Z, respectively. Thus the survivor function of D_1, D_2 is :

$$S_D(d_1, d_2) = P[D_1 > d_1, D_2 > d_2] = \psi(2d_1 + d_2). \tag{4.3}$$

Similar computations can be performed at age h :

$$\begin{aligned} S_{h,D}(d_1,d_2) &= P[D_1 > d_1 + h, D_2 > d_2 + h | D_1 > h] = \psi_h(2d_1 + d_2) \\ &= \psi(2d_1 + d_2 + 2h)/\psi(2h). \end{aligned} \tag{4.4}$$

Thus the properties of (D_1, D_2) are the same than the properties of (X, Y) up to a change of time unit for D_1. In particular they have identical copulas.

4.3 Constrained model

Special specifications can be derived by constraining the joint distribution of the durations D_1, D_2.

Proposition 5 : Let us consider homogenous risks X and Y, such that the joint distribution of X, Y is symmetric continuous. Then $D_1 = \min(X, Y)$ and $D_2 = \max(X, Y) - \min(X, Y)$ are independent if and only if :

$$f(x, y) = \begin{cases} a(x)b(y-x), \text{ if } y > x, \\ \\ a(y)b(x-y), \text{ if } x > y. \end{cases}$$

Then D_1 and D_2 are also independent at any age h.

Proof : see Appendix 5.

Example 5 : If X and Y are independent with identical survivor function $S(x) = \exp - \wedge(x)$, where $\wedge$ is a cumulated hazard function, the condition of Proposition 5 can only be satisfied if $\wedge(x) = \lambda x$. Then we get :

$$\begin{aligned} f(x, y) &= \lambda^2 \exp -\lambda(x+y) \\ \\ &= \lambda^2 \exp -2\lambda x \exp[-\lambda(y-x)]. \end{aligned}$$

This explains the result of Section 4.2

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Appendix 1
Time deformation

The general specification of the MMPH model is such that :

$$P[X^* > x^*|Z] = \exp[-Z \wedge_{x^*}(x^*)],$$

$$P[Y^* > y^*|Z] = \exp[-Z \wedge_{y^*}(y^*)],$$

where $\wedge_{x^*}$ (resp. $\wedge_{y^*}$) summarizes the baseline integrated hazard function for X^* (resp. Y^*) and the effect of the observable individual explanatory variables. Note that the baseline hazard functions can include the effect of individual time invariant covariates. Thus the model of Section 2.3.1 is valid for $X = \wedge_{x^*}(X^*), Y = \wedge_{y^*}(Y^*)$, that is after time deformations of X^* and Y^*.

Appendix 2
Age independent survivor function

The condition can be written as :

$$S(x+h, y+h) = S(h,h)S(x,y), \forall x, y, h. \tag{a.1}$$

i) First note that $(X > h, Y > h) = [\min(X,Y) > h]$, and that the distribution of $\min(X-h, Y-h) = \min(X,Y) - h$ given $\min(X,Y) > h$ is independent of h. Therefore the distribution of the minimum of the durations is an exponential distribution with parameter λ, say, and we have :

$$S(x+h, y+h) = \exp(-\lambda h)S(x,y), \forall x, y, h.$$

ii) In particular if $x > y$, we get :

$$S(x,y) = \exp(-\lambda y)S_x(x-y), \tag{a.2}$$

where S_x denotes the marginal survivor function of X. Similarly if $y > x$, we get :

$$S(x,y) = \exp(-\lambda x)S_y(y-x), \tag{a.3}$$

where S_y is the marginal survivor function of Y.

iii) The joint function S defined by Property 4 satisfies condition (a.1). Let us now check that it is a joint survivor function. For this purpose let us first compute the cross second order derivative of S. If $x > y$, we get :

$$\frac{\partial^2 S(x,y)}{\partial x \partial y} = \exp(-\lambda y)[\lambda f_x(x-y) + f_x'(x-y)],$$

where f_x is the pdf of X. This derivative is nonnegative if and only if $\lambda f_x + f_x' \geq 0 \Leftrightarrow \lambda \geq -\frac{f_x'}{f_x}$.

A similar condition is obtained for the marginal distribution of Y.

These conditions ensure that the continuous part of the joint distribution of (X, Y) is well-defined. However this distribution may also include a degenerate component corresponding to a one dimensional distribution restricted to the 45% line. More precisely we get the composition :

$$S(x,y) = S^c(x,y) + S^d(x,y),$$

where the continuous component is :

$$\begin{aligned} S^c(x,y) &= \exp(-\lambda y) S_x(x-y) + \frac{\lambda - f_x(0) - f_y(0)}{\lambda} \exp(-\lambda x), \text{ if } x > y, \\ &= \exp(-\lambda x) S_y(y-x) + \frac{\lambda - f_x(0) - f_y(0)}{\lambda} \exp(-\lambda y), \text{ if } y > x, \end{aligned}$$

and the degenerate component is :

$$S^d(x,y) = \frac{-\lambda + f_x(0) + f_y(0)}{\lambda} \exp[-\lambda \max(x,y)].$$

The degenerate component corresponds to a nonnegative measure if and only if $\lambda \leq f_x(0) + f_y(0)$, which provides the last condition. Finally it is easily checked that : $S(0,0) = 0, S(\infty, y) = 0, S(x, \infty) = 0$.

Appendix 3
Age independent copula

The condition for age independent copula is :

$$\frac{S(x+h,y+h)}{S(h,h)} = C\left[\frac{S(x+h,h)}{S(h,h)}, \frac{S(h,y+h)}{S(h,h)}\right], \forall x,y,h, \qquad (A.1)$$

where C is the common copula. By derivating with respect to the age we get :

$$\begin{aligned} & S_1(x,y) + S_1(y,x) - S(x,y)2S_1(0,0) \\ = \quad & C_1[S_1(x,0) + S_1(0,x) - S(x,0)2S_1(0,0)] \\ + \quad & C_2[S_1(y,0) + S_1(0,y) - S(0,y)2S_1(0,0)]. \qquad (A.2) \end{aligned}$$

The partial derivatives C_1 and C_2 can be computed by considering condition (A.1) for $h = 0$:

$$S(x,y) = C[S(x,0), S(0,y)].$$

By the implicit function theorem, we get :

$$S_1(x,y) = C_1 S_1(x,0), S_1(y,x) = C_2 S_1(y,0).$$

Thus condition (A.2) becomes :

$$\begin{aligned} & S_1(x,y) + S_1(y,x) - S(x,y)2S_1(0,0) \\ = \quad & \frac{S_1(x,y)}{S_1(x,0)}[S_1(x,0) + S_1(0,x) - S(x,0)2S_1(0,0)] \\ + \quad & \frac{S_1(y,x)}{S_1(y,0)}[S_1(y,0) + S_1(0,y) - S(y,0)2S_1(0,0)], \end{aligned}$$

or equivalently when X, Y have uniform distribution on [0,1] :

$$\begin{aligned} 2S(x,y) \quad = \quad & -S_1(x,y)S_1(0,x) - 2S_1(x,y)(1-x) \\ - \quad & S_1(y,x)S_1(0,y) - 2S_1(y,x)(1-y). \qquad (A.3) \end{aligned}$$

The condition on the survivor copula is directly derived by noting that :

$$C(u, v) = S(1 - u, 1 - v).$$

Appendix 4
Independence of the occurrence of the other event

i) Let us show that X and Y are independent conditionally to $X > h, Y < k$ for any pair (k, h) such that $k \leq h$. For $x \geq 0, y \leq k$ we get :

$$\begin{aligned} & P[X > x + h, Y < k - y | X > x, Y < k] \\ = \; & P[X > x + h | X > h, Y < k - y] P[Y < k - y | X > x, Y < k] \\ = \; & P[X > x + h | X > h, Y < k] P[Y < k - y | X > x, Y < k] \text{ (by assumption).} (a.1) \end{aligned}$$

Because of the independence property, we can write :

$$f(x, y) = \tilde{a}_1(x) \tilde{b}_1(y) \text{ for } x > h, y < k.$$

Since the condition is uniform in $h \geq k$, we deduce :

$$f(x, y) = \tilde{a}_1(x) \tilde{b}_1(y), \text{ for } x < y.$$

ii) The other condition $f(x, y) = a_2(x) b_2(y)$, for $x > y$, is deduced by symmetry.

Appendix 5
Condition for independence between D_1 and D_2

We have :

$$\begin{aligned} & P[D_1 > d_1, D_2 > d_2] \\ = \ & 2P[X > Y, \min(X,Y) > d_1, \max(X,Y) - \min(X,Y) > d_2] \\ = \ & 2P[X > Y, Y > d_1, X - Y > d_2] \\ = \ & 2P[Y > d_1, X - Y > d_2]. \end{aligned}$$

Therefore for $d_2 = x - y > o$, it is equivalent to factorize the joint density of D_1, D_2, or the joint density of $Y, X - Y$.

The result follows.

Finally it is easily checked that the independence between the durations D_1 and D_2 implies also their independence at age h.