Switching VARMA Term Structure Models

Alain MONFORT [(1)] Fulvio PEGORARO [(2)]

First version : February, 2005
This version : January, 2006

Abstract
Switching VARMA Term Structure Models

The purpose of the paper is to propose a global discrete-time modeling of the term structure of interest rates able to capture simultaneously the following important features : (i) interest rates with an historical dynamics involving several lagged values, and switching regimes; (ii) a specification of the stochastic discount factor (SDF) with time-varying and regime-dependent risk-premia; (iii) the possibility to derive explicit or quasi explicit formulas for zero-coupon bond and interest rate derivative prices; (iv) the positiveness of the yields at each maturity. We develop the Switching Autoregressive Normal (SARN) Term Structure model of order p and the Switching Autoregressive Gamma (SARG) Term Structure model of order p. Regime shifts are described by a Markov chain with (historical) state-dependent transition probabilities. In both cases multifactor generalizations are proposed.

Keywords : Affine Term Structure Models, Stochastic Discount Factor, Car processes, Switching Regimes, VARMA processes, Lags, Positiveness, Derivative Pricing.

JEL number : C1, C5, G1

[1]CNAM, Chaire de Modelisation Statistique, 292, rue Saint-Martin 75141 Paris cedex 03 (France); E-mail: monfort@cnam.fr. CREST, Laboratoire de Finance-Assurance, Bureau 1121, Timbre J320, 15, Boulevard Gabriel Péri, 92245 Malakoff Cedex (France); E-mail: monfort@ensae.fr.

[2]CEREMADE, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny 75775 Paris Cedex 16 (France); E-mail: pegoraro@ceremade.dauphine.fr. CREST, Laboratoire de Finance-Assurance, Bureau 1112, Timbre J320, 15, Boulevard Gabriel Péri, 92245 Malakoff Cedex (France); E-mail: pegoraro@ensae.fr.

1 INTRODUCTION

In this paper we propose a global discrete-time modeling of the term structure of interest rates, which captures simultaneously the following important features :

- interest rates with an historical dynamics involving several lagged values, and switching regimes;
- a specification of the stochastic discount factor (SDF) with time-varying and regime-dependent risk-premia;
- the possibility to derive explicit or quasi explicit formulas for zero-coupon bond and interest rate derivative prices;
- the positiveness of the yields at each maturity.

It is well known in the literature that interest rates show an historical dynamics involving lagged values *and* switching regimes [see, among the others, Hamilton (1988), Cai (1994), Driffill and Sola (1994), Garcia and Perron (1996), Gray (1996), Boudoukh, Richardson, Smith, and Whitelaw (1999), Ang and Bekaert (2002a, 2002b), Christiansen (2002), Christiansen and Lund (2005), Cochrane and Piazzesi (2005)]; indeed, changes in the business cycle conditions or monetary policy may affect real rates and expected inflation and cause interest rates to behave quite differently in different time periods. In addition, there is a large empirical literature on bond yields, based in general on the class of Affine Term Structure Models (ATSMs)[3], suggesting that regime switching models describe the term structure of interest rates better than single-regime models [see, for example, Bansal and Zhou (2002), Driffill, Kenc and Sola (2003), Evans (2003), Ang and Bekaert (2005), Dai Singleton and Yang (2005)].

This results lead us to propose dynamic term structure models (DTSMs) where the yield curve is driven by a univariate or multivariate factor (x_t)

[3]The Affine family of dynamic term structure models (DTSMs) is characterized by the fact that the zero-coupon bond yields are affine functions of Markovian state variables, and it gives closed-form expressions for zero-coupon bond prices which greatly facilitates pricing and econometric implementation [see Duffie and Kan (1996), and Dai and Singleton (2003) and Piazzesi (2003) for a survey]. Observe that the Affine Term Structure family is much larger that it has been considered in the literature : indeed, it has been observed recently that the family of Quadratic Term Structure Models (QTSMs) [see Beaglehole and Tenney (1991), Ahn, Dittmar and Gallant (2002), and Leippold and Wu (2002)] is a special case of the Affine class obtained by stacking the factor values and their squares [see Gourieroux and Sufana (2003), Cheng and Scaillet (2005)].

which depends on its p most recent lagged values [X_t, say] and for which all the sensitivity coefficients depend on the present and past values of a latent J-states non homogeneous Markov Chain (z_t) [Z_t, say] describing different regimes in the economy. Consequently, the joint dynamics of (x_t, z_t) is not a Compound Autoregressive (Car) process[4] under the historical probability, and allows for nonlinearities which has been documented in the literature [see Ait-Sahalia (1996), Stanton (1997), Ang and Bekaert (2002b)]. The factor (x_t) is considered as an exogenous variable or an endogenous variable: in the second case the factor is a vector of several yields.

We consider an exponential-affine SDF with time-varying and regime-dependent risk correction coefficients defined as functions of the present and past values of the factor (x_t) and the regime indicator function (z_t). In our models, both factor risk and regime-shift risk are priced, and this is done by taking into account not just the information at date t, that is (x_t, z_t), but a larger information given by (X_t, Z_t). This specification leads to stochastic and regime-dependent risk premia. Observe that, this specification is coherent with the recent empirical literature which suggests to define risk correction coefficients as functions, at the same time, of the state-factor and the volatility-factor, in order to well replicate the observed temporal variation of one-period expected excess returns on zero-coupon bonds [see Ahn, Dittmar and Gallant (2002), Dai and Singleton (2002), Duffee (2002), Duarte (2004)]. Moreover, the fact to consider these coefficients as function of (X_t, Z_t) lead to a multi-lag specification which generalizes the Markovian of order one specifications proposed in the literature [see Dai and Singleton (2000), Duffee (2002), Cheridito, Filipovic and Kimmel (2003), Duarte (2004), Dai, Le and Singleton (2006)].

At the same time, we want to exploit the tractability of Car models, and obtain explicit or quasi explicit formula for zero-coupon bond and interest rate derivative prices. This result is achieved by matching the historical distribution and the SDF in order to get a Car risk-neutral joint dynamics for (x_t, z_t). Moreover, in this paper we deeply use the nice property of the Car family of being able to incorporate lags and switching regimes.

It is now well known [see Gourieroux, Monfort and Polimenis (2005), and Darolles, Gourieroux, Jasiak (2006)] that the class of discrete-time affine (Car) models is much larger than the discrete-time counterparts of the continuous-time affine processes [see Duffie and Kan (1996), Dai and Singleton (2000), and Duffie, Filipovic and Schachermayer (2003)].

[4] A Car (discrete-time affine) process is a Markovian process with an exponential-affine conditional Laplace transform [see Darolles, Gourieroux, Jasiak (2006) for details].

We develop the Switching Autoregressive Normal (SARN) Term Structure model of order p and the Switching Autoregressive Gamma[5] (SARG) Term Structure model of order p, and in both cases we propose multifactor generalizations : the Switching Vector Autoregressive Normal (SVARN) and the Switching Vector Autoregressive Gamma (SVARG) Term Structure models of order p.

Even if the Gaussian family of models does not guarantee the positiveness of the yields for every time to maturity [see, among the others, Vasicek (1977), Dai and Singleton (2000), Bekaert and Grenadier (2001), Ang and Bekaert (2002), Ang and Piazzesi (2003), Ang, Piazzesi and Wei (2005)], we study the SARN(p) Term Structure model (and its multivariate generalization), because it extends many standard models, like the ones just mentioned above and the more recent ones like Dai, Singleton and Yang (2005). Indeed, the historical and risk-neutral dynamics of (x_t) depends from several of their lagged values and from several lagged values of the regime-indicator variable (z_t). In this general setting, we are able to derive formulas, for the yield curve and for the price of derivatives, with simple analytical or quasi explicit representations.

The second kind of models we propose in the paper, based on the (scalar and vector) Switching Autoregressive Gamma process of order p (which has a Regime-Switching AR(p) representation with a martingale difference error), implies the positiveness of the yields for each time to maturity, and regardless of an exogenous or endogenous specification for the factor (x_t). The SARG(p) and the SVARG(p) term structure models give the possibility to replicate complex nonlinear (historical and risk-neutral) factor dynamics and provide explicit or tractable formulas for zero-coupon bond and derivative prices. In a related study, Bansal and Zhou (2002) propose an (approximate, scalar and bivariate) discrete-time Cox-Ingersoll-Ross term structure model with regime shifts. We extend their framework, using the exact discrete-time equivalent of the CIR process (with switching regimes) generalized to an autoregressive order p larger than one (the SARG(p) and the SVARG(p) processes), allowing for a non homogeneous historical transition matrix for (z_t), pricing the regime-shift risk, and providing an exact yield to maturity formula [in Bansal and Zhou (2002), (z_t) is an homogeneous Markov chain, the associated risk correction coefficient is assumed equal to zero, and the term structure formula they provide is based on a log-linear approximation

[5]The Autoregressive Gamma (ARG) process is a Car process, and the ARG(1) specification is the discrete-time counterpart of the Cox-Ingersoll-Ross process [see Gourieroux and Jasiak (2006), Cox, Ingersoll, and Ross (1985)].

applied on the fundamental asset pricing equation].

In a recent paper Dai, Le and Singleton (2006) propose a (discrete-time multivariate) conditionally Gaussian term structure model where nonlinearities are introduced in the (latent) state-factor (historical and risk-neutral) dynamics by means of stochastic volatility factors, for which the risk-neutral conditional distribution is described by a particular VARG(1) process with conditionally independent components. The switching vector Autoregressive Gamma process we use to describe the risk-neutral dynamics of the factor (x_t), in the SVARG(p) term structure model, presents three generalizations with respect to their Markovian of order one specification: a) we consider an autoregressive order p in general larger than one; b) conditionally to the present and past values of x_t and z_t, there is dependence between the components of the factor x_{t+1}; c) the historical and risk-neutral dynamics of x_{t+1} is affected by switching regimes.

The plan of the paper is as follows. In Section 2, we present the Index-Car(p) processes. This family of processes is developed under univariate and multivariate specifications, with and without Switching Regimes. In particular, we study the (scalar and vector) Autoregressive Gaussian of order p models and the (scalar and vector) Autoregressive Gamma of order p models, under single-regime and regime-switching specifications. Then, this class of processes is used, following the SDF modeling principle, to the derive the SARN(p) and the SARG(p) discrete-time term structure models, and their multivariate generalizations. In Section 3 we study the SARN(p) and the SVARN(p) Term Structure models, we derive the Generalized Linear Term Structure formulas and we specify the historical and risk-neutral dynamics of the yield curve processes. These results, presented for an exogenous factor, have also been obtained in the endogenous case. Moreover, we discuss the propagation of shocks on the interest rate surface. Section 4 deal with the SARG(p) and the SVARG(p) Term Structure models. Here, regardless the endogenous or exogenous nature of the factor (x_t), we derive the Generalized Linear Term Structure formulas and the yield curve processes, and we guarantee the positiveness of the yields for each time to maturity. Finally, the pricing methodology proposed in sections 3 and 4, for zero-coupon bonds, is generalized in Section 5 to the case of interest rate derivatives. Section 6 concludes and appendices gather the proofs.

2 LAPLACE TRANSFORMS, CAR(p) PROCESSES AND SWITCHING REGIMES

It is now well documented [see e.g. Darolles, Gourieroux and Jasiak (2006), Gourieroux and Monfort (2006), Gourieroux, Monfort and Polimenis (2002, 2003), Polimenis (2001)] that the Laplace transform (or moment generating function) is a very convenient mathematical tool in many financial domains. It is, in particular, a crucial notion in the theory of Car(p) processes [see Darolles, Gourieroux and Jasiak (2006) for details].

2.1 Definition of a Car(p) process

Definition 1 [Car(p) process]: A n-dimensional process $\tilde{x} = (\tilde{x}_t,\, t \geq 0)$ is a compound autoregressive process of order p [Car(p)] if the distribution of $\tilde{x}_{t+1}$ given the past values $\underline{\tilde{x}_t} = (\tilde{x}_t, \tilde{x}_{t-1}, \ldots)$ admits a real Laplace transform of the following type:

$$\begin{aligned} & E\left[\exp(u'\tilde{x}_{t+1}) \,|\, \underline{\tilde{x}_t}\right] \\ & = E_t[\exp(u'\tilde{x}_{t+1})] \\ & = \exp\left[\tilde{a}_1(u)'\tilde{x}_t + \ldots + \tilde{a}_p(u)'\tilde{x}_{t+1-p} + \tilde{b}(u)\right], \quad u \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $a_i(u)$, $i \in \{1, \ldots, p\}$, and $b(u)$ are nonlinear functions, and where $a_p(u) \neq 0$, $\forall\, u \in \mathbb{R}^n$. The existence of this Laplace transform in a neighborhood of $u = 0$, implies that all the conditional moments exist, and that the conditional expectations and variance-covariance matrices (and all conditional cumulants) are affine functions of $(\tilde{x}'_t, \tilde{x}'_{t-1}, \ldots, \tilde{x}'_{t+1-p})$.

2.2 Univariate Index-Car(p) process

An important class of Car(p) processes are the Index-Car(p) processes, which are built from a Car(1) process. In this section we consider a univariate process x_t and the multivariate case will be considered in sections 2.6 and 2.7.

Definition 2 [Univariate Index-Car(p) process]: Let $\exp[a(u)y_t + b(u)]$ be the conditional Laplace transform of a univariate Car(1) process y_t, the process x_t admitting a conditional Laplace transform defined by:

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[a(u)(\beta_1 x_t + \ldots + \beta_p x_{t+1-p}) + b(u)\right], \quad u \in \mathbb{R}, \tag{2}$$

is called an Univariate Index-Car(p) process.

Note that, if y_t is a positive process and if the parameters $\beta_1, \dots, \beta_p$ are positive, the process x_t will be positive.

Using the notation $\beta = (\beta_1, \dots, \beta_p)'$ and $X_t = (x_t, x_{t-1}, \dots, x_{t+1-p})'$, the Laplace transform (2) can be written as:

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[a(u)\beta' X_t + b(u)\right] . \tag{3}$$

2.3 Examples of Univariate Index-Car(p) processes

a. Gaussian model

If y_t is a Gaussian AR(1) process defined by:

$$y_{t+1} = \nu + \rho y_t + \varepsilon_{t+1}$$

where ε_{t+1} is a gaussian white noise distributed as $\mathcal{N}(0, \sigma^2)$, the conditional Laplace transform of y_{t+1} given $\underline{y_t}$ is:

$$E\left[\exp(uy_{t+1}) \,|\, \underline{y_t}\right] = \exp\left[u\rho y_t + u\nu + \tfrac{\sigma^2}{2}u^2\right] .$$

The process is Car(1) with $a(u) = u\rho$ and $b(u) = u\nu + \frac{\sigma^2}{2}u^2$. The associated Index-Car(p) process has a conditional Laplace transform defined by:

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[u\rho(\beta_1 x_t + \ldots + \beta_p x_{t+1-p}) + u\nu + \tfrac{\sigma^2}{2}u^2\right] ;$$

so, using the notation $\varphi_i = \rho\beta_i$, we see that x_{t+1} is the Gaussian AR(p) process defined by:

$$x_{t+1} = \nu + \varphi_1 x_t + \ldots + \varphi_p x_{t+1-p} + \varepsilon_{t+1} \tag{4}$$

and its conditional Laplace transform becomes:

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[u\varphi' X_t + u\nu + \tfrac{\sigma^2}{2}u^2\right] , \tag{5}$$

where $\varphi = (\varphi_1, \dots, \varphi_p)'$.

b. Gamma model

Let us now consider an autoregressive gamma of order one [ARG(1)] process y_t. The conditional Laplace transform is [see Gourieroux and Jasiak (2005) for details]:

$$E\left[\exp(uy_{t+1}) \,|\, \underline{y_t}\right] = \exp\left[\tfrac{\rho u}{1-u\mu}\, y_t - \nu\log(1-u\mu)\right] , \quad \rho > 0\,, \mu > 0\,, \nu > 0\,,$$

and it is well known that, given y_t, y_{t+1} can be obtained by first drawing a latent variable U_{t+1} in the Poisson distribution $\mathcal{P}(\frac{\rho y_t}{\mu})$ and, then, drawing $\frac{y_{t+1}}{\mu}$ in the gamma distribution $\gamma(\nu+U_{t+1})$. The process y_{t+1} is positive and the associated Index-Car(p) process x_{t+1} is also positive. The conditional Laplace transform of this process is:

$$E\left[\exp(ux_{t+1})\,|\,\underline{x_t}\right] = \exp\left[\tfrac{\rho u}{1-u\mu}(\beta_1 x_t + \ldots + \beta_p x_{t+1-p}) - \nu \log(1-u\mu)\right],$$

with $\beta_i \geq 0$, for $i \in \{1,\ldots,p\}$, or using the same notation as above:

$$E\left[\exp(ux_{t+1})\,|\,\underline{x_t}\right] = \exp\left[\tfrac{u}{1-u\mu}\varphi' X_t - \nu \log(1-u\mu)\right]. \tag{6}$$

Similarly, given X_t, x_{t+1} can be obtained by drawing U_{t+1} in $\mathcal{P}(\frac{\varphi' X_t}{\mu})$ and $\frac{x_{t+1}}{\mu}$ in $\gamma(\nu+U_{t+1})$. It easily seen that the conditional mean and variance of x_{t+1}, given $\underline{x_t}$, are respectively given by $\nu\mu + \varphi' X_t$ and $\nu\mu^2 + 2\mu\varphi' X_t$; so, the process x_{t+1} has the weak AR(p) representation:

$$x_{t+1} = \nu\mu + \varphi' X_t + \varepsilon_{t+1}, \tag{7}$$

where ε_{t+1} is a conditionally heteroscedastic martingale difference, whose conditional variance is $\nu\mu^2 + 2\mu\varphi' X_t$; the process is stationary if and only if $\varphi' e < 1$ [where $e = (1,\ldots,1) \in \mathbb{R}^p$] and, in this case, the process ε_{t+1} has finite unconditional variance given by $\nu\mu^2 + 2\nu\mu^2\frac{\varphi' e}{1-\varphi' e}$. The unconditional mean of x_{t+1} is given by $\frac{\nu\mu}{1-\varphi' e}$.

2.4 Univariate Switching regimes Car(p) process

Let us first consider a J-states homogeneous Markov Chain z_{t+1}, which can take the values $e_j \in \mathbb{R}^J$, $j \in \{1,\ldots,J\}$, where e_j is the j^{th} column of the $(J \times J)$ identity matrix. The transition probability, from state e_i to state e_j is $\pi(e_i, e_j) = Pr(z_{t+1} = e_j \,|\, z_t = e_i)$. It is first worth noting that z_{t+1} is a Car(1) process.

Proposition 1 : The Markov chain process z_{t+1} is a Car(1) process with a conditional Laplace transform given by:

$$E[\exp(v' z_{t+1})|\,\underline{z_t}] = \exp(a_z(v,\pi)' z_t), \tag{8}$$

where

$$a_z(v,\pi) = \left[\log\left(\sum_{j=1}^{J}\exp(v' e_j)\pi(e_1, e_j)\right), \ldots, \log\left(\sum_{j=1}^{J}\exp(v' e_j)\pi(e_J, e_j)\right)\right]'.$$

[Proof : straightforward.]

Let us now consider a univariate Index-Car(p) process with a conditional Laplace transform given by $\exp\left[a(u)\beta' X_t + b(u)\right]$, and let us assume that $b(u)$ can be written:

$$\begin{aligned} b(u) &= \tilde{b}(u)'\lambda \quad \text{where} \\ \tilde{b}(u) &= (b_1(u), \ldots, b_m(u))' \text{ and } \lambda = (\lambda_1, \ldots, \lambda_m)' . \end{aligned} \tag{9}$$

We can generalize this model by assuming that the parameters λ_i are stochastic and linear functions of $Z_t = (z_t', \ldots, z_{t-p}')'$. More precisely, we assume that the conditional distribution of x_{t+1} given $\underline{x_t}$ and $\underline{z_{t+1}}$ has a Laplace transform given by:

$$E[\exp(u x_{t+1}) | \underline{x_t}\,, \underline{z_{t+1}}] = \exp\left[a(u)\beta' X_t + \tilde{b}(u)'\Lambda Z_t\right] , \tag{10}$$

where Λ is a $[m, (p+1)J]$ matrix. Note that we assume no instantaneous causality between x_{t+1} and z_{t+1} and we admit one more lag in Z_t that in X_t [examples given in Section 2.5 show that this assumption may be convenient]; if the process z_t is not observed by the econometrician the no instantaneous causality assumption is not really important at the estimation stage since we could rename z_t as z_{t+1}, however it will be useful at the pricing level in order to obtain simple pricing procedures [Dai, Singleton and Yang (2005) also make this kind of assumption]. The joint process $(x_{t+1}, z_{t+1}')'$ is easily seen to be a Car($p+1$) process.

Proposition 2 : The conditional Laplace transform of $(x_{t+1}, z_{t+1}')'$ given $\underline{x_t}\,, \underline{z_t}$ has the following form:

$$\begin{aligned} & E\left[\exp(u x_{t+1} + v' z_{t+1}) \,|\, \underline{z_t},\, \underline{x_t}\,\right] \\ = \; & \exp\left\{a(u)\beta' X_t + \left[e_1' \otimes a_z(v,\pi)' + \tilde{b}(u)'\Lambda\right] Z_t\right\} , \end{aligned} \tag{11}$$

where e_1 is the first component of the canonical basis in $\mathbb{R}^{p+1}$, and where $\otimes$ denotes the Kronecker product.

[Proof : straightforward.]

2.5 Examples of Univariate Switching regimes Car(p) processes

a. Gaussian case

Let us start from the AR(p) model (4). Its conditional Laplace transform is given by (5):

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[u\varphi' X_t + u\nu + \tfrac{\sigma^2}{2}u^2\right] ,$$

and the function $b(u)$ has the form (9) with $\tilde{b}(u)' = \left(u, \frac{u^2}{2}\right)$ and $\lambda' = (\nu, \sigma^2)$.

If λ is replaced by ΛZ_t, the joint process $(x_{t+1}, z'_{t+1})'$ is Car($p+1$) with a conditional Laplace transform given by:

$$\begin{aligned} & E\left[\exp(ux_{t+1} + v' z_{t+1}) \,|\, \underline{z_t},\, \underline{x_t}\right] \\ = \ & \exp\left[u\varphi' X_t + \left(u, \tfrac{u^2}{2}\right)\Lambda Z_t + a_z(v,\pi) z_t\right] . \end{aligned} \qquad (12)$$

More precisely, the dynamics is given by [using the notation $\Lambda = \begin{pmatrix}\lambda'_1 \\ \lambda'_2\end{pmatrix}$]:

$$x_{t+1} = \lambda'_1 Z_t + \varphi' X_t + (\lambda'_2 Z_t)^{1/2} \varepsilon_{t+1} \,, \qquad (13)$$

where ε_{t+1} is a gaussian white noise distributed as $\mathcal{N}(0, \sigma^2)$, $Z_t = (z'_t, \ldots, z'_{t-p})'$ and z_t is a Markov chain such that $Pr(z_{t+1} = e_j \,|\, z_t = e_i) = \pi(e_i,\, e_j)$.

In particular, let us consider the case:

$$\Lambda = \begin{bmatrix} (1, -\varphi_1, \ldots, -\varphi_p) \otimes \nu^{*'} \\ e'_1 \otimes \sigma^{*2'} \end{bmatrix} \qquad (14)$$

and $\nu^{*'} = (\nu^*_1, \ldots, \nu^*_J)$, $\sigma^{*2'} = (\sigma^{*2}_1, \ldots, \sigma^{*2}_J)$, the conditional distribution of x_{t+1} given $\underline{x_t}$ and $\underline{z_{t+1}}$ is the one corresponding to the switching AR(p) model defined by:

$$x_{t+1} - \nu^{*'} z_t = \varphi_1 \left(x_t - \nu^{*'} z_{t-1}\right) + \ldots + \varphi_p \left(x_{t+1-p} - \nu^{*'} z_{t-p}\right) + (\sigma^{*'} z_t)\varepsilon_{t+1} \,. \qquad (15)$$

b. Gamma case

Let us now start from the ARG(p) process associated with the conditional Laplace transform (6):

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[\tfrac{u}{1-u\mu}\varphi' X_t - \nu \log(1 - u\mu)\right] .$$

Here we have $\tilde{b}(u) = -\log(1-u\mu)$ and $\lambda = \nu$. If ν is replaced by ΛZ_t, where $\Lambda Z_t > 0$, the process x_t has, conditionally to the process z_t, a weak AR(p) representation given by:

$$x_{t+1} = \mu \Lambda Z_t + \varphi_1 \, x_t \; + \; \ldots \; + \; \varphi_p \, x_{t+1-p} \; + \; \zeta_{t+1} \, , \tag{16}$$

where ζ_{t+1} is a conditionally heteroscedastic martingale difference. For instance, we can take :

$$\Lambda = e_1' \otimes \frac{\tilde{\nu}\,'}{\mu} \, , \tag{17}$$

where $\tilde{\nu}\,' = (\tilde{\nu}_1, \ldots, \tilde{\nu}_J)$, $\tilde{\nu}_j \geq 0$. We have $\Lambda Z_t = \frac{\tilde{\nu}\,'}{\mu} z_t$ and, conditionally to the process z_t, the process x_t has a weak AR(p) representation given by:

$$x_{t+1} = \tilde{\nu}\,' z_t + \varphi_1 \, x_t \; + \; \ldots \; + \; \varphi_p \, x_{t+1-p} \; + \; \zeta_{t+1} \, . \tag{18}$$

It is also possible to consider a Λ of the form $(1, -\varphi_1, \ldots, -\varphi_p) \otimes \frac{\tilde{\nu}\,'}{\mu}$ if $\min(\tilde{\nu}_i) > \max(\tilde{\nu}_i) \sum_{i=1}^{J} \varphi_j$, since in this case $\Lambda Z_t = \frac{1}{\mu} \left(\tilde{\nu}\,' z_t - \sum_{i=1}^{J} \varphi_j \, \tilde{\nu}\,' z_{t-i} \right)$ ≥ 0. The weak conditional AR(p) representation is then given by:

$$x_{t+1} - \tilde{\nu}\,' z_t = \varphi_1 \, (x_t - \tilde{\nu}\,' z_{t-1}) \; + \; \ldots \; + \; \varphi_p \, (x_{t+1-p} - \tilde{\nu}\,' z_{t-p}) \; + \; \zeta_{t+1} \, . \tag{19}$$

2.6 Specification of multivariate Car(1) processes

In order to have simple notations we will consider the bivariate case, but all the results are easily extended to the general case. A bivariate Car(1) process $y_t = (y_{1,t}, y_{2,t})'$ will be defined in a recursive way. We consider two univariate exponential affine Laplace transforms :

$$\begin{aligned} & \exp\left[a_1(u_1) w_{1,t} + b_1(u_1)\right] \, , \\ \text{and} \qquad & \exp\left[a_2(u_2) w_{2,t} + b_2(u_2)\right] \, . \end{aligned} \tag{20}$$

Then, we assume that the conditional distribution of $y_{1,t+1}$ given $(y_{2,t+1}, \underline{y_{1,t}}, \underline{y_{2,t}})$ has a Laplace transform given by :

$$\begin{aligned} & E_t[\exp(u_1 y_{1,t+1}) \,|\, y_{2,t+1}, \underline{y_{1t}}, \underline{y_{2t}}] \\ & \quad = \exp\left[a_1(u_1)(\beta_o y_{2,t+1} + \beta_{11} y_{1,t} + \beta_{12} y_{2,t}) + b_1(u_1)\right] \end{aligned} \tag{21}$$

and the conditional distribution of $y_{2,t+1}$, given $(\underline{y_{1,t}}, \underline{y_{2,t}})$, has a Laplace transform given by

$$E_t[\exp(u_2 y_{2,t+1}) \,|\, \underline{y_{1,t}}, \underline{y_{2,t}}] = \exp\left[a_2(u_2)(\beta_{21} y_{1,t} + \beta_{22} y_{2,t}) + b_2(u_2)\right] . \tag{22}$$

Note that, if the Laplace transforms (20) correspond to positive variables and if the parameters $\beta_o, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ are positive the bivariate process y_t has positive components. Moreover, we have the following result :

Proposition 3 : The bivariate process y_t defined by the conditional dynamics (21), (22) is a bivariate Car(1) process with a conditional Laplace transform given by :

$$\begin{aligned} & E[\exp(u_1 y_{1,t+1} + u_2 y_{2,t+1}) | \underline{y_{1,t}}, \underline{y_{2,t}}] \\ = \; & \exp\{[a_1(u_1)\beta_{11} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{21}] y_{1,t} \\ & \quad + [a_1(u_1)\beta_{12} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{22}] y_{2,t} \\ & \qquad + b_1(u_1) + b_2(u_2 + a_1(u_1)\beta_o)\} . \end{aligned} \tag{23}$$

[Proof : see Appendix 1.]

2.7 Specification of multivariate Index-Car(p) processes

We consider a bivariate process $\tilde{x}_t = (x_{1,t}, x_{2,t})'$ and we introduce the notations : $X_{1t} = (x_{1,t}, \ldots, x_{1,t+1-p})'$, $X_{2t} = (x_{2,t}, \ldots, x_{2,t+1-p})'$. Given the univariate Laplace transforms like (20), a bivariate Index-Car(p) is defined in the following way.

Definition 3 : A bivariate Index-Car(p) dynamics is defined by the conditional Laplace transforms:

$$\begin{aligned} & E_t[\exp(u_1 x_{1,t+1}) \,|\, x_{2,t+1}, \underline{x_{1,t}}, \underline{x_{2,t}}] \\ & = \exp\left[a_1(u_1)(\beta_o x_{2,t+1} + \beta'_{11} X_{1t} + \beta'_{12} X_{2t}) + b_1(u_1)\right] , \\ & E_t[\exp(u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}] = \exp\left[a_2(u_2)(\beta'_{21} X_{1t} + \beta'_{22} X_{2t}) + b_2(u_2)\right] , \end{aligned} \tag{24}$$

where the β_{ij} are p-vectors. It is easily seen that the process $\tilde{x}_t$ is a Car(p) process with a conditional Laplace transform given by (23) in which $y_{1,t}$ is replaced by X_{1t} and $y_{2,t}$ by X_{2t} and the β_{ij} by the β'_{ij}, i.e.

$$
\begin{aligned}
& E\left[\exp(u'\tilde{x}_{t+1}) \,|\, \underline{\tilde{x}_t}\right] \\
= \;& \exp\{[a_1(u_1)\beta_{11} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{21}]' X_{1t} \\
& \quad +[a_1(u_1)\beta_{12} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{22}]' X_{2t} \\
& \quad + b_1(u_1) + b_2(u_2 + a_1(u_1)\beta_o)\} \,.
\end{aligned} \tag{25}
$$

From the properties of Car(p) processes we get a representation of the form:

$$
\begin{cases}
x_{1,t+1} = \alpha_1 + \alpha_o x_{2,t+1} + \alpha'_{11} X_{1t} + \alpha'_{12} X_{2t} + \varepsilon_{1,t+1} \\
x_{2,t+1} = \alpha_2 + \alpha'_{21} X_{1t} + \alpha'_{22} X_{2t} + \varepsilon_{2,t+1}
\end{cases} \tag{26}
$$

where the errors terms satisfy :

$$
\begin{aligned}
E[\varepsilon_{1,t+1} \,|\, x_{2,t+1}, \underline{\tilde{x}_t}\,] &= 0 \\
E[\varepsilon_{2,t+1} \,|\, \underline{\tilde{x}_t}] &= 0\,;
\end{aligned} \tag{27}
$$

in particular, we get

$$
\begin{aligned}
E[\varepsilon_{1,t+1} \,|\, \underline{\tilde{x}_t}\,] &= 0 \\
E[\varepsilon_{2,t+1} \,|\, \underline{\tilde{x}_t}] &= 0 \\
Cov(\varepsilon_{1,t+1}, \varepsilon_{2,t+1}) &= E(\varepsilon_{1,t+1}\varepsilon_{2,t+1} \,|\, \underline{\tilde{x}_t}) \\
&= E\left[\varepsilon_{2,t+1} E(\varepsilon_{1,t+1} \mid x_{2,t+1}, \underline{\tilde{x}_t}) \,|\, \underline{\tilde{x}_t}\right] \\
&= 0\,.
\end{aligned} \tag{28}
$$

So, the error terms are non correlated, conditionally heteroscedastic, martingale differences. In particular, in the stationary case, $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are uncorrelated weak white noises and (26) is a weak recursive VAR(p) representation of the process $\tilde{x}_t$.

In the rest of the paper we will consider two important particular cases.

a) Normal VAR(p) or VARN(p) processes

In this case the conditional distributions defined by (20) are gaussian, with affine expectations and fixed variances. In other words:

$$\begin{aligned} a_1(u_1) = \rho_1 u_1 \,,\; b_1(u_1) = \nu_1 u_1 + \tfrac{\sigma_1^2 u_1^2}{2} \\ a_2(u_2) = \rho_2 u_2 \,,\; b_2(u_2) = \nu_2 u_2 + \tfrac{\sigma_2^2 u_2^2}{2} \,. \end{aligned} \tag{29}$$

Using the notations $\varphi_o = \rho_1\beta_o$, $\varphi_{11} = \rho_1\beta_{11}$, $\varphi_{12} = \rho_1\beta_{12}$, $\varphi_{21} = \rho_2\beta_{21}$, $\varphi_{22} = \rho_2\beta_{22}$, we have the following strong VAR(p) recursive representation for the process $\tilde{x}_t = (x_{1,t}, x_{2,t})'$:

$$\left\{ \begin{array}{rcl} x_{1,t+1} & = & \nu_1 + \varphi_o x_{2,t+1} + \varphi_{11}' X_{1t} + \varphi_{12}' X_{2t} + \sigma_1 \eta_{1,t+1} \\ x_{2,t+1} & = & \nu_2 + \varphi_{21}' X_{1t} + \varphi_{22}' X_{2t} + \sigma_2 \eta_{2,t+1} \,, \end{array} \right. \tag{30}$$

where $\eta_t = (\eta_{1,t}, \eta_{2,t})'$ is a bivariate gaussian white noise distributed as $\mathcal{N}(0, I_2)$, where I_2 denotes the (2×2) identity matrix.

b) Gamma VAR(p) or VARG(p) processes

In this case we have:

$$\begin{aligned} a_1(u_1) = \tfrac{\rho_1 u_1}{1-u_1\mu_1} \,,\; b_1(u_1) = -\nu_1 \log(1 - u_1\mu_1) \\ a_2(u_2) = \tfrac{\rho_2 u_2}{1-u_2\mu_2} \,,\; b_2(u_2) = -\nu_2 \log(1 - u_2\mu_2) \,, \end{aligned} \tag{31}$$

and the process $\tilde{x}_t = (x_{1,t}, x_{2,t})'$ has the following weak VAR(p) representation (using the same notation as above, and where all the parameters are positive):

$$\left\{ \begin{array}{rcl} x_{1,t+1} & = & \nu_1\mu_1 + \varphi_o x_{2,t+1} + \varphi_{11}' X_{1t} + \varphi_{12}' X_{2t} + \xi_{1,t+1} \\ x_{2,t+1} & = & \nu_2\mu_2 + \varphi_{21}' X_{1t} + \varphi_{22}' X_{2t} + \xi_{2,t+1} \,, \end{array} \right. \tag{32}$$

where $\xi_{1,t}$ and $\xi_{2,t}$ are non correlated, conditionally heteroscedastic, martingale differences. The conditional variances of $\xi_{1,t+1}$ and $\xi_{2,t+1}$ are given by:

$$\begin{aligned} V[\xi_{1,t+1} \,|\, \underline{\tilde{x}_t}] &= \nu_1\mu_1^2 + 2\mu_1[\varphi_o(\nu_2\mu_2 + \varphi_{21}' X_{1t} + \varphi_{22}' X_{2t}) \\ &\qquad + \varphi_{11}' X_{1t} + \varphi_{12}' X_{2t}] \qquad (33) \\ V[\xi_{2,t+1} \,|\, \underline{\tilde{x}_t}] &= \nu_2\mu_2^2 + 2\mu_2(\varphi_{21}' X_{1t} + \varphi_{22}' X_{2t}) \,. \end{aligned}$$

It is important to stress that the components of this VARG(p) process are positive.

2.8 Switching Multivariate Index-Car processes

Switching regimes can be introduced in a multivariate Index-Car(p) model using a method extending the one retained in the univariate case. If we assume that the functions $b_1(u_1)$, $b_2(u_2)$ appearing in definition 3 can be written, respectively, as $\tilde{b_1}(u_1)'\lambda_1$ and $\tilde{b_2}(u_2)'\lambda_2$, and if we replace λ_1 and λ_2, respectively by $\Lambda_1 Z_t$ and $\Lambda_2 Z_t$, we obtain the following conditional Laplace transform for the distribution of $(x_{1,t+1}, x_{2,t+1}, z_{t+1})$ given $(\underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t})$:

$$
\begin{aligned}
& E[\exp(u_1 x_{1,t+1} + u_2 x_{2,t+1} + v' z_{t+1}) |\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}] \\
= \quad & \exp\{[a_1(u_1)\beta_{11} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{21}]' X_{1t} \\
& \qquad +[a_1(u_1)\beta_{12} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{22}]' X_{2t} \\
& \qquad +[\, e_1' \otimes a_z(v,\pi)' + \tilde{b_1}(u_1)'\Lambda_1 + \tilde{b_2}(u_2 + a_1(u_1)\beta_o)'\Lambda_2] Z_t \Big\} \ ,
\end{aligned} \tag{34}
$$

where $a_z(v,\pi)$ is given in proposition 1. So we obtain a multivariate Car($p+1$) process.

Proposition 4 : The Laplace transform of $(x_{1,t+1}, x_{2,t+1}, z_{t+1})$, conditionally to $(\underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t})$, has the form given in (34) and the process $(x_{1,t}, x_{2,t}, z_t)$ is Car($p+1$).

2.9 Examples of Switching Multivariate Index-Car processes

a. Gaussian case

Taking

$$
a_1(u_1) = \rho_1 u_1,\ b_1(u_1) = \nu_1 u_1 + \tfrac{\sigma_1^2}{2} u_1^2,\ \ \tilde{b_1}(u_1)' = \left(u_1, \tfrac{u_1^2}{2}\right),
$$

$$
a_2(u_2) = \rho_2 u_2,\ b_2(u_2) = \nu_2 u_2 + \tfrac{\sigma_2^2}{2} u_2^2,\ \ \tilde{b_2}(u_2)' = \left(u_2, \tfrac{u_2^2}{2}\right),
$$

$$
\Lambda_1 = \begin{pmatrix} \lambda_{11}' \\ \lambda_{12}' \end{pmatrix},\ \ \Lambda_2 = \begin{pmatrix} \lambda_{21}' \\ \lambda_{22}' \end{pmatrix}\ ,
$$

and using the notations $\varphi_o = \rho_1\beta_o$, $\varphi_{11} = \rho_1\beta_{11}$, $\varphi_{12} = \rho_1\beta_{12}$, $\varphi_{21} = \rho_2\beta_{21}$,

$\varphi_{22} = \rho_2\beta_{22}$, we obtain the Switching VARN(p) model:

$$\left\{\begin{array}{rcl} x_{1,t+1} & = & \lambda'_{11}Z_t + \varphi_o x_{2,t+1} + \varphi'_{11}X_{1t} + \varphi'_{12}X_{2t} + (\lambda'_{12}Z_t)^{1/2}\eta_{1,t+1} \\ x_{2,t+1} & = & \lambda'_{21}Z_t + \varphi'_{21}X_{1t} + \varphi'_{22}X_{2t} + (\lambda'_{22}Z_t)^{1/2}\eta_{2,t+1}\,, \end{array}\right. \tag{35}$$

where $\eta_t = (\eta_{1,t}, \eta_{2,t})'$ is a gaussian white noise distributed as $\mathcal{N}(0, I_2)$, $Z_t = (z'_t, \ldots, z'_{t-p})'$, and where z_t is a homogeneous J-states Markov chain with transition probability $\pi(e_i, e_j)$. Note that (35) can also be written as:

$$\left\{\begin{array}{rcl} x_{1,t+1} & = & \tilde{\lambda}'_{11}Z_t + \tilde{\varphi}'_{11}X_{1t} + \tilde{\varphi}'_{12}X_{2t} + \varphi_o(\lambda'_{22}Z_t)^{1/2}\eta_{2,t+1} + (\lambda'_{12}Z_t)^{1/2}\eta_{1,t+1} \\ x_{2,t+1} & = & \lambda'_{21}Z_t + \varphi'_{21}X_{1t} + \varphi'_{22}X_{2t} + (\lambda'_{22}Z_t)^{1/2}\eta_{2,t+1}\,, \end{array}\right. \tag{36}$$

with $\tilde{\lambda}_{11} = \lambda_{11} + \varphi_o\lambda_{21}$, $\tilde{\varphi}_{11} = \varphi_{11} + \varphi_o\varphi_{21}$, $\tilde{\varphi}_{12} = \varphi_{12} + \varphi_o\varphi_{22}$ or, with obvious notations

$$\tilde{x}_{t+1} = \tilde{\lambda}'Z_t + \tilde{\Phi}'\tilde{X}_t + \begin{bmatrix} (\lambda'_{12}Z_t)^{1/2} & \varphi_o(\lambda'_{22}Z_t)^{1/2} \\ 0 & (\lambda'_{22}Z_t)^{1/2} \end{bmatrix} \eta_{t+1}\,. \tag{37}$$

b. Gamma case

If we take

$$a_1(u_1) = \tfrac{\rho_1 u_1}{1-u_1\mu_1},\ b_1(u_1) = -\nu_1\log(1-u_1\mu_1),\ \tilde{b_1}(u_1) = \log(1-u_1\mu_1),$$

$$a_2(u_2) = \tfrac{\rho_2 u_2}{1-u_2\mu_2},\ b_2(u_2) = -\nu_2\log(1-u_2\mu_2),\ \tilde{b_2}(u_2) = \log(1-u_2\mu_2)\,,$$

we obtain the positive Switching VARG(p) model

$$\left\{\begin{array}{rcl} x_{1,t+1} & = & \mu_1\Lambda'_1 Z_t + \varphi_o x_{2,t+1} + \varphi'_{11}X_{1t} + \varphi'_{12}X_{2t} + \xi_{1,t+1} \\ x_{2,t+1} & = & \mu_2\Lambda'_2 Z_t + \varphi'_{21}X_{1t} + \varphi'_{22}X_{2t} + \xi_{2,t+1}\,, \end{array}\right. \tag{38}$$

where $\xi_{1,t}$ and $\xi_{2,t}$ are non correlated, conditionally heteroscedastic, martingale differences, the conditional variances being respectively given by:

$$\begin{array}{rcl} V[\xi_{1,t+1}\,|\,\underline{\tilde{x}_t}] & = & \Lambda'_1 Z_t\mu_1^2 + 2\mu_1[\varphi_o(\Lambda'_2 Z_t\mu_2 + \varphi'_{21}X_{1t} + \varphi'_{22}X_{2t}) \\ & & \qquad +\varphi'_{11}X_{1t} + \varphi'_{12}X_{2t}] \qquad (39) \\ V[\xi_{2,t+1}\,|\,\underline{\tilde{x}_t}] & = & \Lambda'_2 Z_t\mu_2^2 + 2\mu_2(\varphi'_{21}X_{1t} + \varphi'_{22}X_{2t})\,. \end{array}$$

3 SWITCHING AUTOREGRESSIVE NORMAL (SARN) TERM STRUCTURE MODEL OF ORDER p

We first consider the case of univariate exogenous factor; the endogenous case and the multivariate cases will be discussed, respectively, in sections 3.7 and 3.8.

3.1 The historical dynamics

The first set of assumptions of a SARN(p) Term Structure model deals with the historical dynamics. We assume that the historical dynamics of the exogenous factor x_t is given by

$$x_{t+1} \quad = \quad \nu(Z_t) + \varphi_1(Z_t)x_t + \ldots + \varphi_p(Z_t)x_{t+1-p} + \sigma(Z_t)\varepsilon_{t+1}\,, \tag{40}$$

where ε_{t+1} is a gaussian white noise with $\mathcal{N}(0,1)$ distribution, $Z_t = (z_t', \ldots, z_{t-p}')'$, and z_t is a J-states non-homogeneous Markov chain such that $P(z_{t+1} = e_j \,|\, z_t = e_i; \underline{x_t}) = \pi(e_i, e_j\,; X_t)$ (e_i is the i^{th} column of the identity matrix I_J). Equation (40) will be also written

$$x_{t+1} \quad = \quad \nu(Z_t) + \varphi(Z_t)'X_t + \sigma(Z_t)\varepsilon_{t+1}\,, \tag{41}$$

where $X_t = (x_t, \ldots, x_{t+1-p})'$, $\varphi(Z_t) = (\varphi_1(Z_t), \ldots, \varphi_p(Z_t))'$. This model can also be rewritten in the following vectorial form:

$$X_{t+1} \quad = \quad \Phi(Z_t)X_t + [\nu(Z_t) + \sigma(Z_t)\varepsilon_{t+1}]\, e_1 \tag{42}$$

where

$$\Phi(Z_t) = \begin{bmatrix} \varphi_1(Z_t) & \ldots & \ldots & \varphi_p(Z_t) \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix}$$

is a $(p \times p)$-matrix, and where e_1 is the first column of the identity matrix I_p. Note that, since the coefficients φ_i are allowed to depend on Z_t and since the Markov chain z_t may not be homogeneous, the dynamics of (x_t, z_t) is not Car in general.

3.2 The Stochastic Discount Factor

The second element of a SARN(p) modeling is the SDF. We denote by $M_{t,t+1}$ the stochastic discount factor (SDF) between the date t and $t+1$ and in order to get time-varying risk-premia we specify it as an exponential affine function of the variables $(x_{t+1},\, z_{t+1})$ but with coefficients depending on the information at time t. More precisely we assume that:

$$\begin{aligned} M_{t,t+1} &= \exp\left[-c'X_t - d'Z_t + \Gamma(Z_t, X_t)\,\varepsilon_{t+1}\right. \\ &\qquad \left. -\tfrac{1}{2}\Gamma(Z_t, X_t)^2 - \delta(Z_t, X_t)' z_{t+1}\right] , \end{aligned} \tag{43}$$

where $\Gamma(Z_t, X_t) = \gamma(Z_t) + \tilde{\gamma}'(Z_t)X_t$. Observe that this specification extends to the multi-lag case the one proposed by Dai, Singleton, Yang (2005). It is well known that the existence of a positive stochastic discount factor is equivalent to the absence of arbitrage opportunity condition and that the price p_t at t of a payoff W_{t+1} at $t+1$ is given by:

$$\begin{aligned} p_t &= E[M_{t,t+1}W_{t+1} \,|\, I_t] \\ &= E_t[M_{t,t+1}W_{t+1}] , \end{aligned}$$

where the information I_t, available for the investors at the date t, is given by $(\underline{x_t}, \underline{z_t})$. More generally, the price $p_{t,h}$ at t of an asset paying W_{t+h} at $t+h$ is:

$$p_{t,h} = E_t\left[M_{t,t+1} \cdot \ldots \cdot M_{t+h-1,t+h}W_{t+h}\right] .$$

Using the absence of arbitrage assumption for the short-term interest rate between t and $t+1$, denoted by r_{t+1} and known at t, we get:

$$\begin{aligned} \exp(-r_{t+1}) &= E_t\left(M_{t,t+1}\right) \\ &= \exp\left[-c'X_t - d'Z_t\right] \times \textstyle\sum_{j=1}^{J} \pi\left(e_i, e_j\,; X_t\right) \exp\left[-\delta(Z_t, X_t)'e_j\right] , \end{aligned}$$

and assuming the normalization condition:

$$\textstyle\sum_{j=1}^{J} \pi\left(e_i, e_j\,; X_t\right) \exp\left[-\delta(Z_t, X_t)'e_j\right] = 1 \quad \forall Z_t, X_t\,, \tag{44}$$

we obtain:

$$r_{t+1} = c'X_t + d'Z_t\,. \tag{45}$$

3.3 Risk premia

In this paper we will use the following definition of a risk premium.

Definition 4 : Let p_t the price of a given asset at time t. The risk premium of this asset between t and $t+1$ is $\omega_t = \log(E_t p_{t+1}) - \log p_t - r_{t+1}$.

Using this definition we obtain interpretations of the Γ and δ functions appearing in the SDF which generalize that obtained by Dai, Singleton and Yang (2005).

Proposition 5 : The risk premium between t and $t+1$ of an asset providing the payoff $\exp(-\theta x_{t+1})$ at $t+1$ is :

$$\omega_t(\theta) = \theta\Gamma(X_t, Z_t)\sigma(Z_t)\,. \tag{46}$$

Therefore, θ, $\Gamma(X_t, Z_t)$ and $\sigma(Z_t)$ can be seen respectively as a risk sensitivity of the asset, a risk price and a risk measure. [Proof : see Appendix 2.]

Proposition 6 : If we consider a digital asset providing one money unit at $t+1$ if $z_{t+1} = e_j$, its risk premium between t and $t+1$ is given by :

$$\omega_t(\theta) = \delta_j(X_t, Z_t)\,, \tag{47}$$

and the j^{th} component of δ can be seen as the risk premium associated with the digital asset.[Proof : see Appendix 2.]

We observe that, in general, the magnitude of the risk premium $\omega_t(\theta)$ is not just depending on the currently observed values x_t and z_t, but it reflects the present and past values of both factors, that is, it is a function of the larger information represented by X_t and Z_t.

3.4 Risk-Neutral dynamics

The assumptions on the historical dynamics and on the SDF imply a risk-neutral dynamics. The probability density function of the one-period conditional risk-neutral probability with respect to the corresponding historical probability is $\frac{M_{t,t+1}}{E_t(M_{t,t+1})} = \exp(r_{t+1})M_{t,t+1}$. Note that using $E_t^{\mathbb{Q}}$ as the conditional expectation with respect to this risk-neutral distribution, the risk-premium ω_t can be written $\log(E_t p_{t+1}) - \log(E_t^{\mathbb{Q}} p_{t+1})$.

Proposition 7 : The risk-neutral dynamics of the process (x_t, z_t) is given by:

$$x_{t+1} \overset{\mathbb{Q}}{=} \nu(Z_t) + \gamma(Z_t)\sigma(Z_t) + [\varphi(Z_t) + \tilde{\gamma}(Z_t)\sigma(Z_t)]' X_t + \sigma(Z_t)\xi_{t+1}\,, \tag{48}$$

where $\stackrel{\mathbb{Q}}{=}$ denotes the equality in distribution (associated to the probability $\mathbb{Q}$), ξ_{t+1} is (under $\mathbb{Q}$) a gaussian white noise with $\mathcal{N}(0,1)$ distribution, and where $Z_t = (z_t', \ldots, z_{t-p}')'$, z_t being a Markov chain such that:

$$\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{z_t}; \underline{x_t}) = \pi\,(z_t, e_j\,; X_t) \exp\left[(-\delta(Z_t, X_t))' e_j\right] \,.$$

Note that, from (44), these probabilities add to one. [Proof : see Appendix 3.]

In order to get a generalized linear term structure we impose that the risk-neutral dynamics is switching regime gaussian Car(p). Using (13), this impose that the dynamics has to satisfy the following specification:

$$x_{t+1} \stackrel{\mathbb{Q}}{=} \nu^{*'} Z_t + \varphi^{*'} X_t + (\sigma^{*'} Z_t)\xi_{t+1}\,, \tag{49}$$

where $Z_t = (z_t', \ldots, z_{t-p}')'$, with z_t a J-states Markov chain such that

$$\mathbb{Q}(z_{t+1} = e_j \,|\, z_t = e_i) = \pi^*\,(e_i, e_j) \,. \tag{50}$$

From proposition 7, this implies the following restrictions on the historical dynamics and on the SDF:

i) $\sigma(Z_t) = \sigma^{*'} Z_t$: the historical stochastic volatility must be linear in Z_t;

ii)

$$\gamma(Z_t) \;=\; \frac{\nu^{*'} Z_t - \nu(Z_t)}{\sigma^{*'} Z_t} \;:$$

for a given historical stochastic drift $\nu(Z_t)$ and stochastic volatility $\sigma^{*'} Z_t$, the coefficient $\gamma(Z_t)$ belongs to the previous family indexed by the free parameter vector ν^*.

iii)

$$\tilde{\gamma}(Z_t) \;=\; \frac{\varphi^* - \varphi(Z_t)}{\sigma^{*'} Z_t} \;:$$

for a given historical stochastic slope parameter $\varphi(Z_t)$ and stochastic volatility $\sigma^{*'} Z_t$ the coefficient vector $\tilde{\gamma}(Z_t)$ belongs to the previous family indexed by the free parameter vector φ^*.

iv)

$$\delta_j(X_t, Z_t) \quad = \quad \log\left[\frac{\pi\,(z_t, e_j\,; X_t)}{\pi^*(z_t, e_j)}\right] :$$

for a given historical transition matrix $\pi\,(z_t, e_j\,; X_t)$, the coefficient $\delta_j(X_t, Z_t)$ depend on z_t only and belongs to the previous family indexed by the entries $\pi^*(z_t, e_j)$ of a transition matrix.

Note that condition *iv*) implies that the risk premia coefficients δ_j, $j \in \{1, \ldots, J\}$, cannot be all positive [or all negative] since this would imply $\pi\,(z_t, e_j\,; X_t) > \pi^*(z_t, e_j)$, $\forall j$ [or $\pi\,(z_t, e_j\,; X_t) < \pi^*(z_t, e_j)$, $\forall j$], which is impossible since $\sum_{j=1}^{J} \pi\,(z_t, e_j\,; X_t) = \sum_{j=1}^{J} \pi^*(z_t, e_j) = 1$. Also note that condition *iv*) implies the normalization condition (44).

3.5 The Generalised Linear Term Structure

We have seen in the previous section that the risk-neutral dynamics is defined by relations (49), (50); relation (49) can be rewritten:

$$X_{t+1} \overset{\mathbb{Q}}{=} \Phi^* X_t + \left[\nu^{*'} Z_t + (\sigma^{*'} Z_t)\xi_{t+1}\right] e_1 \tag{51}$$

where

$$\Phi^* = \begin{bmatrix} \varphi_1^* & \ldots & \ldots & \varphi_p^* \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix} \text{ is a } (p \times p) - \text{matrix}\,,$$

$$X_t = (x_t, \ldots, x_{t+1-p})'\,,$$

and where e_1 is the first column of the identity matrix I_p.

Denoting by $B(t,h)$ the price at t of a zero-coupon with residual maturity h, we have the following result.

Proposition 8 : In the univariate SARN(p) Term Structure model the price at date t of the zero-coupon bond with residual maturity h is :

$$B(t,h) = \exp\left(C_h'\, X_t + D_h'\, Z_t\right)\;,\; \text{for } h \geq 1\,, \tag{52}$$

where the vectors C_h and D_h satisfy the following recursive equations :

$$\begin{cases} C_h = \Phi^{*'} C_{h-1} - c \\ \\ D_h = -d\, +\, C_{1,h-1}\,\nu^*\, +\, \frac{1}{2}\, C_{1,h-1}^2\, \sigma^{*2}\, + \tilde{D}_{h-1} + F(D_{1,h-1})\,, \end{cases} \tag{53}$$

where $C_{1,h-1}$ denotes the first component of the p-dimensional vector C_{h-1}, $D_{1,h-1}$ and $D_{2,h-1}$ are, respectively, the first J-dimensional component and the remaining (pJ)-dimensional component of D_{h-1}, i.e. $D_{h-1} = (D'_{1,h-1}, D'_{2,h-1})'$, $\tilde{D}_{h-1} = (D'_{2,h-1}, 0)'$, and where $F(D_{1,h-1}) = e_1 \otimes a_z(D_{1,h-1}, \pi^*)$, e_1 being the vector $(1, 0, \ldots, 0)'$ of size $(p+1)$ and a_z is the J-vector given in proposition 1; σ^{*2} is the vector whose components are the squares of the entries of σ^*. The initial conditions are $C_0 = 0$, $D_0 = 0$ (or $C_1 = -c$, $D_1 = -d$). [Proof : see Appendix 4.]
For clarity we give again the expression of $a_z(D_{1,h-1}, \pi^*)$:

$$
\begin{aligned}
& a_z(D_{1,h-1}, \pi^*) \\
= \; & \left[\log\left(\sum_{j=1}^{J} \exp(D'_{1,h-1} e_j) \pi^*(e_1,\, e_j) \right), \ldots, \log\left(\sum_{j=1}^{J} \exp(D'_{1,h-1} e_j) \pi^*(e_J,\, e_j) \right) \right]'.
\end{aligned}
$$

From proposition 8 we see that the yields to maturity are:

$$
\begin{aligned}
R(t,h) &= -\frac{1}{h} \log B(t,h) \\
&= -\frac{C'_h}{h} X_t - \frac{D'_h}{h} Z_t \,, \quad h \geq 1 \,.
\end{aligned} \tag{54}
$$

So, they are linear functions of the p-dimensional vector X_t and of the $(p+1)J$-dimensional vector Z_t. This means that, the term structure at date t depends on the present and past values of x_t and z_t, and not just on their values in t. Moreover, we observe that there is, in general, instantaneous causality between x_t and z_t.

3.6 The Switching VARMA yield curve process

The result presented in Proposition 8 describes, conditionally to X_t and Z_t, the yields as a deterministic function of the time to maturity h, for a fixed date t. Nevertheless, in many financial and economic contexts one needs, for instance, also to study the effects of a shock, in the state variables, on the yield curve at different future times and for several maturities (e.g.: a Central Bank that needs to set a monetary policy). This means that we are interested in the dynamics of the process $R_{\mathcal{H}} = [\, R(t,h),\, 0 \leq t < T,\, h \in \mathcal{H}\,]$, for a given set of residual time to maturities $\mathcal{H} = (1, \ldots, H)$.

If we consider a fixed h, the process $R = [\, R(t,h),\, 0 \leq t < T\,]$ can be described by the following proposition.

Proposition 9 : For a fixed time to maturity h, the process $R = [\,R(t,h),\, 0 \leq t < T\,]$ is, under the historical probability, a Switching ARMA(p, $p-1$) process of the following type :

$$\begin{aligned} \Psi(L,Z_t)\,R(t+1,h) &= D_h(L)\,\Psi(L,Z_t)\,z_{t+1} + C_h(L)\,\nu(Z_t) \\ &\qquad + C_h(L)[(\sigma^{*'}Z_t)\,\varepsilon_{t+1}]\,. \end{aligned} \tag{55}$$

where

$$\begin{aligned} C_h(L) &= -\frac{1}{h}\,(C_{1,h} + C_{2,h}L + \ldots + C_{p,h}L^{p-1}) \\ D_h(L) &= -\frac{1}{h}\,(D_{1,h} + D_{2,h}L + \ldots + D_{p+1,h}L^{p}) \\ \Psi(L,Z_t) &= 1 - \varphi_1(Z_t)L - \ldots - \varphi_p(Z_t)L^p\,, \end{aligned}$$

are lag polynomials in the lag operator L, and where the AR polynomial $\Psi(L,Z_t)$ applies to t. [Proof : see Appendix 5].

Proposition 10 : For a given set of residual time to maturities $\mathcal{H} = (1,\ldots,H)$, the stochastic evolution of the yield curve process $R_{\mathcal{H}} = [\,R(t,h),\, 0 \leq t < T, h \in \mathcal{H}\,]$ takes the following particular Switching H-variate VARMA(p, $p-1$) representation:

$$\begin{aligned} \Psi(L,Z_t)\begin{pmatrix} R(t+1,1) \\ R(t+1,2) \\ \vdots \\ R(t+1,H) \end{pmatrix} &= \begin{pmatrix} C_1(L) \\ C_2(L) \\ \vdots \\ C_H(L) \end{pmatrix} (\sigma^{*'}Z_t)\varepsilon_{t+1} \\ &\quad + \begin{pmatrix} D_1(L) \\ D_2(L) \\ \vdots \\ D_H(L) \end{pmatrix} \Psi(L,Z_t)\,z_{t+1} + \begin{pmatrix} C_1(L) \\ C_2(L) \\ \vdots \\ C_H(L) \end{pmatrix} \nu(Z_t)\,. \end{aligned} \tag{56}$$

Similar results are easily obtained in the risk-neutral world.

3.7 Endogenous case

In the previous sections the factor x_t was exogenous. It is often assumed, in term structure models, that the factor x_t is the short rate process r_{t+1}. In this case the previous results remain valid, the only modification comes from the absence of arbitrage opportunity condition for r_{t+1}, which imposes:

$$c = e_1\,, d = 0\,, \tag{57}$$

with e_1 the first column of the identity matrix I_p; consequently, the initial conditions in the recursive equations of proposition 8 become:

$$C_1 = -e_1\,, D_1 = 0\,. \tag{58}$$

Moreover, the Switching ARMA(p, $p-1$) representation (55), or its analogous in the risk-neutral world, could be used to analyse how a shock on ε_t, i.e. on $r_{t+1} = R(t,1)$, is propagated on the surface $[\,R(t+\tau,h),\, \tau \in \mathcal{T},\, h \in \mathcal{H}\,]$, where $\mathcal{T} = \{0,\ldots,T-t-1\}$ and $\mathcal{H} = (1,\ldots,H)$ (for instance when the process z_t is exogenous).

3.8 Multi-Factor generalization : the SVARN(p) Term Structure model

For sake of notational simplicity we consider the two factor case but an extension to more that two factors is straightforward. The historical dynamics of $\tilde{x}_t = (x_{1,t}, x_{2,t})'$ is a bivariate SVARN(p) model given by:

$$\left\{\begin{array}{rcl} x_{1,t+1} &=& \nu_1(Z_t) + \varphi_o(Z_t)x_{2,t+1} + \varphi_{11}(Z_t)'X_{1t} \\ && \qquad\qquad + \varphi_{12}(Z_t)'X_{2t} + \sigma_1(Z_t)\varepsilon_{1,t+1} \\ x_{2,t+1} &=& \nu_2(Z_t) + \varphi_{21}(Z_t)'X_{1t} + \varphi_{22}(Z_t)'X_{2t} + \sigma_2(Z_t)\varepsilon_{2,t+1}\,, \end{array}\right. \tag{59}$$

where $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are independent standard normal white noises, $X_{1t} = (x_{1,t},\ldots,x_{1,t+1-p})'$, $X_{2t} = (x_{2,t},\ldots,x_{2,t+1-p})'$, $Z_t = (z'_t,\ldots,z'_{t-p})'$, with z_t a J-states non-homogeneous Markov chain such that $P(z_{t+1} = e_j \,|\, z_t = e_i; \underline{\tilde{x}_t}) = \pi(e_i, e_j\,; \tilde{X}_t)$, and where $\tilde{X}_t = (X'_{1t}, X'_{2t})'$. The recursive form (59) is equivalent to the canonical form :

$$\left\{\begin{array}{rcl} x_{1,t+1} &=& \tilde{\nu}_1(Z_t) + \tilde{\varphi}_{11}(Z_t)'X_{1t} + \tilde{\varphi}_{12}(Z_t)'X_{2t} \\ && \qquad + \sigma_1(Z_t)\varepsilon_{1,t+1} + \varphi_o(Z_t)\sigma_2(Z_t)\varepsilon_{2,t+1} \\ x_{2,t+1} &=& \nu_2(Z_t) + \varphi_{21}(Z_t)'X_{1t} + \varphi_{22}(Z_t)'X_{2t} + \sigma_2(Z_t)\varepsilon_{2,t+1}\,, \end{array}\right. \tag{60}$$

where $\tilde{\nu}_1 = \nu_1 + \varphi_o\nu_2$, $\tilde{\varphi}_{11} = \varphi_{11} + \varphi_o\varphi_{21}$, $\tilde{\varphi}_{12} = \varphi_{12} + \varphi_o\varphi_{22}$ or, with obvious notations:

$$\tilde{x}_{t+1} \;=\; \tilde{\nu}(Z_t) + \tilde{\Phi}(Z_t)\tilde{X}_t + S(Z_t)\varepsilon_{t+1}\,, \tag{61}$$

where

$$S(Z_t) = \begin{bmatrix} \sigma_1(Z_t) & \varphi_o(Z_t)\sigma_2(Z_t) \\ 0 & \sigma_2(Z_t) \end{bmatrix}$$

Using the notation

$$\Gamma(Z_t, \tilde{X}_t) = \left[\Gamma_1(Z_t, \tilde{X}_t),\, \Gamma_2(Z_t, \tilde{X}_t) \right]'$$

where $\Gamma_i(Z_t, \tilde{X}_t) = \gamma_i(Z_t) + \tilde{\gamma}_i(Z_t)'\tilde{X}_t,\ i \in \{1,2\}$ and $\Gamma(Z_t, \tilde{X}_t) = \gamma(Z_t) + \tilde{\Gamma}(Z_t, \tilde{X}_t)\tilde{X}_t$, with $\gamma(Z_t) = [\gamma_1(Z_t), \gamma_2(Z_t)]'$, $\tilde{\Gamma}(Z_t, \tilde{X}_t) = [\tilde{\gamma}_1(Z_t)', \tilde{\gamma}_2(Z_t)']'$, the SDF is defined as :

$$\begin{aligned} M_{t,t+1} \quad = \quad & \exp\Big[-c'\tilde{X}_t - d'Z_t + \Gamma(Z_t, \tilde{X}_t)'\,\varepsilon_{t+1} \\ & \qquad -\tfrac{1}{2}\Gamma(Z_t, \tilde{X}_t)'\Gamma(Z_t, \tilde{X}_t) - \delta(Z_t, \tilde{X}_t)'z_{t+1} \Big] \,. \end{aligned} \tag{62}$$

Assuming the normalization condition (44) and the absence of arbitrage opportunity for r_{t+1} we get:

$$r_{t+1} = c'\tilde{X}_t + d'Z_t \,. \tag{63}$$

It is also easily seen that the risk premium for an asset providing the payoff $\exp(-\theta'\tilde{x}_{t+1})$ at $t+1$ is $\omega(\theta) = \theta' S(Z_t)\Gamma(Z_t, \tilde{X}_t)$ and that the risk premium associated with the digital payoff $\mathbb{I}_{(e_j)}(z_{t+1})$ is unchanged.

Proposition 11 : The risk-neutral dynamics of the process $(\tilde{x}_t, z_t)$ is given by:

$$\tilde{x}_{t+1} \stackrel{\mathbb{Q}}{=} \tilde{\nu}(Z_t) + S(Z_t)\gamma(Z_t) + [\tilde{\Phi}(Z_t) + S(Z_t)\tilde{\Gamma}(Z_t, \tilde{X}_t)]\tilde{X}_t + S(Z_t)\xi_{t+1}\,, \tag{64}$$

where $\stackrel{\mathbb{Q}}{=}$ denotes the equality in distribution (associated to the probability $\mathbb{Q}$), ξ_{t+1} is (under $\mathbb{Q}$) a bivariate gaussian white noise with $\mathcal{N}(0, I_2)$ distribution, and where $Z_t = (z_t', \ldots, z_{t-p}')'$, with z_t a Markov chain such that:

$$\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{z_t}; \underline{\tilde{x}_t}) = \pi(z_t, e_j\,; \tilde{X}_t)\exp\left[(-\delta(Z_t, \tilde{X}_t))'e_j \right] .$$

[Proof : see Appendix 6.]

If we want to obtain a Switching bivariate Car process in the risk-neutral world, we must have using (37) :

i)

$$\begin{aligned}\sigma_1(Z_t) &= \sigma_1^{*'} Z_t \\ \sigma_2(Z_t) &= \sigma_2^{*'} Z_t \\ \varphi_o(Z_t) &= \varphi_o^*,\end{aligned}$$

and, therefore,

$$S(Z_t) = \begin{bmatrix} \sigma_1^{*'} Z_t & \varphi_o^* \sigma_2^{*'} Z_t \\ 0 & \sigma_2^{*'} Z_t \end{bmatrix}$$

ii)

$$\gamma(Z_t) = [S(Z_t)]^{-1}[\nu^* Z_t - \tilde{\nu}(Z_t)],$$

where ν^* is a $(2 \times (p+1)J)$-matrix.

iii)

$$\tilde{\Gamma}(Z_t, \tilde{X}_t) = [S(Z_t)]^{-1}\left[\Phi^* - \tilde{\Phi}(Z_t)\right],$$

where Φ^* is a $(2 \times 2p)$-matrix.

iv)

$$\delta_j(\tilde{X}_t, Z_t) = \log\left[\frac{\pi(z_t, e_j\,; \tilde{X}_t)}{\pi^*(z_t, e_j)}\right].$$

The risk-neutral dynamics can be written:

$$\begin{cases} x_{1,t+1} \overset{\mathbb{Q}}{=} \nu_1^* Z_t + \Phi_1^* \tilde{X}_t + S_1^*(Z_t)\xi_{t+1} \\ x_{2,t+1} \overset{\mathbb{Q}}{=} \nu_2^* Z_t + \Phi_2^* \tilde{X}_t + S_2^*(Z_t)\xi_{t+1}, \end{cases} \tag{65}$$

where ν_i^*, Φ_i^*, S_i^* are the i^{th} row of ν^*, Φ^*, S^*, with $i \in \{1, 2\}$, or

$$\tilde{X}_{t+1} \overset{\mathbb{Q}}{=} \tilde{\Phi}^* \tilde{X}_t + [\nu_1^* Z_t + S_1^*(Z_t)\xi_{t+1}]\, e_1 + [\nu_2^* Z_t + S_2^*(Z_t)\xi_{t+1}]\, e_{p+1},$$

where e_1 (respectively, e_{p+1}) is of size $2p$, with entries equal to zero except the first (respectively, the $(p+1)^{th}$) one which is equal to one, and

$$\tilde{\Phi}^* = \begin{bmatrix} \Phi_{11}^* & \Phi_{12}^* \\ \tilde{I} & \tilde{\mathbf{0}} \\ \Phi_{21}^* & \Phi_{22}^* \\ \tilde{\mathbf{0}} & \tilde{I} \end{bmatrix}$$

where $\Phi_1^* = (\Phi_{11}^*, \Phi_{12}^*)$, $\Phi_2^* = (\Phi_{21}^*, \Phi_{22}^*)$, and where $\tilde{\mathbf{0}}$ is a $[(p-1)\times p]$-matrix of zeros and $\tilde{I}$ is a $[(p-1)\times p]$-matrix equal to $(I_{p-1}, 0)$, where 0 is a vector of size $(p-1)$.
The term structure is given by the following proposition:

Proposition 12 : In the bivariate SVARN(p) Term Structure model the price at date t of the zero-coupon bond with residual maturity h is :

$$B(t,h) = \exp\left(C_h'\,\tilde{X}_t + D_h'\,Z_t\right)\ ,\ \text{for } h \geq 1 \tag{66}$$

where the vectors C_h and D_h satisfy the following recursive equations :

$$\left\{\begin{array}{l} C_h = \tilde{\Phi}^{*\prime} C_{h-1} - c \\ \\ D_h = -d\, +\, C_{1,h-1}\,\nu_1^{*'}\, +\, \frac{1}{2}\,C_{1,h-1}^2(\sigma_1^{*2} + \varphi_o^{*2}\sigma_2^{*2}) \\ \\ \qquad\qquad + C_{p+1,h-1}\,\nu_2^{*'} + \frac{1}{2}\,C_{p+1,h-1}^2\sigma_2^{*2} + \tilde{D}_{h-1} + F(D_{1,h-1})\,, \end{array}\right. \tag{67}$$

where $\tilde{D}_{h-1}$ and $F(D_{1,h-1})$ have the same meaning as in proposition 8, and the initial conditions are $C_0 = 0$, $D_0 = 0$ (or $C_1 = -c$, $D_1 = -d$). [Proof : see Appendix 7.]

So, proposition 12 shows that the yields to maturity are:

$$R(t,h) \;\;=\;\; -\frac{C_h'}{h}\tilde{X}_t - \frac{D_h'}{h}Z_t\,, \quad h \geq 1\,. \tag{68}$$

In the endogenous case we can take $x_{1t} = r_{t+1}$, and $x_{2t} = R(t,H)$ for a given time to maturity H. In this case the absence of arbitrage conditions for r_{t+1} and $R(t,H)$ imply:

$$\begin{aligned} (i)\; C_1 \;&=\; -e_1,\; D_1 = 0\,,\ \text{or}\ \ c = e_1,\; d = 0 \\ \\ (ii)\; C_H \;&=\; -H\,e_{p+1},\; D_H = 0\,. \end{aligned}$$

Using the notations $C_h = (C_{1,h}, C^*_{1,h}, C_{p+1,h}, C^*_{2,h})'$, $\tilde{C}_{1,h} = (C^{*\prime}_{1,h}, 0)'$, $\tilde{C}_{2,h} = (C^{*\prime}_{2,h}, 0)'$ (where the zeros are scalars), and $\tilde{C}_h = (\tilde{C}'_{1,h}, \tilde{C}'_{2,h})'$, it easily seen that the recursive equation $C_h = \tilde{\Phi}^{*\prime} C_{h-1} - c$ can be written :

$$C_h = \Phi_1^{*\prime} C_{1,h-1} + \Phi_2^{*\prime} C_{p+1,h-1} + \tilde{C}_{h-1} - c\,.$$

Conditions (i) are used as initial values in the recursive procedure of proposition 10, and conditions (ii) implies restrictions on the parameters $\tilde{\Phi}^*, \nu_1^*, \nu_2^*, \sigma_1^*, \sigma_2^*, \varphi_o^*, \pi^*(z_t, e_j)$ which must be taken into account at the estimation stage.

4 SWITCHING AUTOREGRESSIVE GAMMA (SARG) TERM STRUCTURE MODEL OF ORDER p

Like for SARN(p) models, we start the description of the SARG(p) modeling by the case of one exogenous factor.

4.1 The historical dynamics

We assume that the Laplace transform of the conditional distribution of x_{t+1}, given $(\underline{x_t}, \underline{z_t})$, is:

$$\begin{aligned} E\left[\exp(u x_{t+1}) \,|\, \underline{x_t}, , \underline{z_t}\right] &= \exp\Big[\tfrac{u}{1-u\mu(X_t,Z_t)}\left[\varphi_1(Z_t)x_t + \ldots + \varphi_p(Z_t)x_{t-p+1}\right] \\ &\qquad - \nu(Z_t)\log(1-u\mu(X_t,Z_t))\Big]\,, \end{aligned} \tag{69}$$

where $Z_t = (z'_t, \ldots, z'_{t-p})'$, with z_t a J-states non-homogeneous Markov chain such that $P(z_{t+1} = e_j \,|\, z_t = e_i; \underline{x_t}) = \pi(e_i, e_j\,; \tilde{X}_t)$, and where $X_t = (x_t, \ldots, x_{t+1-p})'$. Using the notation:

$$\begin{aligned} A[u; \varphi(Z_t), \mu(X_t, Z_t)] &= \tfrac{u}{1-u\mu(X_t,Z_t)}[\varphi_1(Z_t), \ldots, \varphi_p(Z_t)]' = \tfrac{u}{1-u\mu(X_t,Z_t)}\varphi(Z_t) \\ b[u; \nu(Z_t), \mu(X_t, Z_t)] &= -\nu(Z_t)\log(1-u\mu(X_t,Z_t))\,, \end{aligned}$$

relation (69) can be written:

$$\begin{aligned} E\left[\exp(u x_{t+1}) \,|\, \underline{x_t}, , \underline{z_t}\right] &= \exp\{A[u; \varphi(Z_t), \mu(X_t, Z_t)]' X_t \\ &\qquad + b[u; \nu(Z_t), \mu(X_t, Z_t)]\}\,. \end{aligned} \tag{70}$$

The process (x_t) can also be written:

$$
\begin{aligned}
x_{t+1} &= \nu(Z_t)\mu(X_t, Z_t) + \varphi_1(Z_t)x_t + \ldots + \varphi_p(Z_t)x_{t+1-p} + \varepsilon_{t+1} \\
&= \nu(Z_t)\mu(X_t, Z_t) + \varphi(Z_t)'X_t + \varepsilon_{t+1}\,,
\end{aligned}
\tag{71}
$$

where ε_{t+1} is a martingale difference sequence with conditional Laplace transform given by:

$$
\begin{aligned}
E\left[\exp(u\varepsilon_{t+1})\,|\,\underline{x_t},,\underline{z_t}\right] &= \exp\{-u[\nu(Z_t)\mu(X_t, Z_t) + \varphi(Z_t)'X_t] \\
&\qquad + A[u;\varphi(Z_t),\mu(X_t, Z_t)]'X_t \\
&\qquad\qquad + b[u;\nu(Z_t),\mu(X_t, Z_t)]\} \\
&= \exp\{[A[u;\varphi(Z_t),\mu(X_t, Z_t)] - u\varphi(Z_t)]'X_t \\
&\qquad + b[u;\nu(Z_t),\mu(X_t, Z_t)] - u\,\nu(Z_t)\mu(X_t, Z_t)\}\ .
\end{aligned}
\tag{72}
$$

Note that the dynamics of (x_t, z_t) is in general not Car.

4.2 The Stochastic Discount Factor

In the SARG(p) model the SDF is specified in the following way:

$$
\begin{aligned}
M_{t,t+1} &= \exp\{-c'X_t - d'Z_t + \Gamma(Z_t, X_t)\varepsilon_{t+1} + \Gamma(Z_t, X_t)\,[\nu(Z_t)\mu(X_t, Z_t) + \varphi(Z_t)'X_t] \\
&\qquad - A[\Gamma(Z_t, X_t);\varphi(Z_t),\mu(X_t, Z_t)]'X_t \\
&\qquad\quad - b[\Gamma(Z_t, X_t);\nu(Z_t),\mu(X_t, Z_t)] - \delta(Z_t, X_t)'z_{t+1}\}\ ,
\end{aligned}
\tag{73}
$$

where $\Gamma(Z_t, X_t) = \gamma(Z_t) + \tilde{\gamma}'(Z_t)X_t$, or, equivalently

$$
\begin{aligned}
M_{t,t+1} &= \exp\{-c'X_t - d'Z_t + \Gamma(Z_t, X_t)x_{t+1} - A[\Gamma(Z_t, X_t);\varphi(Z_t),\mu(X_t, Z_t)]'X_t \\
&\qquad - b[\Gamma(Z_t, X_t);\nu(Z_t),\mu(X_t, Z_t)] - \delta(Z_t, X_t)'z_{t+1}\}\ ,
\end{aligned}
\tag{74}
$$

Assuming the normalisation condition (44), we get that:

$$
r_{t+1} = c'X_t + d'Z_t\,. \tag{75}
$$

4.3 Useful lemmas

In the subsequent sections we will use several times the following lemmas.
Let us consider the functions:

$$\tilde{a}(u;\rho,\mu)=\frac{\rho u}{1-u\mu}\quad\text{and}\quad\tilde{b}(u;\nu,\mu)=-\nu\log(1-u\mu)\,;$$

we have:

Lemma 1 :

$$\tilde{a}(u+\alpha;\rho,\mu)-\tilde{a}(\alpha;\rho,\mu)=\tilde{a}(u;\rho^*,\mu^*)$$

$$\tilde{b}(u+\alpha;\nu,\mu)-\tilde{b}(\alpha;\nu,\mu)=\tilde{b}(u;\nu,\mu^*)$$

$$\text{with }\rho^*=\frac{\rho}{(1-\alpha\mu)^2},\ \mu^*=\frac{\mu}{1-\alpha\mu}\,,$$

[Proof : see Appendix 8.]

Lemma 1 immediately implies lemma 2.

Lemma 2 :

$$A[u+\alpha;\varphi(Z_t),\mu(X_t,Z_t)]-A[\alpha;\varphi(Z_t),\mu(X_t,Z_t)]=A[u;\varphi^*(Z_t),\mu^*(X_t,Z_t)]$$

$$b[u+\alpha;\nu(Z_t),\mu(X_t,Z_t)]-b[\alpha;\nu(Z_t),\mu(X_t,Z_t)]=b[u;\nu(Z_t),\mu^*(Z_t,X_t)]$$

$$\text{with }\varphi^*(Z_t)=\frac{\varphi(Z_t)}{[1-\alpha\mu(Z_t,X_t)]^2},\ \mu^*(Z_t,X_t)=\frac{\mu(X_t,Z_t)}{1-\alpha\mu(X_t,Z_t)}\,.$$

4.4 Risk-neutral dynamics

The Laplace transform of the risk-neutral conditional distribution of (x_{t+1}, z_{t+1}) is, using the notation $\Gamma_t = \Gamma(X_t, Z_t)$:

$$
\begin{aligned}
& E_t^{\mathbb{Q}}[\exp(u x_{t+1} + v' z_{t+1})] \\
= \;& E_t\{\exp[(u+\Gamma_t)x_{t+1} - A[\Gamma_t; \varphi(Z_t), \mu(X_t, Z_t)]' X_t \\
& \qquad -b[Z_t; \nu(Z_t), \mu(X_t, Z_t)] + (v - \delta(X_t, Z_t))' z_{t+1}]\} \\
= \;& \exp\{[(A[u+\Gamma_t; \varphi(Z_t), \mu(X_t, Z_t)] - A[\Gamma_t; \varphi(Z_t), \mu(X_t, Z_t)])' X_t \\
& \qquad + b[u+\Gamma_t; \nu(Z_t), \mu(X_t, Z_t)] - b[\Gamma_t; \nu(Z_t), \mu(X_t, Z_t)]]\} \\
& \qquad \times \textstyle\sum_{j=1}^{J} \pi(z_t, e_j\,; X_t) \exp[(v - \delta(Z_t, X_t))' e_j]\ ,
\end{aligned}
\tag{76}
$$

and, using lemma 2, (77) can be written:

$$
\begin{aligned}
& E_t^{\mathbb{Q}}[\exp(u x_{t+1} + v' z_{t+1})] \\
= \;& \exp\{A[u; \varphi^*(Z_t), \mu^*(X_t, Z_t)]' X_t + b[u; \nu(Z_t), \mu^*(Z_t, X_t)]\} \\
& \qquad \times \textstyle\sum_{j=1}^{J} \pi(z_t, e_j\,; X_t) \exp[(v - \delta(Z_t, X_t))' e_j]\ ,
\end{aligned}
\tag{77}
$$

with $\varphi^*(Z_t) = \dfrac{\varphi(Z_t)}{[1 - \Gamma_t \mu(Z_t, X_t)]^2}$ and $\mu^*(Z_t, X_t) = \dfrac{\mu(X_t, Z_t)}{1 - \Gamma_t \mu(X_t, Z_t)}$.

So, from (70), we see that the risk-neutral conditional distribution of x_{t+1}, given $(\underline{x_t}, \underline{z_t})$, is in the same class as the historical one and obtained by replacing $\varphi(Z_t)$ with $\varphi^*(Z_t)$, and $\mu(X_t, Z_t)$ with $\mu^*(Z_t, X_t)$.

In order to get a generalize linear term structure we impose that the risk-neutral dynamics is a switching regime Gamma Car(p) process. So, using the results in section 2.5.b, we get that $\varphi^*(Z_t)$ and $\mu^*(Z_t, X_t)$ must be constant, $\nu(Z_t) = \nu^{*'} Z_t$ and $\pi(z_t, e_j\,; X_t) = \pi^*(z_t, e_j) \exp[(\delta(Z_t, X_t))' e_j]$. Also note that μ^* must be positive as well as the components of ν^* and φ^*. This implies the following constraint on the historical dynamics and on the

SDF:

$$\begin{aligned}
\mu(X_t, Z_t) &= \mu^*[1 - \Gamma(X_t, Z_t)\mu(X_t, Z_t)] \\
\varphi(Z_t) &= \varphi^*[1 - \Gamma(X_t, Z_t)\mu(X_t, Z_t)]^2 \\
\nu(Z_t) &= \nu^{*'} Z_t \\
\delta_j(X_t, Z_t) &= \log\left[\tfrac{\pi(z_t, e_j\,; X_t)}{\pi^*(z_t, e_j)}\right].
\end{aligned}$$

We see that $\varphi(Z_t) = \frac{\varphi^*}{\mu^{*2}}\,\mu(X_t, Z_t)^2$, so $\mu(X_t, Z_t)$ must depend only on Z_t, and therefore the same is true for $\Gamma(X_t, Z_t)$. Finally, we have the constraint:

i)

$$\mu(Z_t) = \mu^*[1 - \Gamma(Z_t)\mu(Z_t)]$$

ii)

$$\varphi(Z_t) = \varphi^*[1 - \Gamma(Z_t)\mu(Z_t)]^2$$

iii)

$$\nu(Z_t) = \nu^{*'} Z_t$$

iv)

$$\delta_j(X_t, Z_t) = \log\left[\frac{\pi\,(z_t, e_j\,; X_t)}{\pi^*(z_t, e_j)}\right];$$

In particular, since $\varphi(Z_t) = \frac{\varphi^*}{\mu^{*2}}\,\mu(Z_t)^2$, the random vector must be proportional to a deterministic vector.

Moreover, it is easily seen that the risk premium corresponding to the payoff $\exp(-\theta x_{t+1})$ at $t+1$ is:

$$\begin{aligned}
\omega_t(\theta) &= \{A[-\theta; \varphi(Z_t), \mu(Z_t)] - A[-\theta; \varphi^*, \mu^*]\}' X_t \\
&\quad + b[-\theta; \nu^{*'} Z_t, \mu(Z_t)] - b[-\theta; \nu^{*'} Z_t, \mu^*].
\end{aligned}$$

Like in the gaussian case, we obtain an affine function in X_t also depending on Z_t. The risk premium associated with the digital asset providing one money unit at $t+1$ if $z_{t+1} = e_j$, is still given by (47).

4.5 The Generalised Linear Term Structure

Let us introduce the notations:

$$\begin{aligned} A^*(u) &= A(u;\varphi^*,\mu^*) \\ \tilde{C}_h &= (C_{2,h},\ldots,C_{p,h},0)'\,. \end{aligned} \tag{78}$$

As usual, $B(t,h)$ is the price at t of a zero-coupon bond with residual maturity h.

Proposition 13 : In the univariate SARG(p) Term Structure model the price at date t of the zero-coupon bond with residual maturity h is :

$$B(t,h) = \exp\left(C_h'\,X_t + D_h'\,Z_t\right)\,,\ \text{for } h \geq 1\,, \tag{79}$$

where the vectors C_h and D_h satisfy the following recursive equations :

$$\left\{ \begin{aligned} C_h &= -c + A^*(C_{1,h-1}) + \tilde{C}_{h-1} \\ D_h &= -d\ -\ \nu^*\log(1 - C_{1,h-1}\mu^*) +\ \tilde{D}_{h-1} + F(D_{1,h-1})\,, \end{aligned} \right. \tag{80}$$

where $\tilde{D}_{h-1}$ and $F(D_{1,h-1})$ have the same meaning as in proposition 8; the initial conditions are $C_0 = 0$, $D_0 = 0$ (or $C_1 = -c$, $D_1 = -d$) [Proof : see Appendix 9].

Again, we obtain a generalised linear term structure given by:

$$R(t,h) \ = \ -\frac{C_h'}{h}X_t - \frac{D_h'}{h}Z_t\,, \quad h \geq 1\,, \tag{81}$$

and, in the same spirit of propositions 9 and 10 for the univariate SARN(p) model [see section 3.6], it is easy to verify that the processes $R = [\,R(t,h),\, 0 \leq t < T\,]$ and $R_{\mathcal{H}} = [\,R(t,h),\ 0 \leq t < T,\, h \in \mathcal{H}\,]$ are, respectively, a weak Switching ARMA(p, $p-1$) process and a weak H-variate Switching VARMA(p, $p-1$) process.
In the endogenous case, where $x_t = r_{t+1}$, the previous results remains valid with $C_1 = -e_1$, $D_1 = 0$.

4.6 Positiveness of the yields

Since $r_{t+1} = R(t,1) = c'X_t + d'Z_t$, and since the components of X_t are positive, the short term process will be positive as soon as the components of c and d are nonnegative. The positiveness of r_{t+1} implies that of $R(t,h)$, at any date t and time to maturity h, because $R(t,h) = -\frac{1}{h}\log E_t^{\mathbb{Q}}[\exp(-r_{t+1} - \dots - r_{t+h})]$.

This positiveness can also be observed from the recursive equations of proposition 11. Indeed, using the fact that μ^* and the components of φ^* and ν^* are positive and that $0 < \pi^*_{ij} < 1$, it easily seen that, for any $u < 0$, the components of $A^*(u)$ and $-\nu^* \log(1 - C_{1,h-1}\mu^*)$ are negative and the result follows.

4.7 Multi-Factor generalization : the SVARG(p) Term Structure model

The bivariate process $\tilde{x}_t = (x_{1,t}, x_{2,t})$ is a SVARG(p) model defined by the following conditional Laplace transforms:

$$\begin{aligned} & E_t[\exp(u_1 x_{1,t+1}) \,|\, \underline{x_{2,t+1}}, \underline{x_{1,t}}, \underline{z_t}\,] \\ = \quad & \exp\left\{\frac{u_1}{1 - u_1\mu_1(Z_t)}\left[\varphi_o(Z_t)x_{2,t+1} + \varphi_{11}(Z_t)'X_{1t} + \varphi_{12}(Z_t)'X_{2t}\right]\right. \\ & \left. -\nu_1(Z_t)\log(1 - u_1\mu_1(Z_t))\right\} , \end{aligned} \tag{82}$$

$$\begin{aligned} & E_t[\exp(u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}\,] \\ = \quad & \exp\left\{\frac{u_2}{1 - u_2\mu_2(Z_t)}\left[\varphi_{21}(Z_t)'X_{1t} + \varphi_{22}(Z_t)'X_{2t}\right]\right. \\ & \left. -\nu_2(Z_t)\log(1 - u_2\mu_2(Z_t))\right\} . \end{aligned} \tag{83}$$

We will use the notations:

$$\varphi_o(Z_t) = \varphi_{o,t} ,$$

$$[\,\varphi_{11}(Z_t)', \varphi_{12}(Z_t)'\,] = \varphi'_{1,t} , \;\; [\,\varphi_{21}(Z_t)', \varphi_{22}(Z_t)'\,] = \varphi'_{2,t} ,$$

$$\mu_i(Z_t) = \mu_{i,t} , \;\; \nu_i(Z_t) = \nu_{i,t} , \;\; i \in \{1, 2\} ,$$

and using the functions $\tilde{a}, \tilde{b}, A, B$ defined in lemma 1 and in section 4.1, we will introduce the notations:

$$\begin{aligned} a_{1,t}(u_1) &= \tilde{a}(u_1; \varphi_{o,t}, \mu_{1,t}) \\ b_{1,t}(u_1) &= \tilde{b}(u_1; \nu_{1,t}, \mu_{1,t}) \,, \quad b_{2,t}(u_2) = \tilde{b}(u_2; \nu_{2,t}, \mu_{2,t}) \\ A_{1,t}(u_1) &= A(u_1; \varphi_{1,t}, \mu_{1,t}) \,, \quad A_{2,t}(u_2) = A(u_2; \varphi_{2,t}, \mu_{2,t}) \,. \end{aligned}$$

With these notations, the Laplace transforms (82) and (83) become respectively:

$$\begin{aligned} & E_t[\exp(u_1 x_{1,t+1}) \,|\, \underline{x_{2,t+1}}, \underline{x_{1,t}}, \underline{z_t}\,] \\ = \;& \exp\left[a_{1,t}(u_1) x_{2,t+1} + A_{1,t}(u_1)' \tilde{X}_t + b_{1,t}(u_1)\right] \,, \end{aligned} \tag{84}$$

$$\begin{aligned} & E_t[\exp(u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}\,] \\ = \;& \exp\left[A_{2,t}(u_2)' \tilde{X}_t + b_{2,t}(u_2)\right] \,, \end{aligned} \tag{85}$$

where $\tilde{X}_t = (X_{1t}', X_{2t}')'$. Moreover, the joint conditional Laplace transform of $(x_{1,t+1}, x_{2,t+1})$, given $(\underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t})$, is:

$$\begin{aligned} & E_t[\exp(u_1 x_{1,t+1} + u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}\,] \\ = \;& \exp\left\{[A_{1,t}(u_1) + A_{2,t}(u_2 + a_{1,t}(u_1))]' \tilde{X}_t + b_{1,t}(u_1) + b_{2,t}(u_2 + a_{1,t}(u_1))\right\} \,. \end{aligned} \tag{86}$$

The process z_t is assumed to be a non-homogeneous Markov chain such that $P(z_{t+1} = e_j \,|\, z_t = e_i; \underline{\tilde{x}_t}) = \pi(e_i, e_j\,; \tilde{X}_t)$.

We now introduce the SDF:

$$\begin{aligned} M_{t,t+1} = \; & \exp\{-c'\tilde{X}_t - d'Z_t + \Gamma_{1t} x_{1,t+1} + \Gamma_{2t} x_{2,t+1} \\ & - [A_{1,t}(\Gamma_{1t}) + A_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t}))]' \tilde{X}_t \\ & - [b_{1,t}(\Gamma_{1t}) + b_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t}))] - \delta(Z_t, \tilde{X}_t)' z_{t+1}\} \,, \end{aligned} \tag{87}$$

where $\Gamma_{1t} = \Gamma_1(Z_t)$ and $\Gamma_{2t} = \Gamma_2(Z_t)$.

4.8 Risk-neutral dynamics in the multifactor case

We can now present, using the lemmas presented above, the joint conditional Laplace transform of $(x_{1,t+1}, x_{2,t+1})$ in the risk-neutral world in the following proposition.

Proposition 14 : The joint conditional Laplace transform of $(x_{1,t+1}, x_{2,t+1})$ in the risk-neutral world is given by :

$$\begin{aligned} & E_t^{\mathbb{Q}}[\exp(u_1 x_{1,t+1} + u_2 x_{2,t+1}) \,|\, \underline{x_{1t}}, \underline{x_{2t}}, \underline{z_t}\,] \\ = \; & \exp\Big\{[A^*_{1,t}(u_1) + A^*_{2,t}[u_2 + a^*_{1,t}(u_1)]]'\tilde{X}_t \\ & \qquad + b^*_{2,t}[u_2 + a^*_{1,t}(u_1)] \,+\, b^*_{1,t}(u_1)\Big\} \;, \end{aligned} \tag{88}$$

where

$$\begin{aligned} A^*_{1,t}(u_1) &= A_1(u_1; \varphi^*_{1t}, \mu^*_{1t})\,, \\ A^*_{2,t}[u_2 + a^*_{1,t}(u_1)] &= A\left[u_2 + \tilde{a}(u_1; \varphi^*_{ot}, \mu^*_{1,t}); \varphi^*_{2t}, \mu^*_{2t}\right] \,, \\ b^*_{2,t}[u_2 + a^*_{1,t}(u_1)] &= \tilde{b}\left[u_2 + \tilde{a}(u_1; \varphi^*_{ot}, \mu^*_{1,t}); \nu^*_{2t}, \mu^*_{2t}\right] \,, \\ b^*_{1,t}(u_1) &= \tilde{b}_1(u_1; \nu^*_{1t}, \mu^*_{1t})\,, \end{aligned}$$

and with

$$\begin{aligned} \varphi^*_{ot} &= \frac{\varphi_{ot}}{(1-\Gamma_{1t}\,\mu_{1t})^2}\,, \quad \varphi^*_{1t} = \frac{\varphi_{1t}}{(1-\Gamma_{1t}\,\mu_{1t})^2}\,, \quad \varphi^*_{2t} = \frac{\varphi_{2t}}{\{1-[\Gamma_{2t}+a_{1,t}(\Gamma_{1t})]\mu_{2t}\}^2} \\ \mu^*_{1t} &= \frac{\mu_{1t}}{(1-\Gamma_{1t}\mu_{1t})}\,, \quad \mu^*_{2t} = \frac{\mu_{2t}}{\{1-[\Gamma_{2t}+a_{1,t}(\Gamma_{1t})]\mu_{2t}\}}\,. \end{aligned}$$

So, (88) has exactly the same form as (86) with different parameters. In other words the risk-neutral dynamics belongs to the same class as the historical one. [Proof : see Appendix 10.]

In order to have a Car process in the risk-neutral world, we know from section 2.9 that we must have the following constraint between the SDF and the historical dynamics:

i)

$$\frac{\mu_{1t}}{1-\Gamma_{1t}\mu_{1t}} = \mu_1^*$$

ii)
$$\frac{\varphi_{1t}}{(1-\Gamma_{1t}\mu_{1t})^2} = \varphi_1^*$$

iii)
$$\nu_1(Z_t) = \nu_1^{*'} Z_t$$

iv)
$$\frac{\varphi_{ot}}{(1-\Gamma_{1t}\mu_{1t})^2} = \varphi_o^*$$

v)
$$\frac{\mu_{2t}}{1-[\Gamma_{2t}+a_{1,t}(\Gamma_{1t})]\mu_{2t}} = \mu_2^*$$

vi)
$$\frac{\varphi_{2t}}{(1-[\Gamma_{2t}+a_{1,t}(\Gamma_{1t})]\mu_{2t})^2} = \varphi_2^*$$

vii)
$$\nu_2(Z_t) = \nu_2^{*'} Z_t\,.$$

Moreover, the constraint on the dynamics of the Markov chain are the same as in the gaussian case, namely:

viii)
$$\delta_j(\tilde{X}_t, Z_t) = \log\left[\frac{\pi(z_t, e_j\,; \tilde{X}_t)}{\pi^*(z_t, e_j)}\right].$$

It is worth noting that, if there is no instantaneous causality between $x_{1,t+1}$ and $x_{2,t+1}$, that is if $\varphi_{ot} = 0$, function a_{1t} is also equal to zero and constraint v) and vi) are simpler and become similar to i) and ii).

4.9 The Generalized Linear Term Structure in the multifactor case

Using the notations:

$$\begin{aligned}
a_1^*(u_1) &= \tilde{a}(u_1; \varphi_o^*, \mu_1^*) \\
A_1^*(u_1) &= A(u_1; \varphi_1^*, \mu_1^*) \\
A_2^*(u_2) &= A(u_2; \varphi_2^*, \mu_2^*) \\
\tilde{C}_h &= (C_{2,h}, \ldots, C_{p,h}, 0, C_{p+2,h}, \ldots, C_{2p,h}, 0)'\,,
\end{aligned}$$

we have

Proposition 15 : In the bivariate SVARG(p) Term Structure model the price at date t of the zero-coupon bond with residual maturity h is :

$$B(t,h) = \exp\left(C'_h\,\tilde{X}_t + D'_h\,Z_t\right)\ ,\ \text{for } h \geq 1 \tag{89}$$

where the vectors C_h and D_h satisfy the following recursive equations :

$$\begin{cases} C_h &= -c + A_1^*(C_{1,h-1}) + A_2^*[C_{p+1,h-1} + a_1^*(C_{1,h-1})] + \tilde{C}_{h-1} \\ D_h &= -d - \nu_1^* \log(1 - C_{1,h-1}\mu_1^*) - \nu_2^* \log[1 - (C_{p+1,h-1} + a_1^*(C_{1,h-1}))\mu_2^*] \\ & \qquad + \tilde{D}_{h-1} + F(D_{1,h-1})\,, \end{cases} \tag{90}$$

where $\tilde{D}_{h-1}$ and $F(D_{1,h-1})$ have the same meaning as in proposition 8; the initial conditions are $C_0 = 0$, $D_0 = 0$ (or $C_1 = -c$, $D_1 = -d$) [Proof : see Appendix 11].
So, proposition 15 shows that, also for the SVARG(p) model, yields to maturity are linear functions of $\tilde{X}_t$ and Z_t.

In the endogenous case, we can consider as factors the short rate r_{t+1} and the long rate $R(t,H)$, for a given time to maturity H. Now, if we want to define a joint historical and risk-neutral dynamics for these variables, compatible with the no-arbitrage opportunity condition, we have to take into account domain restrictions on $R(t,H)$: given that the support of r_{t+1} is $\mathcal{D}_1 = (0,+\infty)$, under A.A.O. the support of $R(t,H)$ has to be $\mathcal{D}_H = [\,b,+\infty)$, for some constant $b > 0$ [see Gourieroux, Monfort (2006) for details]. Consequently, the bivariate SVARG(p) process $\tilde{x}_t$, being with support $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_1$, will be specified for $x_{1t} = r_{t+1}$ and $x_{2t} = R(t,H) - b$, and the results presented for the SVARN(p) case [see section 3.8] will apply also in this case.

It is also easily seen that the risk premium of the payoff $p_{t+1} = \exp(-\theta_1 x_{1,t+1} - \theta_2 x_{2,t+1})$ is:

$$\begin{aligned} \omega_t(\theta_1,\theta_2) &= \{A_{2,t}[-\theta_2 + a_{1,t}(-\theta_1)] + A_{1,t}(-\theta_1) \\ &\quad - A_2^*[-\theta_2 + a_1^*(-\theta_1)] - A_1^*(-\theta_1)\}'\,X_t \\ &\quad + b_{2,t}[-\theta_2 + a_{1,t}(-\theta_1)] + b_{1,t}(-\theta_1) \\ &\quad - b_{2,t}^*[-\theta_2 + a_1^*(-\theta_1)] - b_{1,t}^*(-\theta_1)\,, \end{aligned}$$

with

$$\begin{aligned} b_{1,t}(u_1) &= -\nu_1^{*'} Z_t \log(1-u_1\mu_1^*) \\ b_{2,t}(u_2) &= -\nu_2^{*'} Z_t \log(1-u_2\mu_2^*)\,, \end{aligned}$$

and the risk premium of the digital asset is still given by relation (47).

5 DERIVATIVE PRICING

5.1 Generalization of the recursive pricing formula

In the previous sections we have derived recursive formulas for the zero-coupon bond price $B(t,h)$ in various contexts which share the feature that the process $(\tilde{x}_t, z_t)$ is Car in the risk-neutral world. In fact the recursive approach can be generalized to other assets.

Let us consider a class of payoffs $g(\tilde{X}_{t+h}, Z_{t+h})$, (t,h) varying, for a given g function and let us assume that the price at t of this payoff is of the form:

$$P_t(g,h) = \exp\left[C_h(g)'\tilde{X}_t + D_h(g)'Z_t\right]\,. \tag{91}$$

It is clear that:

$$\begin{aligned} &\exp\left[C_h(g)'\tilde{X}_t + D_h(g)'Z_t\right] \\ = \;& E_t\left[M_{t,t+1}\exp\left(C_{h-1}(g)'\tilde{X}_{t+1} + D_{h-1}(g)'Z_{t+1}\right)\right] \\ = \;& \exp(-c'\tilde{X}_t - d'Z_t)E_t^{\mathbb{Q}}\left[\exp\left(C_{h-1}(g)'\tilde{X}_{t+1} + D_{h-1}(g)'Z_{t+1}\right)\right]\;; \end{aligned}$$

so the sequences $C_h(g), D_h(g), h \geq 1$, follow recursive equations which does not depend on g and, therefore, are identical to the case $g = 1$, that is to say to the zero-coupon bond pricing formulas given in the previous sections. The only condition for (91) to be true is to hold for $h = 1$ and, of course, this initial condition depends on g.

Formula (91) is valid for $h = 1$ if $g(\tilde{X}_{t+h}, Z_{t+h}) = \exp(\tilde{u}'\tilde{X}_{t+h} + \tilde{v}'Z_{t+h})$ for some vector $\tilde{u}$ and $\tilde{v}$. Indeed, using the notations

$$\begin{aligned} \tilde{u}'\tilde{X}_{t+1} &= u_1'\tilde{x}_{t+1} + u_{-1}'\tilde{X}_t \\ \tilde{v}'Z_{t+1} &= v_1'z_{t+1} + v_{-1}'Z_t\,, \end{aligned}$$

with $u'_{-1} = (u'_2, \ldots, u'_p, 0)$, $v'_{-1} = (v'_2, \ldots, v'_p, 0)$, we get:

$$\begin{aligned} P_t(\tilde{u}, \tilde{v}; 1) &= \exp(-c'\tilde{X}_t - d'Z_t + u'_{-1}\tilde{X}_t + v'_{-1}Z_t) \\ &\quad \times E_t^{\mathbb{Q}}\left[\exp\left(u'_1 \tilde{x}_{t+1} + v'_1 z_{t+1}\right)\right] , \end{aligned} \tag{92}$$

which, using the Car representation of $(\tilde{x}_{t+1}, z_{t+1})$ under the probability $\mathbb{Q}$, has obviously the exponential linear form (91) and provides the initial conditions of the recursive equations. The standard recursive equations provide the price $P_t(\tilde{u}, \tilde{v}; h)$ at date t for the payoff $\exp(\tilde{u}'\tilde{X}_{t+h} + \tilde{v}'Z_{t+h})$. So we have the following proposition.

Proposition 16 : The price $P_t(\tilde{u}, \tilde{v}; h)$ at time t of the payoff $g(\tilde{X}_{t+h}, Z_{t+h}) = \exp(\tilde{u}'\tilde{X}_{t+h} + \tilde{v}'Z_{t+h})$ has the exponential form (91) where $C_h(g)$ and $D_h(g)$ follow the *same* recursive equations as in the zero-coupon bond case with initial values $C_1(g)$ and $D_1(g)$ given by the coefficients of $\tilde{X}_t$ and Z_t in equation (92).

When $\tilde{u}$ and $\tilde{v}$ have complex components, $P_t(\tilde{u}, \tilde{v}; h)$ provides the complex Laplace transform $E_t[M_{t,t+h} \exp(\tilde{u}'\tilde{X}_{t+h} + \tilde{v}'Z_{t+h})]$.

5.2 Explicit and quasi explicit pricing formulas

The explicit formulas for zero-coupon bond prices also immediately provide explicit formulas for some derivatives like swaps. Moreover, the result of section 5.1, where $\tilde{u}$ and $\tilde{v}$ have complex components, can be used to price payoffs of the form:

$$\left[\exp(\tilde{u}'_1\tilde{X}_{t+h} + \tilde{v}'_1 Z_{t+h}) - \exp(\tilde{u}'_2\tilde{X}_{t+h} + \tilde{v}'_2 Z_{t+h})\right]^+ ,$$

like caps, floors or options on zero-coupon bonds. Let us consider, for instance, the problem to price, at date t, a European call option on the zero-coupon bond $B(t+h, H-h)$, then the pricing relation is :

$$\begin{aligned} p_t(K, h) &= E_t\left[M_{t,t+h}\left(B(t+h, H-h) - K\right)^+\right] \\ &= E_t\left[M_{t,t+h}\left(\exp[-(H-h)R(t+h, H-h)] - K\right)^+\right] , \end{aligned} \tag{93}$$

and, substituting here the yield to maturity formula (68), for the SVARN(p)

model, or formula (89), for the SVARG(p) model, we can write :

$$
\begin{aligned}
p_t(K,h) &= E_t\left[M_{t,t+h}\left(\exp[C'_{H-h}\tilde{X}_{t+h}+D'_{H-h}Z_{t+h}]-K\right)^+\right] \\
&= E_t\left[M_{t,t+h}\left(\exp[C'_{H-h}\tilde{X}_{t+h}+D'_{H-h}Z_{t+h}]-K\right)\mathbb{I}_{[-C'_{H-h}\tilde{X}_{t+h}-D'_{H-h}Z_{t+h}<-\log K]}\right] \\
&= E_t\left[M_{t,t+h}\left(\exp[C'_{H-h}\tilde{X}_{t+h}+D'_{H-h}Z_{t+h}]\right)\mathbb{I}_{[-C'_{H-h}\tilde{X}_{t+h}-D'_{H-h}Z_{t+h}<-\log K]}\right] \\
&\qquad -KE_t\left[M_{t,t+h}\mathbb{I}_{[-C'_{H-h}\tilde{X}_{t+h}-D'_{H-h}Z_{t+h}<-\log K]}\right] \\
&= G_t(C_{H-h},D_{H-h},-C_{H-h},-D_{H-h},-\log K;h) \\
&\qquad -KG_t(\,0,0,-C_{H-h},-D_{H-h},-\log K;h)\,,
\end{aligned}
\tag{94}
$$

where $\mathbb{I}$ denotes the indicator function, and where

$$
\begin{aligned}
&G_t(\tilde{u}_0,\tilde{v}_0,\tilde{u}_1,\tilde{v}_1,K;h) \\
=\ & E_t\left[M_{t,t+h}\left(\exp[\tilde{u}'_0\tilde{X}_{t+h}+\tilde{v}'_0Z_{t+h}]\right)\mathbb{I}_{[-\tilde{u}'_1\tilde{X}_{t+h}-\tilde{v}'_1Z_{t+h}<K]}\right]
\end{aligned}
$$

denotes the truncated real Laplace transform that we can deduce from the (untruncated) complex Laplace transform. More precisely, we have the following formula [see Duffie, Pan, Singleton (2000) for details]:

$$
\begin{aligned}
G_t(\tilde{u}_0,\tilde{v}_0,\tilde{u}_1,\tilde{v}_1,K;h) &= \frac{P_t(\tilde{u}_0,\tilde{v}_0,h)}{2} \\
&\quad -\frac{1}{\pi}\int_0^{+\infty}\left[\frac{Im[P_t(\tilde{u}_0+i\tilde{u}_1y,\tilde{v}_0+i\tilde{v}_1y;h)]\exp(-iyK)}{y}\right]dy
\end{aligned}
\tag{95}
$$

where $Im(z)$ denotes the imaginary part of the complex number z. So, formula (94) is quasi explicit since it only requires a simple (one-dimensional) integration to derives the values of G_t.

6 Conclusions

This paper has developed a general discrete-time modeling of the term structure of interest rates able to take into account at the same time several important features : a) interest rates with an historical dynamics involving several lagged values and the present and past values of the (non homogeneous) regime indicator function (z_t); b) a specification of the exponential-affine stochastic discount factor (SDF) with time-varying coefficients implying stochastic risk premia, functions of the present and past values of the factor (x_t) and the regime indicator function (z_t); c) the possibility to derive explicit or quasi explicit formulas for zero-coupon bond (the Generalized Linear Term Structure formula) and interest rate derivative prices; d) the positiveness of the yields at each maturity (in the Autoregressive Gamma framework), regardless the endogenous or exogenous nature of the factor (x_t).

We have studied, in the Gaussian framework, the SARN(p) and the SVARN(p) Term Structure models, providing a generalization of the recent modelisation proposed by Dai, Singleton and Yang (2005). In the Autoregressive Gamma setting, we have proposed the SARG(p) and the SVARG(p) Term Structure models, extending several discrete time CIR term structure models like Bansal and Zhou (2002).

The purpose of future research will be to test the models presented in the paper, from observations of zero-coupon bond prices, in terms of their ability to fit the data, to replicate observed term structure shapes, and to explain statistical and financial stylized facts [see Dai and Singleton (2003) and Piazzesi (2003)].

Appendix 1

Proof of Proposition 3

$$
\begin{aligned}
& E[\exp(u_1 y_{1,t+1} + u_2 y_{2,t+1})|\, \underline{y_{1,t}}, \underline{y_{2,t}}] \\
= \;& E\left[\exp(u_2 y_{2,t+1}) E\left(\exp(u_1 y_{1,t+1})|\, \underline{y_{1,t}}, \underline{y_{2,t+1}}\right) \,|\, \underline{y_{1,t}}, \underline{y_{2,t}}\right] \\
= \;& \exp\left[a_1(u_1)(\beta_{11} y_{1,t} + \beta_{12} y_{2,t}) + b_1(u_1)\right] E_t\left[(u_2 + a_1(u_1)\beta_o) y_{2,t+1} \,|\, \underline{y_{1,t}}, \underline{y_{2,t}}\right] \\
= \;& \exp\left[a_1(u_1)(\beta_{11} y_{1,t} + \beta_{12} y_{2,t}) + b_1(u_1)\right. \\
& \qquad \left. + a_2(u_2 + a_1(u_1)\beta_o)(\beta_{21} y_{1,t} + \beta_{22} y_{2,t}) + b_2(u_2 + a_1(u_1)\beta_o)\right] \\
= \;& \exp\left\{[a_1(u_1)\beta_{11} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{21}] y_{1,t}\right. \\
& \qquad \left. + [a_1(u_1)\beta_{12} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{22}] y_{2,t} + b_1(u_1) + b_2(u_2 + a_1(u_1)\beta_o)\right\} \;.
\end{aligned}
$$

Appendix 2

Proof of propositions 5 and 6

Proof of Proposition 5 : Let us first consider an asset providing the payoff $\exp(-\theta x_{t+1})$ at $t+1$; the price at t of this asset is

$$\begin{aligned} p_t &= E_t[M_{t,t+1}\exp(-\theta x_{t+1})] \\ &= \exp\left[-r_{t+1} - \theta\nu(Z_t) - \theta\varphi(Z_t)'X_t - \tfrac{1}{2}\Gamma(X_t,Z_t)^2\right] \times \\ &\qquad E_t\left\{\exp\left[\left[\Gamma(X_t,Z_t) - \theta\sigma(Z_t)\right]\varepsilon_{t+1}\right]\right\} \\ &= \exp\left[-r_{t+1} - \theta\nu(Z_t) - \theta\varphi(Z_t)'X_t - \theta\Gamma(X_t,Z_t)\sigma(Z_t) + \tfrac{\theta^2}{2}\sigma^2(Z_t)\right] \ , \end{aligned}$$

and

$$\begin{aligned} E_t p_{t+1} &= E_t[\exp(-\theta x_{t+1})] \\ &= \exp\left[-\theta\nu(Z_t) - \theta\varphi(Z_t)'X_t\right] \times \\ &\qquad E_t\left\{\exp\left[\left[-\theta\sigma(Z_t)\right]\varepsilon_{t+1}\right]\right\} \\ &= \exp\left[-\theta\nu(Z_t) - \theta\varphi(Z_t)'X_t + \tfrac{\theta^2}{2}\sigma^2(Z_t)\right] \ . \end{aligned}$$

Finally, from Definition 4, the risk premium is:

$$\omega_t(\theta) = \theta\Gamma(X_t,Z_t)\sigma(Z_t) \, .$$

Proof of Proposition 6 : Similarly, if we consider a digital asset providing one money unit at $t+1$ if $z_{t+1} = e_j$, we get:

$$\begin{aligned} p_t &= E_t[M_{t,t+1}\mathbb{I}_{(e_j)}(z_{t+1})] \\ &= \exp[-r_{t+1}]\exp[-\delta_j(X_t,Z_t)]\pi\left(z_t, e_j\,; X_t\right) \ , \end{aligned}$$

and

$$\begin{aligned} E_t p_{t+1} &= E_t[\mathbb{I}_{(e_j)}(z_{t+1})] \\ &= \pi\left(z_t, e_j\,; X_t\right) \ . \end{aligned}$$

Therefore, applying Definition 4, the risk premium is :

$$\omega_t(\theta) = \delta_j(X_t,Z_t) \, .$$

Appendix 3

Proof of Proposition 7

The Laplace transform of the one-period conditional risk-neutral probability is:

$$
\begin{aligned}
& E_t^{\mathbb{Q}}[\exp(ux_{t+1} + v'z_{t+1})] \\
= \; & E_t\{\exp[\Gamma(X_t, Z_t)\,\varepsilon_{t+1} - \tfrac{1}{2}\Gamma(X_t, Z_t)^2 - \delta'(Z_t, X_t)z_{t+1} \\
& \qquad + u[\nu(Z_t) + \varphi(Z_t)'X_t + \sigma(Z_t)\varepsilon_{t+1}] + v'z_{t+1}]\} \\
= \; & \exp\left\{ u[\,\varphi'(Z_t)X_t + \Gamma(X_t, Z_t)\sigma(Z_t)] + u\nu(Z_t) + \tfrac{1}{2}u^2\sigma(Z_t)^2 \right\} \times \\
& \qquad \textstyle\sum_{j=1}^{J} \pi\,(z_t, e_j\,; X_t) \exp\left[(v - \delta(Z_t, X_t))'e_j\right] \\
= \; & \exp\left\{ u[\,\varphi(Z_t) + \tilde{\gamma}(Z_t)\sigma(Z_t)]'X_t + u[\nu(Z_t) + \gamma(Z_t)\sigma(Z_t)] + \tfrac{1}{2}u^2\sigma(Z_t)^2 \right\} \times \\
& \qquad \textstyle\sum_{j=1}^{J} \pi\,(z_t, e_j\,; X_t) \exp\left[(v - \delta(Z_t, X_t))'e_j\right] \,.
\end{aligned}
$$

Therefore, we get the result of Proposition 7.

Appendix 4

Proof of Proposition 8

$$\begin{aligned}
B(t,h) &= \exp(C_h' X_t + D_h' Z_t) \\
&= \exp\left(-r_{t+1}\right) E_t^Q \left[B(t+1,h-1)\right] \\
&= \exp\left[-c' X_t - d' Z_t\right] \, E_t^Q \left[\exp\left(C_{h-1}' X_{t+1} + D_{h-1}' Z_{t+1}\right)\right] \\
&= \exp\left[-c' X_t - d' Z_t\right] \times \\
& \qquad E_t^Q \left[\exp\left(C_{h-1}' \left[\Phi^* X_t + \left(\nu^{*'} Z_t + \sigma^{*'} Z_t \, \xi_{t+1}\right) e_1\right] + D_{1,h-1}' z_{t+1} + \tilde{D}_{h-1}' Z_t\right)\right] \\
&= \exp\left[\left(\Phi^{*'} C_{h-1} - c\right)' X_t + \left(-d + C_{1,h-1}\nu^* + \tfrac{1}{2} C_{1,h-1}^2 \sigma^{*2} + \tilde{D}_{h-1}\right)' Z_t\right] \times \\
& \qquad \quad E_t^Q \left[\exp\left(D_{1,h-1}' z_{t+1}\right)\right] \\
&= \exp\left\{\left(\Phi^{*'} C_{h-1} - c\right)' X_t + \right. \\
& \qquad \quad \left. \left[-d + C_{1,h-1}\nu^* + \tfrac{1}{2} C_{1,h-1}^2 \sigma^{*2} + \tilde{D}_{h-1} + F(D_{1,h-1})\right]' Z_t\right\} ,
\end{aligned}$$

and the result follows by identification.

Appendix 5

Proof of Proposition 9

Using the lag polynomials:

$$
\begin{aligned}
C_h(L) &= -\frac{1}{h}\,(C_{1,h} + C_{2,h}L + \ldots + C_{p,h}L^{p-1}) \\
D_h(L) &= -\frac{1}{h}\,(D_{1,h} + D_{2,h}L + \ldots + D_{p+1,h}L^{p}) \\
\Psi(L, Z_t) &= 1 - \varphi_1(Z_t)L - \ldots - \varphi_p(Z_t)L^p\,,
\end{aligned}
$$

we get from (54):

$$
R(t,h) = C_h(L)x_t + D_h(L)'z_t\,,
$$

and

$$
\begin{aligned}
\Psi(L, Z_t)\,R(t+1,h) &= C_h(L)\,\Psi(L,Z_t)\,x_{t+1} + D_h(L)\,\Psi(L,Z_t)\,z_{t+1}, \\
&= D_h(L)\,\Psi(L,Z_t)\,z_{t+1} + C_h(L)\,\nu(Z_t) + C_h(L)[(\sigma^{*'}Z_t)\,\varepsilon_{t+1}]\,.
\end{aligned}
$$

Appendix 6

Proof of Proposition 11

The Laplace transform of the one-period conditional risk-neutral distribution is :

$$
\begin{aligned}
& E_t^{\mathbb{Q}}[\exp(u'\tilde{x}_{t+1} + v'z_{t+1})] \\
= \;& E_t\{\exp[\Gamma(\tilde{X}_t, Z_t)'\,\varepsilon_{t+1} - \tfrac{1}{2}\Gamma(\tilde{X}_t, Z_t)'\Gamma(\tilde{X}_t, Z_t) - \delta'(Z_t, \tilde{X}_t)z_{t+1} \\
& \quad + u'[\tilde{\nu}(Z_t) + \tilde{\Phi}(Z_t)\tilde{X}_t + S(Z_t)\varepsilon_{t+1}] + v'z_{t+1}]\} \\
= \;& \exp\left\{ u'[\,\tilde{\Phi}(Z_t)\tilde{X}_t + S(Z_t)\Gamma(\tilde{X}_t, Z_t)] + u'\tilde{\nu}(Z_t) + \tfrac{1}{2}u'S(Z_t)S(Z_t)'u \right\} \times \\
& \quad \textstyle\sum_{j=1}^{J} \pi(z_t, e_j\,; \tilde{X}_t) \exp\left[(v - \delta(Z_t, \tilde{X}_t))'e_j\right] \\
= \;& \exp\left\{ u'[\,\tilde{\Phi}(Z_t) + S(Z_t)\tilde{\Gamma}(Z_t, \tilde{X}_t)]\tilde{X}_t + u'[\,\tilde{\nu}(Z_t) + S(Z_t)\gamma(Z_t)] + \tfrac{1}{2}u'S(Z_t)S(Z_t)'u \right\} \times \\
& \quad \textstyle\sum_{j=1}^{J} \pi(z_t, e_j\,; \tilde{X}_t) \exp\left[(v - \delta(Z_t, \tilde{X}_t))'e_j\right] .
\end{aligned}
$$

Therefore, we get the result of Proposition 11.

Appendix 7

Proof of Proposition 12

$$
\begin{aligned}
B(t,h) &= \exp(C'_h \tilde{X}_t + d'_h Z_t) \\
&= \exp\left(-r_{t+1}\right) E_t^Q\left[B(t+1,h-1)\right] \\
&= \exp\left[-c'\tilde{X}_t - d'Z_t\right] \, E_t^Q\left[\exp\left(C'_{h-1}\tilde{X}_{t+1} + D'_{h-1}Z_{t+1}\right)\right] \\
&= \exp\left[-c'\tilde{X}_t - d'Z_t\right] \times \\
&\quad E_t^Q\Big[\exp\Big(C'_{h-1}\tilde{\Phi}^*\tilde{X}_t + C_{1,h-1}(\nu_1^* Z_t + S_1^*(Z_t)\xi_{t+1}) \\
&\qquad + C_{p+1,h-1}(\nu_2^* Z_t + S_2^*(Z_t)\xi_{t+1}) + D'_{1,h-1}z_{t+1} + \tilde{D}'_{h-1}Z_t\Big)\Big] \\
&= \exp\Big[\left(\tilde{\Phi}^{*'}C_{h-1} - c\right)' X_t + \Big[-d + C_{1,h-1}\nu_1^{*'} + \tfrac{1}{2}C_{1,h-1}^2(\sigma_1^{*2} + \varphi_o^{*2}\sigma_2^{*2}) \\
&\qquad + C_{p+1,h-1}\nu_2^{*'} + \tfrac{1}{2}C_{p+1,h-1}^2\sigma_2^{*2} + \tilde{D}_{h-1} + F(D_{1,h-1})\Big]' Z_t\Big] \,,
\end{aligned}
$$

and the result follows by identification.

Appendix 8

Proof of Lemma 1

$$
\begin{aligned}
\tilde{a}(u+\alpha;\rho,\mu)-\tilde{a}(\alpha;\rho,\mu) &= \frac{\rho(u+\alpha)}{1-(u+\alpha)\mu}-\frac{\rho\alpha}{1-\alpha\mu} \\
&= \rho\frac{u}{(1-\alpha\mu)^2-u\mu(1-\alpha\mu)} \\
&= \frac{\rho}{(1-\alpha\mu)^2}\,\frac{u}{1-\frac{u\mu}{1-\alpha\mu}} \\
&= \frac{\rho^* u}{1-u\mu^*}=\tilde{a}(u;\rho^*,\mu^*)\,;
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b}(u+\alpha;\nu,\mu)-\tilde{b}(\alpha;\nu,\mu) &= -\nu\log(1-(u+\alpha)\mu)+-\nu\log(1-\alpha\mu) \\
&= -\nu\log\left[\frac{1-(u+\alpha)\mu}{1-\alpha\mu}\right] \\
&= -\nu\log\left[1-\frac{u\mu}{1-\alpha\mu}\right] \\
&= -\nu\log(1-u\mu^*) \\
&= \tilde{b}(u;\nu,\mu^*)\,.
\end{aligned}
$$

Appendix 9

Proof of Proposition 13

$$
\begin{aligned}
B(t,h) &= \exp(C'_h X_t + D'_h Z_t) \\
&= \exp\left[-c' X_t - d' Z_t\right] E_t^Q \left[\exp\left(C'_{h-1} X_{t+1} + D'_{h-1} Z_{t+1}\right)\right] \\
&= \exp\Big(-c' X_t - d' Z_t + \tilde{C}'_{h-1} X_t + \tilde{D}'_{h-1} Z_t\Big) \\
&\qquad E_t^Q \Big[\exp\Big(C_{1,h-1} x_{t+1} + D'_{1,h-1} z_{t+1}\Big)\Big] \\
&= \exp\Big[-c' X_t - d' Z_t + \tilde{C}'_{h-1} X_t + \tilde{D}'_{h-1} Z_t + A^*(C_{1,h-1})' X_t \\
&\qquad -\nu^{*'} Z_t \log(1 - C_{1,h-1}\mu^*) + F'(D_{1,h-1}) Z_t\Big] \,,
\end{aligned}
$$

and the result follows by identification.

Appendix 10

Proof of Proposition 14

The joint conditional Laplace transform of $(x_{1,t+1}, x_{2,t+1})$ in the risk-neutral world is:

$$
\begin{aligned}
& E_t^{\mathbb{Q}}[\exp(u_1 x_{1,t+1} + u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}\,] \\
= \; & \exp\Big\{ A_{2,t}\left[u_2 + \Gamma_{2t} + a_{1,t}(u_1 + \Gamma_{1t})\right]' \tilde{X}_t + b_{2,t}(u_2 + \Gamma_{2t} + a_{1,t}(u_1 + \Gamma_{1t})) \\
& + A_{1,t}(u_1 + \Gamma_{1t})' \tilde{X}_t + b_{1,t}(u_1 + \Gamma_{1t}) \\
& - A_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t}))' \tilde{X}_t - b_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t})) \\
& - A_{1,t}(\Gamma_{1t})' \tilde{X}_t - b_{1,t}(\Gamma_{1t}) \Big\} \; .
\end{aligned}
$$

Using lemma 2 we get:

$$
\begin{aligned}
& A_{2,t}\left[u_2 + \Gamma_{2t} + a_{1,t}(u_1 + \Gamma_{1t})\right] - A_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t})) \\
= \; & A\left[u_2 + a_{1,t}(u_1 + \Gamma_{1t}) - a_{1,t}(\Gamma_{1t}); \varphi_{2t}^*, \mu_{2t}^*\right] \, ,
\end{aligned}
$$

with

$$
\begin{aligned}
\varphi_{2t}^* &= \frac{\varphi_{2t}}{\{1 - [\Gamma_{2t} + a_{1,t}(\Gamma_{1t})]\mu_{2t}\}^2} \\
\mu_{2t}^* &= \frac{\mu_{2t}}{\{1 - [\Gamma_{2t} + a_{1,t}(\Gamma_{1t})]\mu_{2t}\}} \, ,
\end{aligned}
$$

and using lemma 1

$$
\begin{aligned}
& A\left[u_2 + a_{1,t}(u_1 + \Gamma_{1t}) - a_{1,t}(\Gamma_{1t}); \varphi_{2t}^*, \mu_{2t}^*\right] \\
= \; & A\left[u_2 + \tilde{a}(u_1 + \Gamma_{1t}; \varphi_{ot}, \mu_{1,t}) - \tilde{a}(\Gamma_{1t}; \varphi_{ot}, \mu_{1,t}); \varphi_{2t}^*, \mu_{2t}^*\right] \\
= \; & A\left[u_2 + \tilde{a}(u_1; \varphi_{ot}^*, \mu_{1,t}^*); \varphi_{2t}^*, \mu_{2t}^*\right] \\
= \; & A_{2,t}^*[u_2 + a_{1,t}^*(u_1)] \text{ (say)}
\end{aligned}
$$

with

$$\varphi^*_{ot} = \frac{\varphi_{ot}}{(1-\Gamma_{1t}\,\mu_{1t})^2}$$

$$\mu^*_{1t} = \frac{\mu_{1t}}{(1-\Gamma_{1t}\mu_{1t})}\,.$$

Similarly, we get:

$$\begin{aligned} & b_{2,t}\left[u_2+\Gamma_{2t}+a_{1,t}(u_1+\Gamma_{1t})\right]-b_{2,t}(\Gamma_{2t}+a_{1,t}(\Gamma_{1t})) \\ = \ & \tilde{b}\left[u_2+\tilde{a}(u_1;\varphi^*_{ot},\mu^*_{1,t});\nu^*_{2t},\mu^*_{2t}\right] \\ = \ & b^*_{2,t}[u_2+a^*_{1t}(u_1)] \text{ (say)}\,, \end{aligned}$$

$$\begin{aligned} & b_{1,t}(u_1+\Gamma_{1t})-b_{1,t}(\Gamma_{1t}) \\ = \ & \tilde{b}_1(u_1;\nu^*_{1t},\mu^*_{1t}) \\ = \ & b^*_{1,t}(u_1) \text{ (say)}\,, \end{aligned}$$

$$\begin{aligned} & A_{1,t}(u_1+\Gamma_{1t})-\ A_{1,t}(\Gamma_{1t}) \\ = \ & A_1(u_1;\varphi^*_{1t},\mu^*_{1t}) \\ = \ & A^*_{1,t}(u_1) \text{ (say)}\,, \end{aligned}$$

with

$$\varphi^*_{1t} = \frac{\varphi_{1t}}{(1-\Gamma_{1t}\,\mu_{1t})^2}\,.$$

And finally, the joint conditional Laplace transform of $(x_{1,t+1}, x_{2,t+1})$ becomes:

$$\begin{aligned} & E^{\mathbb{Q}}_t[\exp(u_1x_{1,t+1}+u_2x_{2,t+1})\,|\,\underline{x_{1t}},\underline{x_{2t}},\underline{z_t}\,] \\ = \ & \exp\Big\{[A^*_{1,t}(u_1)+A^*_{2,t}[u_2+a^*_{1,t}(u_1)]]'\tilde{X}_t \\ & \qquad\qquad +b^*_{2,t}[u_2+a^*_{1,t}(u_1)]\,+\,b^*_{1,t}(u_1)\Big\}\,, \end{aligned}$$

and the result of Proposition 14 is proved.

Appendix 11

Proof of Proposition 15

$$
\begin{aligned}
B(t,h) &= \exp(C_h'\tilde{X}_t + D_h' Z_t) \\
&= \exp\left[-c'\tilde{X}_t - d'Z_t\right] E_t^Q\left[\exp\left(C_{h-1}'\tilde{X}_{t+1} + D_{h-1}'Z_{t+1}\right)\right] \\
&= \exp\left(-c'\tilde{X}_t - d'Z_t + \tilde{C}_{h-1}'\tilde{X}_t + \tilde{D}_{h-1}'Z_t\right) \\
&\qquad E_t^Q\left[\exp\left(C_{1,h-1}'x_{1,t+1} + C_{p+1,h-1}'x_{2,t+1} + D_{1,h-1}'z_{t+1}\right)\right] \\
&= \exp\Big[-c'\tilde{X}_t - d'Z_t + \tilde{C}_{h-1}'\tilde{X}_t + \tilde{D}_{h-1}'Z_t + A_1^*(C_{1,h-1})'\tilde{X}_t \\
&\qquad -\nu_1^{*'}Z_t\log(1 - C_{1,h-1}\mu_1^*) + A_2^*[C_{p+1,h-1} + a_1^*(C_{1,h-1})]'\tilde{X}_t \\
&\qquad -\nu_2^{*'}Z_t\log[1 - (C_{p+1,h-1} + a_1^*(C_{1,h-1}))\mu_2^*] + F'(D_{1,h-1})Z_t\Big] ,
\end{aligned}
$$

and the result follows by identification.

R E F E R E N C E S

Ahn, D., Dittmar, R., and R. Gallant (2002) : "Quadratic Term Structure Models : Theory and Evidence", Review of Financial Studies, 15, 243-288.

Ang, A., and G. Bekaert, (2002a) : "Regime Switches in Interest Rates", Journal of Business and Economic Statistics, 20, 163-182.

Ang, A., and G. Bekaert, (2002b) : "Short Rate Nonlinearities and Regime Switches", Journal of Economic Dynamics and Control, 26, 1243-1274.

Ang, A., and G. Bekaert, (2005) : "Term Structure of Real Rates and Expected Inflation", Working Paper.

Ang, A., and M. Piazzesi, (2003) : "A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables", Journal of Monetary Economics, 50, 745-787.

Ang, A., Piazzesi, M., and M. Wei (2005) : "What does the Yield Curve tell us about GDP Growth", forthcoming Journal of Econometrics.

Bansal, R., and H. Zhou, (2002) : "Term Structure of Interest Rates with Regime Shifts", Journal of Finance, 57, 1997-2043.

Beaglehole, D. R., and M. S. Tenney (1991) : "General Solutions of Some Interest Rate Contingent Claim Pricing Equations", Journal of Fixed Income, September, 69-83.

Bekaert, G. and S. Grenadier (2001) : "Stock and Bond Pricing in an Affine Equilibrium", Working Paper.

Boudoukh, J., Richardson, M., Smith, T., and R. F. Whitelaw (1999) : "Regime Shifts and Bond Returns", Working Paper.

Cai, J., (1994) : "A Markov Model of Unconditional Variance in ARCH", Journal of Business and Economic Statistics, 12, 309-316.

Campbell, J., and R. J. Shiller (1991) : "Yield Spreads and Interest Rate Movements : A Bird's Eye View", Review of Economic Studies, 58, 495-514.

Cheng, P., and O. Scaillet, (2005) : "Linear-Quadratic Jump-Diffusion modeling with Application to Stochastic Volatility", Working Paper.

Cheridito, R., Filipovic, D., and R. Kimmel (2003) : "Market Prices of Risk in Affine Models : Theory and Evidence", Working Paper.

Christiansen, C., (2002) : "Regime Switching in the Yield Curve", Working Paper.

Christiansen, C., and J. Lund (2005) : "Revisiting the Shape of the Yield Curve : the Effect of Interest Rate Volatility", Working Paper.

Cochrane, J., and M. Piazzesi, (2005) : "Bond Risk Premia", American Economic Review, 95/1, 138-160.

Cox, J.C., Ingersoll, J.E. and S. A. Ross (1985) : "A theory of the term structure of interest rates", Econometrica, 53/2, pp. 385-407.

Dai, Q., and K. Singleton (2000) : "Specification Analysis of Affine Term Structure Models", Journal of Finance, 55, 1943-1978.

Dai, Q., and K. Singleton (2002) : "Expectations Puzzles, Time-varying Risk Premia, and Affine Model of the Term Structure", Journal of Financial Economics, 63, 415-441.

Dai, Q., and K. Singleton (2003) : "Term Structure Dynamics in Theory and Reality", Review of Financial Studies, 16, 631-678.

Dai, Q., Singleton, K., and W. Yang (2005) : "Regime Shifts in a Dynamic Term Structure Model of U.S. Treasury Bond Yields", Working Paper.

Dai, Q., Le, A., and K. Singleton (2006) : "Discrete-time Term Structure Models with Generalized Market Prices of Risk", Working Paper.

Darolles, S., Gourieroux, C. and J. Jasiak (2006) : "Structural Laplace Transform and Compound Autoregressive Models", forthcoming Journal of Time Series Analysis.

Driffill, J., and M. Sola, (1994) : "Testing the Term Structure of Interest Rates using a Stationary Vector Autoregression with Regime Switching", Journal of Economic Dynamics and Control, 18, 601-628.

Driffill, J., T., Kenc, and M. Sola, (2003) : "An Empirical Examination of Term Structure Models with Regime Shifts", Working Paper.

Duarte, J. (2004) : "Evaluating An Alternative Risk Preference in Affine Term Structure Models", Review of Financial Studies, 17, 379-404.

Duffee, G. R., (2002) : "Term Premia and Interest Rate Forecasts in Affine Models", Journal of Finance, 57, 405-443.

Duffie, D., Filipovic, D. and W. Schachermayer (2003) : "Affine processes and applications in finance", The Annals of Applied Probability 13, 3.

Duffie, D., and R., Kan (1996) : "A yield-factor model of interest rates", Mathematical Finance, 6, 379-406.

Duffie, D., J. Pan, and K. Singleton (2000) : "Transform Analysis and Asset Pricing for Affine Jump Diffusions", Econometrica, 68, 1343-1376.

Evans, M. (2003) : "Real Risk, Inflation Risk, and the Term Structure", The Economic Journal, 113, 345-389.

Garcia, R. and P. Perron (1996) : "An Analysis of Real Interest Rate Under Regime Shifts", Reviews of Economics and Statistics, 1, 111-125.

Gourieroux, C., and J. Jasiak (2006) : "Autoregressive Gamma Processes", forthcoming Journal of Forecasting.

Gourieroux, C., Jasiak, J., and R. Sufana (2004) : "A Dynamic Model for Multivariate Stochastic Volatility : The Wishart Autoregressive Process", Working Paper.

Gourieroux, C., and A. Monfort (2006) : "Domain Restrictions on Interest Rates Implied by No Arbitrage", Crest DP.

Gourieroux, C., and A. Monfort (2006) : "Econometric Specifications of Stochastic Discount Factor Models", forthcoming Journal of Econometrics.

Gourieroux, C., Monfort, A. and V. Polimenis (2003) : "Discrete Time Affine Term Structure Models", Crest DP.

Gourieroux, C., Monfort, A. and V. Polimenis (2005) : "Affine Model for Credit Risk Analysis", Crest DP.

Gourieroux, C., and R. Sufana, (2003) : "Wishart Quadratic Term Structure Models", Working Paper.

Gray, S., (1996) : "Modeling the Conditional Distribution of Interest Rates as a Regime Switching Process", Journal of Financial Economics, 42, 27-62.

Hamilton, J. D., (1988) : "Rational-Expectations Econometric Analysis of Changes in Regimes - An Investigation of the Term Structure of Interest Rates", Journal of Economic Dynamics and Control, 12, 385-423.

Jamshidian, F. (1989) : "An Exact Bond Option Formula", Journal of Finance, 44, 205-209.

Piazzesi, M. (2003) : "Affine Term Structure Models", forthcoming Handbook of Financial Econometrics.

Polimenis, V. (2001) : "Essays in Discrete Time Asset Pricing", Ph. D. Thesis, Wharton School, University of Pennsylvania.

Vasicek, O. (1977) : "An Equilibrium Characterization of the Term Structure", Journal of Financial Economics, 5, 177-188.