PRICING AND INFERENCE WITH MIXTURES OF CONDITIONALLY NORMAL PROCESSES

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Abstract

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We consider the problems of inference and derivative pricing when the stochastic discount factor has an exponential-affine form and the geometric return of the underlying asset has a dynamics characterized by a mixture of conditionally normal processes. We consider both the static case in which the underlying process is a white noise distributed as a mixture of gaussian distributions (including extreme risks and jump diffusions) and the dynamic case in which the underlying process is conditionally distributed as a mixture of gaussian distributions. Semi-parametric and non parametric situations are also considered. In all cases the risk neutral processes and explicit pricing formulas are obtained.

Keywords : Derivative Pricing, Stochastic Discount Factor, Implied Volatility, Mixture of Normal Distributions, Mixture of Conditionally Normal Processes, Nonparametric Kernel estimation, Mixed-Normal GARCH processes.

JEL number : C1, C5, G1

1. Introduction

The basic option pricing model, proposed by Black and Scholes (1973), assumes that the logarithmic return of the underlying asset follow a normal white noise. It is well known that the pricing formula derived from this approach is misspecified; in particular that the implied volatilities are not constant, as a function of the strike and of the maturity and, moreover, depend on time. In the literature, two main routes have been followed in order to try to solve these problems.

The first type of solutions consider various generalizations of the historical distribution of the underlying stochastic processes. Among these numerous generalizations are: time varying volatilities [see Merton (1973)], stochastic volatilities [Hull and White (1987), Chesney and Scott (1989), Stein and Stein (1991), Heston (1993), Melino and Turnbull (1995)], or jump components [Merton (1976), Bates (1996)].

The second type of solutions deal directly with the option pricing formula, the implied volatility surfaces or the risk neutral probability [see Madan and Milne (1994), Dumas, Fleming and Whaley (1996), Melick and Thomas (1997), Ait-Sahalia and Lo (1998), Jondeau and Rockinger (2000), Cont (2002)]. In the latter approach the link between the historical and the risk neutral distributions is generally ignored.

In this paper we focus on two important sources of misspecification for the Black-Scholes approach, namely the lack of normality and the dynamics. More precisely we explore various specifications of the historical stochastic processes aiming at solving these problems, while providing at the same time explicit risk neutral distributions of the processes and option pricing formulas. Typically, the risk neutral distributions of the processes will be found to belong to the same class as the historical distribution and the option pricing formulas will be focus to combinations of Black-Scholes formulas. The basic tools are the mixtures of discrete time conditionally normal processes, that is to say processes $\{y_t\}$ such that y_t is gaussian conditionally to its past values and the present value z_t of a discrete value unobservable white noise process. In other words, conditionally to its past only, y_t is distributed as a mixtures of normal distributions. We also use an exponential-affine stochastic discount factor which has proved useful in many circumstances [see Gerber and Shiu (1994), Gourieroux and Monfort (2002), Gourieroux, Monfort and Polimenis (2002, 2003)].

From a probabilistic point of view, we consider the static case, i.e. white noises distributed as a mixture of gaussian distributions, and the dynamic case, i.e. processes conditionally distributed as a mixture of gaussian distributions (possibly non markovian) and, from a statistical point of view, we consider the parametric, semi-parametric and non parametric cases. In the non parametric and semi-parametric cases the normality is introduced in the kernel used at the estimation stage.

The plan of the paper is as follow. In section 2 we review the use of the exponential-affine stochastic discount factor and of the real Laplace transform (or moment generating function). In section 3 we consider the case where the historical process is a white noise distributed as a gaussian mixture and we consider the special cases of extreme risks and jump diffusions. In section 4 we study the non parametric static case, that is the case of a white noise with unspecified distribution. In section 5 we consider the parametric dynamic case; more precisely, several kinds of conditionally mixed-normal GARCH processes are proposed. In section 6 we deal with the semi-parametric dynamic case, in which the distribution of the process standardized by its conditional mean and its conditional standard error is left unspecified. Section 7 concludes and appendices gather the proofs.

2. Pricing with Exponential-affine Stochastic Discount Factor

In order to briefly present the Stochastic Discount Factor (SDF) modelling principle [see Gourieroux and Monfort (2002) for a detailed presentation] we consider a market with a riskfree asset and one risky asset; we denote by r^f_{t+1} the riskfree rate between the dates t and $t+1$ (known at time t, i.e. predetermined) and by $y_{t+1} = \ln(\frac{S_{t+1}}{S_t})$ the geometric return on the risky asset with price S_t.

In this context, the price at t of a european derivative asset paying $g(y_{t+1}, \ldots, y_{t+H})$ at $t+H$, is written (under the historical probability) in the following way :

$$\begin{aligned} C_t(g,H) &= E[M_{t,t+1} \ldots M_{t+H-1,t+H}\, g(y_{t+1}, \ldots, y_{t+H}) |\, I_t], \\ &= E_t[M_{t,t+H} g(y_{t+1}, \ldots, y_{t+H})], \end{aligned} \tag{2.1}$$

where $I_t = \underline{y_t} = (y_t, y_{t-1}, \ldots)$ is the information on the current and lagged values of the state variable available at date t for the investor, and where $M_{t,t+1}$ is the Stochastic Discount Factor between t and $t+1$, which is function of I_{t+1}.

In particular, we present the problem of asset pricing by means of a Stochastic Discount Factor (SDF) $M_{t,t+1}$ characterized by an exponential-affine form :

$$M_{t,t+1} = \exp[\alpha_t y_{t+1} \ + \ \beta_t], \tag{2.2}$$

where the coefficients α_t and β_t can be path dependent, that is function of I_t.

By writing the pricing formula for the riskfree asset and the risky asset at different dates, we obtain two arbitrage free conditions that induce restrictions on the relationship between the SDF and the historical distribution. More precisely, the constraints are :

$$\begin{cases} E_t\left[M_{t,t+1}\exp r^f_{t+1}\right] = 1 \\ E_t\left[M_{t,t+1}\exp y_{t+1}\right] = 1 \end{cases}$$

$$\iff \begin{cases} \exp(r^f_{t+1} + \beta_t)\, E_t[\exp \alpha_t y_{t+1}] = 1 \\ \exp(\beta_t)\, E_t[\exp(\alpha_t + 1) y_{t+1}] = 1 \end{cases}$$

$$\iff \begin{cases} \exp(r^f_{t+1} + \beta_t)\, \varphi_t(\alpha_t) = 1 \\ \exp(\beta_t)\, \varphi_t(\alpha_t + 1) = 1, \end{cases}$$

where $\varphi_t(\alpha)$ is the real conditional Laplace transform (also called moment generating function) of y_{t+1} (given I_t).

The Laplace transform, which is the way we use to describe the conditional historical distribution of y_{t+1}, is defined on a convex set that depends on the tails of the conditional distribution. We assume below that this convex set is not reduced to one point located at the origin.

This system in general admits a unique solution (α_t,β_t) such that:

$$\begin{cases} \varphi_t(\alpha_t + 1) = \exp r^f_{t+1} \varphi_t(\alpha_t) \\ \exp \beta_t = [\varphi_t(\alpha_t + 1)]^{-1}, \end{cases} \tag{2.3}$$

and, consequently, we deduce a unique specification of the SDF (2.2).

The associated unique risk-neutral conditional distribution Q_t of y_{t+1} given I_t has a p.d.f. with respect to the corresponding historical distribution given by $M_{t,t+1}/E_t(M_{t,t+1})$ and Laplace transform given by:

$$\begin{aligned} & E^{Q_t}\left[\exp u y_{t+1}\right] \\ &= E_t\left[\frac{M_{t,t+1}}{E_t(M_{t,t+1})}\exp u y_{t+1}\right] \\ &= \exp(r^f_{t+1} + \beta_t) E_t\left[\exp((\alpha_t + u) y_{t+1})\right] \\ &= \frac{\varphi_t(\alpha_t + u)}{\varphi_t(\alpha_t)}. \end{aligned} \tag{2.4}$$

An asset providing the payoff $g(y_{t+1})$ at time $t+1$ is priced at time t by :

$$\begin{aligned} C_t &= E_t\left[M_{t,t+1} g(y_{t+1})\right] \\ &= \exp(-r^f_{t+1}) E^{Q_t}\left[g(y_{t+1})\right]. \end{aligned} \tag{2.5}$$

With a longer time horizon, the conditional joint risk neutral distribution Q_t^H of $(y_{t+1}, \ldots, y_{t+H})$ given I_t has a p.d.f., with respect to the corresponding historical distribution P_t^H, given by :

$$\frac{dQ_t^H}{dP_t^H} = \frac{M_{t,t+1} \cdot \ldots \cdot M_{t+H-1,t+H}}{E_t(M_{t,t+1}) \cdot \ldots \cdot E_{t+H-1}(M_{t+H-1,t+H})} \tag{2.6}$$

and the associated pricing formula takes the form :

$$\begin{aligned} C_t &= E_t\left[M_{t,t+1} \cdot \ldots \cdot M_{t+H-1,t+H}\, g(y_{t+1}, \ldots, y_{t+H})\right] \\ &= E^{Q_t}\left[E_t(M_{t,t+1}) \cdot \ldots \cdot E_{t+H-1}(M_{t+H-1,t+H}) g(y_{t+1}, \ldots, y_{t+H})\right]. \end{aligned} \tag{2.7}$$

If the short term riskfree rates are known at date t, we get :

$$C_t = \exp\left(-\sum_{i=1}^{H} r_{t+i}\right) E^{Q_t}\left[g(y_{t+1}, \ldots, y_{t+H})\right].$$

3. The static parametric model

In this section we study the pricing problem through a static parametric modelisation. More precisely, in paragraph 3.1 we consider the case where the geometric returns y_t of the risky asset are i.i.d. and distributed as a finite mixture of Normal distributions. So the process y_t can be viewed as gaussian conditionally to the discrete value white noise giving at each date t the index of the relevant gaussian component. The riskless rate is fixed and denoted by r. In paragraph 3.2 we use the mixed normal distribution for modelling extreme risks whereas in paragraph 3.3 we present the jump-diffusion case (infinite countable mixture of Normal distributions).

3.1 Pricing with a Finite Mixture of Normal distributions

The choice of a mixture of Normal distributions for the modelisation of stock returns derive from its ability to describe their empirical distributions in a useful, simple and flexible way.

It is well known from empirical research that the distributions of stock returns are characterized by asymmetries and fat tails [see Mandelbrot (1962, 1963a,b, 1967), Fama (1965)], and in order to capture these features several distributions have been proposed in the literature. Families of distributions that have shown a close fit to data are the Stable Paretian distributions [see, for example, Mandelbrot (1997), Mittnick and Rachev (1993a,b), Mittnick, Paolella and Rachev (1997), Adler et al. (1998)], the Finite Mixture of Normal distributions [see, among the others, Kon (1984), Akgiray and Booth (1987), Tucker and Pond (1988)], the Student distributions [see Bollerslev (1987), Baillie and Bollerslev (1989), Palm and Vlaar (1997), Lambert and Laurent (2000, 2001)] and the Hyperbolic distributions [see Barndorff-Nielsen (1994), Eberlein and Keller (1995), Kuechler et al. (1994)].

Two important elements are in favor of the mixed normal statistical model.

The first element arise from an empirical observation [see, for instance, Campbell, Lo and MacKinlay (1997)] that the Paretian family is not able to reproduce. Empirical analysis has shown that asymmetries and fat-tails are much weaker for low frequency observations (long horizon returns) than for high frequency ones (short horizon returns): therefore, if we want to perform this kind of behavior we need to use distributions characterized by finite moments for which the Central Limit Theorem applies and drives longer-horizon returns towards normality. In particular, empirical researches [see Tucker (1992)] has shown that the general stable Paretian model is dominated by the jump-diffusion model [see Merton (1976) and paragraph 3.3] and by the finite mixture of Normal distributions model; in the latter case it seems that the higher goodness of fit is obtained by a mixture of two distributions.

The second element is linked to an interesting feature of the Mixed Normal modelisation, not shared by other distributions (like, for instance, the Hyperbolic or the Student distribution), that is the economic interpretation that we can give about it and that can be of interest not only for researchers but also for practitioners such as risk managers. A mixture of two or more normal variables allows for an interpretation of two or more groups of investors with different behaviours with respect to risk [see Shleifer and Summer (1990)], or the mixture may also be interpreted as representing market periods with different levels of volatility.

3.1.1 Historical and risk neutral distributions

Since, in this section, we are in a static case we will often drop the index t for sake of notational simplicity. If we consider a geometric return y whose distribution is a mixture of J Normal distributions denoted by $\mathcal{MN}(J, p_j, \mu_j, \sigma_j^2)$, its p.d.f. is given by :

$$f(y) = \sum_{j=1}^{J} p_j \, n(y; \mu_j, \sigma_j^2), \tag{3.1}$$

where, for $j = 1, ..., J$

$$n(y; \mu_j, \sigma_j^2) = \frac{1}{\sigma_j \sqrt{2\pi}} e^{-\frac{1}{2}\frac{(y-\mu_j)^2}{\sigma_j^2}},$$

$$0 \leq p_j \leq 1, \; \sum_{j=0}^{J} p_j = 1,$$

and its mean, variance, skewness and kurtosis are

$$\mu = \sum_{j=1}^{J} p_j \mu_j$$

$$\sigma^2 = \sum_{j=1}^{J} p_j (\sigma_j^2 + \mu_j^2) - \mu^2$$

$$\tilde{\mu}_3 = \frac{1}{\sigma^3} \sum_{j=1}^{J} p_j (\mu_j - \mu)[3\sigma_j^2 + (\mu_j - \mu)^2]$$

$$\tilde{\mu}_4 = \frac{1}{\sigma^4} \sum_{j=1}^{J} p_j [3\sigma_j^4 + 6(\mu_j - \mu)^2 \sigma_j^2 + (\mu_j - \mu)^4].$$

By applying in this static framework the general approach described in section 2 we get the following results. In particular, in Proposition 3 we obtain the pricing formula for a Call option with maturity one and strike K: this derivative asset gives the payoff $(S_{t+1} - K)^+ = S_t \left[\exp y_{t+1} - \kappa\right]^+$ where $\kappa = K/S_t$. Normalizing S_t to one the payoff is $(\exp y_{t+1} - \kappa)^+$.

Proposition 1 : If the historical distribution is a mixture of J Normal distributions $\mathcal{MN}(J, p_j, \mu_j, \sigma_j^2)$ and if the stochastic discount factor is exponential affine, we have a unique solution (α,β) that satisfies system (2.3). The unique value of α is the solution of :

$$\sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j + \sigma_j^2 \frac{\alpha^2}{2}\right) \left[\exp\left(\mu_j + \sigma_j^2 \alpha + \frac{\sigma_j^2}{2}\right) - \exp r\right] = 0. \quad (3.2)$$

Proof : see Appendix 1.

Proposition 2 : The risk-neutral distribution Q is unique and is a mixture of Normal distributions with the following p.d.f. :

$$f^Q(y) = \sum_{j=1}^{J} \nu_j \, n(y; \mu_j + \alpha\sigma_j^2, \sigma_j^2), \quad (3.3)$$

where, for $j = 1, ..., J$

$$\nu_j = \frac{p_j \exp\left(\alpha\mu_j + \sigma_j^2 \frac{\alpha^2}{2}\right)}{\sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j + \sigma_j^2 \frac{\alpha^2}{2}\right)}$$

$$0 \leq \nu_j \leq 1, \ \sum_{j=1}^{J} \nu_j = 1,$$

Proof : see Appendix 2.

The risk-neutral distribution depends not only on the σ_j^2's but also on the μ_j's, denoting an evolution with respect to the Black-Scholes framework.

Proposition 3 : The price of the European Call option with payoff $(\exp y_{t+1} - \kappa)^+$ and maturity one is:

$$\begin{aligned} C(\kappa) &= \exp(-r^f) E^Q[\exp y - \kappa]^+ \\ &= \sum_{j=1}^{J} \nu_j \, \gamma_j \, C_{BS}\left(\sigma_j^2, \frac{\kappa}{\gamma_j}\right), \end{aligned} \tag{3.4}$$

where C_{BS} is the (one-period) Black-Scholes formula with a volatility equal to σ_j^2 and moneyness strike equal to κ/γ_j, and

$$\gamma_j = \exp\left(\mu_j + \alpha\sigma_j^2 - r + \frac{\sigma_j^2}{2}\right).$$

Moreover, it can be shown that

$$0 \leq \nu_j\gamma_j \leq 1, \ \sum_{j=1}^{J} \nu_j\gamma_j = 1.$$

Proof : see Appendix 3.

The propositions presented above consider the case of a one-period geometric return y_{t+1}, but the same results can be specified if we consider the general case of a geometric stock return $y_{t,t+H} = y_{t+1} + \ldots + y_{t+H}$ at horizon

H larger than one; indeed, $y_{t,t+H}$ has a distribution which is once more a mixture of normal distributions with historical p.d.f. :

$$f(y_{t,t+H}) = \sum_{\substack{h_j=0 \\ j=\{1,\dots,J-1\}}}^{H} \frac{H!}{h_1! \cdot \ldots \cdot h_J!} p_1^{h_1} \cdot \ldots \cdot p_J^{h_J} \, n\left(\sum_{j=1}^{J} h_j \mu_j, \sum_{j=1}^{J} h_j \sigma_j^2 \right),$$

with $\sum_{j=1}^{J} h_j = H$.

3.1.2 Generalizing the Black-Scholes framework

The pricing formula that we obtain by the finite mixture of Normal distributions assumption is a generalization of the Black-Scholes pricing formula. It is a linear combination of J Black-Scholes formulas and it depends not only on the variances but also on the means of the gaussian distributions in the mixture. Moreover, it is easily seen that, in the gaussian case (i.e. when $J = 1$) we have $\alpha = -(\sigma^2)^{-1}[\mu - r + (\sigma^2/2)]$, therefore γ_j can be written :

$$\gamma_j = \exp\left[\sigma_j^2(\alpha - \alpha_j)\right],$$

where $\alpha_j = -(\sigma_j^2)^{-1}[\mu_j - r + (\sigma_j^2/2)]$ is the value of α corresponding to the gaussian case associated with the j^{th} component of the mixture.

The mixture of distributions appearing in the historical probability of the returns implies a modified mixture of the BS formulas corresponding to the volatilities of the elements of the mixtures and to modified moneyness strikes. In particular, the modified BS formula $C_{BS}(\sigma_j^2, \kappa/\gamma_j)$ defines an option price larger (resp. smaller) than the BS price $C_{BS}(\sigma_j^2, \kappa)$ if the coefficient γ_j modifying the strike is larger (resp. smaller) than one, that is if α_j is smaller (resp. larger) than α [compare with Ritchey (1990) where the strong assumption of risk neutrality is made].

3.2 Modelling extreme risks

Here we are interested in a particular mixture of two normal distributions, one of which having a small weight associated with a strong variance. This example enables us to take into account the unlikely occurrence of an important shock on the volatility.

More precisely, we consider a geometric return y whose distribution is given by the following p.d.f :

$$f(y) = \lambda \, n(y; \mu, \frac{\sigma_1^2}{\lambda}) + (1-\lambda) \; n(y; \mu, \frac{\sigma_2^2}{1-\lambda}), \tag{3.5}$$

and we consider the situation in which λ tends to zero.

Proposition 4 : The main characteristics of this distribution :

$$\begin{aligned}
E\,(y) &= \mu \\
\sigma^2 = V(y) &= \sigma_1^2 + \sigma_2^2 \\
\frac{E\,(y-\mu)^3}{\sigma^3} &= 0 \\
\frac{E\,(y-\mu)^4}{\sigma^4} &= \frac{3\left[\frac{\sigma_1^4}{\lambda} + \frac{\sigma_2^4}{1-\lambda}\right]}{\left[\sigma_1^2 + \sigma_2^2\right]^2} \quad \to +\infty \quad \textit{when} \quad \lambda \to 0 \\
& \quad y \xrightarrow{D} \mathcal{N}\left(\mu, \sigma_2^2\right)
\end{aligned}$$

These results show that, when λ tends to 0, this distribution is in a sense close to the gaussian distribution, but with a strong kurtosis; it is a case where the convergence in distribution does not imply the convergence of the moments.

We now apply the general propositions of section 3.1.1 to this special case.

Proposition 5 : The value of α given by the solution of equation (3.2) converges to $-\frac{1}{2}$ when $\lambda \to 0$.

Proposition 6 : The risk-neutral distribution Q has the following p.d.f. :

$$f^Q(y) = \nu_1 \, n(y; \mu \, + \, \alpha\frac{\sigma_1^2}{\lambda}, \frac{\sigma_1^2}{\lambda}) + \nu_2 \, n(y; \mu \, + \, \alpha\frac{\sigma_2^2}{1-\lambda}, \frac{\sigma_2^2}{1-\lambda}) \tag{3.6}$$

where

$$\nu_1 \quad = \quad \frac{\lambda \exp\left(\frac{\sigma_1^2}{\lambda}\frac{\alpha^2}{2}\right)}{\lambda \exp\left(\frac{\sigma_1^2}{\lambda}\frac{\alpha^2}{2}\right) + (1-\lambda)\exp\left(\frac{\sigma_2^2}{1-\lambda}\frac{\alpha^2}{2}\right)} \quad \textit{and} \quad \nu_2 = 1 - \nu_1 \, .$$

We note that : $\nu_1 \longrightarrow 1$ when $\lambda \to 0$.

Proposition 7 : The price of the European Call option can be written :

$$C(\kappa) = \nu_1\, \gamma_1\, C_{BS}\left(\frac{\sigma_1^2}{\lambda}, \frac{\kappa}{\gamma_1}\right) + \nu_2\, \gamma_2\, C_{BS}\left(\frac{\sigma_2^2}{1-\lambda}, \frac{\kappa}{\gamma_2}\right) \tag{3.7}$$

with

$$\gamma_1 = \exp\left(\mu + \alpha\frac{\sigma_1^2}{\lambda} - r + \frac{\sigma_1^2}{2\lambda}\right).$$
$$\gamma_2 = \exp\left(\mu + \alpha\frac{\sigma_2^2}{1-\lambda} - r + \frac{\sigma_2^2}{2(1-\lambda)}\right)$$

and :

$$\nu_1\, \gamma_1 \longrightarrow 1$$
$$C(\kappa) \longrightarrow 1, \quad when \quad \lambda \to 0.$$

Proofs : see Appendix 4.

We notice that the first component of the historical mixture distribution has a small weight whereas on the contrary it is associated with a strong weight for the risk-neutral distribution and consequently for the price of the European Call option. The first term of the price tends to 1, that is to say the (normalized) price of the underlying asset, and the second term tends to zero. This may be seen as the effect of strong kurtosis.

3.3 The jump-diffusion model

A general jump diffusion model, as proposed by Merton (1976), is defined as a superposition of two continuous-time processes. The first one is a Brownian motion used classically to model "normal" movements on returns and the second one is constructed on the basis of a Poisson process (each Poisson event causes a normally distributed jump on returns). The latter enables to model "abnormal" movements on returns.

Here we want to work in a discrete-time context and so we define a geometric return distribution for one period. Let us first recall that the continuous-time process is :

$$\ln\left(\frac{S_t}{S_0}\right) = \mu t + \sigma B\left(t\right) + \sum_{j=1}^{N_t} Y_j \tag{3.8}$$

where S_t is the asset price at time t, $B(t)$ is a standard Brownian motion, N_t is a Poisson counting process with parameter λ and Y_j (which measures the j^{th} jump amplitude) are independently and identically normally distributed $\mathcal{N}(\mu_p, \sigma_p^2)$.

The geometric return distribution on one period, i.e the distribution of $y_t = \log\left(\frac{S_t}{S_{t-1}}\right)$, is an infinite countable mixture of Normal distributions with Poisson weights. Its p.d.f. is the following :

$$\sum_{j=0}^{+\infty} e^{-\lambda}\frac{\lambda^j}{j!}\, n(\mu + j\mu_p, \sigma^2 + j\sigma_p^2) \tag{3.9}$$

Therefore we are able to give analoguous formulas to (3.2), (3.3), (3.4).

Proposition 8.a : The unique value of α is the solution of :

$$\mu - r + \left(\alpha + \frac{1}{2}\right)\sigma^2 + \lambda \exp\left(\alpha\mu_p + \sigma_p^2\frac{\alpha^2}{2}\right)\left[\exp\left(\mu_p + \sigma_p^2(\alpha + \frac{1}{2})\right) - 1\right] = 0 \tag{3.10}$$

Proposition 8.b : The risk-neutral distribution is again an infinite countable mixture of Normal distributions with new Poisson weights :

$$f^Q(y) = \sum_{j=0}^{+\infty} \nu_j\, n(y; \mu + \alpha\sigma^2 + j\left(\mu_p + \alpha\sigma_p^2\right), \sigma^2 + j\sigma_p^2), \tag{3.11}$$

where $\nu_j = \exp(-\lambda')\frac{\lambda'^j}{j!}$ with $\lambda' = \lambda \exp\left(\alpha\mu_p + \sigma_p^2\frac{\alpha^2}{2}\right)$.

So the risk neutral process is again of jump diffusion type, with a new drift $\mu' = \mu + \alpha\,\sigma^2$, the same volatility σ^2, a new mean $\mu_p' = \mu_p + \alpha\,\sigma_p^2$ and the same variance for the amplitude of the shock and, finally, a modified Poisson parameter λ'.

Proposition 8.c : The price of the European Call option is an average of the Black-Scholes formulas with Poisson weights :

$$C(\kappa) = \sum_{j=0}^{\infty} \beta_j \, C_{BS} \left(\sigma^2 + j\sigma_p^2, \frac{\kappa}{\gamma_j} \right), \tag{3.12}$$

where $\beta_j = \exp(-\tilde{\lambda}) \frac{\tilde{\lambda}^j}{j!}$ with $\tilde{\lambda} = \lambda \exp\left((\alpha+1)\mu_p + \sigma_p^2 \frac{(\alpha+1)^2}{2}\right)$
and $\gamma_j = \exp\left(\mu + j\mu_p + \alpha(\sigma^2 + j\sigma_p^2) - r + \frac{\sigma^2 + j\sigma_p^2}{2}\right)$.

Proofs : see Appendix 5.

3.4 Implied Black-Scholes Volatility and historical parameters

We have seen in previous sections that the call option pricing formula we propose depends on the parameters $\Lambda = (p_j, \mu_j, \sigma_j^2; j = 1, \ldots, J)$ of the historical distribution instead of the only volatility σ in the BS formula. In particular, the assumption of a mixed normal distribution, able to reproduce asset's returns stylized facts such as asymmetries and fat tails, allows to describe implied volatility curves with smile and skew shapes. These features are presented on Figures 1, 2 and 3.

In particular, in the first case we consider a mixture of two gaussian distributions with the same means and probabilities ($\mu_1 = \mu_2 = 0.03$, $p = 0.50$) and with a global variance fixed to a certain level ($\sigma^2 = 0.04$): this situation allows to isolate and to observe the effect of an increasing kurtosis on the implied BS volatility (for maturity one), starting from the (flat) BS form ($\sigma_1^2 = \sigma_2^2 = \sigma^2 = 0.04$) and applying an increasing variance in the second component of the mixture[1] (σ_1^2 contemporaneously reduces to guarantee a fixed global variance). We observe that the implied volatility induced by our model takes a smile shape when the kurtosis coefficient leaves the BS case and takes higher values; in particular, we are able to reproduce the empirically observed fact that the BS formula tends to underprice out-of-the-money and in-the-money options, while overpricing at-the-money options [see Figure 1].

In the second case we present a more asymmetric smile shape by a mixture of two gaussian random variables with higher means ($\mu_1 = \mu_2 = 0.07$) and

[1]It is easy to verify that, in the case of a mixture of two gaussian random variables with the same means and with a global variance σ^2 fixed to a certain level M, $\tilde{\mu}_3 = 0$ and $\tilde{\mu}_4 = g(M, \sigma_2^2)$ with $(\partial\tilde{\mu}_4/\partial\sigma_2^2) \gtreqless 0$ if and only if $\sigma_2^2 \gtreqless M$.

variances than in the first case, but now the first component has a much higher weight than the second one: as in the previous case we consider a fixed global variance ($\sigma^2 = 0.06$) and we induce an increasing kurtosis by an increasing change in σ_2^2 (σ_1^2 reduces to keep the global variance fixed). This structure describes a market with high expected returns and with a (typical) low volatility scenario that sometimes switch to an high volatility framework: the fact to combine these two features by means of the mixed normal model leads to produce implied BS volatilities with more asymmetric smiles [see Figure 2].

In the third case [see Figure 3] we present the volatility skew, that is an implied volatility shape typical of equity options: indeed, in this situation we have that the underlying is characterized by a left-asymmetric distribution with a left tail fatter than the right one because of the higher probability of large negative returns. In order to reproduce this type of distribution we consider a mixture where the first component has a lower mean but a variance higher than the second one. As in the previous cases, we consider a fixed global variance and we observe the effect of an increasing negative skewness induced by an increasing value in the variance of the first component[2];in particular, Figure 3 shows that as the negative skewness increases with the size of the left tail, the implied volatility takes a more pronounced skewed shape denoting the induced higher value for out-of-the-money Put options and in-the-money Call options.

Another important proof of the ability of the mixed normal models to describe the distribution's properties of stock returns, and in particular the ability to take into account the fact that longer-horizon returns tend towards normality, is given by the implied volatility surface presented in Figure 4 (here the time to maturity, measured in years, changes from 0.25 to 1.5): indeed, we can observe that as the maturity increases the profile of the implied volatility takes a flatter form denoting, by consequence, an increasing presence of normality in the distribution of the underlying.

Now, if we consider the previous volatility skew case without fixing the global variance we can observe the movement of the implied BS volatility in a context characterized at the same time by increasing global variance and

[2]In this situation, the movement on σ_1^2 modifies at the same time the skewness and kurtosis parameters, but the fact to consider a fixed global variance with μ_1 lower than μ_2 guarantee to have $\tilde{\mu}_3 < 0$ for every $\sigma_1^2 > \sigma_2^2 + \frac{1}{3}(\mu_1 - \mu_2)^2(2p-1)$ and the skewness parameter takes higher negative values when the difference between σ_1^2 and the RHS of the inequality increases.

skewness and by changes in the kurtosis coefficient: Figure 5 shows that in this situation the joint movement of the three moments, induced once more by an increasing value of σ_1^2, gives an implied volatility curve that leaves the BS case to take more pronounced skewed shapes at higher value levels of volatility. In other words, in this more general case we have at the same time a level and skew effect.

It is also interesting to study the behaviour of the implied BS volatility of the extreme-risks parameterization presented in section 3.2: here we have that, as λ tends to zero, the geometric return (in the historical framework) converges in distribution to the gaussian law, but with a kurtosis increasing to infinity (convergence in distribution without convergence of the moments): Figure 6 (in which we set $\sigma_1^2 = \sigma_2^2$) shows that starting from the flat implied volatility ($\lambda = 0.5$), if we consider a decreasing value of λ, the induced higher kurtosis leads to a smile shape as presented in Figures 1 and 2, but now, as the kurtosis tends rapidly to infinity when λ tends to zero, the implied BS volatility increase quickly for every moneyness strike [see Figure 7] denoting the effect induced by extreme risks.

As a last example, we can also show the relation of the implied BS volatility, induced by the Jump-Diffusion model presented in section 3.3, with the parameters of the jump component $\Lambda_p = (\mu_p, \sigma_p^2, \lambda)$; the results are presented on Figures 8, 9 and 10. We observe in Figure 8 that an increasing mean amplitude of jumps (μ_p) produces a decreasing value of the implied volatility for small moneyness strike and an increasing value for large moneyness strike, whereas an increasing value of the variance of the jump amplitude distribution σ_p^2 or of the jump rate parameter λ induces a general increasing value in the implied volatility [see Figures 9 and 10].

4 The static non parametric case

We now consider the nonparametric static viewpoint. In this approach, the geometric returns are IID, their distribution is not specified and we estimate it by means of the well-known Gaussian kernel density estimator. It turns out that this estimator is a mixture of normal distributions with the particularity that the variances are constant (equal to h^2 where h is the smoothing parameter (bandwidth)) and that the weights are all equal.

We note $(y_1, y_2, ..., y_J)$ the observations of the geometric return. The gaussian kernel estimator is :

$$f(y) = \sum_{j=1}^{J} \frac{1}{J}\, n(y; y_j, h^2), \tag{4.1}$$

So by a direct application of formulas (3.2), (3.3), (3.4) we obtain the following results.

First, α is the unique solution of :

$$\sum_{j=1}^{J} \exp(\alpha y_j) \left[\exp\left(y_j \,+\, h^2\alpha \,+\, \frac{h^2}{2} \right) - \exp r \right] = 0. \tag{4.2}$$

Second, the risk-neutral distribution is :

$$f^Q(y) = \sum_{j=1}^{J} \nu_j\, n(y; y_j \,+\, \alpha h^2, h^2), \tag{4.3}$$

where, for $j = 1, ..., J$

$$\nu_j = \frac{\exp(\alpha y_j)}{\sum_{j=1}^{J} \exp(\alpha y_j)}$$

Lastly, the price of the European Call option is :

$$C(\kappa) = \sum_{j=1}^{J} \nu_j^{'}\, C_{BS}\left(h^2, \frac{\kappa}{\gamma_j} \right), \tag{4.4}$$

with

$$\nu_j^{'} = \frac{\exp((\alpha+1)\, y_j)}{\sum_{j=1}^{J} \exp((\alpha+1)\, y_j)}$$

and

$$\gamma_j = \exp\left(y_j + \alpha h^2 - r + \frac{h^2}{2} \right).$$

Thus, using the gaussian kernel estimator (and it is well known that the choice of the kernel is not crucial, contrary to the choice of the bandwidth), it is possible to include the non parametric case in the framework of normal mixtures, and to derive general procedures for obtaining the risk neutral distribution and pricing formulas.

5 The Mixed-Normal GARCH model

The aim of this section is to give a dynamic framework to the model presented in section 3.1; in particular, the dynamics is introduced by means of a GARCH-type characterization of the geometric stock return with a conditional distribution defined by a mixture of gaussian random variables.

This specification gives us the possibility to describe in a simple and realistic way not only the typical stylized fact of asset returns as volatility clustering, heavy tails and asymmetries, but also new emerging behaviors like time-varying skewness and kurtosis [see the works of Hansen (1994), Paolella (1999), Harvey and Siddique (1999), Brännäs and Nordman (2001), Rockinger and Jondeau (2002)].

We consider first a mixture of two gaussian distributions with a mean equal to zero, which will be used to derive classes of martingale differences. Such a distribution can be parameterized as :

$$f(u) = \lambda n\left(u; a(1-\lambda), \sigma_1^2\right) + (1-\lambda)n\left(u; -a\lambda, \sigma_2^2\right) \tag{5.1}$$

where $0 \leq \lambda \leq 1$, and $a \in R$.

The mean, variance, skewness and kurtosis are

$$\begin{aligned}
\mu &= 0 \\
\sigma^2 &= [\lambda\sigma_1^2 + (1-\lambda)\sigma_2^2] + a^2\lambda(1-\lambda) \\
\tilde{\mu}_3 &= \frac{1}{\sigma^3}\{a\lambda(1-\lambda)[3(\sigma_1^2 - \sigma_2^2)]\} \\
\tilde{\mu}_4 &= \frac{1}{\sigma^4}\{3[\lambda\sigma_1^4 + (1-\lambda)\sigma_2^4] + 6a^2\lambda(1-\lambda)[(1-\lambda)\sigma_1^2 + \lambda\sigma_2^2] \\
&\qquad + a^4\lambda(1-\lambda)[3\lambda^2 - 3\lambda + 1]\}.
\end{aligned}$$

Normal mixtures in a GARCH context were suggested by Vlaar and Palm (1993, 1997), Bauwens *et al.* (1999) and Lin and Yeh (2000). All these modelisations are special cases of the general specification proposed by Hass, Mittnick and Paolella (2002)[HMP hereafter].

The GARCH characterizations of the dynamics we propose for our pricing framework are of two types, and the first one is a particular case of the HMP model [see Appendix 4].

5.1 The MN-GARCH process of first type

In this case, the conditional distribution of the process (ε_t) is a mixture of two gaussian distributions :

$$f(\varepsilon_{t+1}|\,\underline{\varepsilon_t}) = \lambda n\left(\varepsilon_{t+1}; a(1-\lambda), \sigma^2_{1t+1}\right) + (1-\lambda) n\left(\varepsilon_{t+1}; -a\lambda, \sigma^2_{2t+1}\right), \quad (5.2)$$

where $\underline{\varepsilon_t} := (\varepsilon_t, \varepsilon_{t-1}, \ldots)$ is the information on the current and lagged values of ε_t and where the variances of the mixture components evolve according to GARCH specifications. In order to keep the notation simple we consider GARCH(1,1) specifications (a generalization to GARCH(p,q) structures is straightforward) :

$$\begin{cases} \sigma^2_{1t+1} = \omega_1 + b_1\varepsilon^2_t + c_1\sigma^2_{1t} \\ \\ \sigma^2_{2t+1} = \omega_2 + b_2\varepsilon^2_t + c_2\sigma^2_{2t}, \end{cases} \quad (5.3)$$

with non-negativity conditions $c_i \geq 0, \omega_i > 0$ and $b_i \geq 0, i = 1, 2$, assumed.

Note that, conditionally to the values of a i.i.d. latent process giving the chosen component at each date t, the process is gaussian conditionally to its past. The process ε_t is a martingale difference since $E(\varepsilon_{t+1}|\,\underline{\varepsilon_t}) = 0$ which can be used as a basic building block for more complex models, for instance ARMA-GARCH models or GARCH-M models.

It is possible to verify [see Appendix 6] that ε_t also has a GARCH specification if we impose the contraint $c_1 = c_2 = c$; indeed, in this case we can write :

$$E(\varepsilon^2_{t+1}|\,\underline{\varepsilon_t}) := \sigma^2_{t+1} = \xi + \left[\lambda b_1 + (1-\lambda) b_2\right] \varepsilon^2_t + c\sigma^2_t,$$

with $\xi := (1-c)a^2\lambda(1-\lambda) + \lambda\omega_1 + (1-\lambda)\omega_2$.

In this type of model the conditional distribution $(\frac{\varepsilon_{t+1}}{\sigma_{t+1}} | \underline{\varepsilon_t})$ is a mixture of normal distributions depending on the past, while the second type of MN-GARCH model presented below gives an i.i.d. specification to the normalized process.

5.2 The MN-GARCH process of second type

If we consider again the simple GARCH(1,1) specification, the model takes the following form :

$$\begin{cases} \varepsilon_{t+1} = \sigma_{t+1} u_{t+1} \\ \\ \sigma_{t+1}^2 = \omega + c\varepsilon_t^2 + d\sigma_t^2, \end{cases} \tag{5.4}$$

where (u_t) is a sequence of independent zero mean mixed normal random variables characterized by the following p.d.f. :

$$f(u_t) = \lambda n\left(u_t; a(1-\lambda), \sigma_1^2\right) + (1-\lambda) n\left(u_t; -a\lambda, \sigma_2^2\right), \tag{5.5}$$

and where the conditional distribution of ε_t takes the following form :

$$\begin{aligned} f(\varepsilon_{t+1} | \underline{\varepsilon_t}) &= \lambda n\left(\varepsilon_{t+1}; \sigma_{t+1} a(1-\lambda), \sigma_{t+1}^2 \sigma_1^2\right) \\ &\quad + (1-\lambda) n\left(\varepsilon_{t+1}; -\sigma_{t+1} a\lambda, \sigma_{t+1}^2 \sigma_2^2\right). \end{aligned} \tag{5.6}$$

So, conditionally to the i.i.d. latent process giving the chosen component at each date, the process is also conditionally gaussian. By definition the variables $(\frac{\varepsilon_t}{\sigma_t})$ are i.i.d.. Now the model is overparameterized, and in order to solve the identification problem of this modelisation we propose two possible identification restrictions :

i) Normalization of the variance of u_t

With this first restriction we impose that :

$$V(u_t) = [\lambda\sigma_1^2 + (1-\lambda)\sigma_2^2] + a^2\lambda(1-\lambda) = 1, \tag{5.7}$$

and, consequently, we obtain $V(\varepsilon_{t+1}|\underline{\varepsilon_t}) \equiv \sigma^2_{t+1}$ and $V(\varepsilon_{t+1}) \equiv E(\sigma^2_{t+1})$.

ii) Normalization of $E(\sigma^2_{t+1})$

In this second case, we impose :

$$E(\sigma^2_{t+1}) = \frac{\omega}{1-c-d} = 1, \tag{5.8}$$

or $\omega = 1 - c - d$. So $V(\varepsilon_{t+1}|\underline{\varepsilon_t}) = \sigma^2\sigma^2_{t+1}$ where $\sigma^2 = V(u_{t+1}) = V(\varepsilon_{t+1})$.

The advantage of this modelisation is the possibility to implement a two-step estimation procedures for the model parameters.

a) First step :

We estimate the variance σ^2 of the marginal distribution of (ε_t) by the empirical variance $\hat{\sigma}^2$ and the parameter $\theta_1 = (c, d)$ by a Pseudo-Maximum Likelihood procedure (based on a gaussian GARCH model) applied to $\left(\frac{\varepsilon_t}{\hat{\sigma}}\right)$ and using the restriction $\omega = 1 - c - d$.

b) Second step :

With the estimated values $\hat{\sigma}_t$ we have an estimated sequence of independent mixed normal random variables $\left(\frac{\varepsilon_t}{\hat{\sigma}_t}\right)$ for which we proceed to the estimation of the mixture parameters $\theta_2 = (\lambda, a, \sigma_1^2, \sigma_2^2)$. We can use, for instance, a Maximum Likelihood estimation or a Bayesian Approach estimation using posterior simulation via Markov Chain Monte Carlo (MCMC) methods [see McLachlan and Peel (2000) and the references there in].

Note however that an advantage of the first type of normalization is that it allows for an IGARCH modelisation of $(\varepsilon_{t+1}|\underline{\varepsilon_t})$, whereas the second normalization condition makes (by construction) impossible this specification.

5.3 Computation of the S.D.F. and of the risk neutral distribution

With the definition of the conditional distribution of the stock return $y_{t+1} = \varepsilon_{t+1}$ [3] we have the dynamic structure that gives us the possibility to specify for every date t (conditionally to I_t) the pricing model presented in section 3.1.1. More precisely, we can derive the following results.

Proposition 9 : If the historical distribution of the process y_t is a MN-GARCH of first or second type and if the stochastic discount factor is exponential affine we have for every date t a unique solution $\alpha_t = \alpha(I_t)$ and $\beta_t = \beta(I_t)$ that satisfies the system (2.3). The unique value of α_t is solution of :

$$\lambda \exp\left[\alpha_t a(1-\lambda) + \sigma_{1t}^2 \frac{\alpha_t^2}{2}\right]\left[\exp\left(a(1-\lambda) + \sigma_{1t}^2\alpha_t + \frac{\sigma_{1t}^2}{2}\right) - \exp r_{t+1}^f\right]$$
$$+(1-\lambda)\exp\left[\sigma_{2t}^2\frac{\alpha_t^2}{2} - \alpha_t a\lambda\right]\left[\exp\left(\sigma_{2t}^2\alpha_t + \frac{\sigma_{2t}^2}{2} - a\lambda\right) - \exp r_{t+1}^f\right] = 0, \tag{5.9}$$

for the MN-GARCH of first type, while, if we consider the MN-GARCH of second type, the unique value of α_t is solution of :

$$\lambda \exp\left[\sigma_t\alpha_t\left(a(1-\lambda) + \sigma_1^2\frac{\sigma_t\alpha_t}{2}\right)\right]\left[\exp\left(\sigma_t a(1-\lambda) + \sigma_t^2\sigma_1^2\alpha_t + \frac{\sigma_t^2\sigma_1^2}{2}\right) - \exp r_{t+1}^f\right]$$
$$+(1-\lambda)\exp\left[\sigma_t\alpha_t\left(\sigma_2^2\frac{\sigma_t\alpha_t}{2} - a\lambda\right)\right]\left[\exp\left(\sigma_t^2\sigma_2^2\alpha_t + \frac{\sigma_t^2\sigma_2^2}{2} - \sigma_t a\lambda\right) - \exp r_{t+1}^f\right] = 0. \tag{5.10}$$

Given the unique solution (α_t, β_t) we can compute for every t the unique S.D.F. $M_{t,t+1}$ and, by consequence, the unique mixed normal conditional risk neutral distribution.

Proposition 10 : The conditional risk neutral distribution of the MN-GARCH process is unique and is a mixture of Normal distributions. For the MN-GARCH process of first type the risk neutral p.d.f. is given by :

[3] The modelisation of the stock return dynamics can be obviously generalized by the definition of a dynamic statistical model (for instance, ARMA model) for y_{t+1} in which the mixed-normal process ε_{t+1} is introduced as the noise component.

$$
\begin{aligned}
f_{\mathrm{I}}^{Q}(y_{t+1}) &= \nu_{\mathrm{I}t}\mathcal{N}\left(y_{t+1}; a(1-\lambda)+\alpha_t\sigma_{1t}^2, \sigma_{1t}^2\right) \\
&\qquad +(1-\nu_{\mathrm{I}t})\mathcal{N}\left(y_{t+1}; \alpha_t\sigma_{2t}^2 - a\lambda, \sigma_{2t}^2\right), \quad (5.11)
\end{aligned}
$$

where $0 \leq \nu_{\mathrm{I}t} \leq 1$ and

$$
\nu_{\mathrm{I}t} = \frac{\lambda \exp\left(\alpha_t a(1-\lambda) + \sigma_{1t}^2 \frac{\alpha_t^2}{2}\right)}{\lambda \exp\left(\alpha_t a(1-\lambda) + \sigma_{1t}^2 \frac{\alpha_t^2}{2}\right) + (1-\lambda)\exp\left(\sigma_{2t}^2 \frac{\alpha_t^2}{2} - \alpha_t a\lambda\right)},
$$

while, for the MN-GARCH process of second type the risk neutral p.d.f. is the following :

$$
\begin{aligned}
f_{\mathrm{II}}^{Q}(y_{t+1}) &= \nu_{\mathrm{II}t}\mathcal{N}\left(y_{t+1}; \sigma_t(a(1-\lambda)+\alpha_t\sigma_t\sigma_1^2), \sigma_t^2\sigma_1^2\right) \\
&\qquad +(1-\nu_{\mathrm{II}t})\mathcal{N}\left(y_{t+1}; \sigma_t(\alpha_t\sigma_t\sigma_2^2 - a\lambda), \sigma_t^2\sigma_2^2\right), \quad (5.12)
\end{aligned}
$$

where $0 \leq \nu_{\mathrm{II}t} \leq 1$ and

$$
\nu_{\mathrm{II}t} = \frac{\lambda \exp\left[\sigma_t\alpha_t\left(a(1-\lambda) + \sigma_t\sigma_1^2\frac{\alpha_t}{2}\right)\right]}{\lambda \exp\left[\sigma_t\alpha_t\left(a(1-\lambda) + \sigma_t\sigma_1^2\frac{\alpha_t}{2}\right)\right] + (1-\lambda)\exp\left[\sigma_t\alpha_t\left(\sigma_t\sigma_2^2\frac{\alpha_t}{2} - a\lambda\right)\right]}.
$$

Proposition 11 : The price at the date t of the European Call option with payoff $(\exp y_{t+1} - \kappa_t)^+$ and maturity one is, for the MN-GARCH process of the first type :

$$
C_{\mathrm{I}}(\kappa_t) = \nu_{\mathrm{I}t}\,\gamma_{\mathrm{I}t,1}\,C_{BS}\left(\sigma_{1t}^2, \frac{\kappa_t}{\gamma_{\mathrm{I}t,1}}\right) + (1-\nu_{\mathrm{I}t})\,\gamma_{\mathrm{I}t,2}\,C_{BS}\left(\sigma_{2t}^2, \frac{\kappa_t}{\gamma_{\mathrm{I}t,2}}\right), \quad (5.13)
$$

where the (one-period) Black-Scholes formula is defined for a volatility σ_{jt}^2 and a moneyness strike $\kappa_t/\gamma_{\mathrm{I}t,j}$ $(j=1,2)$, and

$$
\begin{aligned}
\gamma_{\mathrm{I}t,1} &= \exp\left[\alpha_t\sigma_{1t}^2 + \frac{\sigma_{1t}^2}{2} + a(1-\lambda) - r_{t+1}^f\right] \\
\gamma_{\mathrm{I}t,2} &= \exp\left[\alpha_t\sigma_{2t}^2 + \frac{\sigma_{2t}^2}{2} - a\lambda - r_{t+1}^f\right].
\end{aligned}
$$

Moreover, by the procedure given in Appendix 3 it can be shown that :

$$0 \leq \nu_{\mathrm{I}t}\gamma_{\mathrm{I}t,1} \leq 1, \;\; 0 \leq (1-\nu_{\mathrm{I}t})\gamma_{\mathrm{I}t,2} \leq 1,$$

$$\nu_{\mathrm{I}t}\gamma_{\mathrm{I}t,1} + (1-\nu_{\mathrm{I}t})\gamma_{\mathrm{I}t,2} = 1, \; \forall t > 0.$$

If we consider the MN-GARCH process of the second type, the pricing formula is given by :

$$\begin{aligned} C_{\mathrm{II}}(\kappa_t) &= \nu_{\mathrm{II}t}\, \gamma_{\mathrm{II}t,1}\, C_{BS}\left(\sigma_t^2\sigma_1^2, \frac{\kappa_t}{\gamma_{\mathrm{II}t,1}}\right) \\ &\quad + (1-\nu_{\mathrm{II}t})\, \gamma_{\mathrm{II}t,2}\, C_{BS}\left(\sigma_t^2\sigma_2^2, \frac{\kappa_t}{\gamma_{\mathrm{II}t,2}}\right), \end{aligned} \tag{5.14}$$

where the (one-period) Black-Scholes formula is now defined for a volatility $\sigma_t^2\sigma_j^2$ and a moneyness strike $\kappa_t/\gamma_{\mathrm{II}t,j}$ $(j=1,2)$, and

$$\begin{aligned} \gamma_{\mathrm{II}t,1} &= \exp\left[\alpha_t\sigma_t^2\sigma_1^2 + \frac{\sigma_t^2\sigma_1^2}{2} + \sigma_t a(1-\lambda) - r_{t+1}^f\right] \\ \gamma_{\mathrm{II}t,2} &= \exp\left[\alpha_t\sigma_t^2\sigma_2^2 + \frac{\sigma_t^2\sigma_2^2}{2} - \sigma_t a\lambda - r_{t+1}^f\right]. \end{aligned}$$

In addition, we still have that :

$$0 \leq \nu_{\mathrm{II}t}\gamma_{\mathrm{II}t,1} \leq 1, \;\; 0 \leq (1-\nu_{\mathrm{II}t})\gamma_{\mathrm{II}t,2} \leq 1,$$

$$\nu_{\mathrm{II}t}\gamma_{\mathrm{II}t,1} + (1-\nu_{\mathrm{II}t})\gamma_{\mathrm{II}t,2} = 1, \; \forall t > 0.$$

The generalization of the propositions above to the case of a mixture of J components is straightforward.

6 The dynamic semi-parametric case

In this section we develop a semi-parametric analysis of the mixed normal pricing model proposed in the sections above. In particular, we consider that the return satisfies :

$$y_{t+1} = m_{t+1} + \sigma_{t+1}\varepsilon_{t+1},\ \sigma_{t+1} > 0, \tag{6.1}$$

where m_t and σ_t are the location and scale parameters, respectively, that may depend on lagged values of the return and (ε_t) is a sequence of i.i.d. variables.

This approach is very similar to the one proposed by Gourieroux and Monfort (2001) which is based on the estimated empirical distribution of the errors (ε_t). In particular, we consider a parametric specification of m_t and σ_t :

$$m_t = m(y_{t-1};\theta), \sigma_t = \sigma(y_{t-1};\theta), \tag{6.2}$$

and we leave unspecified the distribution of the error term. The available observations on the returns are denoted by $y_1, \ldots, y_T$.

The parameter θ can be consistently estimated from historical data by applying a Pseudo-Maximum Likelihood method; the estimator is given by :

$$\hat{\theta}_T = \arg\max_{\theta} \sum_{t=1}^{T} \left\{ -\log\sigma^2(y_{t-1};\theta) - \frac{[y_t - m(y_{t-1};\theta)]^2}{\sigma^2(y_{t-1};\theta)} \right\};$$

then we compute the residuals :

$$\hat{\varepsilon}_\tau = \frac{r_\tau - m(r_{\tau-1};\hat{\theta}_T)}{\sigma(r_{\tau-1};\hat{\theta}_T)}, \tau = 2, \ldots, T,$$

and the distribution of ε_t can be approximated by a Gaussian Kernel estimator based on the $\hat{\varepsilon}_\tau$'s.

Therefore the p.d.f. of ε_{t+1} is approximated by the gaussian mixture :

$$\frac{1}{T-1}\sum_{\tau=2}^{T} n(\varepsilon;\hat{\varepsilon}_\tau, h^2),$$

where h is the bandwidth.

The conditional historical p.d.f. of y_{t+1} given y_t is approximated by the gaussian mixture :

$$\frac{1}{T-1}\sum_{\tau=2}^{T}\mathcal{N}(y_{t+1};\hat{m}_{t+1}+\hat{\sigma}_{t+1}\hat{\varepsilon}_{\tau},\hat{\sigma}_{t+1}^{2}h^{2}),$$

where $\hat{m}_{t+1}=m(y_t,\hat{\theta}_T),\hat{\sigma}_{t+1}=\sigma(y_t,\hat{\theta}_T)$.

This result allows to derive the dynamic semi-parametric mixed normal pricing model. In particular we obtain the following results.

Proposition 12 : If the conditional historical distribution of y_{t+1} is approximated by a mixture of $(T-1)$ Normal distributions $\mathcal{MN}(T-1;\frac{1}{T-1};\hat{m}_{t+1}+\hat{\sigma}_{t+1}\hat{\varepsilon}_{\tau},\hat{\sigma}_{t+1}^{2}h^{2})$ and if the stochastic discount factor is exponential affine, we have for every t a unique solution $\alpha_t=\alpha(I_t)$ and $\beta_t=\beta(I_t)$ that satisfies the system (2.3). The unique value of α_t is solution of :

$$\sum_{\tau=2}^{T}\exp\left[\alpha_t(\hat{m}_{t+1}+\hat{\sigma}_{t+1}\hat{\varepsilon}_{\tau})+\hat{\sigma}_{t+1}^{2}h^{2}\frac{\alpha_t^2}{2}\right]$$
$$\left[\exp\left((\hat{m}_{t+1}+\hat{\sigma}_{t+1}\hat{\varepsilon}_{\tau})+\hat{\sigma}_{t+1}^{2}h^{2}\alpha_t+\frac{\hat{\sigma}_{t+1}^{2}h^{2}}{2}\right)-\exp r_{t+1}^{f}\right]=0.$$

Proposition 13 : The conditional risk neutral distribution is unique and is approximated by a mixture of Normal distributions with the following p.d.f. :

$$f^{Q}(y_{t+1})=\sum_{\tau=2}^{T}\nu_{\tau}n\left(y_{t+1};\hat{m}_{t+1}+\hat{\sigma}_{t+1}(\hat{\varepsilon}_{\tau}+\alpha_t\hat{\sigma}_{t+1}h^{2}),\hat{\sigma}_{t+1}^{2}h^{2}\right), \qquad (6.3)$$

where, for $\tau=2,\dots,T$

$$\nu_{\tau}=\frac{\exp\left[\alpha_t(\hat{m}_{t+1}+\hat{\sigma}_{t+1}\hat{\varepsilon}_{\tau})+\hat{\sigma}_{t+1}^{2}h^{2}\frac{\alpha_t^2}{2}\right]}{\sum_{\tau=2}^{T}\exp\left[\alpha_t(\hat{m}_{t+1}+\hat{\sigma}_{t+1}\hat{\varepsilon}_{\tau})+\hat{\sigma}_{t+1}^{2}h^{2}\frac{\alpha_t^2}{2}\right]},$$

$$0\leq\nu_{\tau}\leq 1,\ \sum_{\tau=2}^{T}\nu_{\tau}=1.$$

The risk neutral distribution depends, for every t on the estimated location and scale parameters, on the computed residuals and on the smoothing parameter h.

Proposition 14 : The price of the European Call option written on $\exp y_{t+1}$ with payoff $(\exp y_{t+1} - \kappa)^+$ and maturity one is :

$$\begin{aligned} C(\kappa_t) &= \exp(-r^f_{t+1}) E^Q [\exp y_{t+1} - \kappa_t]^+ \\ &= \sum_{\tau=2}^{T} \nu_\tau \, \gamma_\tau \, C_{BS} \left(\hat{\sigma}^2_{t+1} h^2, \frac{\kappa_t}{\gamma_\tau} \right), \end{aligned} \tag{6.4}$$

where C_{BS} is the (one-period) Black-Scholes formula with a volatility equal to $\hat{\sigma}^2_{t+1} h^2$ and moneyness strike equal to κ_t / γ_τ, and

$$\gamma_\tau = \exp \left[\hat{m}_{t+1} + \hat{\sigma}_{t+1} \hat{\varepsilon}_\tau - r^f_{t+1} + \alpha_t \hat{\sigma}^2_{t+1} h^2 + \frac{\hat{\sigma}^2_{t+1} h^2}{2} \right].$$

By the same procedure as in Appendix 3 it can be shown that :

$$0 \leq \nu_\tau \, \gamma_\tau \leq 1, \; \sum_{\tau=2}^{T} \nu_\tau \, \gamma_\tau = 1.$$

These three propositions show the natural flexibility that the mixed normal framework is able to give for the modelisation of the S.D.F. pricing procedure: in particular, the mixed normal nature of the gaussian kernel estimation allows, by the application of the results of section 3.1.1, to rapidly specify a dynamic semi-parametric pricing model.

Appendix 1
Existence and uniqueness of alpha

If we consider the absence of arbitrage condition $\varphi(\alpha+1) = \exp(r)\varphi(\alpha)$ for the particular case of $r = 0$, we find in the mixed-normal framework the following relation :

$$H(\alpha) \equiv \sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j + \sigma_j^2 \frac{\alpha^2}{2}\right)\left[\exp\left(\mu_j + \sigma_j^2\alpha + \frac{\sigma_j^2}{2}\right) - 1\right] = 0, \tag{A.1}$$

and we immediately see the existence of a solution since :

$$\lim_{\alpha\to+\infty} H(\alpha) = +\infty, \quad \lim_{\alpha\to-\infty} H(\alpha) = -\infty.$$

Now, we have to verify that the solution is unique. In order to obtain this result we write the first derivative of function $H(\alpha)$:

$$\begin{aligned} H'(\alpha) \quad = \quad & \sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j + \sigma_j^2\frac{\alpha^2}{2}\right) \\ & \left\{(\mu_j + \sigma_j^2\alpha)\left[\exp\left(\mu_j + \sigma_j^2\alpha + \frac{\sigma_j^2}{2}\right) - 1\right] + \sigma_j^2 \exp\left(\mu_j + \sigma_j^2\alpha + \frac{\sigma_j^2}{2}\right)\right\}. \end{aligned} \tag{A.2}$$

If we consider the function :

$$h(x) = x\left[\exp\left(x + \frac{a}{2}\right) - 1\right] + a\exp\left(x + \frac{a}{2}\right), \tag{A.3}$$

with $a > 0$, it is easy to verify that h is strictly positive for every value of x; consequently, taking $x := \mu_j + \sigma_j^2\alpha$ and $a := \sigma_j^2$, we see that $H'(\alpha)$ is strictly positive for every value of α and $H(\alpha)$ is strictly increasing: therefore, the value of α such that $\varphi(\alpha+1) = \varphi(\alpha)$ is unique.

Now, let us consider the general case of $r > 0$. The relation $H(\alpha)$ takes the following form :

$$H(\alpha) = \sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j \ + \sigma_j^2 \tfrac{\alpha^2}{2}\right) \left[\exp\left(\mu_j \ + \ \sigma_j^2\alpha \ + \ \tfrac{\sigma_j^2}{2}\right) - \exp r\right] = 0; \tag{A.4}$$

we can rewrite relation (A.4) in this way :

$$\exp(r) \sum_{j=0}^{J} p_j \exp\left(\alpha\mu_j \ + \sigma_j^2 \frac{\alpha^2}{2}\right) \left[\exp\left(\mu_j \ - \ r \ + \ \sigma_j^2\alpha \ + \ \frac{\sigma_j^2}{2}\right) - 1\right] = 0$$

which is equivalent to

$$\exp(r + \alpha r) \sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j' \ + \sigma_j^2 \tfrac{\alpha^2}{2}\right) \left[\exp\left(\mu_j' \ + \ \sigma_j^2\alpha \ + \ \tfrac{\sigma_j^2}{2}\right) - 1\right] = 0, \tag{A.5}$$

with $\mu_j' = \mu_j - r$, and we obtain the same relation as in the case of $r = 0$ multiplied by the positive quantity $\exp(r + \alpha r)$. This result leads us to the general conclusion that the value α is unique for every value of $r \geq 0$ and (μ_j, σ_j^2, p_j), $j = 1, \ldots, J$.

Appendix 2
The risk neutral distribution

From asset pricing theory we know that the specification of the risk neutral distribution through the SDF change of measure is given by :

$$f^Q(y) = \frac{M}{E(M)} f(y). \tag{A.6}$$

In our framework this relation take the following form :

$$\begin{aligned} f^Q(y) &= \frac{f(y)\exp(\alpha y)}{\sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j + \alpha^2 \frac{\sigma_j^2}{2}\right)} \\ &= \frac{f(y)\exp(\alpha y)}{\varphi(\alpha)}, \end{aligned} \tag{A.7}$$

where $f(y)$ is given by equation (3.1) and $\varphi(\alpha)$ is the Laplace transform of a mixture of normal distributions. Now, we can write equation (A.7) in the following way :

$$
\begin{aligned}
f^Q(y) &= \frac{\displaystyle\sum_{j=1}^{J} p_j \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left(-\frac{(y-\mu_j)^2}{2\sigma_j^2} + \alpha y\right)}{\varphi(\alpha)} \\
&= \frac{\displaystyle\sum_{j=1}^{J} p_j \frac{1}{\sigma_j\sqrt{2\pi}} \exp\left[-\frac{(y-\mu_j-\alpha\sigma_j^2)^2}{2\sigma_j^2} + \alpha\mu_j + \frac{\alpha^2}{2}\sigma_j^2\right]}{\varphi(\alpha)} \\
&= \sum_{j=1}^{J} \left[\frac{p_j \exp\left(\alpha\mu_j + \frac{\alpha^2}{2}\sigma_j^2\right)}{\displaystyle\sum_{j=1}^{J} p_j \exp\left(\alpha\mu_j + \alpha^2\frac{\sigma_j^2}{2}\right)}\right] \frac{\exp\left[-\frac{(y-\mu_j-\alpha\sigma_j^2)^2}{2\sigma_j^2}\right]}{\sigma_j\sqrt{2\pi}} \\
&= \sum_{j=1}^{J} \nu_j f_j(y;\mu_j+\alpha\sigma_j^2, \sigma_j^2).
\end{aligned}
$$

QED

The risk neutral distribution is still a mixture of normal distributions with new means $\mu_j + \alpha\sigma_j^2$ and the same variances σ_j^2, but characterized by a new mixing distribution (risk-adjusted mixing distribution) ν_j, $j = 1, \ldots, J$.

Appendix 3
The option pricing formula

We have seen in Proposition 2 (and in Appendix 2) that the unique risk-neutral distribution Q associated to the historical distribution (3.1) is once more a mixture of gaussian distributions. This result allows to write the pricing formula as :

$$\begin{aligned} C(\kappa) &= \exp(-r)E^Q[\exp y - \kappa]^+ \\ &= \exp(-r)\sum_{j=1}^{J} \nu_j E[\exp(y_j) - \kappa]^+, \end{aligned}$$

with $y_j \sim \mathcal{N}[\mu_j + \alpha\sigma_j^2, \sigma_j^2]$. Now, this gaussian random variable can be decomposed in the following sum :

$$y_j = z_j + \mu_j + \alpha\sigma_j^2 - r + \frac{\sigma_j^2}{2},$$

with $Z_j \sim \mathcal{N}[r - \frac{\sigma_j^2}{2}, \sigma_j^2]$. This decomposition give us the possibility to write the formula as an average of Black-Scholes pricing formulas :

$$C(\kappa) = \exp(-r)\sum_{j=1}^{J} \nu_j E[\gamma_j \exp(z_j) - \kappa]^+,$$

with $\gamma_j = \exp\left(\mu_j + \alpha\sigma_j^2 + \frac{\sigma_j^2}{2}\right)$, and by consequence

$$\begin{aligned} C(\kappa) &= \exp(-r)\sum_{j=1}^{J} \nu_j\gamma_j E\left[\exp(z_j) - \frac{\kappa}{\gamma_j}\right]^+ \\ &= \sum_{j=1}^{J} \nu_j\, \gamma_j\, C_{BS}\left(\sigma_j^2, \frac{\kappa}{\gamma_j}\right), \end{aligned}$$

QED

with $C_{BS}(\sigma^2, \kappa)$ the one-period Black-Scholes formula

$$C_{BS}(\sigma^2, \kappa) = \left[\Phi\left(-\frac{log(\kappa e^{-r})}{\sigma} + \frac{\sigma}{2} \right) - \kappa e^{-r} \Phi\left(-\frac{log(\kappa e^{-r})}{\sigma} - \frac{\sigma}{2} \right) \right],$$

and

$$\nu_j \gamma_j = \frac{p_j \exp\left(\alpha \mu'_j + \sigma_j^2 \frac{\alpha^2}{2}\right) \exp\left(\mu'_j + \alpha \sigma_j^2 + \frac{\sigma_j^2}{2}\right)}{\sum_{j=1}^{J} p_j \exp\left(\alpha \mu'_j + \sigma_j^2 \frac{\alpha^2}{2}\right)},$$

$$\mu'_j := \mu_j - r.$$

Moreover we have $\sum_{j=1}^{J} \nu_j \gamma_j = 1$. Indeed, because of formula (A.5), we can write :

$$\sum_{j=1}^{J} \nu_j \gamma_j = \frac{\sum_{j=1}^{J} p_j \exp\left(\alpha \mu'_j + \sigma_j^2 \frac{\alpha^2}{2}\right) \exp\left(\mu'_j + \alpha \sigma_j^2 + \frac{\sigma_j^2}{2}\right)}{\sum_{j=1}^{J} p_j \exp\left(\alpha \mu'_j + \sigma_j^2 \frac{\alpha^2}{2}\right)}$$

$$= \frac{\sum_{j=1}^{J} p_j \exp\left(\alpha \mu'_j + \sigma_j^2 \frac{\alpha^2}{2}\right)}{\sum_{j=1}^{J} p_j \exp\left(\alpha \mu'_j + \sigma_j^2 \frac{\alpha^2}{2}\right)} = 1.$$

Appendix 4
Modelling extreme risks

Proof of Proposition 5 : Let us recall the following relation (A.4) of Appendix 1 applied in this particular case :

$$H_\lambda(\alpha) \equiv \lambda \exp\left(\frac{\sigma_1^2}{\lambda}\frac{\alpha^2}{2}\right)\left[\exp\left(\mu + \frac{\sigma_1^2}{\lambda}\alpha + \frac{\sigma_1^2}{2\lambda}\right) - \exp(r)\right] +$$
$$(1-\lambda)\exp\left(\frac{\sigma_2^2}{1-\lambda}\frac{\alpha^2}{2}\right)\left[\exp\left(\mu + \frac{\sigma_2^2}{1-\lambda}\alpha + \frac{\sigma_2^2}{2(1-\lambda)}\right) - \exp(r)\right] = 0\,.$$

In particular,

$$H_\lambda(-\frac{1}{2}+\epsilon) \equiv \lambda \exp\left(\frac{\sigma_1^2}{\lambda}\frac{\left(-\frac{1}{2}+\epsilon\right)^2}{2}\right)\left[\exp\left(\mu + \epsilon\frac{\sigma_1^2}{\lambda}\right) - \exp(r)\right] +$$
$$(1-\lambda)\exp\left(\frac{\sigma_2^2}{1-\lambda}\frac{(-\frac{1}{2}+\epsilon)^2}{2}\right)\left[\exp\left(\mu + \epsilon\frac{\sigma_2^2}{1-\lambda}\right) - \exp(r)\right]\,.$$

If $\epsilon > 0$: $H_\lambda(-\frac{1}{2}+\epsilon) \to +\infty$ when $\lambda \to 0$.
If $\epsilon < 0$: $H_\lambda(-\frac{1}{2}+\epsilon) \to -\infty$ when $\lambda \to 0$.
As H_λ is an increasing function, we can deduce that :
$\forall \epsilon > 0$, for λ sufficiently small, the solution α of the equation lies in the interval $-\frac{1}{2} \pm \epsilon$.
So $\alpha \to -\frac{1}{2}$ when $\lambda \to 0$.

Proof of Proposition 7 : We have written the price of a European Call option as a direct application of formula (3.4) :

$$C(\kappa) = \nu_1\,\gamma_1\,C_{BS}\left(\frac{\sigma_1^2}{\lambda}, \frac{\kappa}{\gamma_1}\right) + \nu_2\,\gamma_2\,C_{BS}\left(\frac{\sigma_2^2}{1-\lambda}, \frac{\kappa}{\gamma_2}\right)$$

with

$$\begin{aligned}\nu_1\gamma_1 &= \frac{\lambda\exp\left(\frac{\sigma_1^2}{2\lambda}(\alpha+1)^2\right)}{\lambda\exp\left(\frac{\sigma_1^2}{2\lambda}(\alpha+1)^2\right) + (1-\lambda)\exp\left(\frac{\sigma_2^2}{2(1-\lambda)}(\alpha+1)^2\right)} \\ &\to \quad 1 \quad \text{when} \quad \lambda \to 0\,.\end{aligned}$$

The second term $\nu_2 \gamma_2 C_{BS}\left(\frac{\sigma_2^2}{1-\lambda}, \frac{\kappa}{\gamma_2}\right)$ tends to 0 because $\nu_2 \gamma_2$ tends to 0 and $C_{BS}\left(\frac{\sigma_2^2}{1-\lambda}, \frac{\kappa}{\gamma_2}\right)$ is bounded. Now for the first term : $\gamma_1 \rightarrow 1$ because $\nu_1 \rightarrow 1$ (see Proposition 6) and $\nu_1\gamma_1 \rightarrow 1$. So, $C_{BS}\left(\frac{\sigma_1^2}{\lambda}, \frac{\kappa}{\gamma_1}\right) \rightarrow 1$. Therefore $C\left(\kappa\right) \rightarrow 1$.

Appendix 5
The jump-diffusion model

Proof of Proposition 8.a :

Using the absence of arbitrage condition $\varphi(\alpha+1) = \exp(r)\,\varphi(\alpha)$ we successively find :

$$\sum_{j=0}^{+\infty} e^{-\lambda}\frac{\lambda^j}{j!}\,\exp\left((\alpha+1)(\mu+j\mu_p)+\left(\sigma^2+j\sigma_p^2\right)\frac{(\alpha+1)^2}{2}\right)$$
$$= \exp(r)\,\sum_{j=0}^{+\infty} e^{-\lambda}\frac{\lambda^j}{j!}\,\exp\left(\alpha(\mu+j\mu_p)+\left(\sigma^2+j\sigma_p^2\right)\frac{\alpha^2}{2}\right)\,,$$

$$\exp\left(\mu+\left(\alpha+\frac{1}{2}\right)\sigma^2\right)\sum_{j=0}^{+\infty}\frac{\left[\lambda\,\exp\left((\alpha+1)\mu_p+\sigma_p^2\frac{(\alpha+1)^2}{2}\right)\right]^j}{j!}$$
$$= \exp(r)\,\sum_{j=0}^{+\infty}\frac{\left[\lambda\,\exp\left(\alpha\mu_p+\sigma_p^2\frac{\alpha^2}{2}\right)\right]^j}{j!}\,,$$

then

$$\begin{aligned}&\exp\left(\mu+\left(\alpha+\tfrac{1}{2}\right)\sigma^2\right)\exp\left[\lambda\,\exp\left((\alpha+1)\mu_p+\sigma_p^2\tfrac{(\alpha+1)^2}{2}\right)\right]\\&= \exp(r)\,\exp\left[\lambda\,\exp\left(\alpha\mu_p+\sigma_p^2\tfrac{\alpha^2}{2}\right)\right]\end{aligned} \tag{A.8}$$

and we finally get

$$\mu - r + \left(\alpha+\frac{1}{2}\right)\sigma^2 + \lambda\exp\left(\alpha\mu_p+\sigma_p^2\frac{\alpha^2}{2}\right)\left[\exp\left(\mu_p+\sigma_p^2\left(\alpha+\frac{1}{2}\right)\right)-1\right] = 0\,.$$

Proof of Proposition 8.b :

Applying formula (3.3) we get the risk-neutral distribution with weights given by :

$$
\begin{aligned}
\nu_j &= \frac{\exp(-\lambda)\frac{\lambda^j}{j!}\exp\left(\alpha\left(\mu + j\mu_p\right) + \left(\sigma^2 + j\sigma_p^2\right)\frac{\alpha^2}{2}\right)}{\sum_{j=0}^{+\infty}\exp(-\lambda)\frac{\lambda^j}{j!}\exp\left(\alpha\left(\mu + j\mu_p\right) + \left(\sigma^2 + j\sigma_p^2\right)\frac{\alpha^2}{2}\right)} \\
&= \frac{\frac{\left(\lambda\exp\left(\alpha\mu_p + \sigma_p^2\frac{\alpha^2}{2}\right)\right)^j}{j!}}{\sum_{j=0}^{+\infty}\frac{\left(\lambda\exp\left(\alpha\mu_p + \sigma_p^2\frac{\alpha^2}{2}\right)\right)^j}{j!}} \\
&= \exp(-\lambda^{'})\frac{\lambda^{'j}}{j!}\ ,
\end{aligned}
$$

with $\lambda^{'} = \lambda\exp\left(\alpha\mu_p + \sigma_p^2\frac{\alpha^2}{2}\right)$.

Proof of Proposition 8.c :

We now apply formula (3.4) to get the price of the European Call option which is an average of the Black-Scholes formulas with weights :

$$\beta_j = \nu_j\,\gamma_j$$

where $\quad \gamma_j = \exp\left(\mu + j\mu_p + \alpha(\sigma^2 + j\sigma_p^2) - r + \frac{\sigma^2 + j\sigma_p^2}{2}\right)$

Using relation (A.8) above we get

$$
\begin{aligned}
\gamma_j &= \exp\left[-\lambda\exp\left((\alpha+1)\mu_p + \sigma_p^2\frac{(\alpha+1)^2}{2}\right) + \lambda\exp\left(\alpha\mu_p + \sigma_p^2\frac{\alpha^2}{2}\right)\right] \\
&\times \left[\exp\left(\mu_p + \alpha\sigma_p^2 + \frac{\sigma_p^2}{2}\right)\right]^j .
\end{aligned}
$$

So,

$$\beta_j = \exp\left(-\lambda\exp\left((\alpha+1)\mu_p + \sigma_p^2\frac{(\alpha+1)^2}{2}\right)\right)\frac{\left[\lambda\exp\left((\alpha+1)\mu_p + \sigma_p^2\frac{(\alpha+1)^2}{2}\right)\right]^j}{j!} .$$

Appendix 6
The MN-GARCH process of first type

If we consider the case of a mixture of J components, the model presented by HMP [Hass, Mittnick and Paolella (2002)] takes the following form :

$$\varepsilon_{t+1}|\underline{\varepsilon_t} \sim \mathcal{MN}\left(J, p_j, \mu_j, \sigma^2_{jt+1}\right) \tag{A.9}$$

where $\mu_J = -\sum_{i=1}^{J-1}(p_i/p_J)\mu_i$ and where the $J\times 1$ vector of variances, denoted by σ^2_{t+1}, evolves according to :

$$\sigma^2_{t+1} = \omega + \sum_{i=0}^{q-1}\mathcal{B}_{i+1}\varepsilon^2_{t-i} + \sum_{j=0}^{p-1}\mathcal{C}_{j+1}\sigma^2_{t-j}, \tag{A.10}$$

with $\sigma^2_{t+1} = \left[\sigma^2_{1t+1}, \ldots, \sigma^2_{Jt+1}\right]'$, $\omega = [\omega_1, \ldots, \omega_J]'$, $\mathcal{B}_i = [b_{i1}, \ldots, b_{iJ}]'$, $i = 1, ..., q$, and $\mathcal{C}_j$, $j = 1, ..., p$, are $J \times J$ matrices with typical element $c_{j,mn}$. Non-negativity conditions on the parameters are assumed.

The special case of $J = 2$ and $p = q = 1$ can be represented in the following way :

$$\begin{bmatrix} \sigma^2_{1t+1} \\ \\ \sigma^2_{2t+1} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \\ \omega_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ \\ b_2 \end{bmatrix} \varepsilon^2_t + \begin{bmatrix} c_{11} & c_{12} \\ \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \sigma^2_{1t} \\ \\ \sigma^2_{2t} \end{bmatrix},$$

which is our specification if we impose on the parameters the constrains $c_{12} = c_{21} = 0$.

In their paper, HMP note that the *Diagonal* MN-GARCH, model with $\mathcal{C}_j$, $j = 1, ..., p$, a diagonal matrix, fits well the data employing various model-selection criteria [see HMP (2002) for details].

If we impose the constraint $c_{11} = c_{22}$, ε_{t+1} has a GARCH structure; indeed, if we have :

$$E(\varepsilon_{t+1}|\,\underline{\varepsilon_t}) \;=\; \lambda a(1-\lambda) - \lambda a(1-\lambda) = 0,$$

and

$$\begin{aligned}
\sigma^2_{t+1} = E(\varepsilon^2_{t+1}|\,\underline{\varepsilon_t}) &= \lambda[a^2(1-\lambda)^2 + \sigma^2_{1t+1}] + (1-\lambda)[a^2\lambda^2 + \sigma^2_{2t+1}] \\
&= a^2\lambda(1-\lambda) + \lambda\sigma^2_{1t+1} + (1-\lambda)\sigma^2_{2t+1} \\
&= \lambda\omega_1 + (1-\lambda)\omega_2 + a^2\lambda(1-\lambda) + [\lambda b_1 + (1-\lambda)b_2]\,\varepsilon^2_t \\
&\quad + c_1\lambda\sigma^2_{1t} + c_2(1-\lambda)\sigma^2_{2t};
\end{aligned}$$

now, if we impose $c_1 = c_2 = c$ we can write :

$$\begin{aligned}
\sigma^2_{t+1} = E(\varepsilon^2_{t+1}|\,\underline{\varepsilon_t}) &= (1-c)a^2\lambda(1-\lambda) + \lambda\omega_1 + (1-\lambda)\omega_2 \\
&\quad + [\lambda b_1 + (1-\lambda)b_2]\,\varepsilon^2_t + c\sigma^2_t,
\end{aligned}$$

QED

with $\xi := (1-c)a^2\lambda(1-\lambda) + \lambda\omega_1 + (1-\lambda)\omega_2$ the (positive) constant term of the relation.

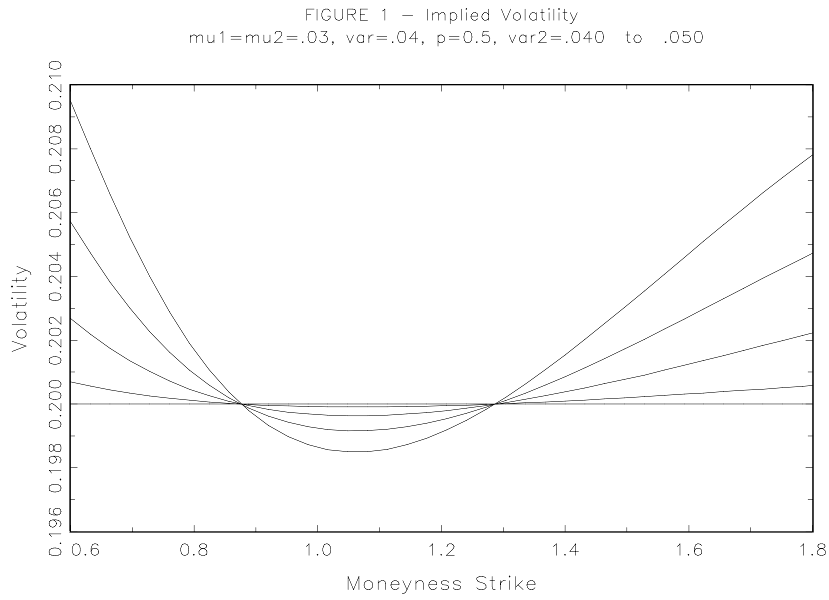


FIGURE 1 — Implied Volatility
mu1=mu2=.03, var=.04, p=0.5, var2=.040 to .050
Volatility
0.196
0.198
0.200
0.202
0.204
0.206
0.208
0.210
0.6
0.8
1.0
1.2
1.4
1.6
1.8
Moneyness Strike

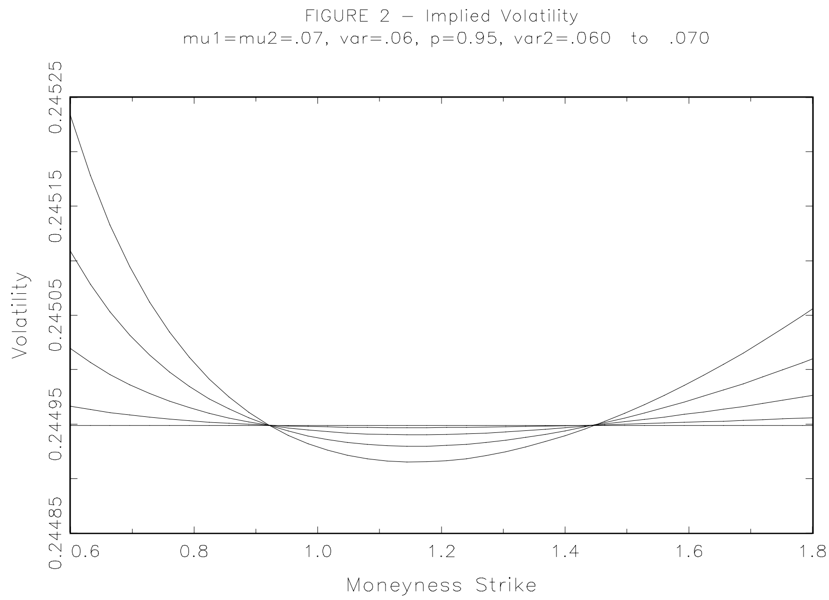


FIGURE 2 — Implied Volatility
mu1=mu2=.07, var=.06, p=0.95, var2=.060 to .070
Volatility
0.24485
0.24495
0.24505
0.24515
0.24525
0.6
0.8
1.0
1.2
1.4
1.6
1.8
Moneyness Strike

FIGURE 3 – Implied Volatility
mu1=.03, mu2=.07, var=.06, p=0.5, var1=.060 to .070

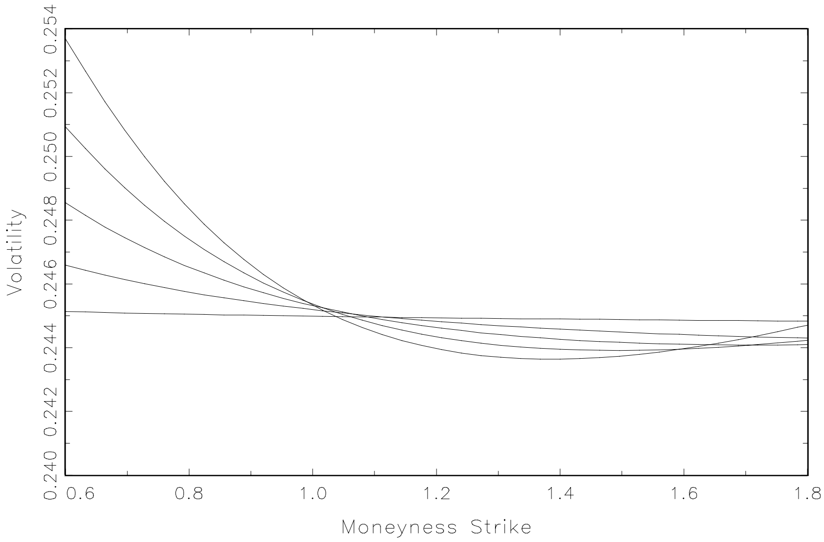


FIGURE 4 – Implied Volatility Surface
mu1=mu2=0.03, var=.04, var2=.0475, p=0.5

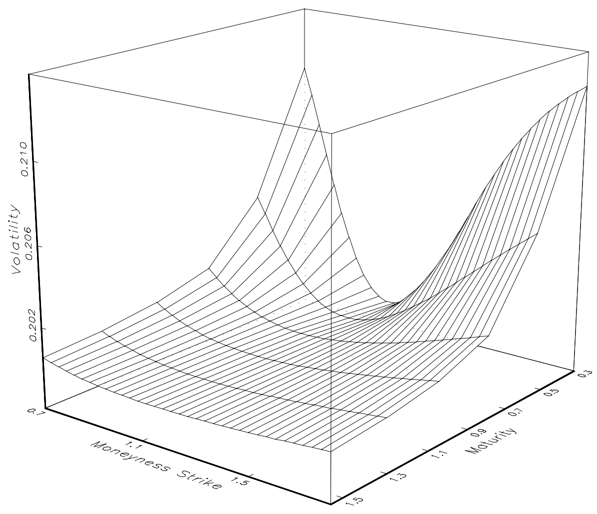


FIGURE 5 – Implied Volatility
mu1=.03, mu2=.07, var2=.04, p=0.5, var1=.040(bottom) to .060(up)

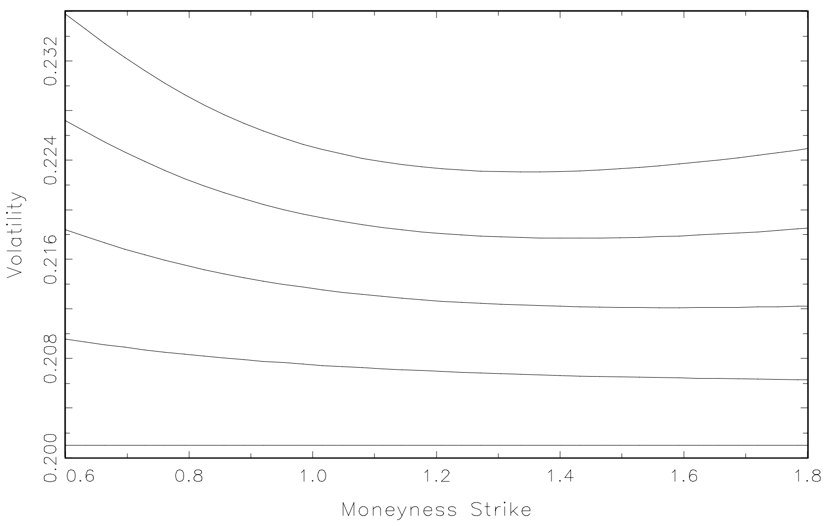


FIGURE 6 – Implied Volatility for Extreme Risks
mu1=mu2=.07, var=.06
lambda = .5001(flat curve) to .001(up; strong kurtosis)

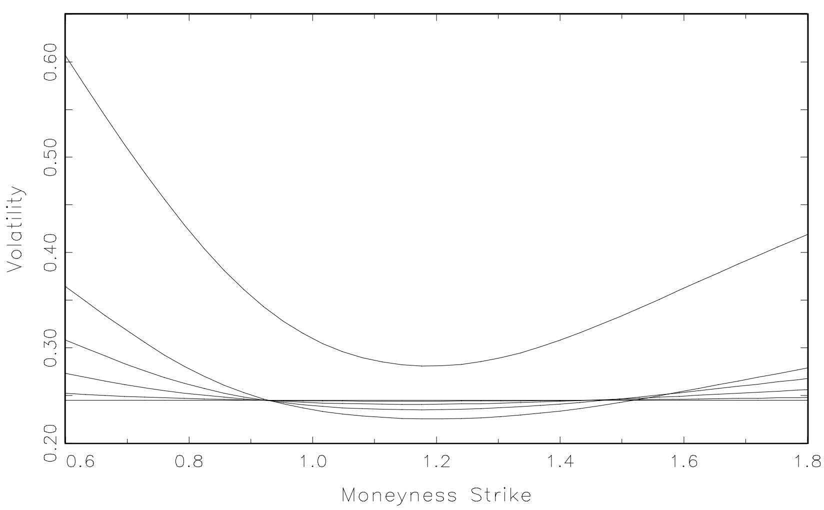


FIGURE 7 – Implied Volatility for Extreme Risks:
relation with lambda (strong kurtosis effect)
mu1=mu2=.07, var=.06, k=1

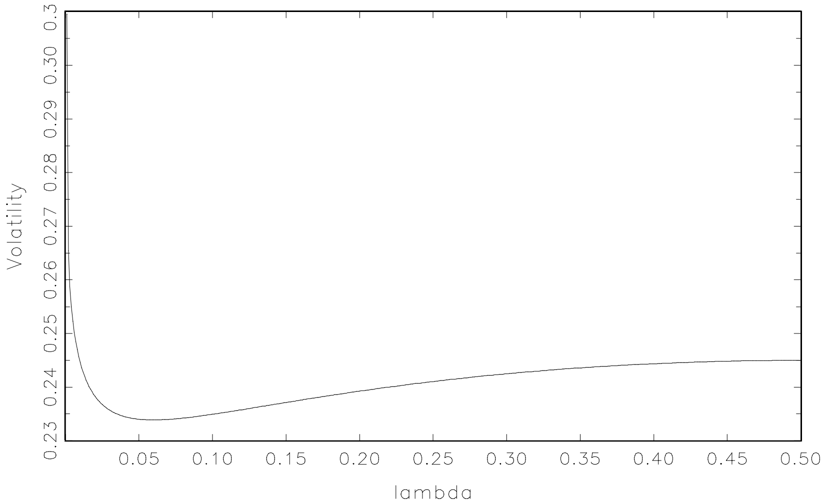


FIGURE 8 – Implied Volatility and Jump–Diffusion: relation with mu_p
mu=.07, var=.06, lambda=20, var_p=0.002,
mu_p=.005 to .009

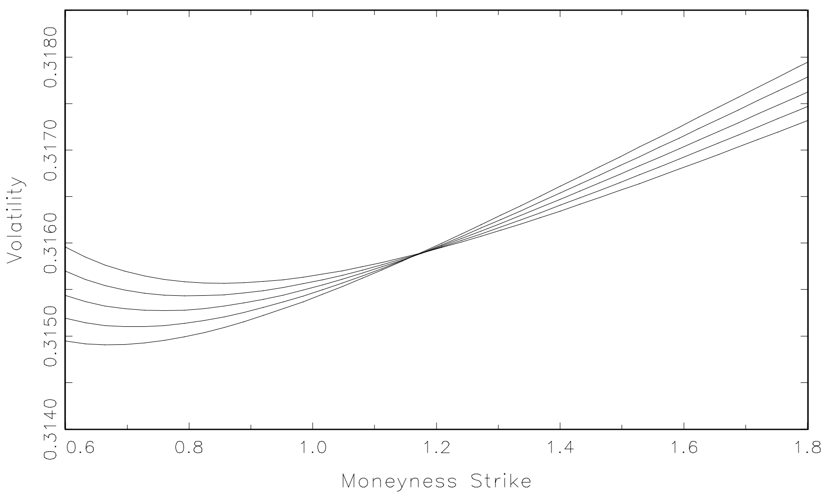


FIGURE 9 – Implied Volatility and Jump–Diffusion: relation with var_p
mu=.07, var=.06, lambda=20, mu_p=.005
var_p=.002 to .004

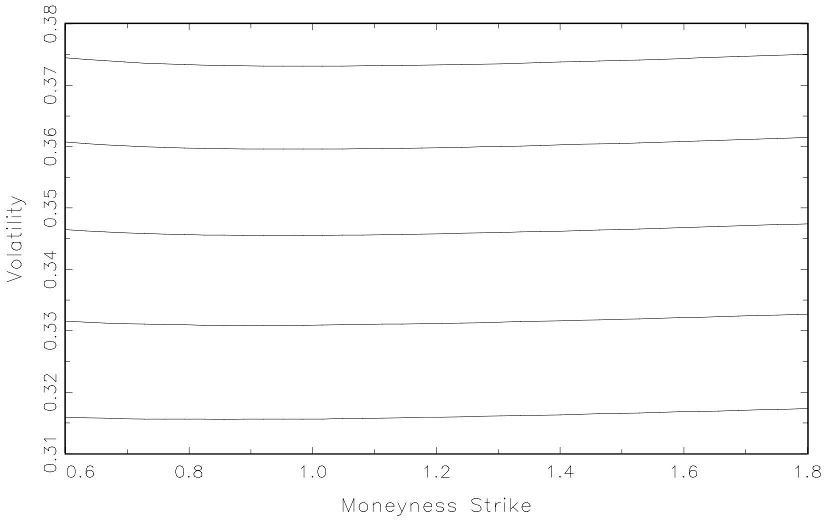
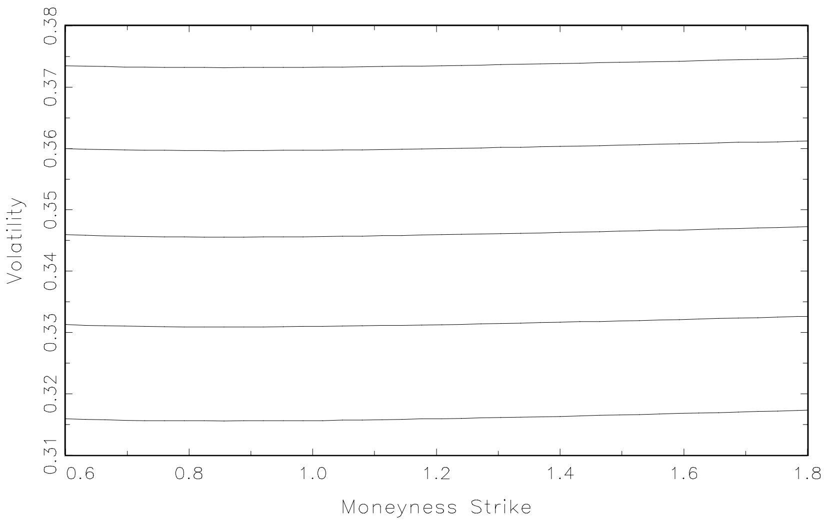


FIGURE 10 – Implied Volatility and Jump–Diffusion: relation with lambda
mu=.07, var=.06, mu_p=.005, var_p=.002
lambda= 20(bottom) to 40(up)



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