ECONOMETRIC SPECIFICATIONS OF STOCHASTIC DISCOUNT FACTOR MODELS

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Abstract
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We consider the problem of derivative pricing when the stochastic discount factors are exponential-affine functions of underlying factors. In particular we discuss the conditionally gaussian framework and introduce semi-parametric pricing methods for models with path dependent location and scale parameters. This approach is also applied to more complicated frameworks, such as the pricing of a derivative written on an index, when the interest rate is stochastic.

Keywords : Derivative Pricing, Esscher Transform, Stochastic Discount Factor, Implied Volatility, Variance-gamma Model, Semi-Parametric Pricing.

Résumé
Econométrie des modèles à facteurs d'escompte stochastique

Nous considérons le problème de valorisation de produits dérivés, lorsque les facteurs d'escompte stochastiques sont fonctions exponentielles-affines de facteurs sous-jacents. En particulier nous discutons le cas conditionnellement gaussien et développons des méthodes de valorisation semi-paramétriques, pour des modèles où les paramètres d'échelle et de position dépendent de l'historique. Cette approche est également appliquée à des contextes plus complexes, tel la valorisation d'un dérivé sur indice en présence de taux d'intérêt stochastique.

Mots clés : Valorisation, transformée d'Esscher, facteur d'escompte stochastique, volatilité implicite, modèle variance-gamma, valorisation semi-paramétrique.

JEL number : C1, C5, G1

1. Introduction

The pricing of derivatives is generally based on continuous time models, which may include latent stochastic factors, or jumps. This approach provides formulas for derivative prices, expressed either as conditional expectations, or solutions of partial differential equations. However the formulas are often difficult to implement, since both data and hedging strategies are in discrete time. This explains why standard continuous time models have a rather simple structure [see e.g. Black, Scholes (1973), Hull, White (1987), Melino, Turnbull (1990), Heston (1993)]. In particular the number of underlying factors is generally small, the risk premia associated with these factors are assumed constant or even equal to zero, and the term structure of interest rates is often treated independently of the index derivatives . However these restrictions, that simplify the implementation of the derivative pricing formulas, often induce a poor fit when time series data, or cross-section data on derivative prices are considered.

The aim of this paper is to address the problem of derivating pricing using the stochastic discount factor introduced in [Harrison, Kreps (1979), Garman, Ohlson (1980), Hansen, Richard (1987), Hansen, Jagannathan (1991), Bansel, Viswanathan (1993), Cochrane (2001)]. It is known that, if agents make their investments at date t ($t \in \mathbb{N}$) based on an information set J_t, and if we assume the absence of arbitrage, the prices of actively traded assets satisfy a linear valuation formula. More precisely there exists a stochastic discount factor (sdf) $M_{t,t+1}$ function of the updated information set J_{t+1}, such that the price of an asset that provides the payoff g_{t+1} at date $t+1$ is :

$$C_t(g) = E\left[M_{t,t+1} g_{t+1} | J_t\right]. \tag{1.1}$$

Since the market is incomplete in discrete time, there exists a multiplicity of stochastic discount factors that are compatible with the valuation formula (1.1) for the actively traded assets.

In this situation we can simply note that multiplicity of prices for illiquid (derivative) assets and refuse to price them (or propose very large admissible price intervals) [see the discussion by Delbaen, Schachermayer and Schweizer (1994)]. We could also select the (martingale) minimal price with the risk of underestimating the values. However these strategies are generally not followed in accounting, economics and finance. Let us for instance consider accounting. The banks and financial institutions report regularly their balance sheets. These balance sheets include a number of real or financial assets, which are not liquid, but nevertheless have to be priced. This is typically the case of illiquid derivatives, but also of patents, machines used for productions, or stock of unsold outputs. Let us focus on machines. Generally there does not exist an active second hand market for such machines and second hand market prices are not observed. The accounting practice consists in introducing a convention for explaining how the value of a machine evolves with its age, that is an amortizing scheme. This convention is only one value evolution among an infinite number of admissible ones. For practical purpose two or three alternative conventions have been retained for accounting, with for instance the possibility to choose the amortizing time.

There is also a need for such conventions whenever "prices" are used for comparing illiquid assets [3]. This occurs if we have to construct a balance sheet, to determine the capital required to balance the risk included in a portfolio of illiquid derivative assets (the so-called Value-at-Risk), to compare competing investment strategies on the basis of future profit... In our framework the conventions consist in restricting a priori the set of admissible stochastic discount factors. These conventions[4] have to be compatible with some available price data, have to include conventions previously used in the incomplete market framework, have to provide rather simple pricing formula and have to be easy to understand.

In this paper we consider a class of sdf, that are exponential functions of an affine combination of factors. This specification corresponds to the Esscher transform used in insurance [see e.g. Esscher (1932)], and for derivative pricing [see the pioneer paper by Gerber, Shiu (1994) a,b, and applications in

[3]Another solution proposed in the theoretical literature, but also by financial practitioners, is to create new financial markets, and then to price marked-to-market. Typically since individual corporate default is difficult to price, markets for default derivatives (such as Credit Default Swap) are currently created. However these markets are illiquid and the quoted prices not representative. To take a comparison, the stock of unsold cars by a car producer is generally priced marked-to-market, that is at the official price proposed by the producer. This "price" is clearly too high in a period of increasing stock.

[4]In the standard statistical terminology the observed derivative prices are not sufficient to identify the sdf. The convention is simply an identifiability constraint. However the selection of the identifiability constraint is not innocuous since the pricing of other financial assets involves the whole sdf, not only its identifiable components.

Buhlman et alii (1996), (1998), Shyraev (1999), Gourieroux, Monfort (2001), Yao (2001)]. Next we discuss the restrictions on the sdf implied by the valuation formula (1.1).

In Section 2, we review standard pricing formulas derived from either an equilibrium condition [as the consumption based CAPM, or the model with recursive utility], or the condition of no arbitrage in a continuous time framework [see e.g. Hull, White (1987), Heston (1993)]. The aim is to justify the exponential-affine specification of the sdf and to give examples of possible factors. In Section 3, we consider a simple framework in which the riskfree rate is constant (equal to zero) and the information set includes the returns on actively traded assets only. We prove that there exists a single sdf compatible with the exponential-affine specification. Then we discuss in Section 4 the conditionally gaussian framework and the semi-parametric models with path dependent location and scale parameters. These examples are used to derive a semi-parametric pricing method and to discuss the patterns of implied (Black-Scholes) volatility surfaces in terms of departures from time independence and conditional normality. Section 5 extends the basic approach to the framework of stochastic interest rates and unobservable factors. As illustration we discuss the conditionally gaussian factors framework and explain how to price a derivative written on an index when the interest rate is stochastic. Section 6 concludes.

2. Examples of stochastic discount factors

In financial theory the expressions of stochastic discount factors are generally derived under either an equilibrium condition, or arbitrage free restrictions in a continuous time framework. We provide below some examples to show that stochastic discount factors considered in the literature are usually exponential-affine functions of underlying state variables.

Example 1 : Consumption based CAPM (CCAPM)

In the standard CCAPM, an agent maximizes his expected intertemporal utility expressed in terms of a physical good [Lucas (1978)]. The intertemporal transfers are ensured by means of investments on a financial market. Let us denote by U the utility function, by δ the intertemporal psychological discount rate, and assume the existence of a representative agent. At equilibrium we get the relation :

$$p_t = E_t \left[p_{t+1} \frac{q_t}{q_{t+1}} \delta \frac{\frac{dU}{dc}(C_{t+1})}{\frac{dU}{dc}(C_t)} \right],$$

where p_t is the vector of prices of financial assets, q_t the price of the consumption good and C_t the quantity consumed at date t.

i) Thus, for a power utility function, the sdf associated with the consumption based CAPM is :

$$\begin{aligned} M_{t,t+1} &= \frac{q_t}{q_{t+1}} \delta \left(\frac{C_{t+1}}{C_t} \right)^{\gamma} \\ &= \exp \left[\log \delta - \log \frac{q_{t+1}}{q_t} + \gamma \log \frac{C_{t+1}}{C_t} \right]. \end{aligned}$$

It is an exponential-affine function of two state variables, which are the inflation rate and the rate of change in consumption. It does not depend on the asset returns. However the asset returns, that can be considered as additional state variables, are generally included in the available information set, and thus also influence the derivative prices.

ii) Note that the exponential affine specification of the sdf is not a consequence of the choice of a power utility function. Indeed in the general case we get :

$$M_{t,t+1} = \exp[\log \delta - \log \frac{q_{t+1}}{q_t} + \log[\frac{dU}{dc}(C_{t+1}) / \frac{dU}{dc}(C_t)]].$$

For instance for a CARA utility funciton we get :

$$\log[\frac{dU}{dc}(C_{t+1}) / \frac{dU}{dc}(C_t)] = -A(C_{t+1} - C_t),$$

where A denotes the risk-aversion. Thus the choice of a utility function is equivalent to the selection of an appropriate transformation of the consumption as state variable.

Example 2 : Recursive utility

Similar results can be derived, when the representative agent maximizes a recursive utility [see Epstein-Zin (1991) and Weil (1989)]. If the utility is a power function and the aggregator admits a Cobb-Douglas form, the sdf is given by :

$$\begin{aligned} M_{t,t+1} &= \delta^{\alpha/\rho}\left(\frac{C_{t+1}}{C_t}\right)^{-(1-\rho)\alpha/\rho}\left(\frac{W_t}{W_{t+1}}\right)^{1-\alpha/\rho}\left(\frac{q_{t+1}}{q_t}\right)^{-\alpha/\rho} \\ &= \exp\left\{\frac{\alpha}{\rho}\log\delta - \frac{(1-\rho)\alpha}{\rho}\log\frac{C_{t+1}}{C_t} - (1-\alpha/\rho)\log\frac{W_{t+1}}{W_t} - \frac{\alpha}{\rho}\log\frac{q_{t+1}}{q_t}\right\}. \end{aligned}$$

The exponential-affine function now involves a third state variable $\log\dfrac{W_{t+1}}{W_t}$, that represents the return on the market portfolio.

Example 3 : Stochastic volatility model

Standard stochastic volatility models [Hull, White (1987), Heston (1993)] are of the type :

$$\begin{cases} dS_t = \mu S_t dt + \sigma_t S_t dW_t^S, \\ df(\sigma_t) = a(\sigma_t)dt + b(\sigma_t)dW_t^\sigma, \end{cases}$$

where S_t is the asset price, σ_t the stochastic volatility and $(W_t^S), (W_t^\sigma)$ two independent brownian motions. For a constant riskfree rate r, the discount factor is deduced from Girsanov's theorem and given by :

$$\begin{aligned} M_{t,t+1} &= \exp(-r)\exp\left\{-(\mu-r)\int_t^{t+1}\frac{dW_\tau^S}{\sigma_\tau} - \frac{1}{2}(\mu-r)^2\int_t^{t+1}\frac{d\tau}{\sigma_\tau^2}\right\} \\ &\exp\left\{-\int_t^{t+1}\nu_\tau dW_\tau^\sigma - \frac{1}{2}\int_t^{t+1}\nu_\tau^2 d\tau\right\}, \end{aligned}$$

where ν_τ is the risk premium associated with the stochastic volatility. This premium can be selected arbitrarily as a function of the past due to the incomplete market framework. By approximating the integrals by their discrete time counterpart, we note that :

$$M_{t,t+1} \simeq \exp(-r)\exp\left\{-(\mu-r)\frac{\varepsilon_{t+1}^S}{\sigma_t} - \frac{1}{2}(\mu-r)^2\frac{1}{\sigma_t^2} - \nu_t\varepsilon_{t+1}^\sigma - \frac{1}{2}\nu_t^2\right\}.$$

It is an exponential affine function of the innovations $\varepsilon^S_{t+1}, \varepsilon^\sigma_{t+1}$ corresponding to both return and volatility processes, with path dependent coefficients.

The examples above show :

i) the convenience of exponential-affine specifications, which ensure both the positivity and the tractability of the sdf[5] ;

ii) the various candidates for state variables, including financial returns, innovations on financial returns or volatilities, rate of change in consumption, inflation rate, etc. These state variables can be observable or latent, financial or real ;

iii) the multiplicity of specifications in the incomplete market framework due to the arbitrary choice of the risk premium for non traded factors.

In the next section, we adopt an approach, in which we specify a priori admissible forms of the sdf. Then this set is structured by taking into account the arbitrage free restrictions.

3. SDF modelling : the principle

To present the modelling principle, let us first consider the framework of a riskfree asset with zero riskfree rate [6] and several risky assets with prices $p_{j,t}, j = 1, \ldots, J$ and (geometric) returns : $r_{j,t+1} = \log(p_{j,t+1}/p_{j,t})$. We assume that these assets are actively traded on the markets, and that the different prices are observable for the investor.

3.1 The historical distribution

The (conditional) historical distribution of the return $r_{t+1} = (r_{1,t+1}, \ldots, r_{J,t+1})'$ is defined by means of its (conditional) Laplace transform (also called moment generating function) supposed to belong to a parametric set :

[5] Exponential affine specifications also arise when the risk neutral distribution is at the minimum Kullback-Leibler distance of the historical distribution and satisfies the constraints inferred by option prices [see e.g. Stutzer (1995), (1996), Buchen, Kelly (1996)]

[6] If the future evolution of the riskfree rate is known at date t, it is possible to get a zero riskfree rate by a deterministic change of numeraire. The case of a predetermined stochastic interest rate is considered in Section 5.5.

$$\begin{aligned} & E[\exp u' r_{t+1} | \underline{r_t}] \\ = & \exp \psi_t(u;\theta), \qquad \text{(say)}. \end{aligned}$$

The Laplace transform is defined on a convex set, that depends on the tails of the conditional distribution. We assume below that this convex set is not reduced to one point located at the origin. Note that the returns can be serially dependent [see Gerber, Shiu (1994) a) for examples in the i.i.d. case].

3.2 The stochastic discount factor

We assume a priori that the sdf can be written under an exponential-affine form :

$$M_{t,t+1} = \exp\left(\alpha_t' r_{t+1} + \beta_t\right), \tag{3.1}$$

with $r_{j,t+1}, j = 1, \ldots, J$ as the J state variables and coefficients α_t and β_t, that can be path dependent, that is function of the return history : $\underline{r_t} = (r_t, r_{t-1}, \ldots)$.

By writing the pricing formula for the riskfree asset and the J risky assets, we get $J+1$ restrictions on the relationship between the sdf and the historical distribution. More precisely the constraints induced by the arbitrage free conditions are :

$$\begin{cases} E\left(M_{t,t+1} | \underline{r_t}\right) = 1, \\ E\left[M_{t,t+1} \dfrac{p_{j,t+1}}{p_{j,t}} | \underline{r_t}\right] = E\left[M_{t,t+1} \exp r_{j,t+1} | \underline{r_t}\right] = 1, j = 1, \ldots, J, \end{cases}$$

$$\Longleftrightarrow \begin{cases} E\left[\exp\left(\alpha_t' r_{t+1} + \beta_t\right) | \underline{r_t}\right] = 1, \\ E\left[\exp\left(\alpha_t' r_{t+1} + e_j' r_{t+1} + \beta_t\right) | \underline{r_t}\right] = 1, j = 1, \ldots, J, \end{cases}$$

where $e_j = (0, \ldots 0, 1, 0, \ldots 0)'$, with 1 as component of order j,

$$\iff \begin{cases} \exp\left[\psi_t(\alpha_t;\theta)+\beta_t\right]=1, \\ \\ \exp\left[\psi_t(\alpha_t+e_j;\theta)+\beta_t\right]=1, \forall j=1,\ldots,J, \end{cases}$$

$$\iff \begin{cases} \beta_t=-\psi_t(\alpha_t;\theta), \\ \\ \psi_t(\alpha_t+e_j;\theta)-\psi_t(\alpha_t;\theta)=0, \forall j=1,\ldots,J. \end{cases}$$

This system of $J+1$ equations for $J+1$ unknown parameters generally admits a unique solution :

$$\begin{cases} \alpha_t=\alpha(\underline{r_t};\theta), \\ \\ \beta_t=\beta(\underline{r_t};\theta)=-\psi_t[\alpha(\underline{r_t};\theta),\theta], \text{ say.} \end{cases}$$

Then we deduce a unique form of the exponential-affine sdf that satisfies the condition of no arbitrage. The associated risk neutral distribution Q admits the conditional Laplace transform :

$$\begin{aligned} & \overset{Q}{E}\left[\exp u' r_{t+1} | \underline{r_t}\right] \\ = \; & E\left[M_{t,t+1}\exp u' r_{t+1} | \underline{r_t}\right] / E\left[M_{t,t+1} | \underline{r_t}\right] \\ = \; & E\left[\exp(\alpha_t' r_{t+1}+\beta_t)\exp u' r_{t+1} | \underline{r_t}\right] \\ = \; & \exp\left[\psi_t(\alpha_t+u;\theta)+\beta_t\right] \\ = \; & \exp\left[\psi_t(\alpha_t+u;\theta)-\psi_t(\alpha_t;\theta)\right]. \end{aligned}$$

This result is summarized in the proposition below.

Proposition 1 : For a zero riskfree rate and an exponential-affine sdf with state variable r_{t+1}, there exists in general a unique admissible risk-neutral distribution. Its conditional Laplace transform is given by :

$$\overset{Q}{E}\left[\exp u' r_{t+1} | \underline{r_t}\right] = \exp\left[\psi_t(\alpha_t+u;\theta)-\psi_t(\alpha_t;\theta)\right],$$

where α_t is the solution of :

$$\psi_t(\alpha_t + e_j;\theta) = \psi_t(\alpha_t;\theta), j = 1,\ldots,J,$$

and ψ_t denotes the conditional historical Log-Laplace transform.

In discrete time the market is incomplete . Therefore there exists an infinity of admissible risk neutral distributions. The uniqueness condition in Proposition 3.1 (that is the non ambiguous convention) is due to the constrained exponential-affine form of the sdf [7] [8]. We will explain in the next section how to introduce a multiplicity of sdf by means of unobservable stochastic factors with parameterized risk premia.

By using the sdf $M_{t,t+1} = M_{t,t+1}(\underline{r}_{t+1};\theta)$, we can propose a price for any derivative written on r_t [9]. Let us denote by $g(\underline{r_{t+h}}; t+h), h = 1,\ldots,H$ the payoffs provided at $t+h, h = 1,\ldots,H$. The derivative price is :

$$C_t(g) = \sum_{h=1}^{H} E\left[M_{t,t+1}(\underline{r_{t+1}};\theta)\ldots M_{t+h-1,t+h}(\underline{r_{t+h}};\theta)g(\underline{r_{t+h}};t+h)|\underline{r_t}\right]. \quad (3.2)$$

In practice this price cannot be computed analytically and is approximated by simulations. The simulations can be performed under the historical probability, or under a modified probability [see Gourieroux, Jasiak (2001), Chapter 13, Section 5.2, for a discussion].

3.3 Information and time aggregation

[7] Some other constraints could have been imposed on $M_{t,t+1}$. For instance [Hansen, Jagannathan (1997), Cochrane (2000)] assume an affine form $M_{t,t+1} = \alpha_t' r_{t+1} + \beta_t$. This convention does not ensure the positivity of the underlying state prices and does not extend the usual specifications seen in Section 2.

[8] The dynamics of the return under the historical and risk neutral distributions can be very different. For instance they can feature stationarity or nonstationarity under the risk neutral distribution whereas the historical world is stationary. This question is difficult to study in the general framework of this paper [see the discussion in Gourieroux,Monfort, Polimenis (2001)b for affine models of interest rates].

[9] On the market the derivatives are usually written on prices and not directly on returns. However, let us consider a European call for instance, with payoff $(S_{t+1}-K)^+$ at date $t+1$. This derivative is S_t times the derivative with payoff $(\exp r_{t+1} - k_t)^+$, where $k_t = K/S_t$ is the moneyness strike. Thus we are just assuming a preliminary transformation of the payoff.

The assumption of exponential-affine sdf depends on the selected information set and time horizon. To illustrate this dependence let us consider below two examples.

i) If we are interested on derivatives based on a subset of risky assets with returns r^*_{t+1}, the basic pricing formula (1.1) applied to $g(r^*_{t+1})$ can be written as :

$$\begin{aligned} C_t(g) &= E[M_{t,t+1} g(r^*_{t+1}) | \underline{r_t}] \\ &= E[M^*_{t,t+1} g(r^*_{t+1}) | \underline{r_t}], \end{aligned}$$

where $M^*_{t,t+1} = E[M_{t,t+1} | r^*_{t+1}, \underline{r_t}]$.

This modified sdf does not admit in general an exponential-affine expression $M^*_{t,t+1} = \exp(\alpha^{*'}_t r^*_{t+1} + \beta^*_t)$, even if the initial sdf does. Thus the information set matters.

ii) Moreover the assumption is not stable by a change of time unit. Let us consider the pricing at horizon 2 of a European derivative written on the short rate . The price is given by :

$$\begin{aligned} C_t(g) &= E[M_{t,t+1} M_{t+1,t+2} g(r_{t+2}) | \underline{r_t}] \\ &= E[E[M_{t,t+1}, M_{t+1,t+2} | r_{t+2}, \underline{r_t}] g(r_{t+2}) | \underline{r_t}]. \end{aligned}$$

Neither $M_{t,t+1} M_{t+1,t+2}$, nor $E[M_{t,t+1} M_{t+1,t+2} | r_{t+2}, \underline{r_t}]$ is exponential-affine w.r.t. r_{t+2} even if $M_{t,t+1}$ is [10].

Therefore in practice it will be necessary to check for the right information set and horizon, before applying the approach of exponential-affine sdf, or to fix by convention the horizon and the information set [This strategy is typically followed by the Basle Committee for credit portfolios).

4. Examples

The aim of this section is to illustrate the approach of exponential-affine sdf [see Gerber, Shiu (1994) a, Yao (2001) for other examples in the i.i.d case].

[10] In the standard continuous time stochastic volatility model considered in example 3, the sdf is exponential-affine at infinitesimal horizon, and is not exponential-affine for more realistic horizons.

We first describe the conditionally gaussian framework and the variance-gamma model, previously introduced in the literature. Then we consider a semi-parametric specification with path dependent location and scale parameters. In this case we develop a semi-parametric pricing method and discuss the different causes for the smiles and asymmetries observed on Black-Scholes implied volatilities.

4.1 The conditionally gaussian framework

If the conditional historical distribution of r_{t+1} given $\underline{r_t}$ is gaussian, with mean m_t and variance-covariance matrix Σ_t, the conditional log-Laplace transform is given by :

$$\psi_t(u;\theta) = u' m_t(\theta) + \frac{1}{2} u' \Sigma_t(\theta) u. \tag{4.1}$$

The risk correction term α_t satisfies :

$$(\alpha_t' + e_j') m_t(\theta) + \frac{1}{2}(\alpha_t' + e_j')\Sigma_t(\theta)(\alpha_t + e_j)$$

$$-\alpha_t' m_t(\theta) - \frac{1}{2}\alpha_t' \Sigma_t(\theta)\alpha_t = 0, \forall j = 1, \dots, J,$$

$$\iff \quad e_j' m_t(\theta) + e_j' \Sigma_t(\theta)\alpha_t + \frac{1}{2} e_j' \Sigma_t(\theta) e_j = 0, \forall j = 1, \dots J,$$

$$\iff \quad m_t(\theta) + \Sigma_t(\theta)\alpha_t + \frac{1}{2} \; vdiag \; \Sigma_t(\theta) = 0,$$

where $vdiag \; \Sigma_t(\theta)$ is the vector, whose components are the diagonal elements of $\Sigma_t(\theta)$. We deduce that :

$$\alpha_t = -\Sigma_t(\theta)^{-1}[m_t(\theta) + \frac{1}{2} \; vdiag \; \Sigma_t(\theta)]. \tag{4.2}$$

Then the conditional log-Laplace transform of the risk-neutral distribution given in Proposition 1 is :

$$\begin{aligned}
& \psi_t(\alpha_t+u;\theta)-\psi_t(\alpha_t;\theta) \\
= & \; u'\left[m_t(\theta)+\Sigma_t(\theta)\alpha_t\right]+\frac{1}{2}u'\Sigma_t(\theta)u \qquad (4.3) \\
= & \; -\frac{1}{2}u' \; vdiag \; \Sigma_t(\theta)+\frac{1}{2}u'\Sigma_t(\theta)u.
\end{aligned}$$

The risk-neutral distribution is also conditionally gaussian, with the same variance-covariance matrix as the conditional historical distribution, and with a conditional mean function of $\Sigma_t(\theta)$, which is introduced to ensure the arbitrage-free condition. These formulas can be applied to a large class of models including for instance the conditionally gaussian multivariate ARCH models [see e.g. Duan (1995)] [11]. In particular they do not require a Markov condition for (r_{t+1}), that is the fact that m_t and Σ_t depend on the past through r_t only.

4.2 Variance-gamma model

This model has been introduced in Madan, Seneta (1990), Madan, Milne (1991), Madan, Carr, Chang (1998). There is only one risky asset and the historical log-Laplace transform of its return is :

$$\psi_t(u)=\nu_t \log\left(1-um_t-u^2\frac{\sigma_t^2}{2}\right),$$

where $\nu_t>0$; it corresponds to a time deformed gaussian model. The correcting factor is :

$$\alpha_t=-\frac{1}{\sigma_t^2}\left(m_t+\frac{\sigma_t^2}{2}\right),$$

whereas the risk-neutral log-Laplace transform is :

$$\psi_t^Q(u)=\nu_t \log\left(1-um_t^*-u^2\frac{\sigma_t^{*2}}{2}\right),$$

where : $m_t^*=\dfrac{m_t+\alpha_t\sigma_t^2}{1-\alpha_t m_t-\alpha_t^2\dfrac{\sigma_t^2}{2}}, \sigma_t^{*2}=\dfrac{\sigma_t^2\alpha_t^2}{1-\alpha_t m_t-\dfrac{\alpha_t^2\sigma_t^2}{2}}.$

[11] and of course the Black-Scholes model [Black, Scholes (1973)].

Thus the (conditional) parameter ν_t, that determines the time deformation is unchanged, whereas the drift m_t and volatility σ_t^2 are modified. It is easy to check that the Laplace transforms of the historical and risk neutral distributions are both defined on non degenerate intervals.

4.3 Path dependent location and scale parameters

In this subsection let us consider a single risky asset ($J = 1$) and assume that the return satisfies :

$$r_{t+1} = m_t + \sigma_t \varepsilon_{t+1}, \sigma_t > 0, \tag{4.4}$$

where m_t and σ_t are the location and scale parameters, respectively, that may depend on lagged values of the return, and (ε_t) is a sequence of i.i.d. variables with Laplace transform :

$$E(\exp u\varepsilon_{t+1}) = \exp \psi(u), \qquad \text{(say)}. \tag{4.5}$$

This model is convenient to show the consequence of the conditional normality assumption. Indeed it includes the case of gaussian errors, but may also allow for other types of conditional distributions with heavier tails, such as stable or Laplace distribution.

i) The stochastic discount factor and the risk neutral distribution.

The conditional log-Laplace transform of r_{t+1} is given by :

$$\psi_t(u) = m_t u + \psi(\sigma_t u).$$

From Proposition 1, the log-Laplace transform of the risk-neutral distribution is given by :

$$\begin{aligned} \psi_t^Q(u) &= \psi_t(\alpha_t + u) - \psi_t(\alpha_t) \\ &= m_t u + \psi[\sigma_t(\alpha_t + u)] - \psi[\sigma_t \alpha_t], \end{aligned}$$

where α_t is solution of :

$$\begin{aligned} \psi_t(\alpha_t + 1) &= \psi_t(\alpha_t) \\ \Longleftrightarrow -m_t &= \psi[\sigma_t(\alpha_t + 1)] - \psi[\sigma_t \alpha_t]. \end{aligned} \tag{4.6}$$

Proposition 2 : There exists a unique solution α_t to equation (4.6), if the log-Laplace transform ψ is strictly convex and tends to infinity at boundaries of its domain.

Proof : Indeed the mapping $\alpha \to \psi[\sigma_t(\alpha+1)] - \psi(\sigma_t \alpha)$ is continuous, strictly increasing, with range $(-\infty, +\infty)$. The result follows directly.

QED

Various examples are provided in Table 1. They include standard models such as the Black-Scholes model with the gaussian error, the binomial tree with dichotomous errors, or the Laplace models [Gourieroux, Monfort (2001)] and a number of other specifications.

Table 1 : Historical and risk neutral distributions

Distribution	p.d.f.	ψ	$\frac{d^2\psi}{du^2}$	α_t	R.N. Distribution ψ_t^Q
$N(0,1)$	$\frac{1}{\sqrt{2\pi}} \exp -\frac{\varepsilon^2}{2}$	$\frac{u^2}{2}$ $u \in \mathbb{R}$	$1 > 0$	$-\frac{1}{2} - \frac{m_t}{\sigma_t^2}$	$-\frac{\sigma_t^2}{2}u + \frac{\sigma_t^2}{2}u^2$
symmetrical exponential	$\frac{1}{2} \exp -\lvert x \rvert$	$-\log(1-u^2)$ $u \in [-1,1]$	$2\frac{1+u^2}{1-u^2} > 0$	$-\frac{1}{2}$ if $m_t = 0$ $\sigma_t < 2$	skewed Laplace distribution with p.d.f. : $\pi(y)$ $= \frac{1}{2}(1 - \frac{\sigma_t^2}{4}) \exp(\frac{1}{\sigma_t} - \frac{1}{2})y$, if $y \leq 0$. $= \frac{1}{2}(1 - \frac{\sigma_t^2}{4}) \exp(\frac{1}{\sigma_t} + \frac{1}{2})y$, if $y \geq 0$.
$u_{[-1,1]}$	$\frac{1}{2}1_{[-1,+1]}(x)$	$\log \frac{shu}{u}$ $u \in \mathbb{R}$	$\frac{1}{u^2} - \frac{1}{sh^2 u} > 0$		
$\frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$	$P[\varepsilon = \pm 1] = \frac{1}{2}$	$\log ch(u)$ $u \in \mathbb{R}$	$[ch(u)]^{-2} > 0$	(*)	$\log\{p_t \exp u(m_t + \sigma_t)$ $+(1-p_t)\exp u(m_t - \sigma_t)\}$ where : $p_t = \frac{\exp(-\sigma_t) - \exp(-m_t)}{\exp(-\sigma_t) - \exp(\sigma_t)}$
$\sum_{j=1}^{J} p_j \delta_{\varepsilon_j}$	$P[\varepsilon = \varepsilon_j] = p_j$	$\log(\sum_{j=1}^{J} p_j \exp u\varepsilon_j)$			$\log[\sum_{j=1}^{J} p_{j,t} \exp u(m_t + \sigma_t \varepsilon_j)]$ where : $p_{j,t} = \frac{p_j \exp(\sigma_t \alpha_t \varepsilon_j)}{\sum_{j=1}^{J} p_j \exp(\sigma_t \alpha_t \varepsilon_j)}$

(*) $\alpha_t = \frac{1}{2\sigma_t} \log \left\{ \frac{\exp(-\sigma_t) - \exp(-m_t)}{\exp(-m_t) - \exp(\sigma_t)} \right\}$, if $|m_t| < \sigma_t$.

ii) Semi-parametric pricing

The approach above can in particular be applied to the sample distribution of residuals deduced from a given model of returns. As an illustration, let us consider parametric specifications of the location and scale parameters :

$$m_t = m(r_t; \theta), \sigma_t = \sigma(r_t; \theta), \qquad \text{(say)}, \tag{4.7}$$

and let us leave unspecified the distribution of the error term. The available observations on the returns are denoted by $r_1, \ldots, r_T$.

Step 1 : Calibration

The parameter θ can be consistently estimated from historical data by applying a quasi-maximum likelihood method; the estimator is given by :

$$\hat{\theta}_T = \arg\max_{\theta} \sum_{t=1}^{T} \left\{ -\log \sigma(r_{t-1};\theta) - \frac{1}{2} \frac{[r_t - m(r_{t-1};\theta)]^2}{\sigma^2(r_{t-1};\theta)} \right\}.$$

Step 2 : Estimation of the error distribution

Then we compute the residuals :

$$\hat{\varepsilon}_\tau = \frac{r_\tau - m(r_{\tau-1};\hat{\theta}_T)}{\sigma(r_{\tau-1};\hat{\theta}_T)}, \tau = 2,\ldots,T,$$

and the empirical distribution of the residuals :

$$\hat{P} = \frac{1}{T-1} \sum_{\tau=2}^{T} \delta_{\hat{\varepsilon}_\tau},$$

where δ_ε denotes the point mass at ε.

Its log-Laplace transform is given by :

$$\hat{\psi}_T(u) = \log \left(\frac{1}{T-1} \sum_{\tau=2}^{T} \exp u\hat{\varepsilon}_\tau \right), u \text{ varying},$$

and is a consistent estimator of the unknown log-Laplace transform ψ.

Step 3 : Determination of the risk correction factor

The correcting term α_t associated with the true distribution ψ can be consistently approximated by the correcting term $\hat{\alpha}_t$ associated with $\hat{\psi}_T$. The correcting term $\hat{\alpha}_t$ is the solution of :

$$\sum_{\tau=2}^{T} \exp[\sigma(r_t;\hat{\theta}_T)\hat{\alpha}_t\hat{\varepsilon}_\tau]\{\exp[m(r_t;\hat{\theta}_T) + \sigma(r_t;\hat{\theta}_T)\hat{\varepsilon}_\tau] - 1\} = 0.$$

Step 4 : Determination of the SDF

The true underlying sdf is approximated by :

$$
\begin{aligned}
\hat{M}_{t,t+1} &= \exp\left[\hat{\alpha}_t[r_{t+1} - m(r_t;\hat{\theta}_T)] - \hat{\psi}_T(\hat{\sigma}_t\hat{\alpha}_t)\right] \\
&= \exp\hat{\alpha}_t[r_{t+1} - m(r_t;\hat{\theta}_T)]\left[\frac{1}{T-1}\sum_{\tau=2}^{T}\exp\hat{\sigma}_t\hat{\alpha}_t\hat{\varepsilon}_\tau\right]^{-1}. \\
&= \hat{m}_{t,t+1}(r_{t+1}, r_t), \qquad \text{(say)}.
\end{aligned}
$$

Step 5 : Pricing

Finally the price of a derivative can easily be approximated. For instance the price of a derivative with residual maturity 1 and payoff $g(r_{t+1})$ is approximated by :

$$
\begin{aligned}
\hat{C}_T(g) &= \hat{E}\left[\hat{m}_{T,T+1}(r_{T+1}, r_T)g(r_{T+1})\right] \\
&= \hat{E}\left\{g[m(r_T;\hat{\theta}_T) + \sigma(r_T;\hat{\theta}_T)\varepsilon_{T+1}]\exp[\hat{\alpha}_T\sigma(r_T;\hat{\theta}_T)\varepsilon_{T+1}]\right\} \\
&\quad \left[\frac{1}{T-1}\sum_{\tau=2}^{T}\exp(\hat{\alpha}_T\hat{\sigma}_T\hat{\varepsilon}_\tau)\right]^{-1} \\
&= \frac{\sum_{\tau=2}^{T}\exp(\hat{\alpha}_T\hat{\sigma}_T\hat{\varepsilon}_\tau)g(\hat{m}_T + \hat{\sigma}_T\hat{\varepsilon}_\tau)}{\sum_{\tau=2}^{T}\exp(\hat{\alpha}_T\hat{\sigma}_T\hat{\varepsilon}_\tau)},
\end{aligned}
$$

by the last line of Table 1.

When the derivative to be priced corresponds to a maturity H much larger than 1, different steps of the algorithm above have to be modified. For instance the correcting factors α and the sdf's have to be computed for several future dates $T, T+1, \ldots T+H-1$, and the pricing formula modified according to (3.2). Finally the expectation in the price formula has to be approximated by a sample average over simulated paths $r_{T+1}, \ldots, r_{T+H-1}$ drawn from the sample distribution.

iii) Implied volatility surfaces

In practice the implied Black-Scholes volatilities depend on the moneyness strike and residual maturity contrary to what is assumed in the Black-Scholes model. They can feature smiles, asymmetries or sneers (see e.g. Gourieroux, Jasiak (2002) for a discussion). These patterns are due to misspecification of the Black-Scholes model, which assumes i.i.d. gaussian returns. The aim of this section is to consider conditionally heteroscedastic models of the type :

$$r_{t+1} = \sigma(r_t)\varepsilon_{t+1},$$

to derive the price of european calls for varying moneyness strike and horizon by the approach of the section above and to describe how the implied volatility surface depends on the functional form of the volatility and on the error distribution. We consider the following four experiments :

experiment (1) : ARCH model

$$\sigma(r_t) = (10^{-3} + 0.95 r_t^2)^{1/2}, \varepsilon_t \sim N(0,1)$$

experiment (2) : Symmetrical piecewise linear volatility

$$\sigma(r_t) = (0.1\ \mathbb{1}_{r_t>0} + 0.01\ \mathbb{1}_{r_t<0})^{1/2}, \varepsilon_t \sim N(0,1)$$

experiment (3) : Asymmetrical piecewise linear volatility

$$\sigma(r_t) = (0.005\ \mathbb{1}_{|r_t|>0.05} + 0.001\ \mathbb{1}_{|r_t|<0.05})^{1/2}, \varepsilon_t \sim N(0,1)$$

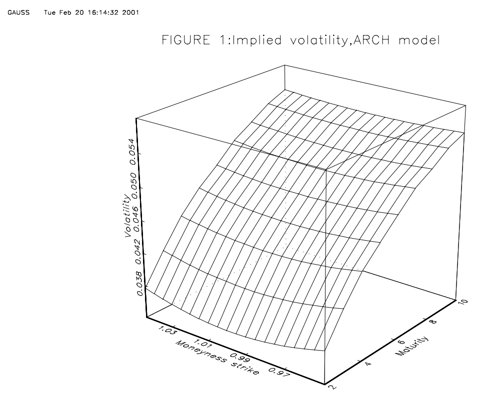
experiment (4) : $\sigma(r_t) = 0.1, \varepsilon_t$ i.i.d. symmetrical exponential.

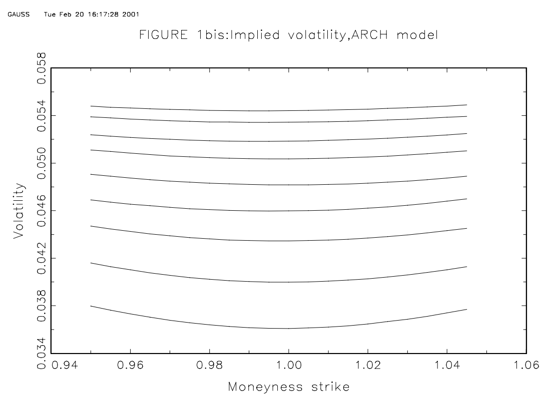
As expected, smile or sneer effects are observed. The importance of smile effect is a function of how the volatility depends on the past (experiment (2)) and of the tail magnitude of the error distribution (experiment (4)). A sneer effect is obtained by introducing either an asymmetric volatility function (experiment (3)), or a skewed distribution of the error. Finally the smile and sneer effects diminish, when maturity increases. This is a consequence of central limit theorem, which can be applied to the return between t and $t+H$, given by :

$$r_{t,t+H} = r_{t+1} + r_{t+2} + \ldots + r_{t+H},$$

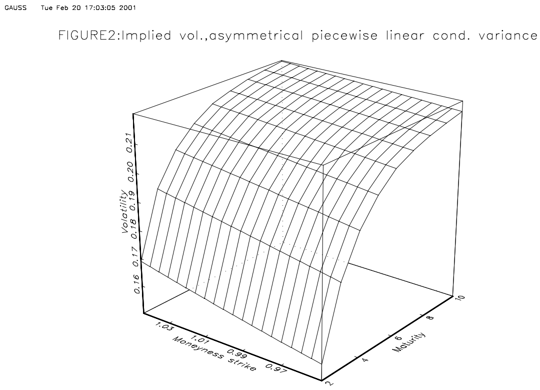
and is valid for return processes that are stationary under the risk neutral distribution.

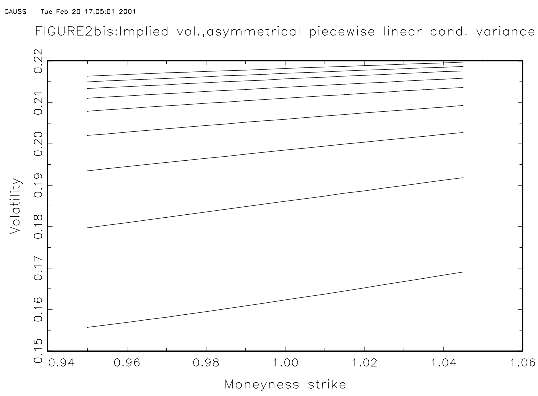
[Insert Figure 1 and 1 bis : Arch model]



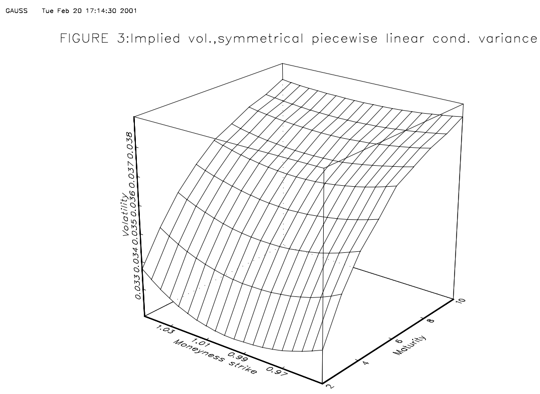


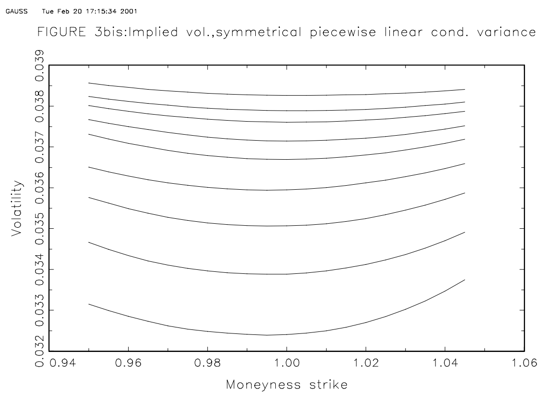
[Insert Figure 2 and 2 bis : Black-Scholes implied volatilities, asymmetrical piecewise linear conditional variance]



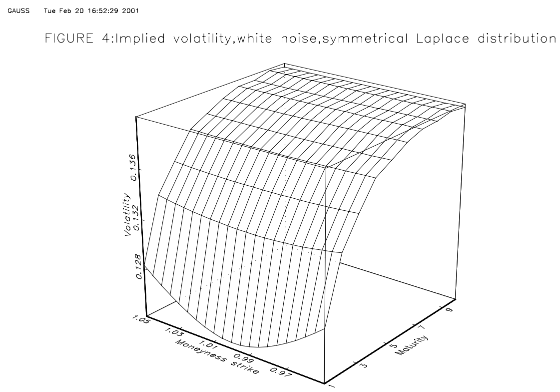


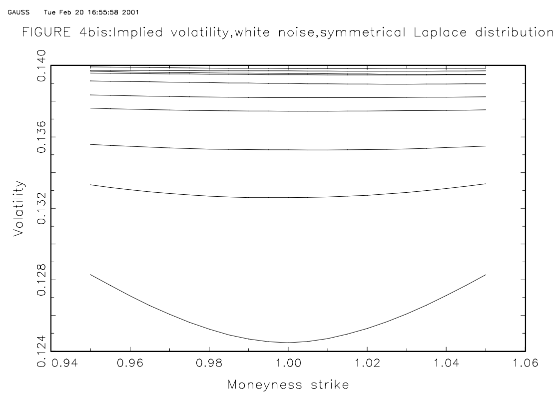
[Insert Figure 3 and 3 bis : Black-Scholes implied volatilities, symmetrical piecewise linear conditional variance]





[Insert Figure 4 and 4 bis : Black-Scholes implied volatilities, symmetrical exponential]





5. General specification

In this section the basic approach to sdf modelling is extended in two directions. First, the information used by the agents to fix their asset demands does not only include the asset returns, but also variables from the real sector of the economy and additional factors. These factors are not directly observed by the econometrician. Second we introduce a time varying riskfree rate r^f_{t+1}. This rate is predetermined, that is known at time t, but assumed stochastic and in particular unknown at time $t-1$. This allows for a joint analysis of derivatives written on the risky assets and on the short term interest rate. In particular we are interested in the analysis of the term structure of interest rates and in the effect of the stochastic interest rate on the price of a European call with maturity H written on a stock index, for instance [12].

5.1 The investor's information and the sdf

The investors' information at date t is denoted by J_t. It includes the data used by investors, when they rebalance their portfolios, submit their orders and decide the volumes to be traded. This information includes :

i) the current riskfree rate and its lagged values, $\underline{r^f_{t+1}}$, say;

ii) the lagged returns on J risky assets, $\underline{r_t}$, say ;

iii) the lagged changes of real economic variables, $\underline{x_t}$, say, that may be macrovariables as the GNP, the retail price index..;

iv) the values of additional factors, $\underline{f_t}$, say.

At this stage the additional factors cannot be interpreted. However they can be partly recovered through their effect on prices of basic and derivative assets.

[12] The standard financial literature often treats separately the analysis of the term structure and the pricing of options written on a stock. Typically the term structure is constructed using only the information contained in the past interest rates, whereas the standard option pricing formula assumes that the interest rate r^f_{t+1} is deterministic, that is known in advance for all future dates.

Thus the investor's information is :

$$J_t = \left(\underline{r_{t+1}^f}, \underline{r_t}, \underline{x_t}, \underline{f_t}\right). \qquad (5.1)$$

The sizes of the different vectors are 1, J, L and K, respectively.

In the sequel we are interested in pricing derivatives written on the basic riskfree and risky assets [13]. If the derivative provides a payoff $g\left(\underline{r_{t+2}^f}, \underline{r_{t+1}}\right)$ at date $t+1$, its price at date t is given by [14]. :

$$C_t(g) = E\left[M_{t,t+1}\, g\left(\underline{r_{t+2}^f}, \underline{r_{t+1}}\right) | J_t\right], \qquad (5.2)$$

where the stochastic discount factor $M_{t,t+1}$ depends on J_{t+1}. As before the sdf is constrained to be exponential-affine :

$$M_{t,t+1} = \exp\left(\alpha_{o,t} r_{t+2}^f + \alpha_t' r_{t+1} + \gamma_t' x_{t+1} + \delta_t' f_{t+1} + \beta_t\right), \qquad (5.3)$$

where the different coefficients depend on the investors' information J_t. Let us denote :

$$F_{t+1} = \left[r_{t+2}^f, x_{t+1}', f_{t+1}'\right]', \Delta_t = (\alpha_{o,t}, \gamma_t', \delta_t')',$$

such that the sdf becomes :

$$M_{t,t+1} = \exp\left[\alpha_t' r_{t+1} + \Delta_t' F_{t+1} + \beta_t\right]. \qquad (5.4)$$

5.2 The historical distribution

[13] It is also possible to include real variables in the contractual payoff as for French Treasury Bonds indexed on inflation.

[14] The role of conditioning information is important in derivative pricing (see e.g. Hansen, Richard (1987)). For instance the pricing formula (5.2) can also be written as :

$$\begin{aligned} C_t(g) &= E\left[E(M_{t,t+1} | r_{t+2}^f, r_{t+1}, J_t) g(\underline{r_{t+2}^f}, \underline{r_{t+1}}) | J_t\right] \\ &= E\left[M_{t,t+1}^* g(\underline{r_{t+2}^f}, \underline{r_{t+1}}) | J_t\right]. \end{aligned}$$

Clearly $M_{t,t+1}^*$ is another admissible sdf, but does not admit an exponential-affine expression with respect to r_{t+1}, r_{t+2}^f, once the factors x_{t+1}, f_{t+1} have been integrated out. Thus the constraint of exponential-affine function is associated with the basic traders' information J_{t+1}.

This distribution has to be considered for all variables in the investors' information set. The conditional Laplace transform of $(r'_{t+1}, F'_{t+1})'$ given J_t is :

$$E\left[\exp(u' r_{t+1} + v' F_{t+1}) | J_t\right] = \exp \psi_t(u, v; \theta). \tag{5.5}$$

It depends on lagged values of the asset returns $\underline{r_t}$, of the real variables $\underline{x_t}$, of the unobservable factors $\underline{f_t}$ and on current and lagged values of the riskfree rate $\underline{r^f_{t+1}}$. θ is a vector of parameters that characterizes the conditional distribution.

The riskfree rate has a particular status. It is not only predetermined, but also constrained to be positive, whereas r_t and x_t are generally of any sign due to their interpretations as changes (see the example of CCAPM for real variables) .

Example 4 : Stochastic volatility model

It is interesting to discuss the notion of factor for the discretized stochastic volatility model :

$$\begin{cases} S_{t+1} - S_t = \mu S_t + \sigma_t S_t \varepsilon^S_{t+1}, \\ \\ f(\sigma_{t+1}) - f(\sigma_t) = a(\sigma_t) + b(\sigma_t)\varepsilon^\sigma_{t+1}. \end{cases}$$

A possible expression of the sdf is given in Example 3 as an exponential-affine function of the standardized innovations ε^S_{t+1} and ε^σ_{t+1}. However this sdf can also be expressed as an exponential-affine function of $S_{t+1}, f(\sigma_{t+1})$, or of $(S_{t+1} - S_t)/S_t, (f(\sigma_{t+1}) - f(\sigma_t))/f(\sigma_t)$ with path dependent coefficients. Therefore the factors are not uniquely defined. In the example, factors can be either innovations, or levels of variables, or associated returns. In particular by selecting the innovation processes, we can assume white noise properties of the factors under the historical distribution.

5.3 Constraints on the S.D.F.

By writing the pricing conditions for the riskfree asset and the J risky assets, we get the equations :

$$\begin{cases} E\left[M_{t,t+1}\exp r^f_{t+1}|J_t\right] = 1, \\ E\left[M_{t,t+1}\exp r_{j,t+1}|J_t\right] = 1, \forall j = 1,\dots,J, \end{cases}$$

$$\iff \begin{cases} \psi_t(\alpha_t, \Delta_t;\theta) + \beta_t + r^f_{t+1} = 0, \\ \psi_t(\alpha_t + e_j, \Delta_t;\theta) + \beta_t = 0, \forall j = 1,\dots,J. \end{cases}$$

We get a system of $J+1$ equations for $J+L+K+2$ unknown risk adjustment parameters, the solutions of which satisfy :

$$\begin{cases} \alpha_t = \alpha(\Delta_t, \underline{r_t}, \underline{F_t};\theta), \\ \beta_t = \beta(\Delta_t, \underline{r_t}, \underline{F_t};\theta). \end{cases} \tag{5.6}$$

Since the additional variables do not correspond to returns on assets traded at t, the sensitivity coefficient Δ_t can be fixed arbitrarily as a function of the information set (see the stochastic volatility model (Example 3) for a similar case). Thus the undeterminacy of the risk correction Δ_t still exists, despite the a priori constraint on the specification of the sdf. The results are summarized in the proposition below.

Proposition 3 : For an exponential-affine sdf with state variables r_{t+1}, F_{t+1}, there exists a multiplicity of admissible risk-neutral distributions. Their conditional Laplace transforms are given by :

$$\begin{aligned} &\overset{Q}{E}\left[\exp(u'r_{t+1} + v'F_{t+1})|J_t\right] \\ &= E\left[M_{t,t+1}\exp(u'r_{t+1} + v'F_{t+1})|J_t\right] / E\left[M_{t,t+1}|J_t\right] \\ &= E[M_{t,t+1}\exp r^f_{t+1}\exp(u'r_{t+1} + v'F_{t+1})|J_t] \\ &= \exp\left\{\psi_t(\alpha_t + u, \Delta_t + v;\theta) - \psi_t(\alpha_t, \Delta_t;\theta)\right\}, \end{aligned}$$

where α_t is the solution of :

$$\psi_t(\alpha_t + e_j, \Delta_t;\theta) - \psi_t(\alpha_t, \Delta_t;\theta) - r^f_{t+1} = 0, \forall j = 1,\dots,J,$$

and the risk correction Δ_t can be chosen arbitrarily.

The dimension of incompleteness is equal to the dimension $K+L+1$ of Δ_t. It corresponds to the number of real economy state variables (x_t) and factors (f_t) plus the future riskfree rate (r^f_{t+2}) introduced in the sdf.

5.4 The conditionally gaussian framework.

Since the riskfree interest rate r^f_{t+1} admits positive values, the conditionally gaussian framework can only be applied in the framework of Section 5.1 if the future path of the riskfree rate is completely known at date t. Thus this subsection extends subsection 4.1, when there are additional unobservable factors.

When $\left(r'_{t+1}, F'_{t+1}\right)'$ are conditionally gaussian under the historical distribution, the log-Laplace transform is :

$$\psi_t(u,v;\theta) = (u',v')m_t(\theta) + \frac{1}{2}(u',v')\Sigma(\theta)\begin{pmatrix} u \\ v \end{pmatrix}.$$

The log-Laplace transforms of the risk-neutral distributions are :

$$\begin{aligned}\psi^Q_t(u,v;\theta) &= \psi_t(\alpha_t+u, \Delta_t+v;\theta) - \psi_t(\alpha_t,\Delta_t;\theta) \\ &= (u',v')\left[m_t(\theta) + \Sigma_t(\theta)\begin{pmatrix} \alpha_t \\ \Delta_t \end{pmatrix}\right] + \frac{1}{2}(u',v')\Sigma_t(\theta)\begin{pmatrix} u \\ v \end{pmatrix},\end{aligned}$$

where Δ_t can be chosen arbitrarily.

They correspond to gaussian distributions with the same conditional variance-covariance matrix as the historical distribution and a conditional mean adjusted for the risk. The expression of this mean depends on the solution α_t of the pricing restrictions.

Let us block decompose the conditional mean and variance as :

$$m_t = \begin{pmatrix} m_{r,t} \\ \\ m_{F,t} \end{pmatrix}, \Sigma_t = \begin{pmatrix} \Sigma_{rr,t} & \Sigma_{rF,t} \\ \\ \Sigma_{Fr,t} & \Sigma_{FF,t} \end{pmatrix}.$$

The restrictions of Proposition 3 are :

$$(e_j', 0)\left[m_t + \Sigma_t \begin{pmatrix} \alpha_t \\ \Delta_t \end{pmatrix}\right] + \frac{1}{2}(e_j', 0)\Sigma_t \begin{pmatrix} e_j \\ 0 \end{pmatrix} - r_{t+1}^f = 0, \forall j = 1, \ldots, J,$$

$$\iff e_j' m_{r,t} + e_j' \Sigma_{rr,t}\alpha_t + e_j'\Sigma_{rF,t}\Delta_t + \frac{1}{2}e_j'\Sigma_{rr,t}e_j - r_{t+1}^f = 0, \forall j,$$

$$\iff m_{r,t} + \Sigma_{rr,t}\alpha_t + \Sigma_{rF,t}\Delta_t + \frac{1}{2}\ vdiag\ \Sigma_{rr,t} - r_{t+1}^f e = 0,$$

where $e = (1, \ldots, 1)'$.

We deduce that :

$$\alpha_t = \Sigma_{rr,t}^{-1}\left[m_{r,t} - r_{t+1}^f e + \frac{1}{2}\ vdiag\ \Sigma_{rr,t} + \Sigma_{rF,t}\Delta_t\right], \tag{5.7}$$

which extends equation (3.4).

Thus the conditional mean of the risk-neutral distribution is :

$$m_t + \Sigma_t \begin{pmatrix} \alpha_t \\ \Delta_t \end{pmatrix}$$

$$= \begin{pmatrix} m_{r,t} + \Sigma_{rr,t}\alpha_t + \Sigma_{r,F,t}\Delta_t \\ m_{F,t} + \Sigma_{Fr,t}\alpha_t + \Sigma_{FF,t}\Delta_t \end{pmatrix}$$

$$= \begin{bmatrix} r_{t+1}^f e - \frac{1}{2}\ vdiag\ \Sigma_{rr,t} \\ m_{F,t} - \Sigma_{Fr,t}\Sigma_{rr,t}^{-1}\left(m_{r,t} - r_{t+1}^f e + \frac{1}{2}\ vdiag\ \Sigma_{rr,t}\right) + \left(\Sigma_{FF,t} - \Sigma_{Fr,t}\Sigma_{rr,t}^{-1}\Sigma_{rF,t}\right)\Delta_t \end{bmatrix}.$$

It is easy to recognize the residual conditional mean $m_{F,t} - \Sigma_{Fr,t}\Sigma_{rr,t}^{-1}m_{r,t}$ and variance $\Sigma_{FF,t} - \Sigma_{F,r,t}\Sigma_{rr,t}^{-1}\Sigma_{rF,t}$. If $\Delta_t = 0$, the correction for risk is performed on the returns of traded assets, and there is no correction at all on the residual part, that is on the component of F_{t+1} which is not conditionally linked with the asset returns. This result can be considered as a direct extension of the pricing formula derived by Stapleton, Subrahmanyam (1984).

5.5 Models with stochastic interest rate

The pricing approach can be extended to account for stochastic interest rate (see Yao (2001) for another specification).

We first consider a model without unobservable factor, where the short term zero-coupon bond is the only tradable asset. By selecting an autoregressive gamma model for the historical distribution of the riskfree rate (Darolles, Gourieroux, Jasiak (2001)), we get a direct extension of the Cox-Ingersoll-Ross model (Cox, Ingersoll, Ross (1985)). Then we discuss the pricing of an index derivative, when the interest rate is stochastic.

5.5.1. Model with a stochastic interest rate only

i) Stochastic discount factor

When the short term zero-coupon bond is the only tradable asset and the sdf is given by :

$$M_{t,t+1} = \exp(\alpha_o r^f_{t+2} + \beta_t), \tag{5.8}$$

the arbitrage free condition becomes :

$$\beta_t = -r^f_{t+1} - \psi_t(\alpha_o), \tag{5.9}$$

where : $\psi_t(u) = E[\exp u r^f_{t+2} | \underline{r^f_{t+1}}]$ and α_o is a time independent parameter.

As an illustration let us assume that the conditional distribution of r^f_{t+2} is a noncentered gamma distribution [15] with log-Laplace transform :

$$\psi_t(u) = -\nu \log(1-uc) + \frac{\rho u}{1-uc} r^f_{t+1}, \tag{5.10}$$

where ν, c, ρ are positive parameters. ν gives the degrees of freedom, c is a scale parameter, whereas the noncentrality parameter of the conditional distribution is : $\rho r^f_{t+1}/c$. We deduce that :

[15] The variable r^f_{t+2} follows the gamma distribution with degrees of freedom ν, scale parameter c and noncentrality parameter $\rho r^f_{t+1}/c$, if and only if :

r^f_{t+2}/c follows a gamma distribution $\gamma(\nu + Z_t)$, where Z_t is drawn in the Poisson distribution $\mathcal{P}[\rho r^f_{t+1}/c]$. It is easily checked that this conditional distribution corresponds to a discretized version of the Cox, Ingersoll, Ross model, if $0 < \rho < 1$, (see e.g Gourieroux, Jasiak (2000)), and that its log-Laplace transform is given by (5.10).

$$\begin{aligned}
\beta_t &= -r^f_{t+1} - \psi_t(\alpha_o) \\
&= -r^f_{t+1} + \nu \log(1-\alpha_o c) - \frac{\rho\alpha_o}{1-\alpha_o c} r^f_{t+1}, \\
M_{t,t+1} &= \exp\left[\alpha_o r^f_{t+2} - \left(1 + \frac{\rho\alpha_o}{1-\alpha_o c}\right) r^f_{t+1} + \nu \log(1-\alpha_o c)\right].
\end{aligned}$$

ii) Risk neutral distribution.

Finally the log-Laplace transform of the risk-neutral distribution is given by :

$$\begin{aligned}
& \overset{Q}{E_t} [\exp u r^f_{t+2}] \\
= \; & \exp[\psi_t(u+\alpha_o) - \psi_t(\alpha_o)] \\
= \; & \exp[-\nu \log\left(\frac{1-(u+\alpha_o)c}{1-\alpha_o c}\right) + \frac{\rho(u+\alpha_o)}{1-(u+\alpha_o)c} r^f_{t+1} - \frac{\rho\alpha_o}{1-\alpha_o c} r^f_{t+1}] \\
= \; & \exp[-\nu \log(1-uc^*) + \frac{\rho^* u}{1-uc^*} r^f_{t+1}],
\end{aligned}$$

where : $c^* = c/(1-\alpha_o c), \rho^* = \rho/(1-\alpha_o c)^2$.

For the risk neutral distribution to be defined, the risk correcting parameter α_o has to be chosen such that $1 - \alpha_o c > 0 \iff \alpha_o < 1/c$.

Thus the risk-neutral conditional distributions belong to the same family of noncentered gamma distributions as the historical conditional distribution. The degrees of freedom are the same for the historical and risk-neutral distributions, whereas parameters c^* and ρ^* depend on the risk correction α_o, associated with the future short term interest rate. The standard Cox, Ingersoll, Ross model corresponds to a zero risk premium $\alpha_o = 0$.

iii) Derivative prices

Let us now consider the derivatives with exponential payoffs $\exp(ur^f_{t+2}), u$ varying. Since the Laplace transform characterizes the distribution, these derivatives define a generating system for all European derivatives (see the discussions in Duffie, Pan, Singleton (2000), Polimenis (2001), Gourieroux, Monfort, Polimenis (2002)).

The price $C_t(u,1)$ of the European derivative providing payoff $\exp(ur^f_{t+2})$ at date $t+1$ is :

$$C_t(u,1) = \exp(-r^f_{t+1}) \overset{Q}{E_t} (\exp ur^f_{t+2}).$$

This price depends on r^f_{t+1} and α_o :

$$C_t(u,1) = \gamma(\alpha_o, r^f_{t+1}; u), \qquad \text{(say)}.$$

It is easy to check that we get a semi-interval of admissible prices, when α_o varies, that is : $(\gamma^*_t(u), +\infty)$, where : $\gamma^*_t(u) = \min_{\alpha_o<1/c} \gamma(\alpha_o, r^f_{t+1}; u)$.

Explicit formulas can also be derived for pricing European derivatives at any horizon. We still consider the generating system of derivatives with exponential payoffs.

Proposition 4 : Let us denote by $C_t(u,h)$ the price at t of the European derivative, that provides payoff $\exp(ur^f_{t+h+1})$ at $t+h$. We get :

$$C_t(u,h) = \exp[a(h,u)r^f_{t+1} + b(h,u)],$$

where functions a and b satisfy the recursive equations :

$$a(h,u) = -1 - \frac{\rho\alpha_0}{1-\alpha_0 c} + \rho\frac{\alpha_0 + a(h-1,u)}{1-[\alpha_0 + a(h-1,u)]c},$$

$$b(h,u) = \nu\log(1-\alpha_o c) - \nu\log[1-[\alpha_o + a(h-1,u)]c] + b(h-1,u),$$

for $h \geq 2$.

Proof : We get :

$$
\begin{aligned}
C_t(u,h) &= E_t[M_{t,t+1}C_{t+1}(u,h-1)] \\
&= E_t\Big\{\exp[\alpha_o r^f_{t+2} - \Big(1+\frac{\rho\alpha_o}{1-\alpha_o}\Big) r^f_{t+1} + \nu\log(1-\alpha_o c) \\
&\quad +a(h-1,u)r^f_{t+2} + b(h-1,u)]\Big\} \\
&= \exp\Big\{-\Big(1+\frac{\rho\alpha_o}{1-\alpha_o c}\Big) r^f_{t+1} + \nu\log(1-\alpha_o c) + b(h-1,u)\Big\} \\
&\quad E_t\Big\{\exp[\alpha_o + a(h-1,u)]r^f_{t+2}\Big\} \\
&= \exp\Big\{-\Big(1+\frac{\rho\alpha_o}{1-\alpha_o c}\Big) r^f_{t+1} + \nu\log(1-\alpha_o c) - \nu\log[1-(\alpha_o + a(h-1,u))c] \\
&\quad +b(h-1,u) + \rho\frac{\alpha_o + a(h-1,u)}{1-(\alpha_o+a(h-1,u))c} r^f_{t+1}\Big\}.
\end{aligned}
$$

The result follows by identification.

QED

The nonlinear recursive equation does not depend on the argument u, that is on the payoff to be priced [16]. This argument has an effect through the initial condition only, since : $a(1,u) = -1 + \frac{\rho^* u}{1-uc^*}, b(1,u) = -\nu\log(1-uc^*)$. Once functions a and b are known, it is easy to derive the joint conditional forward risk neutral distribution $Q^*_{t,h}$ between $t+1$ and $t+h$ [see El Karoui, Geman, Rochet (1995)]. Indeed this distribution has a pdf with respect to the historical pdf equal to :

$$
M_{t,t+h}/E_t M_{t,t+h} = M_{t,t+1}\dots M_{t+h-1,t+h}/E_t[M_{t,t+1}\dots M_{t+h-1,t+h}].
$$

Therefore we get :

[16] This property is analogous to a standard property for continuous time models, saying that the price of a European derivative satisfies a partial differential equation, which is independent of the payoff.

$$\begin{aligned} C_t(u,h) &= E_t[M_{t,t+h} \exp(ur_{t+h+1})], \\ &= E_t(M_{t,t+h}) \overset{Q^*_{t,h}}{E} [\exp(ur_{t+h+1})] \\ &= C_t(0,h) \exp \psi^{Q^*}_{t,h}(u), \end{aligned}$$

where $\psi^{Q^*}_{t,h}$ is the log-Laplace transform of $Q^*_{t,h}$. The log-Laplace transform is given by :

$$\begin{aligned} \psi^{Q^*}_{t,h}(u) &= \log C_t(u,h) - \log C_t(0,h) \\ &= [a(h,u) - a(h,0)] r^f_{t+1} + b(h,u) - b(h,0). \end{aligned}$$

It is possible to get the explicit expressions of the coefficient $a(h,u)$ (and $b(h,u)$). Indeed the series $a(h,u)$ satisfies a rational recursive equation, which is equivalent to :

$$\frac{a(h,u)-\gamma_1}{a(h,u)-\gamma_2} = \frac{1+\gamma_1}{\gamma_1} \frac{\gamma_2}{1+\gamma_2} \frac{a(h-1,u)-\gamma_1}{a(h-1,u)-\gamma_2},$$

where γ_1 and γ_2 are distinct real roots of the second degree polynomial : $c^* \gamma^2 + \gamma[\rho^* + c^* - 1] - 1 = 0$.

Thus we get :

$$\frac{a(h,u)-\gamma_1}{a(h,u)-\gamma_2} = \left[\frac{1+\gamma_1}{\gamma_1} \frac{\gamma_2}{1+\gamma_2} \right]^{h-1} \frac{a(1,u)-\gamma_1}{a(1,u)-\gamma_2}.$$

This approach can be extended to multifactor models of interest rates and the derivation of affine term structure models [see Polimenis (2001), Gourieroux, Monfort, Polimenis (2002)] .

ii) Risky asset and stochastic interest rate

The approach above can be extended to joint analysis of a risky asset with return r_{t+1} and a stochastic interest rate r^f_{t+1}. A simple specification of the historical distribution assumes independence between the riskfree rate process (r^f_{t+1}) and the excess return process $(r^*_{t+1} = r_{t+1} - r^f_{t+1})$. Then by selecting appropriate distributions for both components, we can take into account the positivity constraint on the riskfree rate.

As an illustration, let us consider a gaussian autoregressive model for (r^*_{t+1}) and an autoregressive gamma model for (r^f_{t+1}). Thus the joint conditional log-Laplace transform is :

$$\begin{aligned}
\psi_t(u,v) &= \log E[\exp(u r_{t+1} + v r^f_{t+2}) | J_t] \\
&= \log E(\exp u r^*_{t+1} | \underline{r^*_t}) + u r^f_{t+1} + \log E(\exp v r^f_{t+2} | \underline{r^f_{t+1}}) \\
&= u(a r^*_t + b) + \frac{\sigma^2 u^2}{2} + u r^f_{t+1} - \nu \log(1 - vc) + \frac{\rho v}{1 - vc} r^f_{t+1} \\
&= \psi_{1t}(u) + u r^f_{t+1} + \psi_{2t}(v), \qquad \text{(say)}, \qquad (5.11)
\end{aligned}$$

where $a r^*_t + b$ denotes the one-step ahead prediction of excess return.

The stochastic discount factor is set as :

$$\begin{aligned}
M_{t,t+1} &= \exp[\alpha_o r^f_{t+2} + \alpha_t r^*_{t+1} + \beta_t] \\
&= \exp[\alpha_o r^f_{t+2} + \alpha_t (r_{t+1} - r^f_{t+1}) + \beta_t].
\end{aligned} \qquad (5.12)$$

Then the arbitrage free conditions imply :

$$\left\{ \begin{array}{l} E_t(M_{t,t+1} \exp r^f_{t+1}) = 1, \\ \\ E_t[M_{t,t+1} \exp(r^*_{t+1} + r^f_{t+1})] = 1, \end{array} \right.$$

$$\iff \left\{ \begin{array}{l} \psi_{1,t}(\alpha_t) + \psi_{2,t}(\alpha_o) + r^f_{t+1} + \beta_t = 0, \\ \\ \psi_{1,t}(\alpha_t + 1) + \psi_{2,t}(\alpha_o) + r^f_{t+1} + \beta_t = 0. \end{array} \right.$$

Thus the risk correcting factor α_t satisfies :

$$\begin{aligned}
& \psi_{1t}(\alpha_t + 1) = \psi_{1t}(\alpha_t) \\
\iff \quad & \alpha_t = -\frac{1}{2} - \frac{a r^*_t + b}{\sigma^2}.
\end{aligned} \qquad (5.13)$$

The log-Laplace transforms of the conditional risk neutral distributions at horizon 1 are given by :

$$\psi_t^Q(u,v) = \psi_{1t}(u+\alpha_t) - \psi_{1t}(\alpha_t) + ur_{t+1}^f + \psi_{2t}(v+\alpha_o) - \psi_{2t}(\alpha_o), \quad (5.14)$$

where α_o can be chosen arbitrarily.

The processes (r_{t+1}^*) and (r_{t+1}^f) are still independent in the risk neutral world, for any value of α_o. However the joint distribution of (r_{t+1}, r_{t+1}^f) has to be used for pricing a standard European call written on r_{t+1}, since the payoff depends jointly on r_{t+1}^* and r_{t+1}^f : :

$$(\exp r_{t+1} - k)^+ = [\exp(r_{t+1}^* + r_{t+1}^f) - k]^+.$$

This standard call can be considered as a quanto-option, for which the riskfree rate provides the exchange rate between the money units of dates t and $t+1$, respectively.

6. Concluding remarks

The aim of this paper is to analyze the pricing models with exponential-affine stochastic discount factor and to derive the joint specifications of the historical and risk neutral distributions. This approach underlies different pricing methods introduced in the literature, such as the pricing with ARCH models, the affine term structure model or the variance-gamma model. Its general use is promising as illustrated by the various applications given in the paper, that are the semi-parametric pricing for model with path dependent scale and location parameters, the discussion of smiles and asymmetries of the Black-Scholes implied volatility surfaces or the pricing models for stochastic riskfree interest rate. In particular exponential affine sdf underlies the affine models currently introduced for credit risk, that is for a coherent analysis of the prices of T-bonds and corporate bonds (Duffie, Lando (2001), Gourieroux, Monfort, Polimenis (2002) b).

This specification of the stochastic discount factor can lead to parametric or semi-parametric models for asset prices. This specification step is necessary before considering statistical inference, that is the estimation of both the historical distribution and the stochastic discount factor from prices data on underlying assets and derivatives.

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