Econometric Asset Pricing Modelling

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First version : December, 2006
This version : March, 2007

Abstract
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The purpose of this paper is to propose a general econometric approach to asset pricing modelling based on three main ingredients : (i) the historical discrete-time dynamics of the factor representing the information, (ii) the Stochastic Discount Factor (SDF), and (iii) the discrete-time risk-neutral (R.N.) factor dynamics. Retaining an exponential-affine specification of the SDF, its modelling is equivalent to the specification of the factor loading vector and of the short rate, if the latter is neither exogenous nor a known function of the factor. In this general framework, we distinguish three modelling strategies: the Direct Modelling, the Risk-Neutral Constrained Direct Modelling and the Back Modelling. In all the approaches we study the internal consistency constraints, implied by the absence of arbitrage opportunity (AAO) assumption, and the identification problem. We also propose interpretations of the factor loading vector in terms of market price of risk. The general modelling strategies are applied to two important cases: security market models and term structure of interest rates models. In the context of security market models, we show the relevance of our methods for various kinds of specifications: switching regime models, stochastic volatility models, Gaussian and Inverse Gaussian GARCH-type models (with or without regime-switching). In the interest rates modelling context, we consider several illustrations: VAR modelling, Switching VAR modelling and Wishart modelling. We also propose, using a Gaussian VAR(1) approach, an example of joint modelling of geometric returns, dividends and short rate. In these contexts we stress the usefulness of the Risk-Neutral Constrained Direct Modelling approach and of the Back Modelling approach, both allowing to conciliate a flexible historical dynamics and a Car R.N. dynamics leading to explicit or quasi explicit pricing formulas for various derivative products. Moreover, we highlight the possibility to specify asset pricing models able to accommodate non-affine historical and R.N. factor dynamics with tractable pricing formulas.

Keywords : Direct Modelling, Risk-Neutral Constrained Direct Modelling, Back Modelling, absence of arbitrage opportunities, identification problem, Car and Extended Car processes, Laplace Transform.

JEL number : C1, C5, G12.

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1 Introduction

The purpose of this paper is to build a bridge between econometric modelling and asset pricing. More precisely, we propose a framework which is able to produce models dealing at the same time with usual econometric problems (historical analysis, prediction ...) and pricing of derivative assets. This general econometric approach is based on three main ingredients: i) the discrete-time historical dynamics of the factor representing the economy, ii) the Stochastic Discount Factor (SDF), and iii) the discrete-time risk-neutral (R.N.) factor dynamics. The central mathematical tool used in the specification of the historical and R.N. dynamics is the conditional Laplace transform. The SDF is assumed to be exponential-affine [see Gourieroux and Monfort (2007)], and its specification is equivalent to the specification of a factor loading vector and of the short rate, if the latter is neither exogenous nor a known function of the factor.

The three elements can not be defined independently and we distinguish three modelling strategies according to the retained basic elements. In the Direct Modelling strategy, we specify the historical dynamics and the SDF, that is to say, the factor loading vector and, possibly, the short rate. The R.N. dynamics is obtained as a by-product. In the Risk-Neutral Constrained Direct Modelling strategy, we specify the historical dynamics and we constrain the R.N. dynamics to belong to a given family, typically the family of Compound Autoregressive (Car) processes proposed by Darolles, Gourieroux and Jasiak (2006)[4]. In this case, the factor loading vector characterizing the SDF is obtained as a by-product. Finally, in the Back Modelling strategy, we specify the R.N. dynamics and, possibly, the short rate, as well as the factor loading and, consequently, the historical dynamics is obtained as by-product. Thus, we get three kinds of Econometric Asset Pricing Models (EAPMs).

In the Back Modelling approach, and in the Risk-Neutral Constrained Direct Modelling approach, the factor is, in general, assumed to be a Car process under the R.N. probability in order to facilitate pricing implementation and econometric analysis. However, we are able to derive asset pricing models where, even if the historical and R.N. dynamics of the factor is not affine, the introduction of a new variable, function of the initial factor, defines a new (extended) factor which turns out to be Car at least under the R.N. probability and, therefore, explicit or quasi explicit pricing formulas can be obtained. This extended factor will be called (historical or risk-neutral) Extended Car process.

For all the strategies, we discuss basic problems of econometric modelling like parameterization, identification and internal consistency with the Absence of Arbitrage Opportunity (AAO) assumption. We also propose interpretations of the factor loading vector in terms of market price of risk. The general modelling strategies are applied to two important cases: discrete-time security market models (the basis of option pricing models) and term structure of interest rates models[5]. In the

[4]A Car (discrete-time affine) process is a Markovian process with an exponential-affine conditional Laplace transform [see Darolles, Gourieroux, Jasiak (2006) for details].

[5]As far as the discrete-time (security market and) option pricing literature is concerned see, among the others, Bertholon, Monfort and Pegoraro (2006), Christoffersen, Elkamhi and Jacobs (2005), Christoffersen, Jacobs and Wang (2006a, 2006b), Duan (1995), Duan, Popova and Ritchken (2002), Duan, Ritchken and Sun (2005, 2006), Garcia and Renault (1998), Garcia, Luger and Renault (2001, 2003), Garcia, Ghysels and Renault (2003), Heston and Nandi (2000) and Leon, Mencia and Sentana (2006). With regard to the discrete-time term structure modelling see, for instance, Ang and Bekaert (2005), Ang and Piazzesi (2003), Ang, Piazzesi and Wei (2006), Audrino and De Giorgi (2006), Backus, Foresi and Telmer (1998), Bansal and Zhou (2002), Bansal, Tauchen and Zhou (2004), Dai, Le and Singleton (2006), Dai, Singleton and Yang (2006), Filipovic and Zabczyk (2002), Gourieroux, Monfort and Polimenis

context of security market models, we show the relevance of our methods for various kinds of specifications: switching regime models, stochastic volatility models, Gaussian and Inverse Gaussian GARCH-type models (with or without regime-switching). In the interest rates modelling context, we consider several illustrations: VAR modelling, Switching VAR modelling and Wishart modelling. We also propose, using a Gaussian VAR(1) approach, an example of joint modelling of geometric returns, dividends and short rate. In these contexts we stress the usefulness of the Risk-Neutral Constrained Direct Modelling approach and of the Back Modelling approach, both allowing to conciliate a flexible historical dynamics and a Car risk-neutral dynamics leading to explicit or quasi explicit pricing formulas for various derivative products. Moreover, we highlight the possibility to specify asset pricing models able to accommodate non-affine historical and R.N. factor dynamics with tractable pricing formulas.

The paper is organized as follows. In Section 2 we define the historical dynamics, the SDF and the R.N. dynamics. In Section 3 we propose financial interpretation of the factor loading vector in two important cases : a) when the factor is a vector of geometric returns and b) when the factor is a vector of yields. In Section 4 we discuss the status of the short rate, we describe the various modelling strategies for the specification of an EAPM, and we present the associated inference problem. Sections 5 and 6 consider, respectively, applications to Econometric Security Market Models and to Econometric Term Structure Models, while, in Section 7 we present an example of Security Market Model with stochastic dividends and short rate. Section 8 concludes, and the proofs are gathered in the appendices.

2 Historical and Risk-Neutral Dynamics

2.1 Information and Historical Dynamics

We consider an economy between dates 0 and T. The new information in the economy at date t is denoted by w_t and the overall information at date t is $\underline{w}_t = (w_t, w_{t-1}, ..., w_0)$. The random variable w_t is called a factor or a state vector, and it may be observable, partially observable or unobservable by the econometrician. The size of w_t is K.

The historical dynamics of w_t is defined by the joint distribution of $\underline{w}_T$, denoted by $\mathbb{P}$, or by the conditional p.d.f. (with respect to some measure):

$$f_t(w_{t+1}|\underline{w}_t)\,,$$

or by the conditional Laplace transform (L.T.):

$$\varphi_t(u|\underline{w}_t) = E[\exp(u'w_{t+1})|\underline{w}_t]\,,$$

which is assumed to be defined in an open convex set of $\mathbb{R}^K$ (containing zero). We also introduce the conditional Log-Laplace transform :

$$\psi_t(u|\underline{w}_t) = \mathrm{Log}[\varphi_t(u|\underline{w}_t)]\,.$$

The conditional expectation operator, given $\underline{w}_t$, is denoted by E_t. $\varphi_t(u|\underline{w}_t)$ and $\psi_t(u|\underline{w}_t)$ will be also denoted by $\varphi_t(u)$ and $\psi_t(u)$.

(2003), Gourieroux and Sufana (2003), Monfort and Pegoraro (2006a, 2006b, 2007). See also Gourieroux, Monfort and Polimenis (2006) for a discrete-time approach to Credit Risk analysis.

2.2 The Stochastic Discount Factor (SDF)

Let us denote by L_{2t} the (Hilbert) space of square integrable functions[6] $g(\underline{w}_t)$. Following Hansen and Richard (1987) we consider the following assumptions :

A1 (Existence and uniqueness of a price): Any payoff $g(\underline{w}_s)$ of L_{2s}, delivered at s, has a unique price at any $t < s$ for each $\underline{w}_t$, denoted by $p_t[g(\underline{w}_s)]$, function of $\underline{w}_t$.

A2 (Linearity and continuity):

- $p_t\left[\lambda_1 g_1(\underline{w}_s) + \lambda_2 g_2(\underline{w}_s)\right] = \lambda_1 p_t\left[g_1(\underline{w}_s)\right] + \lambda_2 p_t\left[g_2(\underline{w}_s)\right]$ (law of one price)
- if $g_n(\underline{w}_s) \xrightarrow[n\to\infty]{L_{2s}} 0$, $p_t\left[g_n(\underline{w}_s)\right] \underset{n\to\infty}{\longrightarrow} 0$.

A3 (Absence of Arbitrage Opportunity): For every date $s > t$, $s, t \in \{0, \ldots, T\}$, and for every $\underline{w}_t$, there is no payoff $g(\underline{w}_s) \in L_{2s}$ satisfying $g(\underline{w}_s) \geq 0, \mathbb{P}\left[g(\underline{w}_s) > 0 | \underline{w}_t\right] > 0$, and $p_t\left[g(\underline{w}_s)\right] \leq 0$.

Under **A1**, **A2** and **A3**, a conditional version of the Riesz representation theorem implies, for each $t \in \{0, \ldots, T-1\}$, the existence and uniqueness of the stochastic discount factor $M_{t,t+1}(\underline{w}_{t+1})$, belonging to $L_{2,t+1}$, such that the price at date t of the payoff $g(\underline{w}_s)$ delivered at $s > t$ is given by [see Appendix 1] :

$$p_t\left[g(\underline{w}_s)\right] = E_t\left[M_{t,t+1} \ldots M_{s-1,s} g(\underline{w}_s)\right] . \tag{1}$$

Moreover, under **A3**, $M_{t,t+1}$ is positive for each $t \in \{0, \ldots, T-1\}$. The process $M_{0t} = \prod_{j=0}^{t-1} M_{j,j+1}$ is called the state price deflator over the period $\{0, \ldots, t\}$.

Since $L_{2,t+1}$ contains 1, the price at t of a zero-coupon bond maturing at $t+1$ is :

$$B(t, 1) = \exp(-r_{t+1}) = E_t(M_{t,t+1}), \tag{2}$$

where r_{t+1} is the (geometric) short rate, between t and $t+1$, known at t. The bank account is :

$$R_{t+1} \quad = \quad \exp(r_1 + \ldots + r_{t+1}) = \frac{1}{E_0(M_{01}) \ldots E_t(M_{t,t+1})}.$$

For any price process p_t we have :

$$p_t = E_t(M_{t,t+1} \ldots M_{s-1,s}\, p_s)\,, \quad \forall s > t\,, s, t \in \{0, \ldots, T\}\,,$$

or :

$$M_{0t} p_t = E(M_{0s} p_s)\,,$$

so $M_{0t} p_t$ and, in particular, $M_{0t} R_t$ are $\mathbb{P}$-martingales.

[6] We do not distinguish functions which are equal almost surely.

2.3 Exponential-affine SDF

We assume that $M_{t,t+1}(\underline{w}_{t+1})$ has an exponential-affine form :

$$M_{t,t+1} = \exp\left[\alpha_t(\underline{w}_t)' w_{t+1} + \beta_t(\underline{w}_t)\right] , \tag{3}$$

where α_t is the "factor loading" or "sensitivity" vector.

The justification of this exponential-affine specification is now well documented in the asset pricing literature. First, this form naturally appears in equilibrium models like CCAPM [see e.g. Lucas (1978) and Cochrane (2005)], Consumption-based asset pricing models with habit formation or with Epstein-Zin preferences [see, among the others, Campbell and Cochrane (1999), Collard, Feve and Ghattassi (2006), Garcia, Meddahi and Tedongap (2006), Garcia, Renault and Semenov (2006)]. Second, in general continuous time security market models the discretized version of the SDF is exponential-affine [see Gourieroux and Monfort (2007)]. Third, the exponential-affine specification is particularly well adapted to the Laplace Transform which is a central tool in discrete-time asset pricing theory [see e.g. Bertholon, Monfort and Pegoraro (2006), Darolles, Gourieroux and Jasiak (2006), Gourieroux, Jasiak and Sufana (2004), Gourieroux, Monfort and Polimenis (2003, 2006), Monfort and Pegoraro (2006a, 2006b, 2007), Pegoraro (2006), Polimenis (2001)].

Since $\exp(-r_{t+1}) = E_t(M_{t,t+1}) = \exp\left[\psi_t(\alpha_t \mid \underline{w}_t) + \beta_t\right]$, the SDF can also be written :

$$M_{t,t+1} = \exp\left[-r_{t+1}(\underline{w}_t) + \alpha_t'(\underline{w}_t) w_{t+1} - \psi_t(\alpha_t|\underline{w}_t)\right] . \tag{4}$$

2.4 Risk-Neutral Dynamics

The joint historical distribution of $\underline{w}_t$, denoted by $\mathbb{P}$, is defined by the conditional distribution of w_{t+1} given $\underline{w}_t$, characterized either by the p.d.f. $f_t(w_{t+1}|\underline{w}_t)$ or the Laplace transform $\varphi_t(u|\underline{w}_t)$, or the Log-Laplace transform $\varphi_t(u|\underline{w}_t)$.

The Risk-Neutral (R.N.) dynamics is another joint distribution of $\underline{w}_t$, denoted by $\mathbb{Q}$, defined by the conditional p.d.f., with respect to the corresponding conditional historical probability, given by :

$$\begin{aligned} d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) &= \frac{M_{t,t+1}(\underline{w}_{t+1})}{E_t\left[M_{t,t+1}(\underline{w}_{t+1})\right]} \\ &= \exp(r_{t+1}) M_{t,t+1}(\underline{w}_{t+1}). \end{aligned} \tag{5}$$

So, the R.N. conditional p.d.f. (with respect to the same measure as the corresponding conditional historical probability) is :

$$f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = f_t(w_{t+1}|\underline{w}_t) d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t),$$

and the conditional p.d.f. of the conditional historical distribution with respect to the R.N. one is given by :

$$d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \frac{1}{d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t)}. \tag{6}$$

The joint p.d.f. of $\mathbb{Q}$ with respect to $\mathbb{P}$ is :

$$\begin{aligned} \xi_T = \frac{d\mathbb{Q}}{d\mathbb{P}} &= \prod_{t=0}^{T-1} d_t^{\mathbb{Q}}(w_{t+1}/\underline{w}_t) \\ &= \prod_{t=0}^{T-1} \exp(r_{t+1}) M_{t,t+1}. \end{aligned} \tag{7}$$

Note that the p.d.f. of the R.N. distribution of $\underline{w}_t$ with respect to the corresponding historical one is :

$$\xi_t = \prod_{t=0}^{t-1} d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = R_t M_{0t} = E_t \xi_T \, , \tag{8}$$

and, therefore, ξ_t is a $\mathbb{P}$-martingale.

The basic pricing formula (1) can be rewritten

$$p_t \left[g(\underline{w}_s) \right] = E_t^{\mathbb{Q}} \left[\exp(-r_{t+1} - ... - r_s) g(\underline{w}_s) \right] \tag{9}$$

or :

$$p_t = E_t^{\mathbb{Q}} \left(\frac{R_t}{R_s} \, p_s \right) \, ,$$

showing that (p_t/R_t) is a $\mathbb{Q}$-martingale. In particular, for any price process p_t, we have

$$p_t = \exp(-r_{t+1}) E_t^{\mathbb{Q}}(p_{t+1})$$

or, using, the arithmetic return $\rho_{A,t+1} = \frac{p_{t+1} - p_t}{p_t}$, and the arithmetic short rate $r_{A,t+1} = [\exp(r_{t+1}) - 1]$, we get:

$$E_t^{\mathbb{Q}}(\rho_{A,t+1}) = r_{A,t+1}. \tag{10}$$

Thus, the excess return $(\rho_{A,t+1} - r_{A,t+1})$ is a $\mathbb{Q}$-martingale difference, and, therefore $\mathbb{Q}$-uncorrelated.

When the SDF is exponential-affine, we have convenient additional results:

$$\begin{aligned} d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) &= \frac{\exp(\alpha_t' w_{t+1} + \beta_t)}{E_t \exp(\alpha_t' w_{t+1} + \beta_t)} \\ &= \exp\left[\alpha_t' w_{t+1} - \psi_t(\alpha_t) \right] , \end{aligned} \tag{11}$$

so $d_t^{\mathbb{Q}}$ is also exponential-affine. The conditional R.N. Laplace transform of the factor w_{t+1}, given $\underline{w}_t$, is :

$$\begin{aligned} \varphi_t^{\mathbb{Q}}(u|\underline{w}_t) &= E_t^{\mathbb{Q}} \left[\exp(u' w_{t+1}) \right] \\ &= E_t \exp\left[(u + \alpha_t)' w_{t+1} - \psi_t(\alpha_t) \right] \\ &= \frac{\varphi_t(u + \alpha_t)}{\varphi_t(\alpha_t)} \end{aligned} \tag{12}$$

and, consequently, the associated conditional R.N. Log-Laplace transform is :

$$\psi_t^{\mathbb{Q}}(u) = \psi_t(u + \alpha_t) - \psi_t(\alpha_t) \, . \tag{13}$$

Conversely, we get :

$$d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \exp\left[-\alpha_t' w_{t+1} + \psi_t(\alpha_t) \right] \tag{14}$$

and, taking $u = -\alpha_t$ in $\psi_t^{\mathbb{Q}}(u)$, we can write :

$$\psi_t^{\mathbb{Q}}(-\alpha_t) = -\psi_t(\alpha_t) \tag{15}$$

and, replacing u by $u - \alpha_t$, we obtain :

$$\psi_t(u) = \psi_t^{\mathbb{Q}}(u - \alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t). \tag{16}$$

We also have :

$$\begin{aligned} d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) &= \exp\left[-\alpha_t' w_{t+1} - \psi_t^{\mathbb{Q}}(-\alpha_t)\right], \\ d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) &= \exp\left[\alpha_t' w_{t+1} + \psi_t^{\mathbb{Q}}(-\alpha_t)\right]. \end{aligned} \tag{17}$$

3 Risk Premia and Market Price of Risk

3.1 Geometric and Arithmetic Risk Premia

Let p_t be the price at t of any given asset. The geometric return between t and $t+1$ is

$$\rho_{G,t+1} = \text{Log}\left(\frac{p_{t+1}}{p_t}\right),$$

whereas the arithmetic return is :

$$\rho_{A,t+1} = \frac{p_{t+1}}{p_t} - 1 = \exp(\rho_{G,t+1}) - 1\,.$$

In particular, for the risk-free asset we have :

$$\begin{aligned} \rho^f_{G,t+1} &= r_{t+1}\,, \\ \rho^f_{A,t+1} &= \exp(r_{t+1}) - 1 = r_{A,t+1}\,. \end{aligned}$$

So, we can define two risk premia of the given asset :

$$\begin{aligned} \pi_{Gt} &= E_t(\rho_{G,t+1}) - r_{t+1}\,, \\ \pi_{At} &= E_t(\rho_{A,t+1}) - r_{A,t+1} = E_t\left[\exp(\rho_{G,t+1})\right] - \exp(r_{t+1})\,. \end{aligned} \tag{18}$$

Note that the arithmetic risk premia have the advantage to satisfy $\pi_{At}(\lambda) = \Sigma_{j=1}^J \lambda_j \pi_{At,j}$, if $\pi_{At}(\lambda)$ is the risk premium of the portfolio defined by the shares in value λ_j for the asset j. Let us now consider two important particular cases in order to have more explicit forms of these risk premia and to obtain intuitive interpretations of the factor loading vector α_t [see also Dai, Le and Singleton (2006) for a similar analysis].

3.2 The factor is a vector of geometric returns

If w_{t+1} is a K-vector of geometric returns, we have vectors of risk premia π_{Gt} and π_{At} whose entries are :

$$\pi_{Gt,i} = e_i' \psi_t^{(1)}(0) - r_{t+1}, \quad i \in \{1, ..., K\},$$

(where $\psi_t^{(1)}$ is the gradient of ψ_t and e_i is the i^{th} column of the identity matrix I_K),

$$\pi_{At,i} = \varphi_t(e_i) - \exp(r_{t+1}), \quad i \in \{1, ..., K\}.$$

Moreover, we have the pricing identities :

$$1 = E_t \left\{ \exp \left[e_i' w_{t+1} + \alpha_t' w_{t+1} - r_{t+1} - \psi_t(\alpha_t) \right] \right\}, \quad i \in \{1, ..., K\}, \tag{19}$$

that is

$$\exp(r_{t+1}) = \frac{\varphi_t(\alpha_t + e_i)}{\varphi_t(\alpha_t)} = \varphi_t^{\mathbb{Q}}(e_i), \tag{20}$$

or

$$r_{t+1} = \psi_t(\alpha_t + e_i) - \psi_t(\alpha_t) = \psi_t^{\mathbb{Q}}(e_i). \tag{21}$$

So, for each $i \in \{1, ..., K\}$, the risk premia can be written :

$$\pi_{Gt,i} = e_i' \psi_t^{(1)}(0) - \psi_t(\alpha_t + e_i) + \psi_t(\alpha_t) \tag{22}$$

$$\pi_{At,i} = \varphi_t(e_i) - \frac{\varphi_t(\alpha_t + e_i)}{\varphi_t(\alpha_t)}. \tag{23}$$

Note that, for $\alpha_t = 0$, i.e. when the historical and the R.N. dynamics are identical, we have :

$$\pi_{Gt,i} = m_{it} - \psi_t(e_i) \neq 0, \quad i \in \{1, ..., K\},$$

(m_{it} denotes the conditional mean of $w_{i,t+1}$ given $\underline{w_t}$) and

$$\pi_{At,i} = 0, \qquad i \in \{1, ..., K\}.$$

So the arithmetic risk premia seem to have more natural properties. Moreover, considering first order expansions around $\alpha_t = 0$ and neglecting conditional cumulants of order strictly larger than 2 (which are zero in the conditionally gaussian case), we get [see Appendix 2] :

$$\pi_{Gt} \simeq -\frac{1}{2} vdiag(\Sigma_t) - \Sigma_t \alpha_t \tag{24}$$

$$\pi_{At} \simeq -diag \left[\varphi_t(e_i) \right] \Sigma_t \alpha_t \, . \tag{25}$$

where $[\varphi_t(e_i)] := (\varphi_t(e_1), \ldots, \varphi_t(e_K))'$, and $vdiag(\Sigma_t)$ is the vector whose entries are the diagonal terms of Σ_t, and Σ_t is the conditional variance-covariance matrix of w_{t+1} given $\underline{w_t}$. The terms $\varphi_t(e_i) = E_t \exp(w_{i,t+1})$ are obviously positive.

So, α_t can be viewed as the opposite of a market price of risk vector. We will see that the expression of π_{Gt} is exact in the conditionally Gaussian case [see Section 5.2].

3.3 The factor is a vector of yields

Let us denote by $r(t, h)$ the yield at t with residual maturity h; if $B(t, h)$ denotes the price at t of the zero coupon bond with time to maturity h, we have :

$$r(t, h) = -\frac{1}{h} \text{ Log } \left[B(t, h) \right] .$$

We assume that the components of w_{t+1} are :

$$w_{t+1,i} = h_i \, r(t+1, h_i) \, , \qquad i \in \{1, ..., K\} \, ,$$

where h_i are various integer residual maturities; this definition of $w_{t+1,i}$ leads to simpler notations than the equivalent definition $w_{t+1,i} = r(t+1, h_i)$. The payoffs $B(t+1, h_i) = \exp(-w_{t+1,i})$ have price at t equal to

$$B(t, h_i + 1) = \exp\left[-(h_i + 1) r(t, h_i + 1)\right].$$

So, we have

$$1 = E_t \left\{\exp\left[-w_{t+1,i} + (h_i + 1) r(t, h_i + 1) + \alpha_t' w_{t+1} - r_{t+1} - \psi_t(\alpha_t)\right]\right\}, \ i \in \{1, ..., K\}, \tag{26}$$

that is :

$$r_{t+1} = \psi_t(\alpha_t - e_i) - \psi_t(\alpha_t) + (h_i + 1) r(t, h_i + 1), \tag{27}$$

or :

$$\exp(r_{t+1}) = \frac{\varphi_t(\alpha_t - e_i)}{\varphi_t(\alpha_t)} \exp\left[(h_i + 1) r(t, h_i + 1)\right]. \tag{28}$$

The risk premia associated to the geometric returns :

$$Log\left[\frac{B(t+1, h_i)}{B(t, h_i + 1)}\right] = -w_{t+1,i} + (h_i + 1) r(t, h_i + 1) \tag{29}$$

are the vectors with components :

$$\begin{aligned} \pi_{Gt,i} &= -E_t(w_{t+1,i}) + (h_i + 1) r(t, h_i + 1) - r_{t+1} \\ &= -e_i' \psi_t^{(1)}(0) - \psi_t(\alpha_t - e_i) + \psi_t(\alpha_t), \end{aligned} \tag{30}$$

and :

$$\begin{aligned} \pi_{At,i} &= \exp\left[(h_i + 1) r(t, h_i + 1)\right] \varphi_t(-e_i) - \exp(r_{t+1}) \\ &= \exp\left[(h_i + 1) r(t, h_i + 1)\right] \left[\varphi_t(-e_i) - \tfrac{\varphi_t(\alpha_t - e_i)}{\varphi_t(\alpha_t)}\right]. \end{aligned} \tag{31}$$

Expanding relations (30) and (31) around $\alpha_t = 0$, and neglecting conditional cumulants of order strictly larger than 2, we get :

$$\pi_{Gt} \simeq -\frac{1}{2} \, vdiag(\Sigma_t) + \Sigma_t \alpha_t \tag{32}$$

$$\pi_{At} \simeq diag\left[\varphi_t(-e_i) \exp\left((h_i + 1) r(t, h_i + 1)\right)\right] \Sigma_t \alpha_t, \tag{33}$$

where Σ_t is the conditional variance-covariance matrix of w_{t+1} given $\underline{w_t}$. So, α_t can be viewed as a market price of risk vector. Moreover, the formula for π_{Gt} is exact in the conditionally gaussian case [see Appendix 2].

4 Econometric Asset Pricing Models (EAPMs)

The true value of the various mathematical tools introduced in Section 2, for instance ψ_t, $M_{t,t+1}$ or $\psi_t^{\mathbb{Q}}$, are unknown by the econometrician and, therefore, they have to be specified and parameterized. In other words, we have to specify an Econometric Asset Pricing Model (EAPM). We are going to present three ways of specifying an EAPM : the Direct Modelling, the R.N. Constrained Direct Modelling and the Back Modelling. In all approaches, we first need to make more precise the status of the short rate r_{t+1}.

4.1 The status of the short rate

The short rate r_{t+1} is a function of $\underline{w}_t$. This function may be known or unknown by the econometrician. It is known in two main cases :

i) r_{t+1} is exogenous, i.e. $r_{t+1}(\underline{w}_t)$ does not depend on $\underline{w}_t$, and, therefore, $r_{t+1}(.)$ is a known constant function of $\underline{w}_t$;

ii) r_{t+1} is an endogenous factor , i.e. r_{t+1} is a component of w_t.

If the function $r_{t+1}(\underline{w}_t)$ is unknown, it has to be specified parametrically. So we assume that the unknown function belongs to a family :

$$\left\{ r_{t+1}(\underline{w}_t, \tilde{\theta}), \tilde{\theta} \in \tilde{\Theta} \right\} , \tag{34}$$

where $r_{t+1}(.,.)$ is a known function.

4.2 Direct Modelling

In the Direct Modelling approach we first specify the historical dynamics, i.e. we choose a parametric family for the conditional Log-Laplace transform $\psi_t(u \,|\, \underline{w}_t)$:

$$\{\psi_t(u \,|\, \underline{w}_t, \theta_1), \theta_1 \in \Theta_1\} \ . \tag{35}$$

Then, we have to specify the SDF

$$\begin{aligned} M_{t,t+1} &= \exp\left[\alpha_t(\underline{w}_t)' w_{t+1} + \beta_t(\underline{w}_t)\right] \\ &= \exp\left[-r_{t+1}(\underline{w}_t) + \alpha_t'(\underline{w}_t) w_{t+1} - \psi_t(\alpha_t | \underline{w}_t)\right] \ . \end{aligned}$$

Once r_{t+1} has been specified, according to its status described in Section 4.1, as well as ψ_t, the remaining function to be specified is $\alpha_t(\underline{w}_t)$. We assume that $\alpha_t(\underline{w}_t)$ belongs to a parametric family :

$$\{\alpha_t(\underline{w}_t, \theta_2), \theta_2 \in \Theta_2\} \ . \tag{36}$$

Finally, $M_{t,t+1}$ is specified as :

$$M_{t,t+1}(\underline{w}_{t+1}, \theta) = \exp\left\{ -r_{t+1}(\underline{w}_t, \tilde{\theta}) + \alpha_t'(\underline{w}_t, \theta_2) w_{t+1} - \psi_t\left[\alpha_t(\underline{w}_t, \theta_2) | \underline{w}_t, \theta_1\right] \right\} , \tag{37}$$

where $\theta = (\tilde{\theta}', \theta_1', \theta_2')' \in \tilde{\Theta} \times \Theta_1 \times \Theta_2 = \Theta$; note that $\tilde{\Theta}$ may be reduced to one point.

This kind of modelling may have to satisfy some internal consistency conditions. Indeed, for any payoff $g(\underline{w}_s)$ delivered at $s > t$, that has a price $p(\underline{w}_t)$ at t which is a known function of $\underline{w}_t$, we must have :

$$p(\underline{w}_t) = E\left\{ M_{t,t+1}(\theta) ... M_{s-1,s}(\theta)\, g(\underline{w}_s) \,|\, \underline{w}_t, \theta_1 \right\} \quad \forall\ \underline{w}_t, \theta \, . \tag{38}$$

These AAO pricing conditions may imply strong constraints on the parameter θ, for instance when components of w_t are returns of some assets or interest rates with various maturities [see Sections 5 and 6].

The specification of the historical dynamics (35) and of the SDF (37) obviously implies the specification of the R.N. dynamics :

$$\psi_t^{\mathbb{Q}}(u | \underline{w}_t, \theta_1, \theta_2) = \psi_t\left[u + \alpha_t(\underline{w}_t, \theta_2) | \underline{w}_t, \theta_1\right] - \psi_t\left[\alpha_t(\underline{w}_t, \theta_2) | \underline{w}_t, \theta_1\right] \ . \tag{39}$$

4.3 R.N. Constrained Direct Modelling

In the previous kind of modelling, the family of R.N. dynamics $\psi_t^{\mathbb{Q}}(u|\underline{w}_t)$ is obtained as a by-product and therefore, in general, is not controlled.

In some cases it may be important to control the family of R.N. dynamics and, possibly, the specification of the short rate, if we want to have explicit or quasi-explicit formulas for the price of some derivatives. For instance, it is often convenient to impose that the R.N. dynamics be described by a Car (Compound Autoregressive) process. Indeed, a Car process is characterized by an exponential-affine multi-horizon (complex) Laplace transform and, consequently, multi-horizon pricing formulas for derivative products are easily derived and implemented using the Transform Analysis [see Duffie, Pan and Singleton (2000), and Gourieroux, Monfort and Polimenis (2003)]. If we want, at the same time, to control the historical dynamics, for instance to have good fitting when w_t is observable, the by-product of the modelling becomes the factor loading vector $\alpha_t(\underline{w}_t)$. More precisely, we may wish to choose a family $\{\psi_t(u|\underline{w}_t, \theta_1), \theta_1 \in \Theta_1\}$ and a family $\{\psi_t^{\mathbb{Q}}(u|\underline{w}_t, \theta^*), \theta^* \in \Theta^*\}$ such that, for any pair $(\psi_t^{\mathbb{Q}}, \psi_t)$ belonging to these families, there exists a unique function $\alpha_t(\underline{w}_t)$ denoted by $\alpha_t(\underline{w}_t, \theta_1, \theta^*)$ satisfying :

$$\psi_t^{\mathbb{Q}}(u|\underline{w}_t) = \psi_t\left[u + \alpha_t(\underline{w}_t)|\underline{w}_t\right] - \psi_t\left[\alpha_t(\underline{w}_t)|\underline{w}_t\right] .$$

In fact, this condition may be satisfied only for a subset of pairs (θ_1, θ^*). In other words (θ_1, θ^*) belongs to Θ_1^* strictly included in $\Theta_1 \times \Theta^*$, but such that any $\theta_1 \in \Theta_1$ and any $\theta^* \in \Theta^*$ can be reached [see Section 5]. Once the parameterization $(\tilde{\theta}, \theta_1, \theta^*) \in \tilde{\Theta} \times \Theta_1^*$ is defined, internal consistency conditions similar to (38) may be imposed.

4.4 Back Modelling

The final possibility is to parameterize, first the R.N. dynamics [and possibly $r_{t+1}(\underline{w}_t)$]

$$\psi_t^{\mathbb{Q}}(u|\underline{w}_t, \theta_1^*)$$

taking into account, if relevant, internal consistency conditions of the form :

$$p(\underline{w}_t) = E_t^{\mathbb{Q}}\left[\exp(-r_{t+1}(\underline{w}_t, \tilde{\theta}) - ... - r_s(\underline{w}_s, \tilde{\theta}))g(\underline{w}_s)|\underline{w}_t, \theta_1^*\right] , \ \forall \underline{w}_t\, , \tilde{\theta}\, , \theta_1^*. \tag{40}$$

Once this is done, the specification of $\alpha_t(\underline{w}_t)$ is chosen, without any constraint, providing the family $\{\alpha_t(\underline{w}_t, \theta_2^*), \theta_2^* \in \Theta_2^*\}$, and the historical dynamics is a by-product :

$$\psi_t(u|\underline{w}_t, \theta_1^*, \theta_2^*) = \psi_t^{\mathbb{Q}}\left[u - \alpha_t(\underline{w}_t, \theta_2^*)|\underline{w}_t, \theta_1^*\right] - \psi_t^{\mathbb{Q}}\left[-\alpha_t(\underline{w}_t, \theta_2^*)|\underline{w}_t, \theta_1^*\right] . \tag{41}$$

Also note that, if the R.N. conditional p.d.f. $f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t, \theta_1^*)$ is known in (quasi) closed form, the same is true for the historical conditional p.d.f.

$$f_t(w_{t+1}|\underline{w}_t, \theta_1^*, \theta_2^*) = f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t, \theta_1^*) \exp\left\{-\alpha_t'(\underline{w}_t, \theta_2^*)w_{t+1} - \psi_t^{\mathbb{Q}}\left[-\alpha_t(\underline{w}_t, \theta_2^*)|\underline{w}_t, \theta_1^*\right]\right\} . \tag{42}$$

In particular, if w_t is observable we can compute the likelihood function. However the identification of the parameters (θ_1^*, θ_2^*), from the dynamics of the observable components of w_t must be carefully studied (see examples in Sections 5 and 6) and observations of derivative prices may be necessary to reach identifiability.

4.5 Inference in an Econometric Asset Pricing Model

In order to estimate an EAPM, we assume that the econometrician observes, at dates $t \in \{0, \dots, T\}$, a set of prices corresponding to payoffs $g_i(\underline{w}_s), i \in \{1, ..., J_t\}$, given by (using the parameter notations of Direct Modelling) :

$$q_{ti}(\underline{w}_t, \theta) = E\left[g_i(\underline{w}_s) M_{t,s}(\underline{w}_s, \theta) | \underline{w}_t, \theta_1\right] \ , \ i \in \{1, ..., J_t\} \, . \tag{43}$$

Therefore, we have two kinds of equations representing respectively the historical dynamics of the factors and the observations:

$$w_t = \tilde{q}_t(\underline{w}_{t-1}, \varepsilon_{1t}, \theta_1) \, , \, \text{(say)} \tag{44}$$

$$x_t = q_t(\underline{w}_t, \theta) \, , \tag{45}$$

where the first equation is a rewriting of the conditional historical distribution of w_t given $\underline{w}_{t-1}$, and ε_{1t} is a white noise (which can be chosen Gaussian without loss of generality).

Note that, if r_{t+1} is not a known function of $\underline{w}_t$, we must have $r_{t+1} = r_{t+1}(\underline{w}_t, \tilde{\theta})$ among equations (45), and that if some components of w_t are observed they should appear in (45) without parameters.

System (44)-(45) is a nonlinear state space model and appropriate econometric methods may be used for inference in this system (in particular, Maximum Likelihood methods based on Kalman filter, Kitagawa-Hamilton filter, Simulations-based methods or Indirect Inference).

For given x_t's, equations (45) may have no solutions in w_t's and, in this case, an additional white noise is often introduced leading to

$$x_t = q_t(\underline{w}_t, \theta) + \varepsilon_{2t} \, . \tag{46}$$

Moreover, when w_t is (partially) observable, θ_1 may be identifiable from (44) and in this case a two step method is available : i) ML estimation of θ_1 from (44); ii) estimation of θ_2, and possibly $\tilde{\theta}$ by Nonlinear Least Square using (46) in which θ_1 is replaced by its ML estimator (and, possibly, unobserved components of w_t by their smoothed value).

5 Applications to Econometric Security Market Modelling

5.1 General Setting

In an Econometric Security Market Model we assume that the short rate r_{t+1} is exogenous and that the first K_1 components of w_t, denoted by y_t, are observable geometric returns of K_1 basic assets. The remaining $K_2 = K - K_1$ components of w_t, denoted by z_t, are factors not observed by the econometrician. Since the payoffs $\exp(y_{j,t+1})$ delivered at $t+1$, for each $j \in \{1, ..., K_1\}$, have a price at t which are known function of $\underline{w}_t$, namely 1, we have to guarantee internal consistency conditions. In the Direct Modelling approach, and in the Risk-Neutral Constrained Direct Modelling one, these conditions are [using the notation of the (unconstrained) direct approach] :

$$1 = E_t \left\{ \exp(y_{j,t+1} - r_{t+1} + \alpha_t(\underline{w}_t, \theta_2)' w_{t+1} - \psi_t \left[\alpha_t(\underline{w}_t, \theta_2) | \underline{w}_t, \theta_1\right] \right\} \ , \ j \in \{1, ..., K_1\} \tag{47}$$

or :

$$r_{t+1} = \psi_t\left[\alpha_t(\underline{w}_t, \theta_2) + e_j | \underline{w}_t, \theta_1\right] - \psi_t\left[\alpha_t(\underline{w}_t, \theta_2) | \underline{w}_t, \theta_1\right] \,,\; \forall\; \underline{w}_t, \theta_1, \theta_2\,;\, j \in \{1, ..., K_1\}\,. \tag{48}$$

In the Back Modelling approach, these conditions are :

$$r_{t+1} = \psi_t^{\mathbb{Q}}(e_j | \underline{w}_t, \theta_1^*)\,,\;\; \forall \underline{w}_t, \theta_1^*\,;\, j \in \{1, ..., K_1\}\,. \tag{49}$$

If we consider the case where the factor w_{t+1} is a R.N. Car(1) process (the generalization to the case of a Car(p) process is straightforward), with conditional R.N. Log-Laplace transform $\psi_t^{\mathbb{Q}}(u \,|\, \underline{w}_t) = a^{\mathbb{Q}}(u)' w_t + b_t^{\mathbb{Q}}(u)$, the internal consistency conditions (48) or (49) are given by (using the Back Modelling notation):

$$\left\{ \begin{array}{lcl} a^{\mathbb{Q}}(e_j, \theta_1^*) & = & 0\,, \\ & & \\ b_t^{\mathbb{Q}}(e_j, \theta_1^*) & = & r_{t+1}\,,\;\; \forall \theta_1^*\,;\, j \in \{1, ..., K_1\}\,. \end{array} \right. \tag{50}$$

5.2 Conditionally Gaussian Models

In this setting we assume that all the components of w_t are geometric returns ($K_1 = K$), that is, we consider $w_t = y_t$. In the Direct Modelling approach, we specify that the conditional distribution of y_{t+1}, given $\underline{y}_t$, is $N[m_t(\underline{y}_t, \theta_1), \Sigma_t(\underline{y}_t, \theta_1)]$ or, equivalently, that :

$$y_{t+1} = m_t(\underline{y}_t, \theta_1) + \Sigma_t^{1/2}(\underline{y}_t, \theta_1)\varepsilon_{t+1}\,,\;\; \varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0, I_K)\,,$$

$$\text{or} \qquad \psi_t(u | \underline{y}_t, \theta_1) = u' m_t(\underline{y}_t, \theta_1) + \frac{1}{2} u' \Sigma_t(\underline{y}_t, \theta_1) u\,.$$

For a given function $\alpha_t(\underline{w}_t)$, the internal consistency conditions are :

$$r_{t+1} = e_j' m_t(\underline{y}_t, \theta_1) + \frac{1}{2}(\alpha_t + e_j)' \Sigma_t(\underline{y}_t, \theta_1)(\alpha_t + e_j) - \frac{1}{2}\alpha_t' \Sigma_t(\underline{y}_t, \theta_1)\alpha_t \tag{51}$$

giving :

$$\alpha_t = \Sigma_t^{-1}(\underline{y}_t, \theta_1)\left[r_{t+1} e - m_t(\underline{y}_t, \theta_1) - \frac{1}{2}\; vdiag\; \Sigma_t(\underline{y}_t, \theta_1)\right]\,. \tag{52}$$

So, α_t is uniquely defined and no additional parameterization is needed. Moreover, the vector of geometric risk premia is $m_t - r_{t+1} e = -\frac{1}{2}\, vdiag\, \Sigma_t - \Sigma_t \alpha_t = \pi_{Gt}$ (where e denotes the K-dimensional unitary vector). The conditional R.N. distribution of y_{t+1}, given $\underline{y}_t$, is readily seen to be :

$$\begin{array}{ll} & N\left[r_{t+1} e - \frac{1}{2}\; vdiag\; \Sigma_t(\underline{y}_t, \theta_1)\,,\; \Sigma_t(\underline{y}_t, \theta_1)\right] \\ & \\ \text{or} & y_{t+1} = r_{t+1} e - \frac{1}{2}\; vdiag\; \Sigma_t(\underline{y}_t, \theta_1) + \Sigma_t^{1/2}(\underline{y}_t, \theta_1)\xi_{t+1}\,, \end{array} \tag{53}$$

where $\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0, I_K)$, and we get $\xi_{t+1} = \Sigma_t^{-1/2}(m_t - r_{t+1} e + \frac{1}{2}\; vdiag\; \Sigma_t) + \varepsilon_{t+1}$.

In the Back Modelling approach, we specify :

$$y_{t+1}|\,\underline{y}_t \overset{\mathbb{Q}}{\sim} N\left[m_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*), \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*)\right] ,$$

$$\text{or} \qquad \psi_t^{\mathbb{Q}}(u|\underline{y}_t, \theta_1^*) = u' m_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*) + \frac{1}{2} u' \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*) u \ .$$

The internal consistency conditions are (with obvious notations) :

$$r_{t+1} = m_{jt}^{\mathbb{Q}}(\underline{y}_t, \theta_1^*) + \frac{1}{2}\Sigma_{jj,t}^{\mathbb{Q}}\,,\ j \in \{1, \ldots, K\}\,, \tag{54}$$

and we find again that the conditional R.N. distribution is :

$$N\left[r_{t+1}e - \frac{1}{2}\ vdiag\ \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*), \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*)\right]$$

i.e.

$$\psi_t^{\mathbb{Q}}(u|\underline{y}_t, \theta_1^*) = u' r_{t+1} e - \frac{1}{2} u'\ vdiag\ \Sigma_t^{\mathbb{Q}} + \frac{1}{2} u' \Sigma_t^{\mathbb{Q}} u\,,$$

from which, choosing any $\alpha_t(\underline{y}_t, \theta_2^*)$, we deduce the historical dynamics :

$$\begin{aligned}\psi_t(u|\underline{y}_t, \theta_1^*, \theta_2^*) \quad &= \quad u'\left[r_{t+1}e - \tfrac{1}{2}\ vdiag\ \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*)\right. \\ &\qquad \left. - \ \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*)\alpha_t(\underline{y}_t, \theta_2^*)\right] + \tfrac{1}{2} u' \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*) u\,.\end{aligned}$$

In other words :

$$y_{t+1}\,|\,\underline{y}_t \overset{\mathbb{P}}{\sim} N\left[r_{t+1}e - \frac{1}{2}\ vdiag\ \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*) - \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*)\alpha_t(\underline{y}_t, \theta_2^*),\ \Sigma_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*)\right]\,. \tag{55}$$

Thus, for a given R.N. dynamics, we can reach any conditional historical mean of the factor, whereas the historical conditional variance-covariance matrix is the same as the R.N. one. Moreover θ_1^* and θ_2^* can be identified from the dynamics of y_t only.

This modelling generalizes the basic Black-Scholes framework to the multivariate case, with conditional mean and variance-covariance matrices depending on the past. Options with maturity equal to one have standard Black-Scholes prices whereas the price of options with larger maturities are easily obtained by simulation.

5.3 Direct Modelling of Switching Regime Models

The class of conditionally Mixed-Normal models contains many static, dynamic, parametric, semi-parametric or nonparametric models [see Bertholon, Monfort, Pegoraro (2006)]. Let us consider, for instance, the switching regime models. The factor w_t is equal to $(y_t, z_t')'$, where y_t is an observable geometric return and z_t is a J-state homogeneous Markov chain, valued in $(e_1, ..., e_J)$, and unobservable by the econometrician.

In the Direct Modelling approach, we first define the historical dynamics by :

$$y_{t+1} = \mu_t(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_{11}) + \sigma_t(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_{11})\varepsilon_{t+1}\,, \tag{56}$$

where $\varepsilon_{t+1} \,|\, \underline{\varepsilon}_t, \underline{z}_{t+1} \overset{\mathbb{P}}{\sim} N(0,1)$ and $\mathbb{P}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{P}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}$ [the free parameters π_{ij} are denoted by θ_{12} and thus, following the notation of Section 4.2, we have $\theta_1 = (\theta'_{11}, \theta'_{12})'$].

So, we assume that, conditionally to the past $\underline{w}_t$, y_{t+1} follows a mixture of J Gaussian distributions. The conditional Laplace transform is:

$$\begin{aligned} \varphi_t(u,v) &= E_t \exp(u y_{t+1} + v' z_{t+1}) \\ &= \Lambda(u, v, \underline{y}_t, z_t, \theta_1)' z_t = \textstyle\sum_{i=1}^J \Lambda_i(u, v, \underline{y}_t, e_i, \theta_1) \mathbb{1}_{e_i}(z_t)\,, \end{aligned} \tag{57}$$

where

$$\Lambda_i(u, v, \underline{y}_t, e_i, \theta_1) = \sum_{j=1}^J \pi_{ij} \exp\left[v' e_j + u \mu_t(\underline{y}_t, e_i, e_j, \theta_{11}) + \frac{1}{2} u^2 \sigma_t^2(\underline{y}_t, e_i, e_j, \theta_{11}) \right] ,$$

and the Log-Laplace transform is, therefore, given by :

$$\psi_t(u,v) = \text{Log}(\Lambda(u, v, \underline{y}_t, z_t, \theta_1)' z_t) = \sum_{i=1}^J \text{Log}\left(\Lambda_i(u, v, \underline{y}_t, e_i, \theta_1)\right) \mathbb{1}_{e_i}(z_t)\,. \tag{58}$$

The SDF is specified as :

$$M_{t,t+1} = \exp\left[-r_{t+1} + \gamma_t(\underline{w}_t, \theta_2)\, y_{t+1} + \delta_t(\underline{w}_t, \theta_2)' z_{t+1} - \psi_t(\gamma_t, \delta_t) \right] . \tag{59}$$

It is easily seen that, if δ_t is replaced by $\delta_t + \eta_t e$ (where η_t is a scalar function), $M_{t,t+1}$ is unchanged; therefore we can impose, for instance, $\delta_{Jt} = 0$.

The internal consistency condition is:

$$\varphi_t(\gamma_t + 1, \delta_t) = \exp(r_{t+1}) \varphi_t(\gamma_t, \delta_t)\,. \tag{60}$$

For a given δ_t, this equation has a unique solution in γ_t; so $\gamma_t(\underline{w}_t, \theta_2)$ can be written $\gamma_t\left[\delta_t(\underline{w}_t, \theta_2)\right]$ and we only have to specify $\delta_{1,t}(\underline{w}_t, \theta_2), ..., \delta_{J-1,t}(\underline{w}_t, \theta_2)$.

The R.N. dynamics is defined by

$$\varphi_t^{\mathbb{Q}}(u,v) = \frac{\varphi_t(u + \gamma_t, v + \delta_t)}{\varphi_t(\gamma_t, \delta_t)}$$

and we find :

$$\varphi_t^{\mathbb{Q}}(u,v) = \Lambda^{\mathbb{Q}}(u, v, \underline{y}_t, z_t, \theta_1, \theta_2)' z_t$$

where the i^{th} component of the J-dimensional vector $\Lambda^{\mathbb{Q}}(u, v, \underline{y}_t, z_t, \theta_1, \theta_2)$ is :

$$\Lambda_i^{\mathbb{Q}}(u, v, \underline{y}_t, e_i, \theta_1, \theta_2) = \sum_{j=1}^J \pi^*_{ij,t} \exp\left\{ v' e_j + u\left[\mu_t(\underline{y}_t, e_i, e_j, \theta_{11}) + \gamma_t \sigma_t^2(\underline{y}_t, e_i, e_j, \theta_{11}) \right] + \frac{1}{2} u^2 \sigma_t^2(\underline{y}_t, e_i, e_j, \theta_{11}) \right\} ,$$

with

$$\pi^*_{ij,t} = \frac{\pi_{ij} \exp\left[\delta'_t e_j + \gamma_t \mu_t(\underline{y}_t, e_i, e_j; \theta_{11}) + \frac{1}{2} \gamma_t^2 \sigma_t^2(\underline{y}_t, e_i, e_j; \theta_{11}) \right]}{\sum_{j=1}^J \pi_{ij} \exp\left[\delta'_t e_j + \gamma_t \mu_t(\underline{y}_t, e_i, e_j; \theta_{11}) + \frac{1}{2} \gamma_t^2 \sigma_t^2(\underline{y}_t, e_i, e_j; \theta_{11}) \right]}\,. \tag{61}$$

In other words, the R.N. dynamics is defined by :

$$y_{t+1} = \mu_t(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_{11}) + \gamma_t \left[\delta_t(\underline{w}_t, \theta_2)\right] \sigma_t^2(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_{11}) + \sigma_t(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_{11})\xi_{t+1} \qquad (62)$$

where

$$\begin{aligned} &\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1)\,, \\ &\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \pi^*_{ij,t}\,. \end{aligned} \qquad (63)$$

In particular, we get :

$$\varepsilon_{t+1} = \gamma_t \left[\delta_2(\underline{w}_t, \theta_2)\right] \sigma_t(\underline{y}_t, \underline{z}_t, z_{t+1}, \theta_{11}) + \xi_{t+1}\,. \qquad (64)$$

5.4 Back Modelling of Switching Regime Models

The Direct Modelling has two main drawbacks. First, equation (60) must be solved numerically in γ_t, for any t. Second, the R.N. dynamics is not Car in general, and the pricing of derivatives needs simulations which, in turn, imply to solve (60) for any t and any path.

Let us consider now the Back Modelling approach, starting from a Car R.N. dynamics defined by :

$$y_{t+1} = \nu_t + \rho y_t + \nu_1' z_t + \nu_2' z_{t+1} + (\nu_3' z_{t+1})\xi_{t+1}\,, \qquad (65)$$

where ν_t is a deterministic function of t and where:

$$\begin{aligned} &\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1) \\ &\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j \,|\, z_t = e_i) = \pi^*_{ij}\,. \end{aligned} \qquad (66)$$

In other words, z_t is an exogenous Markov chain in the risk-neutral world. The conditional R.N. Laplace transform is given by :

$$\begin{aligned} \varphi_t^{\mathbb{Q}}(u,\nu) &= E_t^{\mathbb{Q}} \exp(u y_{t+1} + v' z_{t+1}) \\ &= \exp\left[u(\nu_t + \rho y_t + \nu_1' z_t)\right] E_t^{\mathbb{Q}} \exp\left[\left(u\nu_2 + \tfrac{1}{2}u^2\nu_3^2 + v\right)' z_{t+1}\right]\,, \end{aligned}$$

[ν_3^2 is the vector containing the square of the components in ν_3] and we get :

$$\begin{aligned} \psi_t^{\mathbb{Q}}(u,v) &= \operatorname{Log} \varphi_t^{\mathbb{Q}}(u,v) \\ &= u(\nu_t + \rho y_t + \nu_1' z_t) + \Lambda'(u, v, \nu_2, \nu_3, \pi^*) z_t\,, \end{aligned}$$

where the i^{th} component of $\Lambda(u, v, \nu_2, \nu_3, \pi^*)$ is :

$$\Lambda_i(u, v, \nu_2, \nu_3, \pi^*) = \operatorname{Log} \sum_{j=1}^{J} \pi^*_{ij} \exp\left(u\nu_{2j} + \tfrac{1}{2}u^2\nu_{3j}^2 + v_j\right)\,.$$

So, as announced, the joint R.N. dynamics of the process $(y_t, z_t')'$ is Car since :

$$\psi_t^{\mathbb{Q}}(u,v) = a^{\mathbb{Q}}(u,v)' w_t + b_t^{\mathbb{Q}}(u,v)$$

with

$$
\begin{aligned}
a^{\mathbb{Q}}(u,v)' &= [u\rho, u\nu_1' + \Lambda'(u,v,\nu_2,\nu_3,\pi^*)]\,, \\
b_t^{\mathbb{Q}}(u,v) &= u\nu_t\,.
\end{aligned}
\tag{67}
$$

The internal consistency condition is :

$$
\psi_t^{\mathbb{Q}}(1,0) = r_{t+1}
$$

that is :

$$
-r_{t+1} + \nu_t + \rho y_t + \nu_1' z_t + \lambda'(\nu_2,\nu_3,\pi^*) z_t = 0 \qquad \forall\, y_t,\, z_t\,,
\tag{68}
$$

and where the i^{th} component of $\lambda(\nu_2,\nu_3,\pi^*)$ is

$$
\lambda_i(\nu_2,\nu_3,\pi^*) = \text{ Log } \sum_{j=1}^{J} \pi_{ij}^* \exp\left(\nu_{2j} + \frac{1}{2}\nu_{3j}^2\right)\,.
$$

Condition (68) implies, since r_{t+1} and ν_t are deterministic functions of time :

$$
\left\{
\begin{array}{l}
\rho = 0\,, \\
\nu_1 = -\lambda(\nu_2,\nu_3,\pi^*)\,, \\
\nu_t = r_{t+1}\,.
\end{array}
\right.
\tag{69}
$$

Finally, the R.N. dynamics compatible with the AAO conditions is :

$$
y_{t+1} = r_{t+1} - \lambda'(\nu_2,\nu_3,\pi^*) z_t + \nu_2' z_{t+1} + (\nu_3' z_{t+1})\xi_{t+1}\,,
\tag{70}
$$

where

$$
\begin{aligned}
&\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1) \\
&\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j \,|\, z_t = e_i) = \pi_{ij}^*\,.
\end{aligned}
\tag{71}
$$

Note that, if ν_2 is replaced by $\nu_2 + c\,e$, $\nu_2' z_{t+1}$ is replaced by $\nu_2' z_{t+1} + c$ and $-\lambda' z_t$ by $-\lambda' z_t - c$, so the RHS of (70) is unchanged and therefore we can impose, for instance, $\nu_{2J} = 0$.

The historical dynamics can then be deduced by specifying $\gamma_t(\underline{w}_t, \theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$ without any constraints (and assuming, for instance, $\delta_{Jt} = 0$) and we get the Log-Laplace transform :

$$
\psi_t(u,v) = \psi_t^{\mathbb{Q}}(u - \gamma_t, v - \delta_t) - \psi_t^{\mathbb{Q}}(-\gamma_t, -\delta_t)\,,
$$

where

$$
\psi_t^{\mathbb{Q}}(u,v) = u(r_{t+1} - \lambda' z_t) + \Lambda'(u,v) z_t\,.
$$

We get

$$
\psi_t(u,v) = u(r_{t+1} - \lambda' z_t) + [\Lambda(u - \gamma_t, v - \delta_t) - \Lambda(-\gamma_t, -\delta_t)]' \, z_t\,,
$$

where

$$\Lambda_i(u-\gamma_t, v-\delta_t) - \Lambda_i(-\gamma_t, -\delta_t) =$$

$$= \text{Log}\, \frac{\sum_{j=1}^{J} \pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_t^2 \nu_{3j}^2 - \delta_{jt}\right) \exp\left[u(\nu_{2j} - \gamma_t \nu_{3j}^2) + \frac{1}{2}u^2 \nu_{3j}^2 + v_j\right]}{\sum_{j=1}^{J} \pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_j^2 \nu_{3j}^2 - \delta_{jt}\right)}$$

$$= \text{Log}\, \sum_{j=1}^{J} \pi_{ij,t} \exp\left[u(\nu_{2j} - \gamma_t \nu_{3j}^2) + \frac{1}{2}u^2 \nu_{3j}^2 + v_j\right]$$

with

$$\pi_{ij,t} = \frac{\pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_t^2 \nu_{3j}^2 - \delta_{jt}\right)}{\sum_{j=1}^{J} \pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_t^2 \nu_{3j}^2 - \delta_{jt}\right)}.$$

Therefore, the historical dynamics is :

$$y_{t+1} = r_{t+1} - \lambda'(\nu_2, \nu_3, \pi^*) z_t + (\nu_2 - \gamma_t \nu_3^2)' z_{t+1} + (\nu_3' z_{t+1}) \varepsilon_{t+1} \tag{72}$$

where

$$\varepsilon_{t+1} \,|\, \underline{\varepsilon_t}, \underline{z_{t+1}} \overset{\mathbb{P}}{\sim} N(0,1)$$

$$\mathbb{P}(z_{t+1} = e_j \,|\, \underline{y_t}, \underline{z_{t-1}}, z_t = e_i) = \pi_{ij,t}$$

$$\lambda_i(\nu_2, \nu_3, \pi^*) = \text{Log} \sum_{j=1}^{J} \pi_{ij}^* \exp\left(\nu_{2j} + \frac{1}{2}\nu_{3j}^2\right),$$

and

$$\varepsilon_{t+1} = \xi_{t+1} + \gamma_t\, (\nu_3' z_{t+1})\,. \tag{73}$$

Conditionally to $\underline{w_t}$, the historical distribution of y_{t+1} is a mixture of J Gaussian distributions with means $(r_{t+1} - \lambda' z_t + \nu_{2j} - \gamma_t \nu_{3j}^2)$ and variances ν_{3j}^2, and with weights given by $\pi_{ij,t}$, $j \in \{1, \ldots, J\}$, when $z_t = e_i$.

Since γ_t and δ_t are arbitrary functions of $\underline{w_t}$ (assuming, for instance, $\delta_{Jt} = 0$), we obtain a large class of historical switching regime dynamics which can be matched with a Car switching regime R.N. dynamics.

As mentioned in Section 4.4, the identification problem must be discussed. Let us consider the case where γ and δ are constant. In this case, the parameters π_{ij} are constant and the identifiable parameters are the π_{ij}, ν_3, the vector of the J coefficients of z_{t+1} in (72), and $(J-1)$ coefficients of z_t [assuming, for instance, $\lambda_J = 0$], i.e. $J(J-1) + 3J - 1 = J(J+2) - 1$ parameters, whereas the parameters to be estimated are the π_{ij}^*, ν_2 (with $\nu_{2J} = 0$), ν_3, γ, δ (with $\delta_J = 0$) i.e. $J(J+2) - 1$ parameters also. So all the parameters might be estimated from the observations of the $y_t's$.

5.5 Back Modelling of Stochastic Volatility Models

We focus on the Back Modelling, starting from a Car representation of the R.N. dynamics of the factor $w_t = (y_t, \sigma_t^2)$, where y_t is an observable geometric return, whereas σ_t^2 is an unobservable stochastic variance. More precisely the R.N. dynamics is assumed to satisfy :

$$y_{t+1} = \lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2 + (\lambda_3 \sigma_t)\xi_{t+1}\,, \tag{74}$$

where λ_t is a deterministic function of t and

$$\begin{aligned} \xi_{t+1} \,|\, \underline{\xi_t}, \underline{\sigma_{t+1}^2} &\overset{\mathbb{Q}}{\sim} N(0,1) \\ \sigma_{t+1}^2 \,|\, \underline{\xi_t},\, \underline{\sigma_t^2} &\overset{\mathbb{Q}}{\sim} ARG(1,\nu,\rho) \end{aligned} \tag{75}$$

and where the conditional $ARG(1,\nu,\rho)$ distribution [characterizing an Autoregressive Gamma process of order one (ARG(1)) with unit scale parameter[7]] is defined by the affine conditional R.N. Log-Laplace transform :

$$\psi_t^{\mathbb{Q}}(v) = a^{\mathbb{Q}}(v)\sigma_t^2 + b^{\mathbb{Q}}(v)\,,$$

where $a^{\mathbb{Q}}(v) = \frac{\rho v}{1-v}$, $b^{\mathbb{Q}}(v) = -\nu \text{Log}(1-v)$, $v < 1$, $\rho > 0$, $\nu > 0$. The conditional R.N. Log-Laplace transform of $(y_{t+1}, \sigma_{t+1}^2)$ is :

$$\psi_t^{\mathbb{Q}}(u,v) = (\lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2)u + \frac{1}{2}\lambda_3^2 \sigma_t^2 u^2 + a^{\mathbb{Q}}(v)\sigma_t^2 + b^{\mathbb{Q}}(v)\,. \tag{76}$$

In this modelling there is no instantaneous causality between y_{t+1} and σ_{t+1}^2, but it is straightforward to extend the results to this case.

The internal consistency condition is :

$$\psi_t^{\mathbb{Q}}(1,0) = r_{t+1}$$

or

$$r_{t+1} = \lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2 + \frac{1}{2}\lambda_3^2 \sigma_t^2\,,$$

which implies :

$$\lambda_t = r_{t+1},\ \lambda_1 = 0,\ \lambda_2 = -\frac{1}{2}\lambda_3^2\,.$$

So, the R.N. dynamics compatible with the AAO restriction is given by (75) and :

$$y_{t+1} = r_{t+1} - \frac{1}{2}\lambda_3^2 \sigma_t^2 + \lambda_3 \sigma_t \xi_{t+1}\,, \tag{77}$$

that is

$$\psi_t^{\mathbb{Q}}(u,v) = \left(r_{t+1} - \frac{1}{2}\lambda_3^2 \sigma_t^2\right) u + \frac{1}{2}\lambda_3^2 \sigma_t^2 u^2 + a^{\mathbb{Q}}(v)\sigma_t^2 + b^{\mathbb{Q}}(v)\,. \tag{78}$$

[7]See Darolles, Gourieroux and Jasiak (2006), Gourieroux and Jasiak (2006) and Monfort and Pegoraro (2006b) for a presentation of single regime and regime-switching (scalar and vector) Autoregressive Gamma processes.

The historical dynamics is defined by specifying $\gamma_t(\underline{w}_t\theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$, and we get :

$$\begin{aligned}\psi_t(u,v) &= \psi_t^{\mathbb{Q}}(u-\gamma_t, v-\delta_t) - \psi_t^{\mathbb{Q}}(-\gamma_t, -\delta_t) \\ &= \left(r_{t+1} - \tfrac{1}{2}\lambda_3^2\sigma_t^2\right)u - \lambda_3^2\sigma_t^2\gamma_t u + \tfrac{1}{2}\lambda_3^2\sigma_t^2 u^2 \\ &\qquad + \left[a^{\mathbb{Q}}(v-\delta_t) - a^{\mathbb{Q}}(-\delta_t)\right]\sigma_t^2 + b^{\mathbb{Q}}(v-\delta_t) - b^{\mathbb{Q}}(-\delta_t) \\ &= \left(r_{t+1} - \tfrac{1}{2}\lambda_3^2\sigma_t^2 - \lambda_3^2\sigma_t^2\gamma_t\right)u + \tfrac{1}{2}\lambda_3^2\sigma_t^2 u^2 + a_t(v)\sigma_t^2 + b_t(v)\,,\end{aligned}$$

with

$$\begin{aligned}a_t(v) &= \frac{\rho_t v}{1-v\mu_t}\,,\ b_t(v) = -\nu \mathrm{Log}(1-v\mu_t)\,, \\ \rho_t &= \frac{\rho}{(1+\delta_t)^2}\,,\ \mu_t = \frac{1}{1+\delta_t}\,.\end{aligned}$$

So, the only conditions, when we define the historical dynamics, are $\mu_t > 0$, i.e. $\delta_t > -1$, and $v < 1/\mu_t$. The historical dynamics can be written:

$$y_{t+1} = r_{t+1} - \frac{1}{2}\lambda_3^2\sigma_t^2 - \lambda_3^2\sigma_t^2\gamma_t + \lambda_3\sigma_t\varepsilon_{t+1} \tag{79}$$

where

$$\begin{aligned}\varepsilon_{t+1}\,|\,\underline{\varepsilon_t}, \underline{\sigma_{t+1}^2} &\overset{\mathbb{P}}{\sim} N(0,1) \\ \sigma_{t+1}^2\,|\,\underline{\varepsilon_t}, \underline{\sigma_t^2} &\overset{\mathbb{P}}{\sim} ARG(\mu_t, \nu, \rho_t)\,.\end{aligned} \tag{80}$$

Note that, the conditional historical distribution of σ_{t+1}^2, given $(\underline{y_t}, \underline{\sigma_t^2})$, is given by the Log-Laplace Transform

$$\psi_t(v) = \frac{\rho_t v}{1-v\mu_t}\,\sigma_t^2 - \nu\mathrm{Log}(1-v\mu_t)$$

which is not affine in σ_t^2, except in the case where δ_t is constant (or a deterministic function of t). Moreover we have :

$$\varepsilon_{t+1} = \xi_{t+1} + (\lambda_3\sigma_t)\gamma_t\,. \tag{81}$$

If γ_t and δ_t are constant, the identifiable parameters are the coefficients of σ_t^2 and $\sigma_t\varepsilon_{t+1}$ in (79) as well as the two parameters of the ARG dynamics (with unit scale). So, we have four identifiable parameters. The parameters to be estimated are λ_3, ν, ρ, γ, δ, i.e. five parameters. So these parameters are not identifiable from the dynamics of the $y_t's$. Observations of derivative prices must be added.

5.6 Back Modelling of Switching GARCH Models with leverage effect: a first application of Extended Car Processes

In this section, following a Back Modelling approach, we consider specifications generalizing those proposed by Heston and Nandi (2000) [see also Elliot, Siu and Chan (2006)].

In particular, like in Section 5.4, we assume $w_t = (y_t, z_t')'$, where y_t is an observable geometric return and z_t an unobservable J-state homogeneous Markov chain valued in $\{e_1, ..., e_J\}$. The new feature is the introduction of a GARCH effect (with leverage). More precisely, the R.N. dynamics is assumed to be of the following type :

$$y_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \nu_3' z_{t+1} + \nu_4 \sigma_{t+1}^2 + \sigma_{t+1}\xi_{t+1} \tag{82}$$

where ν_t is a deterministic function of t and

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma_t^2 \,,$$

and

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^* \,.$$

Note that σ_{t+1}^2 is a deterministic function of $(\underline{\xi}_t, \underline{z}_t)$, and therefore of $\underline{w}_t = (\underline{y}_t, \underline{z}_t)$. Also note that, following Heston and Nandi (2000), in this switching GARCH(1,1) model, ξ_t replaces the usual term $\sigma_t\ \xi_t$ in the R.H.S. of the equation giving σ_{t+1}^2 and the term $\alpha_2\sigma_t$ captures an asymmetric or "leverage" effect.

It is easily seen that the R.N. conditional Log-Laplace transform of (y_{t+1}, z_{t+1}) is :

$$\begin{aligned} \psi_t^{\mathbb{Q}}(u,v) &= \operatorname{Log} E_t^{\mathbb{Q}} \exp(u y_{t+1} + v' z_{t+1}) \\ &= (\nu_t + \nu_1 y_t + \nu_2' z_t + \nu_4 \sigma_{t+1}^2) u + \tfrac{1}{2}\sigma_{t+1}^2 u^2 + \Lambda'(u, v, \nu_3, \pi^*) z_t \,, \end{aligned} \tag{83}$$

where the i^{th} component of $\Lambda(u, v, \nu_3, \pi^*)$ is :

$$\Lambda_i(u, v, \nu_3, \pi^*) = \operatorname{Log} \sum_{j=1}^{J} \pi_{ij}^* \exp(u\nu_{3j} + v_j) \,. \tag{84}$$

The internal consistency condition, or AAO constraint, is :

$$\psi_t^{\mathbb{Q}}(1,0) = r_{t+1} \quad \forall \underline{w}_t \,,$$

implying

$$r_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \nu_4 \sigma_{t+1}^2 + \frac{1}{2}\sigma_{t+1}^2 + \lambda'(\nu_3, \pi^*) z_t \,,$$

where the i^{th} component of $\lambda(\nu_3, \pi^*)$ is given by:

$$\lambda_i(\nu_3, \pi^*) = \operatorname{Log} \sum_{j=1}^{J} \pi_{ij}^* \exp(\nu_{3j}) \tag{85}$$

and, therefore, the arbitrage restriction implies:

$$\begin{cases} \nu_1 = 0 \,, \\ \nu_2 = -\lambda(\nu_3, \pi^*) \,, \\ \nu_4 = -\frac{1}{2} \,, \\ \nu_t = r_{t+1} \,. \end{cases}$$

Thus, equation (82) becomes:

$$y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t - \frac{1}{2}\sigma_{t+1}^2 + \nu_3' z_{t+1} + \sigma_{t+1}\xi_{t+1} \tag{86}$$

with

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma_t^2$$

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*\,,$$

(again, we can take $\nu_{3J} = 0$) which gives the R.N. dynamics compatible with the AAO restriction. The corresponding Log-Laplace transform is :

$$\psi_t^{\mathbb{Q}}(u,v) = \left(r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma_{t+1}^2\right) u + \frac{1}{2}\sigma_{t+1}^2 u^2 + \Lambda'(u, v, \nu_3, \pi^*) z_t \tag{87}$$

The historical dynamics is obtained by specifying $\gamma_t(\underline{w}_t, \theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$, with, for instance $\delta_{Jt} = 0$, and in particular we have :

$$\psi_t(u,v) = \psi_t^{\mathbb{Q}}(u - \gamma_t, v - \delta_t) - \psi_t^{\mathbb{Q}}(-\gamma_t, -\delta_t)\,.$$

We obtain :

$$\begin{aligned} \psi_t(u,v) \;&=\; \left(r_{t+1} - \lambda' z_t - \tfrac{1}{2}\sigma_{t+1}^2 - \gamma_t\sigma_{t+1}^2\right) u + \tfrac{1}{2}\sigma_{t+1}^2 u^2 \\ &\quad + \left[\Lambda(u - \gamma_t, v - \delta_t, \nu_3, \pi^*) - \Lambda(-\gamma_t, -\delta_t, \nu_3, \pi^*)\right]' z_t \end{aligned} \tag{88}$$

where

$$\begin{aligned} &\Lambda_i(u - \gamma_t, v - \delta_t, \nu_3, \pi^*) - \Lambda_i(-\gamma_t, -\delta_t, \nu_3, \pi^*) = \mathrm{Log}\Sigma_{j=1}^J \pi_{ij,t} \exp(u\nu_{3j} + v_j) \\ &\text{with} \quad \pi_{ij,t} = \frac{\pi_{ij}^* \exp(-\gamma_t\nu_{3j} - \delta_{jt})}{\Sigma_{j=1}^J \pi_{ij}^* \exp(-\gamma_t\nu_{3j} - \delta_{jt})}\,. \end{aligned} \tag{89}$$

So the non-affine historical dynamics is given by :

$$\begin{aligned} &y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t - \tfrac{1}{2}\sigma_{t+1}^2 - \gamma_t(\underline{w}_t, \theta_2^*)\sigma_{t+1}^2 + \nu_3' z_{t+1} + \sigma_{t+1}\varepsilon_{t+1} \\ &\varepsilon_{t+1} \,|\, \underline{\varepsilon}_t, \underline{z}_{t+1} \overset{\mathbb{P}}{\sim} N(0,1)\,, \end{aligned} \tag{90}$$

with

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma_t^2$$

$$\mathbb{P}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \pi_{ij,t}\,.$$

Comparing (86) and (90) we get :

$$\xi_{t+1} = \varepsilon_{t+1} - \gamma_t\sigma_{t+1}\,,$$

and, therefore, the equation giving σ_{t+1}^2 can be rewritten :

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1\left[\varepsilon_t - (\alpha_2 + \gamma_t)\sigma_t\right]^2 + \alpha_3\sigma_t^2\,. \tag{91}$$

One may observe, from (87), that $w_{t+1} = (y_{t+1}, z'_{t+1})$ does not have a Car R.N. dynamics. So, the pricing seems a priori difficult. Fortunately, it can be shown [see Appendix 3] that the (extended) factor $w^e_{t+1} := (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is R.N. Car, and therefore the pricing methods based on Car dynamics apply. In particular, the R.N. conditional Log-Laplace transform of w^e_{t+1}, given $\underline{w^e_t}$, is :

$$\psi^{\mathbb{Q}}_t(u,v,\tilde{v}) = a^{\mathbb{Q}}_1(u,v,\tilde{v})' z_t + a^{\mathbb{Q}}_2(u,\tilde{v})\sigma^2_{t+1} + b^{\mathbb{Q}}_t(u,\tilde{v})\,, \tag{92}$$

where

$$\begin{aligned}
a^{\mathbb{Q}}_1(u,v,\tilde{v}) &= \tilde{\Lambda}(u,v,\tilde{v},\nu_3,\omega,\pi^*) - \lambda(\nu_3,\pi^*)\,u \\
\text{with} \qquad & \tilde{\Lambda}_i(u,v,\tilde{v},\nu_3,\omega,\pi^*) = \mathrm{Log}\sum_{j=1}^{J}\pi^*_{ij}\exp(u\nu_{3j}+v_j+\tilde{v}\omega_j)\,,\ i\in\{1,\ldots,J\}\,, \\
a^{\mathbb{Q}}_2(u,\tilde{v}) &= -\frac{1}{2}u + \tilde{v}(\alpha_1\alpha_2^2+\alpha_3) + \frac{(u-2\alpha_1\alpha_2\tilde{v})^2}{2(1-2\alpha_1\tilde{v})} \\
b^{\mathbb{Q}}_t(u,\tilde{v}) &= ur_{t+1} - \frac{1}{2}\mathrm{Log}(1-2\alpha_1\tilde{v})\,,
\end{aligned}$$

which is affine in $(z'_t, \sigma^2_{t+1})'$, with an intercept deterministic function of time.

Finally, let us consider the identification problem from the historical dynamics when functions γ and δ are constant. In this case, we can identify from (90) J coefficients of z_{t+1}, $(J-1)$ coefficients of z_t, the coefficient of σ^2_{t+1}, ω, α_1, $(\alpha_2+\gamma)$, α_3, and π_{ij}, i.e. $3J+3+J(J-1) = J(J+2)+3$ parameters. The parameters to be estimated are ν_3 (with $\nu_{3J}=0$), $\omega, \alpha_1, \alpha_2, \alpha_3$, π^*_{ij}, γ, δ (with $\delta_J = 0$), that is, $2(J-1)+J+4+J(J-1) = J(J+2)+2$ parameters. Therefore, the historical model is over identified.

5.7 Back Modelling of Switching IG GARCH Models : a second application of Extended Car Processes

The purpose of this section is to introduce, following the Back Modelling approach, several generalizations of the Inverse Gaussian[8] (IG) GARCH model proposed by Christoffersen, Heston and Jacobs (2006). First, we consider switching regimes in the (historical and risk-neutral) dynamics of the geometric return y_t and in the GARCH variance σ^2_{t+1}. Second, we price not only the factor risk but also the regime-shift risk and, third, risk correction coefficients are in general time-varying. The factor is given by $w_t = (y_t, z'_t)'$, where z_t is the unobservable J-state homogeneous Markov chain valued in $\{e_1, ..., e_J\}$. The R.N. dynamics is given by:

$$y_{t+1} = \nu_t + \nu_1 y_t + \nu'_2 z_t + \nu'_3 z_{t+1} + \nu_4\sigma^2_{t+1} + \eta\xi_{t+1} \tag{93}$$

[8]The strictly positive random variable y has an Inverse Gaussian distribution with parameter $\delta > 0$ [denoted IG(δ)] if and only if its distribution function is given by $F(y;\delta) = \int_0^y \frac{\delta}{\sqrt{2\pi z^3}} e^{-(\sqrt{z}-\delta/\sqrt{z})^2/2} dz$. The generalized Laplace transform is $E[\exp(\varphi y + \theta/y)] = \frac{\delta}{\sqrt{\delta^2-2\theta}}\exp\left(\delta - \sqrt{(\delta^2-2\theta)(1-2\varphi)}\right)$ and $E(y) = V(y) = \delta$ [see Christoffersen, Heston and Jacobs (2006) for further details].

where ν_t is a deterministic function of t and

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} IG\left(\frac{\sigma^2_{t+1}}{\eta^2}\right)$$

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1 \sigma^2_t + \alpha_2 \xi_t + \alpha_3 \frac{\sigma^4_t}{\xi_t}\,,$$

with

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi^*_{ij}\,.$$

The R.N. conditional Log-Laplace transform of (y_{t+1}, z_{t+1}) is :

$$\begin{aligned} \psi^{\mathbb{Q}}_t(u,v) &= \operatorname{Log} E^{\mathbb{Q}}_t \exp(u y_{t+1} + v' z_{t+1}) \\ &= (\nu_t + \nu_1 y_t + \nu'_2 z_t + \nu_4 \sigma^2_{t+1}) u + \Lambda'(u,v,\nu_3,\pi^*) z_t + \frac{\sigma^2_{t+1}}{\eta^2}\left[1 - (1-2u\eta)^{1/2}\right]\,, \end{aligned} \tag{94}$$

where the i^{th} component of $\Lambda(u,v,\nu_3,\pi^*)$ is given by (84). The absence of arbitrage constraint is $\psi^{\mathbb{Q}}_t(1,0) = r_{t+1}\,,\ \forall \underline{w}_t\,,$ implying

$$r_{t+1} = \nu_t + \nu_1 y_t + \nu'_2 z_t + \lambda'(\nu_3,\pi^*) z_t + \sigma^2_{t+1}\left(\nu_4 + \frac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\right)\,,$$

with the i^{th} component of $\lambda(\nu_3,\pi^*)$ given by (85). Therefore, the arbitrage restriction implies:

$$\begin{cases} \nu_1 = 0\,, \\ \nu_2 = -\lambda(\nu_3,\pi^*)\,, \\ \nu_4 = -\frac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\,, \\ \nu_t = r_{t+1}\,. \end{cases}$$

Thus, equation (93) becomes:

$$y_{t+1} = r_{t+1} - \lambda(\nu_3,\pi^*)' z_t - \frac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\sigma^2_{t+1} + \nu'_3 z_{t+1} + \eta \xi_{t+1} \tag{95}$$

with

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1 \sigma^2_t + \alpha_2 \xi_t + \alpha_3 \frac{\sigma^4_t}{\xi_t}$$

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} IG\left(\frac{\sigma^2_{t+1}}{\eta^2}\right)\,,$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi^*_{ij}\,,$$

(again, we can take $\nu_{3J} = 0$) which gives the R.N. dynamics compatible with the AAO restriction. The corresponding Log-Laplace transform is :

$$\begin{aligned} \psi^{\mathbb{Q}}_t(u,v) &= \left(r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\sigma^2_{t+1}\right) u + \\ &\qquad \Lambda'(u,v,\nu_3,\pi^*) z_t + \frac{\sigma^2_{t+1}}{\eta^2}\left[1-(1-2u\eta)^{1/2}\right]\,. \end{aligned} \tag{96}$$

Given the specification of $\gamma_t(\underline{w}_t, \theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$ (with, for instance, $\delta_{Jt} = 0$), the conditional historical Log-Laplace transform of the factor is given by:

$$\begin{aligned}
\psi_t(u,v) &= \Big(r_{t+1} - \lambda' z_t - \tfrac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\sigma_{t+1}^2\Big)\,u \\
&\quad + \left[\Lambda(u-\gamma_t, v-\delta_t, \nu_3, \pi^*) - \Lambda(-\gamma_t, -\delta_t, \nu_3, \pi^*)\right]' z_t \\
&\quad + \tfrac{\sigma_{t+1}^2}{\eta^2}\left[(1+2\gamma_t\eta)^{1/2} - [1-2(u-\gamma_t)\eta]^{1/2}\right] \\
&= \Big(r_{t+1} - \lambda' z_t - \tilde{\eta}_t^{-3/2}\eta^{-1/2}\left[1-(1-2\eta)^{1/2}\right]\tilde{\sigma}_{t+1}^2\Big)\,u \\
&\quad + \left[\Lambda(u-\gamma_t, v-\delta_t, \nu_3, \pi^*) - \Lambda(-\gamma_t, -\delta_t, \nu_3, \pi^*)\right]' z_t \\
&\quad + \tfrac{\tilde{\sigma}_{t+1}^2}{\tilde{\eta}_t^2}\left[1-(1-2u\tilde{\eta}_t)^{1/2}\right],
\end{aligned} \tag{97}$$

with $\Lambda_i(u-\gamma_t, v-\delta_t) - \Lambda_i(-\gamma_t, -\delta_t)$ specified by (89), and where $\tilde{\eta}_t = \frac{\eta}{1+2\gamma_t\eta}$ and $\tilde{\sigma}_{t+1}^2 = \sigma_{t+1}^2\left(\frac{\tilde{\eta}_t}{\eta}\right)^{3/2}$. So, the non-affine historical dynamics is given by :

$$\begin{aligned}
& y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t + \nu_3' z_{t+1} - \tilde{\eta}_t^{-3/2}\eta^{-1/2}\left[1-(1-2\eta)^{1/2}\right]\tilde{\sigma}_{t+1}^2 + \tilde{\eta}_t\varepsilon_{t+1} \\
& \varepsilon_{t+1}\,|\,\underline{\varepsilon}_t, \underline{z}_{t+1} \sim IG\left(\tfrac{\tilde{\sigma}_{t+1}^2}{\tilde{\eta}_t^2}\right),
\end{aligned} \tag{98}$$

with, using (95) and (98), $\eta\xi_{t+1} = \tilde{\eta}_t\varepsilon_{t+1}$ and

$$\tilde{\sigma}_{t+1}^2 = \tilde{\omega}_t' z_t + \tilde{\alpha}_{1,t}\tilde{\sigma}_t^2 + \tilde{\alpha}_{2,t}\varepsilon_t + \tilde{\alpha}_{3,t}\tfrac{\tilde{\sigma}_t^4}{\varepsilon_t}$$

$$\mathbb{P}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \pi_{ij,t}\,,$$

where $\tilde{\omega}_t = \omega(\tilde{\eta}_t/\eta)^{3/2}$, $\tilde{\alpha}_{1,t} = \alpha_1(\tilde{\eta}_t/\tilde{\eta}_{t-1})^{3/2}$, $\tilde{\alpha}_{2,t} = \alpha_2(\tilde{\eta}_t^{3/2}\tilde{\eta}_{t-1}/\eta^{5/2})$ and $\tilde{\alpha}_{3,t} = \alpha_3\tilde{\eta}_t^{3/2}/(\tilde{\eta}_{t-1}^4\eta^{-5/2})$.

As in the previous section, the factor $w_{t+1} = (y_{t+1}, z_{t+1}')'$ is not a R.N. Car process, but it can be verified [see Appendix 4] that the factor $w_{t+1}^e = (y_{t+1}, z_{t+1}', \sigma_{t+2}^2)'$ is R.N. Car. Indeed, the R.N. conditional Log-Laplace transform of w_{t+1}^e, given $\underline{w}_t^e$, is :

$$\psi_t^{\mathbb{Q}}(u,v,\tilde{v}) = a_1^{\mathbb{Q}}(u,v,\tilde{v})' z_t + a_2^{\mathbb{Q}}(u,\tilde{v})\sigma_{t+1}^2 + b_t^{\mathbb{Q}}(u,\tilde{v})\,, \tag{99}$$

where

$$\begin{aligned}
a_1^{\mathbb{Q}}(u,v,\tilde{v}) &= \tilde{\Lambda}(u,v,\tilde{v},\nu_3,\omega,\pi^*) - \lambda(\nu_3,\pi^*)\,u \\
&\text{with} \qquad \tilde{\Lambda}_i(u,v,\tilde{v},\nu_3,\omega,\pi^*) = \mathrm{Log}\sum_{j=1}^{J}\pi_{ij}^*\exp(u\nu_{3j} + v_j + \tilde{v}\omega_j)\,,\ i \in \{1,\ldots,J\}\,, \\
a_2^{\mathbb{Q}}(u,\tilde{v}) &= \tilde{v}\alpha_1 - \frac{1}{\eta^2}\left(u\left[1-(1-2\eta)^{1/2}\right] + 1 - \sqrt{(1-2\tilde{v}\alpha_3\eta^4)\,(1-2(u\eta+\tilde{v}\alpha_2))}\right)\,, \\
b_t^{\mathbb{Q}}(u,\tilde{v}) &= ur_{t+1} - \frac{1}{2}\mathrm{Log}(1-2\tilde{v}\alpha_3\eta^4)\,,
\end{aligned}$$

which is affine in $(z_t', \sigma_{t+1}^2)'$, with an intercept deterministic function of time.

As far as the identification problem is concerned, with functions γ and δ constant, we can identify, from the historical dynamics (98), $3J + J(J-1) + 4$ coefficients, while the parameters to be estimated are ν_3 (with $\nu_{3J} = 0$), $\omega, \alpha_1, \alpha_2, \alpha_3$, π_{ij}^*, γ, δ (with $\delta_J = 0$), and η, that is, $2(J-1) + J + 5 + J(J-1) = 3J + J(J-1) + 3$ parameters. Thus, as in the previous section, the historical model is over identified.

6 Applications to Econometric Term Structure Modelling

It is well known that, if the R.N. dynamics of w_t is Car and if r_{t+1} is an affine function of w_t, the term structure of interest rates $[r(t,h), h \in \{1, ..., H\}]$ is easily determined recursively and is affine in w_t [see Gourii¿½oux, Monfort and Polimenis (2003)]. Indeed, if :

$$\psi_t^{\mathbb{Q}}(u|w_t; \theta_1^*) = a^{\mathbb{Q}}(u, \theta_1^*)' w_t + b^{\mathbb{Q}}(u, \theta_1^*)$$

and $r_{t+1} = \tilde{\theta}_1 + \tilde{\theta}_2' w_t$, then

$$r(t,h) = -\frac{c_h'}{h} w_t - \frac{d_h}{h}\,, \tag{100}$$

where

$$\begin{cases} c_h = -\tilde{\theta}_2 + a^{\mathbb{Q}}(c_{h-1}) \\ d_h = d_{h-1} - \tilde{\theta}_1 + b^{\mathbb{Q}}(c_{h-1}) \\ c_0 = 0,\ d_0 = 0\,. \end{cases} \tag{101}$$

Moreover, applying the transform analysis, various interest rates derivatives have quasi explicit pricing formulas. Note that if the i^{th} component of w_t is a rate $r(t, h_i)$, $i \in \{1, ..., K_1\}$, we must satisfy the internal consistency conditions:

$$c_{h_i} = -h_i e_i,\ d_{h_i} = 0, \quad i \in \{1, ..., K_1\}\,. \tag{102}$$

Therefore, it is highly desirable to have a Car R.N. dynamics and this specification is obtained by one of the three modelling strategies described in Section 4. Let us consider some examples.

6.1 Direct Modelling of the VAR(p) Factor-Based Term Structure Model

For sake of notational simplicity we consider the one factor case, but the results can be extended to the multivariate case [see Monfort and Pegoraro (2006a)]. We assume that the factor w_t, which may be observable or unobservable, has a historical dynamics given by a Gaussian AR(p) model:

$$\begin{aligned} w_{t+1} &= \nu + \varphi_t w_t + ... + \varphi_p w_{t+1-p} + \sigma \varepsilon_{t+1} \\ &= \nu + \varphi' W_t + \sigma \varepsilon_{t+1} \end{aligned} \tag{103}$$

where $\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,1)$, $\varphi = (\varphi_1, ..., \varphi_p)'$ and $W_t = (w_t, ..., w_{t+1-p})'$. This dynamics can also be written :

$$W_{t+1} = \tilde{\nu} + \Phi W_t + \sigma \tilde{\varepsilon}_{t+1}$$

where $\tilde{\nu} = \nu e_1$, $\tilde{\varepsilon}_{t+1} = \varepsilon_{t+1} e_1$ [e_1 denotes the first column of the identity matrix I_p] and

$$\Phi = \begin{bmatrix} \varphi_1 & \dots & \dots & \varphi_{p-1} & \varphi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} \text{ is a } (p \times p) \text{ matrix}.$$

The SDF takes the following exponential-affine form :

$$M_{t,t+1} = \exp\left[-r_{t+1} + \alpha_t w_{t+1} - \psi_t(\alpha_t)\right], \tag{104}$$

with

$$\begin{aligned} \psi_t(u) &= (\nu + \varphi' W_t)u + \tfrac{1}{2}\sigma^2 u^2, \\ \alpha_t &= \alpha_0 + \alpha' W_t \\ &= \alpha_0 + \alpha_1 w_t + ... + \alpha_p w_{t+1-p}, \end{aligned}$$

and the short rate is given by :

$$r_{t+1} = \tilde{\theta}_1 + \tilde{\theta}_2' W_t. \tag{105}$$

If $r_{t+1} = w_t$, we have $\tilde{\theta}_2 = e_1$ and $\tilde{\theta}_1 = 0$.

The conditional R.N. Log-Laplace transform is given by :

$$\begin{aligned} \psi_t^{\mathbb{Q}}(u) &= \psi_t(u + \alpha_t) - \psi_t(\alpha_t) \\ &= (\nu + \varphi' W_t)u + \sigma^2 \alpha_t u + \tfrac{1}{2}\sigma^2 u^2 \\ &= [\nu + \sigma^2 \alpha_0 + (\varphi + \sigma^2 \alpha)' W_t]u + \tfrac{1}{2}\sigma^2 u^2. \end{aligned}$$

Therefore, the R.N. dynamics of the factor is given by :

$$w_{t+1} = (\nu + \sigma^2 \alpha_0) + (\varphi + \sigma^2 \alpha)' W_t + \sigma \xi_{t+1} \tag{106}$$

where $\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$. Moreover, we have that $\varepsilon_{t+1} = \xi_{t+1} + \sigma(\alpha_0 + \alpha' W_t)$.

The yield-to-maturity formula at date t is given by [see Monfort and Pegoraro (2006a) for the proof] :

$$r(t,h) = -\frac{c_h'}{h} W_t - \frac{d_h}{h}, \quad h \geq 1, \tag{107}$$

with

$$\begin{cases} c_h = -\tilde{\theta}_2 + \Phi' c_{h-1} + c_{1,h} \sigma^2 \alpha \\ d_h = -\tilde{\theta}_1 + c_{1,h-1}(\nu + \sigma^2 \alpha_0) + \frac{1}{2} c_{1,h-1}^2 \sigma^2 + d_{h-1} \\ c_0 = 0 \;\; d_0 = 0. \end{cases} \tag{108}$$

6.2 R.N. Constrained Direct Modelling of the Switching VAR(p) Factor-Based Term Structure Model

Again for sake of simplicity we consider the univariate case [see Monfort and Pegoraro (2007) for extensions] where the factor is given by $w_t = (x_t, z_t')'$, with z_t a J-state non-homogeneous Markov chain valued in $\{e_1, ..., e_J\}$. The first component x_t is observable or unobservable, z_t is unobservable and the historical dynamics is given by :

$$x_{t+1} = \nu(Z_t) + \varphi_1(Z_t)x_t + ... + \varphi_p(Z_t)x_{t+1-p} + \sigma(Z_t)\varepsilon_{t+1} \tag{109}$$

where

$$\begin{aligned}
&\varepsilon_{t+1} \,|\, \underline{\varepsilon_t}, \underline{z}_{t+1} \overset{\mathbb{P}}{\sim} N(0,1) \\
&\mathbb{P}(z_{t+1} = e_j | \underline{x}_t \; \underline{z}_{t-1}, z_t = e_i) = \pi(e_i, e_j; X_t) \\
&Z_t = (z_t', ..., z_{t-p}') \\
&X_t = (x_t, ..., x_{t+1-p})' \,.
\end{aligned}$$

Observe that the joint historical dynamics of $(x_t, z_t')'$ is not Car. Functions ν, $\varphi_1, \ldots, \varphi_p$, σ and π are parameterized using a parameter θ_1.

We specify the SDF in the following way :

$$M_{t,t+1} = \exp\left[-r_{t+1} + \Gamma(Z_t, X_t)\varepsilon_{t+1} - \frac{1}{2}\Gamma(Z_t, X_t)^2 - \delta(Z_t, X_t)'z_{t+1}\right] , \tag{110}$$

with $\Gamma(Z_t, X_t) = \gamma(Z_t) + \tilde{\gamma}(Z_t)'X_t$ and, in order to ensure that $E_t M_{t,t+1} = \exp(-r_{t+1})$, we add the condition :

$$\sum_{j=1}^{J} \pi(e_i, e_j, X_i) \exp[-\delta(Z_t, X_t)'e_j] = 1, \;\; \forall Z_t, X_t \,. \tag{111}$$

The short rate is given by :

$$r_{t+1} = \tilde{\theta}_1' X_t + \tilde{\theta}_2' Z_t \,, \tag{112}$$

and, in the observable factor case ($x_t = r_{t+1}$), we have $\tilde{\theta}_1 = e_1$ and $\tilde{\theta}_2 = 0$.

It is easily seen that the R.N. dynamics is given by :

$$\begin{aligned}
&x_{t+1} = \nu(Z_t) + \gamma(Z_t)\sigma(Z_t) + [\varphi(Z_t) + \tilde{\gamma}(Z_t)\sigma(Z_t)]'X_t + \sigma(Z_t)\xi_{t+1} \\
&\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1) \\
&\mathbb{Q}(z_{t+1} = e_j | \underline{x}_t, \underline{z}_{t-1}, z_t = e_i) = \pi(e_i, e_j, ; X_t) \exp[-\delta(Z_t, X_t)'e_j] \,.
\end{aligned} \tag{113}$$

So, if we want the R.N. dynamics of w_t to be Car, we have to impose :

$$
\begin{aligned}
&\text{i) } \sigma(Z_t) = \sigma^{*'} Z_t \ \text{ (linearity in } z_t, ..., z_{t-p}) \\
&\text{ii) } \gamma(Z_t) = \tfrac{\nu^{*'} Z_t - \nu(Z_t)}{\sigma^{*'} Z_t} \\
&\text{iii) } \tilde{\gamma}(Z_t) = \tfrac{\varphi^{*} - \varphi(Z_t)}{\sigma^{*'} Z_t} \\
&\text{iv) } \delta_j(Z_t, X_t) = \text{Log}\left[\frac{\pi(z_t, e_j, X_t)}{\pi^*(z_t, e_j)}\right] ,
\end{aligned}
\tag{114}
$$

where σ^*, ν^*, φ^* are free parameters, $\pi^*(e_i, e_j)$ are the entries of an homogeneous transition matrix. All of these parameters constitute the parameter $\theta^* \in \Theta^*$ introduced in Section 4.3. Also note that, because of constraints $(114-\text{i})$ above, θ and θ^* do not vary independently.
So the R.N. dynamics is :

$$
\begin{aligned}
&X_{t+1} = \Phi^* X_t + [\nu^{*'} Z_t + (\sigma^{*'} Z_t)\xi_{t+1}]e_1 \,, \\
&\Phi^* = \begin{bmatrix} \varphi_1^* & \dots & \dots & \varphi_{p-1}^* & \varphi_p^* \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} \text{ is a } (p \times p) \text{ matrix} \,, \\
&\xi_{t+1} \,|\, \underline{\xi_t}, \underline{z_{t+1}} \overset{\mathbb{Q}}{\sim} N(0,1) \,, \\
&\mathbb{Q}(z_{t+1} = e_j | \underline{x_t}, \underline{z_{t-1}}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*
\end{aligned}
\tag{115}
$$

and the affine (in X_t and Z_t) term structure of interest rates is easily derived [see Monfort and Pegoraro (2007) for the proof, and Dai, Singleton and Yang (2006) for the case $p = 1$].

6.3 Back Modelling of the VAR(p) Factor-Based Term Structure Model

Let us consider the (bivariate) case where w_t is given by $[r(t,1), r(t,2)]'$. We want to impose the following Gaussian VAR(1) R.N. dynamics:

$$
w_{t+1} = \nu + \Phi w_t + \xi_{t+1} \,, \tag{116}
$$

where $\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0, \Sigma)$. In this case, the internal consistency conditions are satisfied if we impose, in (100) and (101), $\tilde{\theta}_1 = 0$, $\tilde{\theta}_2' = (1, 0)$, $c_2 = -2e_2$ and $d_2 = 0$, or :

$$
\begin{cases}
-2e_2 = a^{\mathbb{Q}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \\[2ex]
0 = b^{\mathbb{Q}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} ,
\end{cases}
\tag{117}
$$

where $a^{\mathbb{Q}}(u) = \Phi' u$ and $b^{\mathbb{Q}}(u) = u'\nu + \frac{1}{2}u'\Sigma u$. So, relation (117) becomes, with obvious notations:

$$\begin{aligned} &\begin{bmatrix}\varphi_{11}\\ \varphi_{12}\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}, \\ &\nu_1 = \tfrac{1}{2}\sigma_1^2, \end{aligned} \tag{118}$$

and (116) must be written:

$$\begin{cases} r(t+1,1) = \frac{1}{2}\sigma_1^2 - r(t,1) + 2r(t,2) + \xi_{1,t+1} \\ \\ r(t+1,2) = \nu_2 + \varphi_{21} r(t,1) + \varphi_{22} r(t,2) + \xi_{2,t+1}, \end{cases} \tag{119}$$

with $\xi_t \overset{\mathbb{Q}}{\sim} IIN(0,\Sigma)$. Consequently, the R.N. conditional Log-Laplace transform of w_{t+1}, compatible with the AAO restrictions is:

$$\psi_t^{\mathbb{Q}}(u) = u' \left[\begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} w_t \right] + \frac{1}{2} u' \Sigma u \,. \tag{120}$$

Now, if we move back to the historical conditional Log-Laplace transform, we get:

$$\begin{aligned} \psi_t^{\mathbb{P}}(u) &= \psi_t^{\mathbb{Q}}(u - \alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t) \\ &= u' \left[\begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} w_t \right] - u'\Sigma\alpha_t + \tfrac{1}{2} u' \Sigma u \,. \end{aligned} \tag{121}$$

If we assume $\alpha_t = \gamma + \Gamma w_t$, we get:

$$\psi_t^{\mathbb{P}}(u) \;=\; u' \left\{ \begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} - \Sigma\gamma + \left[\begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} - \Sigma\Gamma \right] w_t \right\} + \frac{1}{2} u' \Sigma u \,,$$

or, equivalently

$$w_{t+1} \;=\; \begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} - \Sigma\gamma + \left[\begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} - \Sigma\Gamma \right] w_t + \varepsilon_{t+1} \,, \tag{122}$$

where $\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,\Sigma)$ and $\varepsilon_t = \xi_t + \Sigma(\gamma + \Gamma w_t)$. If $\Gamma = 0$, the historical dynamics of w_t is constrained, the parameters Σ, φ_{12} and φ_{22} are identifiable from the observations on w_t, whereas γ and ν_2 are not. If $\Gamma \neq 0$, the historical dynamics of w_t is not constrained and only Σ is identifiable from the observations on w_t.

6.4 Direct Modelling of Wishart Term Structure Models and Quadratic Term Structure Models: a third application of Extended Car Processes

The Wishart Quadratic Term Structure model, proposed by Gourieroux and Sufana (2003), is characterized by an unobservable factor W_t which follows a Wishart autoregressive (WAR) process, that is, a process valued in the space of $(n \times n)$ symmetric positive definite matrices; its conditional historical Log-Laplace transform is given by:

$$\begin{aligned} \psi_t^{\mathbb{P}}(\Gamma) &= \text{Log}\{E_t \exp(Tr \Gamma W_{t+1})\} \\ &= Tr\left[M'\Gamma(I_n - 2\Sigma\Gamma)^{-1} M W_t\right] - \tfrac{K}{2} \text{Log det}[(I_n - 2\Sigma\Gamma)]\,, \end{aligned} \tag{123}$$

where Γ is a $(n \times n)$ matrix of coefficients, which can be chosen symmetric [since, with obvious notations, $Tr(\Gamma W_{t+1}) = \sum_{i\ j} \Gamma_{ij} W_{ij,t+1} = \sum_{i \leq j} (\Gamma_{ij} + \Gamma_{ji}) W_{ij,t+1}$]. This dynamics is Car(1) and, if K is integer, it can be defined as:

$$\begin{aligned} & W_t = \sum_{k=1}^{K} x_{k,t} x'_{k,t}\,, \quad (K \geq n) \\ & x_{k,t+1} = M x_{k,t} + \varepsilon_{k,t+1}\,, \;\; k \in \{1, \ldots, K\} \\ & \varepsilon_{k,t+1} \overset{\mathbb{P}}{\sim} IIN(0, \Sigma)\,, \;\; k \in \{1, \ldots, K\}\,, \;\; \text{independent.} \end{aligned} \tag{124}$$

Since W_t is not observed, it can be normalized by $\Sigma = I_n$. The SDF is defined by:

$$M_{t,t+1} = \exp\left[Tr(C W_{t+1}) + d\right]\,, \tag{125}$$

where C is a $(n \times n)$ symmetric matrix and d is a scalar.
The associated R.N. dynamics is defined by:

$$\begin{aligned} \psi_t^{\mathbb{Q}}(\Gamma) &= Tr\left[M'\left\{(C+\Gamma)[I_n - 2(C+\Gamma)]^{-1} - C(I_n - 2C)^{-1}\right\} M W_t\right] \\ & \qquad -\frac{K}{2} \text{Log}\det[(I_n - 2(I_n - 2C)^{-1}\Gamma)]\,, \end{aligned} \tag{126}$$

which is also Car(1). The term structure of interest rates at date t is affine in W_t and given by:

$$\begin{aligned} r(t,h) &= -\frac{1}{h} Tr[A(h) W_t] - \frac{1}{h} b(h)\,, \;\; h \geq 1 \\ A(h) &= M'[C + A(h-1)]\left\{I_n - 2[C + A(h-1)]\right\}^{-1} M \\ b(h) &= d + b(h-1) - \frac{K}{2} \text{Log}\det[I_n - 2(C + A(h-1))] \\ A(0) &= 0\,, \;\; b(0) = 0\,. \end{aligned}$$

In particular, if K is integer, we get:

$$\begin{aligned} r(t,h) &= -\frac{1}{h} Tr[\sum_{k=1}^{K} A(h) x_{k,t} x'_{k,t}] - \frac{1}{h} b(h)\,, \;\; h \geq 1 \\ &= -\frac{1}{h} \sum_{k=1}^{K} x'_{k,t} A(h) x_{k,t} - \frac{1}{h} b(h)\,, \;\; h \geq 1\,, \end{aligned} \tag{127}$$

which is a sum of quadratic forms in $x_{k,t}$. If $K = 1$, we get the standard Quadratic Term Structure Model which is, therefore, a special affine model [see Beaglehole and Tenney (1991), Ahn, Dittmar and Gallant (2002), Leippold and Wu (2002), and Cheng and Scaillet (2006)].

We can also define a quadratic term structure model with a linear term, if the historical dynamics of x_{t+1} is given by the following Gaussian VAR(1) process:

$$\begin{aligned} x_{t+1} &= m + Mx_t + \varepsilon_{t+1}\,, \\ \varepsilon_{t+1} &\overset{\mathbb{P}}{\sim} IIN(0,\Sigma)\,. \end{aligned} \tag{128}$$

Indeed, the factor $w_t = [x_t', vech(x_t x_t')']'$ can be shown to be Car(1), that is, w_t is an Extended Car process in the historical world [see Appendix 5 for the proof]. Moreover, choosing:

$$\begin{aligned} M_{t,t+1} &= \exp\left[C'x_{t+1} + Tr(Cx_{t+1}x_{t+1}') + d\right] \\ &= \exp(C'x_{t+1} + x_{t+1}'Cx_{t+1} + d)\,,\ (C \text{ is a symmetric } (n \times n) \text{ matrix})\,, \end{aligned} \tag{129}$$

the process w_t is also Extended Car in the risk-neutral world. The term structure at date t is affine in w_t, that is, of the form:

$$r(t,h) = x_t'\Lambda(h)x_t + \mu(h)'x_t + \nu(h)\,,\ h \geq 1\,, \tag{130}$$

where $\Lambda(h), \mu(h)$ and $\nu(h)$ follow recursive equations [see also Gourieroux and Sufana (2003), Cheng and Scaillet (2006) and Jiang and Yan (2006)].

7 An Example of Back Modelling for a Security Market Model with Stochastic Dividends and Short Rates

The purpose of this section is to consider an Econometric Security Market Model where the risky assets are dividend-paying assets and the short rate is endogenous. More precisely, the factor is given by $w_t = (y_t, \delta_t, r_{t+1})'$, where:

- $y_t = (y_{1,t}, \ldots, y_{K_1,t})'$ denotes, for each date t, the K_1-dimensional vector of geometric returns associated to cum dividend prices $S_{j,t}$, $j \in \{1, \ldots, K_1\}$;
- $\delta_t = (\delta_{1,t}, \ldots, \delta_{K_1,t})$ is the associated K_1-dimensional vector of (geometric) dividend yields and, denoting $\tilde{S}_{j,t}$ as the ex dividend price of the j^{th} risky asset, we have $S_{j,t} = \tilde{S}_{j,t} \exp(\delta_{j,t})$;
- r_{t+1} denotes the (predetermined) stochastic short rate for the period $[t, t+1]$;

Observe that, compared to the setting of Section 5.1 (where r_{t+1} was exogenous), this model proposes a more general K-dimensional factor w_t (with $K = (2K_1 + 1)$), where we jointly specify y_t, δ_t (which is considered as an observable factor), and the short rate r_{t+1}. It would be straightforward to add an unobservable factor z_t.

Following the Back Modelling approach, we propose a R.N. Gaussian VAR(1) dynamics for the factor and the conditional distribution of w_{t+1}, given $\underline{w_t}$, is assumed to be Gaussian with mean vector $(A_0 + A_1\, w_t)$ and variance-covariance matrix Σ. The process w_{t+1} is, therefore, a Car(1) process with a conditional R.N. Laplace Transform given by :

$$\varphi_t^{\mathbb{Q}}(u \,|\, \underline{w}_t) = E_t^{\mathbb{Q}}\left[\exp\left(u'\, w_{t+1}\right)\right] = \exp\left[a^{\mathbb{Q}}(u)'\, w_t \,+\, b^{\mathbb{Q}}(u)\right]$$

where the functions $a^{\mathbb{Q}}$ and $b^{\mathbb{Q}}$ are the following :

$$\left\{\begin{array}{lcl} a^{\mathbb{Q}}(u) & = & {A_1}'\, u \\ \\ b^{\mathbb{Q}}(u) & = & {A_0}'\, u + \frac{1}{2}u'\Sigma u\,. \end{array}\right.$$

The R.N. dynamics can also be written:

$$\begin{array}{l} w_{t+1} = A_0 + A_1\, w_t + \xi_{t+1} \\ \\ \xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,\Sigma)\,. \end{array} \qquad (131)$$

The AAO restrictions, applied to the K_1-dimensional vector y_{t+1}, are given by :

$$\begin{array}{rl} & E_t^{\mathbb{Q}}\left[\exp[\mathrm{Log}\left(\frac{S_{j,t+1}}{\tilde{S}_{j,t}}\right)\right] = \exp(r_{t+1})\,,\ \ j \in \{1,\ldots,K_1\}\,, \\ \\ \Longleftrightarrow & E_t^{\mathbb{Q}}\left[\exp(y_{j,t+1})\right] = \exp(r_{t+1} - \delta_{j,t})\,,\ \ j \in \{1,\ldots,K_1\}\,, \\ \\ \Longleftrightarrow & \left\{\begin{array}{lclcl} a^{\mathbb{Q}}(e_j) & = & {A_1}'\, e_j & = & e_K - e_{j+K_1}\,,\ \ j \in \{1,\ldots,K_1\}\,, \\ \\ b^{\mathbb{Q}}(e_j) & = & {A_0}'\, e_j + \frac{1}{2}e_j'\Sigma e_j \ = \ 0\ ,\ \ j \in \{1,\ldots,K_1\}. & & \end{array}\right. \end{array} \qquad (132)$$

This means that the first K_1 rows of A_1 and the first K_1 components of A_0 are, for $j \in \{1,\ldots,K_1\}$, respectively given by $(e_K - e_{j+K_1})'$ and $-\frac{1}{2}\sigma_j^2$ [where e_K and e_{j+K_1} denote, respectively, the K^{th} and the $(j+K_1)^{th}$ column of the Identity matrix I_K, while σ_j^2 is the (j,j)-term of Σ]. In other words, the K_1 first equations of (131) are:

$$y_{j,t+1} = -\frac{1}{2}\sigma_j^2 + r_{t+1} - \delta_{j,t} + \xi_{j,t+1}\,,\ \ \ j \in \{1,\ldots,K_1\}\,.$$

Then, coming back to the historical dynamics of w_t, we get :

$$\begin{array}{rcl} \psi_t(u) & = & \psi_t^{\mathbb{Q}}(u-\alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t) \\ \\ & = & \left(a^{\mathbb{Q}}(u-\alpha_t) - a^{\mathbb{Q}}(-\alpha_t)\right)'\, w_t + b^{\mathbb{Q}}(u-\alpha_t) - b^{\mathbb{Q}}(-\alpha_t) \\ \\ & = & u'A_1 w_t + u'A_0 + \frac{1}{2}(u-\alpha_t)'\Sigma(u-\alpha_t) - \frac{1}{2}\alpha_t'\Sigma\alpha_t \\ \\ & = & u'\left(A_0 + A_1 w_t - \Sigma\alpha_t\right) + \frac{1}{2}u'\Sigma u\,. \end{array} \qquad (133)$$

So, if we impose $\alpha_t = (\alpha_0 + \alpha\, w_t)$, the historical dynamics of the factor is also Gaussian VAR(1) with a modified mean vector equal to $[A_0 - \Sigma\alpha_0 + (A_1 - \Sigma\alpha)w_t]$ and the same variance-covariance matrix Σ, that is:

$$w_{t+1} = A_0 - \Sigma\alpha_0 + (A_1 - \Sigma\alpha)w_t + \varepsilon_{t+1}\,,\ \ \varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,\Sigma)\,,$$

and $\varepsilon_{t+1} = \xi_{t+1} + \Sigma(\alpha_0 + \alpha\, w_t)\,.$

We notice that, under the historical probability, any VAR(1) distribution can be reached, but only Σ is identifiable. If we add the constraint $\alpha = 0$, then the historical dynamics of w_t is constrained, and A_0 and α_0 are not identifiable.

8 Conclusions

In this paper we have proposed a general econometric approach to asset pricing modelling based on three main elements : (i) the historical discrete-time dynamics of the factor representing the information, (ii) the Stochastic Discount Factor (SDF), and (iii) the risk-neutral (R.N.) factor dynamics. We have presented three modelling strategies : the Direct Modelling, the R.N. Constrained Direct Modelling and the Back Modelling. In all the approaches we have considered the internal consistency conditions, induced by the AAO restrictions, and the identification problem. These three approaches are studied for several discrete-time security market models and affine term structure models. In all cases, we have indicated the important role played by the R.N. Constrained Direct Modelling and the Back Modelling strategies in determining, at the same time, flexible historical dynamics and Car R.N. dynamics leading to explicit or quasi explicit pricing formulas for various contingent claims. Moreover, we have shown the possibility to derive asset pricing models able to accommodate non-affine historical and risk-neutral factor dynamics with tractable pricing formulas. This result is achieved when the starting R.N. non-affine factor can be modified to be a R.N. Extended Car process. These strategies, already implicitly adopted in several papers, clearly could be the basis for the specification of new asset pricing models leading to promising empirical analysis.

Appendix 1

Proof of the existence and uniqueness of $M_{t,t+1}$ and of the pricing formula (1)

Using A1 and A2, the Riesz representation theorem implies :

$$\forall s > t,\ \forall \underline{w_t}, \exists\ M_{t,s}(\underline{w_s}), \text{unique, such that } \forall g(\underline{w_s}) \in L_{2s}$$

$$p_t[g(\underline{w_s})] = E[M_{t,s}(\underline{w_s})\, g(\underline{w_s}) \,|\, \underline{w_t}].$$

In particular, the price at t of a zero-coupon bond with maturity s is $E[M_{t,s}(\underline{w_s}) \,|\, \underline{w_t}]$. A3 implies that $\mathbb{P}[M_{t,s} > 0 \,|\, \underline{w_t}] = 1$, $\forall t, s \in \{0, \ldots, T\}$, since otherwise the payoff $\mathbb{1}_{(M_{t,s} \leq 0)}$ at s, would be such that $\mathbb{P}[\mathbb{1}_{(M_{t,s} \leq 0)} > 0 \,|\, \underline{w_t}] > 0$ and $p_t\left[\mathbb{1}_{(M_{t,s} \leq 0)}\right] = E_t[M_{t,s}\mathbb{1}_{(M_{t,s} \leq 0)}] \leq 0$, contradicting A3.

Relation (1) will be shown if we prove that, $\forall\ r < t < s$, $\underline{w_t}$, $g(\underline{w_s}) \in L_{2s}$ we have:

$$p_r[g(\underline{w_s})] = p_r\{p_t[g(\underline{w_s})]\}.$$

Let us show, for instance, that if (with obvious notations) $p_r(g_s) > p_r[p_t(g_s)]$, we can construct a strictly positive payoff at s with price zero at r, contradicting A3. The payoff at s is defined by the following trading strategy:

at r: buy p_t, (short) sell g_s, buy $\dfrac{p_r(g_s) - p_r[p_t(g_s)]}{E[M_{r,s} \,|\, \underline{w_r}]}$ zero-coupon bonds with maturity s, generating a zero net profit;

at t: buy g_s and sell p_t, generating a zero net profit;

at s: the net payoff is $g_s - g_s + \dfrac{p_r(g_s) - p_r[p_t(g_s)]}{E[M_{r,s} \,|\, \underline{w_r}]} > 0.$

A similar argument shows that $p_r(g_s) < p_r[p_t(g_s)]$ contradicts A3 and, therefore, relation (1) is proved.

Appendix 2
Geometric and Arithmetic Risk Premia

In this appendix $[f_t(e_i)]$ will denote, for given scalar or row K-vectors $f_t(e_i)$, $i \in \{1, \dots, K\}$, the K-vector or the $K \times K$ matrix $(f_t(e_1)', \dots, f_t(e_K)')'$ with rows $f_t(e_i)$, $i \in \{1, \dots, K\}$; e will denote the K-dimensional unitary vector.

i) w_{t+1} is a K-vector of geometric returns

We have seen in Section 3.2 that the geometric risk premium can be written as:

$$\pi_{Gt} = \psi_t^{(1)}(0) - [\psi_t(\alpha_t + e_i)] + \psi_t(\alpha_t)e\,.$$

Using a first order expansion of $\pi_{Gt} = \pi_{Gt}(\alpha_t)$ around $\alpha_t = 0$ we obtain :

$$\pi_{Gt} \simeq \psi_t^{(1)}(0) - [\psi_t(e_i)] - [\psi_t^{(1)}(e_i)']\alpha_t + (\psi_t^{(1)}(0)'\alpha_t)e\,,$$

and neglecting conditional cumulants of order ≥ 3 we can write :

$$\begin{aligned}\pi_{Gt} &\simeq m_t - m_t - \tfrac{1}{2}vdiag\Sigma_t - (m_t'\alpha_t)e - \Sigma_t\alpha_t + (m_t'\alpha_t)e \\ &\simeq -\tfrac{1}{2}v\,diag\,\Sigma_t - \Sigma_t\alpha_t\,.\end{aligned}$$

If we consider now the arithmetic risk premium, and we apply the same procedure, we get:

$$\begin{aligned}\pi_{At} &= [\varphi_t(e_i)] - \left[\tfrac{\varphi_t(\alpha_t+e_i)}{\varphi_t(\alpha_t)}\right] \\ &\simeq \left[\varphi_t(e_i) - \varphi_t(e_i)\left(1 + \tfrac{\varphi_t^{(1)}(e_i)'\alpha_t}{\varphi_t(e_i)} - \varphi_t^{(1)}(0)'\alpha_t\right)\right] \\ &\simeq [-\varphi_t(e_i)(\psi_t^{(1)}(e_i)'\alpha_t - \varphi^{(1)}(0)'\alpha_t)] \\ &\simeq -diag[\varphi_t(e_i)]((m_t'\alpha_t)e + \Sigma_t\alpha_t - (m_t'\alpha_t)e) \\ &\simeq -diag[\varphi_t(e_i)]\Sigma_t\alpha_t\,.\end{aligned}$$

In the conditionally Gaussian case, where

$$\varphi_t(u) = \exp\left(m_t'u + \frac{1}{2}u'\Sigma_t u\right),\ \psi_t(u) = m_t'u + \frac{1}{2}u'\Sigma_t u\,,$$

the geometric risk premium becomes

$$\begin{aligned}\pi_{Gt} &= \psi_t^{(1)}(0) - [\psi_t(\alpha_t + e_i)] + \psi_t(\alpha_t)e \\ &= m_t - \left[m_t'(\alpha_t + e_i) + \tfrac{1}{2}(\alpha_t + e_i)'\Sigma_t(\alpha_t + e_i) - m_t'\alpha_t - \tfrac{1}{2}\alpha_t'\Sigma\alpha_t\right] \\ &= -\frac{1}{2}vdiag\,\Sigma_t - \Sigma_t\alpha_t\,,\end{aligned}$$

while, the arithmetic risk premium is

$$
\begin{aligned}
\pi_{At} &= [\varphi_t(e_i)] - \left[\tfrac{\varphi_t(\alpha+e_i)}{\varphi_t(\alpha_t)}\right] \\
&= \left[\exp\left(m_{it} + \frac{1}{2}\Sigma_{ii,t}\right) - \exp\left(m_{it} + \frac{1}{2}\Sigma_{ii,t} + e_i'\Sigma_t\alpha_t\right)\right] \\
&= \left[\exp\left(m_{it} + \frac{1}{2}\Sigma_{ii,t}\right)(1 - \exp(e_i'\Sigma_t\alpha_t))\right] \\
&\simeq -diag\left[\exp\left(m_{it} + \frac{1}{2}\Sigma_{ii,t}\right)\right] \Sigma_t\alpha_t\,.
\end{aligned}
$$

ii) $\underline{w_{t+1} \text{ is a } K\text{-vector of yields}}$

Following the same procedure, the geometric risk premium associated to $w_{t+1} = (h_1 r(t+1, h_1), \ldots, h_K r(t+1, h_K))'$ can be written as

$$
\begin{aligned}
\pi_{Gt} &= -\psi_t^{(1)}(0) - [\psi_t(\alpha_t - e_i)] + \psi_t(\alpha_t)e \\
&\simeq -\psi_t^{(1)}(0) - [\psi_t(-e_i) + \psi_t^{(1)}(-e_i)'\alpha_t] + (\psi_t^{(1)}(0)'\alpha_t)e \\
&\simeq -m_t - \left(-m_t + \tfrac{1}{2}\, vdiag\, \Sigma_t + (m_t'\alpha_t)e - \Sigma_t\alpha_t\right) + (m_t'\alpha_t)e \\
&\simeq -\tfrac{1}{2}\, vdiag\, \Sigma_t + \Sigma_t\alpha_t\,,
\end{aligned}
$$

while, the arithmetic risk premium is

$$
\begin{aligned}
\pi_{At} &= \left[\exp((h_i+1)r(t, h_i+1))\left(\varphi_t(-e_i) - \tfrac{\varphi_t(\alpha_t - e_i)}{\varphi_t(\alpha_t)}\right)\right] \\
&\simeq [\exp((h_i+1)r(t, h_i+1))(\varphi_t(-e_i) - \varphi_t(-e_i)(1 + \psi_t^{(1)}(-e_i)'\alpha_t - \varphi_t^{(1)}(0)'\alpha_t))] \\
&\simeq -diag[\varphi_t(-e_i)\exp((h_i+1)r(t, h_i+1))][\psi_t^{(1)}(-e_i)'\alpha_t - \varphi_t^{(1)}(0)'\alpha_t] \\
&\simeq diag[\varphi_t(-e_i)\exp((h_i+1)r(t, h_i+1))]\Sigma_t\alpha_t\,.
\end{aligned}
$$

In the conditionally Gaussian case, we have:

$$
\begin{aligned}
\pi_{Gt} &= -\frac{1}{2} vdiag\, \Sigma_t + \Sigma_t\alpha_t \\
\pi_{At} &\simeq diag\left[\exp\left(-m_{it} + \frac{1}{2}\Sigma_{ii,t}\right)\exp((h_i+1)r(t, h_i+1))\right]\Sigma_t\alpha_t\,.
\end{aligned}
$$

Appendix 3
Switching GARCH Models and Extended Car processes

The purpose of this appendix is to show, in the context of Section 5.6, that under the R.N. probability, even if $w_{t+1} = (y_{t+1}, z'_{t+1})'$ is not a Car process, the extended factor $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is Car. The proof of this result is based on the following two lemmas.

Lemma 1: For any vector $\mu \in \mathbb{R}^n$ and any symmetric positive definite $(n \times n)$ matrix Q, the following relation holds:

$$\int_{\mathbb{R}^n} \exp(-u'Qu + \mu' u)du = \frac{\pi^{n/2}}{(\det\, Q)^{1/2}} \exp\left(\frac{1}{4}\mu' Q^{-1}\mu\right) .$$

Proof: The LHS of the previous relation can be written as

$$\begin{aligned} & \int_{\mathbb{R}^n} \exp\left[-\left(u - \frac{1}{2}Q^{-1}\mu\right)' Q \left(u - \frac{1}{2}Q^{-1}\mu\right)\right] \exp\left(\frac{1}{4}\mu' Q^{-1}\mu\right) du \\ = \quad & \frac{\pi^{n/2}}{(\det Q)^{1/2}} \exp\left(\frac{1}{4}\mu' Q^{-1}\mu\right) \end{aligned}$$

given that the n-dimensional Gaussian distribution $N\left(\frac{1}{2}Q^{-1}\mu,\, (2Q)^{-1}\right)$ admits unit mass.

Lemma 2: If $\varepsilon_{t+1} \sim N(0, I_n)$, we have

$$\begin{aligned} & E_t\left\{\exp[\lambda' \varepsilon_{t+1} + \varepsilon'_{t+1} V \varepsilon_{t+1}]\right\} \\ = \; & \frac{1}{[\det(I - 2V)]^{1/2}} \exp\left[\frac{1}{2}\lambda'(I - 2V)^{-1}\lambda\right] . \end{aligned}$$

Proof: From the Lemma 1, we have that:

$$\begin{aligned} & E_t\left\{\exp(\lambda' \varepsilon_{t+1} + \varepsilon'_{t+1} V \varepsilon_{t+1})\right\} \\ = \; & \tfrac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left[-u'\left(\frac{1}{2}I - V\right)u + \lambda' u\right] du \\ = \; & \tfrac{1}{2^{n/2}\left[\det\left(\frac{1}{2}I - V\right)\right]^{1/2}} \exp\left[\tfrac{1}{4}\lambda'\left(\tfrac{1}{2}I - V\right)^{-1}\lambda\right] \\ = \; & \tfrac{1}{[\det(I-2V)]^{1/2}} \exp\left[\tfrac{1}{2}\lambda'(I - 2V)^{-1}\lambda\right] . \end{aligned}$$

Proposition : In the context of Section 5.6, the process $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is Car(1) under the R.N. probability.

Proof : We have:

$$
\begin{aligned}
& y_{t+1} = r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma^2_{t+1} + \nu'_3 z_{t+1} + \sigma_{t+1}\xi_{t+1} \\
& \xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1) \\
& \sigma^2_{t+1} = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma^2_t \\
& \mathbb{Q}\left(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i\right) = \pi^*_{ij}\,.
\end{aligned}
$$

So, the conditional R.N. Laplace transform of $(y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is:

$$
\begin{aligned}
\varphi^{\mathbb{Q}}_t(u, v, \tilde{v}) &= E^{\mathbb{Q}}_t \exp\left(u y_{t+1} + v' z_{t+1} + \tilde{v}\sigma^2_{t+2}\right) \\
&= E^{\mathbb{Q}}_t \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma^2_{t+1} + \nu'_3 z_{t+1} + \sigma_{t+1}\xi_{t+1}\right)\right. \\
&\qquad \left. + v' z_{t+1} + \tilde{v}[\omega' z_{t+1} + \alpha_1(\xi_{t+1} - \alpha_2\sigma_{t+1})^2 + \alpha_3\sigma^2_{t+1}]\right\} \\
&= \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma^2_{t+1}\right) + \tilde{v}\alpha_1\alpha_2^2\sigma^2_{t+1} + \tilde{v}\alpha_3\sigma^2_{t+1}\right\} \\
&\qquad E^{\mathbb{Q}}_t \exp[\xi_{t+1}\sigma_{t+1}(u - 2\alpha_1\alpha_2\tilde{v}) + \tilde{v}\alpha_1\xi^2_{t+1} + (u\nu_3 + v + \tilde{v}\omega)' z_{t+1}]\,.
\end{aligned}
$$

Using Lemma 2:

$$
\begin{aligned}
\varphi^{\mathbb{Q}}_t(u, v, \tilde{v}) &= \exp[u(r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma^2_{t+1}) + \tilde{v}\alpha_1\alpha_2^2\sigma^2_{t+1} + \tilde{v}\alpha_3\sigma^2_{t+1}] \\
&\qquad \times \exp\left[-\frac{1}{2}\mathrm{Log}(1 - 2\alpha_1\tilde{v}) + \frac{(u - 2\alpha_1\alpha_2\tilde{v})^2}{2(1 - 2\alpha_1\tilde{v})}\sigma^2_{t+1} + \tilde{\Lambda}'(u, v, \tilde{v}, \omega, \nu_3, \pi^*) z_t\right]\,,
\end{aligned}
$$

where the i^{th} component of $\tilde{\Lambda}(u, v, \tilde{v}, \omega, \nu_3, \pi^*)$ is given by:

$$
\tilde{\Lambda}_i(u, v, \tilde{v}, \omega, \nu_3, \pi^*) = \mathrm{Log}\sum_{j=1}^{J} \pi^*_{ij} \exp(u\nu_{3j} + v_j + \tilde{v}\omega_j)\,,
$$

and relation (92) is proved.

Appendix 4
Switching IG GARCH Models and Extended Car processes

In this appendix we show, in the context of Section 5.7, that under the R.N. probability, even if $w_{t+1} = (y_{t+1}, z'_{t+1})'$ is not a Car process, the extended factor $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is Car.

Proposition : In the context of Section 5.7, the process $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is Car(1) under the R.N. probability.

Proof : Let us recall equation (95) :

$$y_{t+1} = r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\sigma^2_{t+1} + \nu'_3 z_{t+1} + \eta\xi_{t+1}$$

with

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1\sigma^2_t + \alpha_2\xi_t + \alpha_3\tfrac{\sigma^4_t}{\xi_t}$$

$$\xi_{t+1}|\,\underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} IG\left(\tfrac{\sigma^2_{t+1}}{\eta^2}\right),$$

$$\mathbb{Q}(z_{t+1} = e_j|\underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j|z_t = e_i) = \pi^*_{ij},$$

So, the conditional R.N. Laplace transform of $(y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is:

$$\begin{aligned}
\varphi^{\mathbb{Q}}_t(u,v,\tilde{v}) &= E^{\mathbb{Q}}_t \exp\left(uy_{t+1} + v'z_{t+1} + \tilde{v}\sigma^2_{t+2}\right)\\
&= E^{\mathbb{Q}}_t \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\sigma^2_{t+1} + \nu'_3 z_{t+1} + \eta\xi_{t+1}\right)\right.\\
&\qquad \left. + v'z_{t+1} + \tilde{v}\left[\omega' z_{t+1} + \alpha_1\sigma^2_{t+1} + \alpha_2\xi_{t+1} + \alpha_3\frac{\sigma^4_{t+1}}{\xi_{t+1}}\right]\right\}\\
&= \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\sigma^2_{t+1}\right) + \tilde{v}\alpha_1\sigma^2_{t+1}\right\}\\
&\qquad E^{\mathbb{Q}}_t \exp[(u\eta + \tilde{v}\alpha_2)\,\xi_{t+1} + \frac{\tilde{v}\alpha_3\sigma^4_{t+1}}{\xi_{t+1}} + (u\nu_3 + v + \tilde{v}\omega)' z_{t+1}]\,.
\end{aligned}$$

Using the formula of the generalized Laplace transform of an Inverse Gaussian distribution given in footnote 8 (section 5.7):

$$\begin{aligned}
\varphi^{\mathbb{Q}}_t(u,v,\tilde{v}) &= \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1-(1-2\eta)^{1/2}\right]\sigma^2_{t+1}\right) + \tilde{v}\alpha_1\sigma^2_{t+1}\right\}\\
&\times \exp\left[-\frac{1}{2}\mathrm{Log}(1-2\tilde{v}\alpha_3\eta^4) + \frac{1}{\eta^2}\left(1-\sqrt{(1-2\tilde{v}\alpha_3\eta^4)\,(1-2(u\eta+\tilde{v}\alpha_2))}\right)\sigma^2_{t+1}\right.\\
&\left. + \tilde{\Lambda}'(u,v,\tilde{v},\nu_3,\omega,\pi^*) z_t\right],
\end{aligned}$$

where the i^{th} component of $\tilde{\Lambda}(u, v, \tilde{v}, \nu_3, \omega, \pi^*)$ is given by:

$$\tilde{\Lambda}_i(u, v, \tilde{v}, \nu_3, \omega, \pi^*) = \text{Log} \sum_{j=1}^{J} \pi_{ij}^* \exp(u\nu_{3j} + v_j + \tilde{v}\omega_j) \, ,$$

and relation (99) is proved.

Appendix 5
Quadratic Term Structure Models and Extended Car processes

Given the Gaussian VAR(1) process defined by relation (128), we have that, for any real symmetric matrix V, the conditional historical Laplace transform of $(x_{t+1}, x_{t+1}\, x_{t+1}')$ is given by:

$$\begin{aligned}
& E_t \, \exp[u'x_{t+1} + Tr(Vx_{t+1}x_{t+1}')] \\
= \; & \exp\{u'm + u'Mx_t + TrV[mm' + Mx_tx_t'M' + mx_t'M' + Mx_tm']\} \\
& \qquad E_t \, \exp\{u'\varepsilon_{t+1} + TrV[\varepsilon_{t+1}\varepsilon_{t+1}' + m\varepsilon_{t+1}' + \varepsilon_{t+1}m' + Mx_t\varepsilon_{t+1}' + \varepsilon_{t+1}x_t'M']\} \\
= \; & \exp\{u'm + u'Mx_t + m'Vm + 2m'VMx_t + Tr(M'VMx_tx_t')\} \\
& \qquad E_t \exp\{[u' + 2(m + Mx_t)'V]\varepsilon_{t+1} + \varepsilon_{t+1}'V\varepsilon_{t+1}\}
\end{aligned}$$

and, using Lemma 2 in Appendix 3, we can write:

$$\begin{aligned}
& E_t \, \exp[u'x_{t+1} + Tr(Vx_{t+1}x_{t+1}')] \\
= \; & \exp\{u'm + m'Vm + (M'u + 2M'Vm)'x_t + x_t'M'VMx_t \\
& \qquad + \tfrac{1}{2}[u' + 2(m + Mx_t)'V](I - 2V)[u + 2V(m + Mx_t)] - \tfrac{1}{2}\,\text{Log}\, det(I - 2V)\} \, ,
\end{aligned}$$

which is exponential-affine in $[x_t', vech(x_tx_t')']'$.

R E F E R E N C E S

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