Love and Death: A Freund Model with Frailty

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Abstract

We introduce new models for analyzing the mortality dependence between individuals in a couple. The mortality risk dependence is usually taken into account in the actuarial literature by introducing an Archimedean copula. This practice implies symmetric effects on the remaining lifetime of the surviving spouse. The new model allows for both asymmetric reactions by means of a Freund model, and risk dependence by means of an unobservable common risk factor (or frailty). These models allow for distinguishing in the lifetime dependence the component due to common lifetime (frailty) from the broken-heart syndrome (Freund model). The model is applied to insurance products such as joint life policy, last survivor insurance, or contracts with reversionary annuities.

Keywords: Life Insurance, Coupled Lives, Frailty, Freund Model, Broken-Heart, Copula, Last Survivor Insurance, Competing Risks.
1 Introduction

This paper introduces new models for analyzing the mortality dependence between individuals in a couple. This type of model is needed for risk management and pricing of life insurance products written on two lives, such as joint life policy, last survivor insurance policy, or contract with reversionary annuities.

The basic actuarial literature usually assumed the independence between the spouses’ mortality risks. Recently the mortality risk dependence has been introduced by means of Archimedean copulas [see e.g. Frees et al. (1996), Carrière (2000), Youn and Shemyakin (1999), Demiit et al. (2001), Shemyakin and Youn (2006), Luciano et al. (2008), Luciano et al. (2010)], and the effect of this dependence on the risk premia starts to be measured. However, ordinary copula models imply symmetric reactions of the mortality of a member of the couple when the other dies. An alternative consists in introducing jumps in mortality intensity (the Freund model) at the time of death of the a spouse, to capture the broken-heart syndrome [see e.g. Spreeuw and Wang (2008), Ji et al. (2011), Spreeuw and Owadally (2013)]. Our paper extends this literature by mixing the Freund model, which allows for asymmetric reactions of the mortality intensities at a death event, with unobservable common factor (or frailty), which underlies usual Archimedean copulas.

The basic Freund model and its properties in terms of conditional intensities are presented in Section 2. This model allows for jump in the mortality intensity of a given spouse when the other spouse dies. The magnitude of this jump and its variation with respect to the age of the couple is the basis for constructing a convenient association measure, useful to analyze the broken-heart syndrome. The Freund model is extended in Section 3 to include common unobserved static frailty. In particular we discuss the properties of Freund models with latent intensities which are exponential affine functions of the frailty. These models are used in Section 4 to derive the prices of various contracts written on two lives. We consider these prices at the inception of the contract as well as during its lifetime. We emphasize the effect of the dependence between the mortality risks of the two spouses on these prices. Section 5 provides insights on the statistical inference of previously introduced models. Section 6 concludes. Proofs are gathered in appendices.

2 The basic Freund model

This type of model has been introduced by Freund (1961) to construct bivariate survival models for dependent duration variables, while still featuring the lack of memory property. It has been noted by Toschi and Holmes (1980) that such models...
have an interpretation in terms of latent variables. We follow this interpretation. The model is written for a given couple, without specifying the index of the couple and possibly its observed characteristics such as the birth dates of the spouses, the difference between their ages [Youn and Shemyakin (1999)], or their age at the day of their marriage or common law. In the application, such static couple characteristics will be introduced to capture the generation effects. The analysis is in continuous time and the lifetime variables are continuous variables.

2.1 The latent model

Let us consider a given couple with two spouses 1 and 2. The potential lifetimes of individuals 1 and 2, when both are alive, are denoted by $X_1$ and $X_2$, respectively. To get a unique time origin for the two members of the couple, these latent lifetimes are measured since the beginning of the common life. A first individual in the couple dies at date $\min(X_1, X_2)$. He/she is individual 1 (resp. individual 2), if $\min(X_1, X_2) = X_1$ [resp. $\min(X_1, X_2) = X_2$]. After this event, there can be a change in the potential residual lifetime distribution of the surviving individual. The potential residual lifetime of individual 1 (resp. individual 2) after the death of individual 2 (resp. individual 1) is denoted by $X_3$ (resp. $X_4$).

The joint distribution of the four latent variables is characterized by

i) the joint survival function of $(X_1, X_2)$:

$$S_{12}(x_1, x_2) = \mathbb{P}[X_1 > x_1, X_2 > x_2];$$  \hspace{1cm} (2.1)

ii) the survival function of $X_3$ given $X_2 = \min(X_1, X_2) = z$:

$$S_3(x_3; z) = \mathbb{P}[X_3 > x_3 | X_2 = \min(X_1, X_2) = z].$$ \hspace{1cm} (2.2)

iii) The survival function of $X_4$ given $X_1 = \min(X_1, X_2) = z$:

$$S_4(x_4; z) = \mathbb{P}[X_4 > x_4 | X_1 = \min(X_1, X_2) = z].$$ \hspace{1cm} (2.3)

These three joint and conditional survival functions, defined on $(0, \infty)$, characterize the latent model for the analysis of the mortality in the couple. In this model there exist at least three generation effects corresponding to the generations of each spouse, and to the generation of the couple, respectively.
2.2 Individual lifetimes

2.2.1 Link between the individual lifetimes and the latent variables

The lifetimes of individuals 1 and 2 (since the beginning of the common life) are denoted by $Y_1$ and $Y_2$. They can be expressed in terms of the latent variables as:

\[
\begin{align*}
Y_1 &= X_1 \mathbb{1}_{Y_1 < Y_2} + (X_2 + X_3) \mathbb{1}_{X_2 < X_1} = \min(X_1, X_2) + X_3 \mathbb{1}_{X_2 < X_1}, \\
Y_2 &= X_2 \mathbb{1}_{X_2 < X_1} + (X_1 + X_4) \mathbb{1}_{X_1 < X_2} = \min(X_1, X_2) + X_4 \mathbb{1}_{X_1 < X_2}.
\end{align*}
\]

(2.4)

This system can be partially solved. First, the $X_1, X_2$ variables are related to variables $(Y_1, Y_2)$ since:

\[
\min(Y_1, Y_2) = \min(X_1, X_2), \quad \text{and} \quad Y_1 > Y_2, \quad \text{if and only if} \quad X_1 > X_2.
\]

Then the variables $X_3$ and $X_4$ can be deduced in some regimes\footnote{There are two regimes, corresponding respectively to the case $Y_1 < Y_2$ and $Y_2 < Y_1$.} since:

\[
X_3 \mathbb{1}_{Y_2 < Y_1} = Y_1 - \min(Y_1, Y_2) \quad \text{and} \quad X_4 \mathbb{1}_{Y_1 < Y_2} = Y_2 - \min(Y_1, Y_2).
\]

As noted in Norberg (1989), the observed model can be interpreted in terms of a chain with four possible states\footnote{In their analysis Ji et al. (2011) consider also the possibility of a direct transition from state 1 to state 4 to account for catastrophic events (car accidents, plane crash) implying simultaneous deaths. They use a 5-day cutoff to account for a possible lag in reporting.} that are:

- state 1: both spouses are alive,
- state 2: husband dead, wife alive,
- state 3: husband alive, wife dead,
- state 4: both spouses are dead,

and transitions can only arise between states 1 and 2, 1 and 3, 2 and 4, and 3 and 4. Since the mortality intensity of a spouse can depend not only on the current state, but potentially on the time elapsed since the death of the other spouse, we get an example of a semi-Markov chain.
2.2.2 The joint density function and its decomposition

The joint probability density function (pdf) of \((Y_1, Y_2)\) is easily derived from the distribution of the latent variables. We have (see Appendix 1):

\[
f(y_1, y_2) = \left[ -\frac{\partial S_{12}}{\partial x_1}(y_1, y_1) \right] \left[ -\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1) \right], \quad \text{if } y_2 > y_1, \tag{2.5}
\]

\[
f(y_1, y_2) = \left[ -\frac{\partial S_{12}}{\partial x_2}(y_2, y_2) \right] \left[ -\frac{\partial S_3}{\partial x_3}(y_1 - y_2; y_2) \right], \quad \text{if } y_1 > y_2.
\]

Therefore, the joint density function can feature a discontinuity when \(y_1 = y_2\).

Let us consider the case \(y_2 > y_1\). The density can also be written as:

\[
f(y_1, y_2) = -\frac{\partial S^*}{\partial y}(y_1) \left[ \frac{\partial S_{12}}{\partial x_1}(y_1, y_1)/\partial S^*(y_1) \right] \left[ -\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1) \right], \tag{2.6}
\]

where \(S^*(y) = S_{12}(y, y)\) is the survival function of \(\min(X_1, X_2)\) and

\[
\frac{\partial S^*}{\partial y}(y) = \frac{\partial S_{12}}{\partial x_1}(y, y) + \frac{\partial S_{12}}{\partial x_2}(y, y). \quad \text{Thus, the decomposition of the bivariate density involves three components: }
\]

\[
i) \left[ -\frac{\partial S^*}{\partial y}(y_1) \right] \text{ is the density of the first death event; }
\]

\[
ii) \text{the ratio } \left[ \frac{\partial S_{12}}{\partial x_1}(y_1, y_1)/\partial S^*(y_1) \right] \text{ is the probability that individual 1 dies at this first death event. It is equal to: }
\]

\[
\mathbb{P}[Y_1 < Y_2|\min(Y_1, Y_2) = y_1],
\]

\[
iii) \left[ -\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1) \right] \text{ is the density of the residual lifetime after this event. }
\]

2.2.3 Individual mortality intensities

Let us now derive the individual mortality intensities given the current information concerning the couple. Their expressions depend on the state either alive, or dead, of the other spouse.

i) Let us first consider a date \(y\) at which both individuals are still alive, that is, such that \(Y_1 \geq y, Y_2 \geq y\). The mortality intensity of individual 1 is defined by:
\[
\lambda_1(y|Y_1 \geq y, Y_2 \geq y) = \lim_{dy \to 0^+} \left\{ \frac{1}{dy} P[y \leq Y_1 \leq y + dy|Y_1 \geq y, Y_2 \geq y] \right\}
= \int_y^\infty f(y, y_2)dy_2/S^*(y). \tag{2.7}
\]

After replacing the bivariate density by its expression (2.5) for \(y_2 > y_1\) and computing the integral, we get:

\[
\lambda_1(y|Y_1 \geq y, Y_2 \geq y) = \left[ -\frac{\partial S_{12}}{\partial x_1}(y, y) \right] / S^*(y). \tag{2.8}
\]

This is the crude intensity function of individual 1 involved in the decomposition of the joint density function.

Similarly, we have:

\[
\lambda_2(y|Y_1 \geq y, Y_2 \geq y) = \lim_{dy \to 0^+} \left\{ \frac{1}{dy} P[y \leq Y_2 \leq y + dy|Y_1 \geq y, Y_2 \geq y] \right\}
= \int_y^\infty f(y_1, y)dy_1/S^*(y). \tag{2.9}
\]

\[
= \left[ -\frac{\partial S_{12}}{\partial x_2}(y, y) \right] / S^*(y).
\]

ii) The expression of the mortality intensities can change if one of the individuals dies exactly at date \(y\). The mortality intensity of individual 1 at date \(y\), if individual 2 dies at date \(y\), becomes:

\[
\lambda_{1\mid 2}(y|Y_1 \geq y, Y_2 = y)
= \lim_{dy \to 0^+} \left[ \frac{1}{dy} P(y < Y_1 \leq y + dy|Y_1 \geq y, Y_2 = y) \right]
= [f(y, y)] / \left[ -\frac{\partial S_{12}}{\partial x_2}(y, y) \right]
= -\frac{\partial S_3}{\partial x_3}(0, y), \tag{2.10}
\]

by applying the expression of the joint density (2.5) with \(y_1 = y_2 = y\).

Similarly, we get:
\[
\lambda_{2|1}(y|Y_1 = y, Y_2 \geq y) = \lim_{dy \to 0^+} \left\{ \frac{1}{dy} P[y \leq Y_2 \leq y + dy|Y_1 = y, Y_2 \geq y] \right\} = -\frac{\partial S_4}{\partial x_4}(0, y). \tag{2.11}
\]

Note that \( S_3(0, y) = S_4(0, y) = 1 \). Therefore we also have:
\[
\lambda_{1|2}(y|Y_1 \geq y, Y_2 > y) = -\frac{\partial \log S_2}{\partial x_3}(0, y),
\]
and \( \lambda_{2|1}(y|Y_1 = y, Y_2 \geq y) = -\frac{\partial \log S_4}{\partial x_4}(0, y), \)
which are the expected expressions of the intensities in terms of survival functions.

iii) Finally, we can also consider the mortality intensity of spouse 1, when the other spouse is dead since a given time. We have, for \( y > y^* \):
\[
\lambda_{1|2}(y|Y_1 \geq y, Y_2 = y^*)
\]
\[
= \lim_{dy \to 0^+} \frac{1}{dy} P[y < Y_1 < y + dy|Y_1 \geq y, Y_2 = y^*]
\]
\[
= f(y, y^*)/ \int_y^\infty f(u, y^*)du
\]
\[
= -\frac{\partial \log S_3}{\partial x_3}(y - y^*, y^*),
\]
which is just the intensity of the residual lifetime \( X_3 \) given the date of the first death.

2.2.4 Dependence and Jump in Intensities

It has been suggested in [Clayton (1978)] to measure the dependence between duration variables by considering the jump in intensities following the news of a death. We get a functional measure of dependence function of the age \( y \) of the couple, which is especially appropriate for following the dependence phenomenon during the couple life. These per-cent jumps are the following ones:

When individual 2 dies at date \( y \), the jump at this date of the mortality intensity of individual 1 is:

\[
\]
\[
\gamma_{1|2}(y) = \frac{\lambda_{1|2}(y|Y_1 \geq y, Y_2 = y)}{\lambda_1(y|Y_1 \geq y, Y_2 \geq y)} = \left\{ \left[ -\frac{\partial S_3}{\partial x_3}(0; y) \right] S^*(y) \right\} / \left[ -\frac{\partial S_{12}}{\partial x_1}(y, y) \right].
\]

(2.12)

Symmetrically, we get:

\[
\gamma_{2|1}(y) = \frac{\lambda_{2|1}(y|Y_1 = y, Y_2 \geq y)}{\lambda_2(y|Y_1 \geq y, Y_2 \geq y)} = \left\{ \left[ -\frac{\partial S_4}{\partial x_4}(0; y) \right] S^*(y) \right\} / \left[ -\frac{\partial S_{12}}{\partial x_2}(y, y) \right].
\]

(2.13)

In the standard literature on bivariate survival models, the bivariate density function is continuous at \(y_1 = y_2 = y\). Then, the two measures \(\gamma_{1|2}(y)\) and \(\gamma_{2|1}(y)\) coincide for any age \(y\) [see the discussion in Sections 3.2, 3.2.2]. This regularity assumption is not necessarily satisfied in a Freund model. We can observe different reactions of a spouse at the death of the other spouse in the couple.

**Definition 1**: We have the broken-heart syndrome for spouse 1 (resp. 2) at date \(y\), if \(\gamma_{1|2}(y) > 1\) [resp. \(\gamma_{2|1}(y) > 1\)].

We can have the broken-heart syndrome (or the reverse broken-heart syndrome when the directional measure of association is strictly smaller than 1), with different magnitude according to the age and spouse. We can even observe reactions in different directions. This arises when the wife is devastated by the death of her husband, with an increase of her mortality intensity, whereas the death of the wife may provide more freedom to her husband and possibly a decrease of his mortality rate. This is the "love and death" phenomenon with the fact that love is not always shared and can be age-dependent.

There exist a few studies trying to measure the effect and showing a positive estimated broken-heart syndrome [see e.g. Parkes et al. (1969), Jagger and Sutton (1991), Ji et al. (2011)]. Moreover it is shown that the broken-heart syndrome affects widowers more than widows [see Spreew and Owadally (2013)]. However, by neglecting the frailty effect discussed in Section 3, the estimates may suffer from an omitted heterogeneity bias.

### 2.3 Observed and latent intensities

Let us now link the distributions of the observed and latent variables. Since \((X_1, X_3)\) and \((X_2, X_4)\) cannot be simultaneously observed, let us assume that these
two couples are independent. Then the distribution of the latent variables is characterized by the following latent intensities:

i) the latent intensity of $X_1$ denoted by $a_1(x_1)$;
ii) the latent intensity of $X_2$ denoted by $a_2(x_2)$;
iii) the latent intensity of $X_3$ given $X_2 = \min(X_1, X_2) = z$, denoted by $a_3(x_3; z)$;
iv) the latent intensity of $X_4$ given $X_1 = \min(X_1, X_2) = z$, denoted by $a_4(x_4; z)$.

The associated cumulated intensities, that are their primitives with respect to the $x$ argument, are denoted by $A_1(x_1), A_2(x_2), A_3(x_3; z), A_4(x_4; z)$, respectively. We deduce that:

\[
S_{12}(x_1, x_2) = \exp\{-[A_1(x_1) + A_2(x_2)]\}, \quad S_3(x_3; z) = \exp[-A_3(x_3; z)],
\]

\[
S_4(x_4; z) = \exp[-A_4(x_4; z)]
\]

Then, the expression (2.5) of the bivariate probability density function becomes:

\[
f(y_1, y_2) = a_1(y_1) \exp\{-[A_1(y_1) + A_2(y_2)]\} a_4(y_2 - y_1; y_1) \exp[-A_4(y_2 - y_1; y_1)], \quad \text{if } y_2 > y_1,
\]

\[
= a_2(y_2) \exp\{-[A_1(y_1) + A_2(y_2)]\} a_3(y_1 - y_2; y_2) \exp[-A_3(y_1 - y_2; y_2)], \quad \text{if } y_1 > y_2.
\]

(2.14)

Similarly the directional measures of association can be written in terms of the latent intensities by using the expressions (2.12)-(2.13).

**Property 1**:

The directional measures of association are:

\[
\gamma_{1|2}(y) = a_3(0; y)/a_1(y), \quad \gamma_{2|1}(y) = a_4(0; y)/a_2(y).
\]

(2.15)

## 3 Freund model with static frailty

The notion of (shared) frailty is first introduced by [Vaupel et al. (1979)](1979). The idea is to use unobserved heterogeneity (or frailty) in bivariate duration models in order
to create an additional dependence between lifetimes. In the basic specification, this frailty is static, since it depends on the couple only, neither on time, nor age. It represents the effect of common lifestyle, or common disasters encountered by the couple. In the extended model, the dependence between the lifetimes are due to either the exogenous shock (the frailty), or to the so-called contagion effects, that are the jumps in the intensities at the time of default. This type of specification allows to disentangle these two effects. We first extend the Freund model of Section 2.4 to include unobserved frailty. Then, we discuss special cases.

3.1 The model

Let us denote by $F$ the frailty variable, possibly multivariate. We consider a Freund model with the structure introduced in Section 2.4, where $X_1$ and $X_2$ are independent conditional on $F$, with latent intensities conditional on $F$ given by: $a_1(x_1; F), a_2(x_2; F), a_3(x_3; z; F), a_4(x_4; z, F)$. Let us now derive the latent survival functions $S_{12}(x_1, x_2), S_3(x_3; z), S_4(x; z)$, when frailty $F$ has been integrated out. We have:

$$S_{12}(x_1, x_2) = \mathbb{E}\left[\mathbb{P}[X_1 \geq x_1, X_2 \geq x_2 | F]\right] = \mathbb{E}\{\exp[-A_1(x_1; F) + A_2(x_2; F)]\},$$

where the expectation is taken with respect to the distribution of $F$.

Similarly we get:

$$S_3(x_3; z) = \mathbb{P}[X_3 > x_3 | X_2 = \min(X_1, X_2) = z] = \mathbb{P}[X_3 > x_3 | X_2 = z, X_1 > z] = \frac{\mathbb{E}[a_2(z, F) \exp(-[A_1(z, F) + A_2(z; F) + A_3(x_3; z; F)])]}{\mathbb{E}[a_2(z; F) \exp(-[A_1(z; F) + A_2(z; F)])]}.$$

These formulas can be used as inputs to derive the bivariate observed density (2.5) and the directional measures of association (2.12)-(2.13). For instance, we have by (2.12):

$$\gamma_{1|2}(y) = \frac{\mathbb{E}\{a_3(0; y; F)a_2(y, F) \exp(-[A_1(y; F) + A_2(y; F)])\} \mathbb{E}[\exp(-[A_1(y; F) + A_2(y; F)])]}{\mathbb{E}\{a_2(y; F) \exp(-[A_1(y; F) + A_2(y; F)])\} \mathbb{E}\{a_1(y; F) \exp[-A_1(y; F) + A_2(y; F)]\}}.$$

We deduce the property below.

---

6Note that the model has two layers of latent variables, first $F$, second $X_1, X_2, X_3, X_4$. 
Property 2:

\[
\gamma_{1|2}(y) = \frac{Q_y}{E[a_2(y; F)]} \frac{Q_y}{E[a_1(y; F)]} \frac{Q_y}{E[a_2(y; F)]}, \tag{3.1}
\]

where \( Q_y \) denotes the probability distribution with density:

\[
q_y(F) = \exp\{-[A_1(y) + A_2(y)]F\}/E[\exp(-(A_1(y) + A_2(y))F)],
\]

with respect to the distribution of \( F \).

The change of probability is due to the aging of the heterogeneity structure in the population of surviving couples, called Population-at-Risk (PaR) at age \( y \) [see e.g. Vaupel et al. (1979), eq. (5)].

Since the conditional directional measure of association is [see (2.15)]:

\[
\gamma_{1|2}(y; F) = a_3(0, y; F)/a_1(y, F),
\]

we can also write the corresponding unconditional measure as:

\[
\gamma_{1|2}(y) = \frac{Q_y}{E[a_2(y; F)]} \frac{Q_y}{E[a_1(y; F)]} \frac{Q_y}{E[a_2(y; F)]} \frac{Q_y}{E[a_1(y; F)]} \frac{Q_y}{E[a_2(y; F)]},
\]

where:

\[
dQ_y = \frac{a_1(y; F)a_2(y; F)}{E[a_1(y; F)]E[a_2(y; F)]}dQ_y.
\]

Thus the unconditional directional measure of association \( \gamma_{1|2}(y) \) is an average of the conditional directional measures of association with respect to a modified probability distribution, and adjusted for the dependence between \( a_1(y; F) \) and \( a_2(y; F) \), since the adjustment term equals 1, when these variables are not correlated under \( Q_y \).

3.2 Single proportional frailty

Following Vaupel et al. (1979), it is usual to consider a single positive frailty with the same effect on all latent intensities. This implies an Archimedean copula for the
bivariate latent variables $X_1$ and $X_2$ see [Oakes (1989)], but not for the observed variables $Y_1, Y_2$, due to the changes in intensities after the first death event. More precisely, if:

$$a_1(x_1; F) = a_1(x_1)F, a_2(x_2; F) = a_2(x_2)F, a_3(x_3; z; F) = a_3(x_3)F; a_4(x_4; z; F) = a_4(x_4)F,$$

we deduce from Property 2 eq.(3.1) that:

$$
\begin{align*}
\gamma_{1|2}(y) &= \frac{a_3(0; y) \mathbb{E}(F^2)}{a_1(y) \mathbb{E}(F^2)^2}, \\
\gamma_{2|1}(y) &= \frac{a_4(0; y) \mathbb{E}(F^2)}{a_2(y) \mathbb{E}(F)^2}.
\end{align*}
$$

(3.2)

In this simple case, the directional measures of association given $F$ are [see (2.15)]:

$$
\begin{align*}
\gamma_{1|2}(y; F) &= \frac{a_3(0; y) F}{a_1(y)} = \frac{a_3(0; y)}{a_1(y)}, \\
\gamma_{2|1}(y; F) &= \frac{a_4(0; y)}{a_2(y)}.
\end{align*}
$$

They are independent of the frailty $F$, but not necessarily equal, which allows for asymmetric reactions.

The omitted heterogeneity introduces a positive bias on these measures. Indeed, we have $\frac{\mathbb{E}(F^2)}{[\mathbb{E}(F)]^2} \geq 1$, by Cauchy-Schwartz inequality and more generally the property below:

**Property 3**: In a Freund model with single proportional frailty the unconditional directional measures of association are larger than the conditional ones. They are equal if and only if frailty $F$ is constant, that is, if there is no omitted heterogeneity:

$$
\gamma_{1|2}(y) \geq \gamma_{1|2}(y; F), \gamma_{2|1}(y) \geq \gamma_{2|1}(y; F), \forall F.
$$

However the per-cent adjustment for omitted heterogeneity is independent of age $y$ and of the direction, which is considered. In particular the symmetry condition between spouses is preserved since:

$$
\gamma_{1|2}(y; F) = \gamma_{2|1}(y; F) \iff \gamma_{1|2}(y) = \gamma_{2|1}(y).
$$
3.3 The actuarial literature

The models with mortality dependence considered in the actuarial literature are often special cases of the single proportional frailty model of Section 3.2.1, assuming moreover the continuity of the latent intensities:

**Continuity assumption of the latent intensities**

\[ a_3(x; z) = a_1(x + z), \forall x, z, \]

\[ a_4(x; z) = a_2(x + z), \forall x, z. \]

Under the continuity assumption, the lifetimes \( Y_1, Y_2 \) are independent given the shared frailty \( F \), with joint conditional survivor function:

\[ S_{12}(y_1, y_2 | F) = \exp[-[A_1(y_1) + A_2(y_2)]]. \]

To ensure the positivity of the intensity, the frailty \( F \) has to be positive. Let us denote by \( \psi \) its Laplace transform defined for positive arguments \( u \) by:

\[ \psi(u) = \mathbb{E}[\exp(-uF)]. \] (3.3)

By integrating out the frailty, we deduce the joint survivor function:

\[ S_{12}(y_1, y_2) = \psi[A_1(y_1) + A_2(y_2)]. \] (3.4)

A similar computation can be performed to derive the marginal survivor functions. We get:

\[ S_1(y_1) = \psi[A_1(y_1)], S_2(y_2) = \psi[A_2(y_2)]. \] (3.5)

Since the Laplace transform of \( F \) is continuous and strictly increasing, it is invertible. We deduce the expression of \( S_{12} \) in terms of \( S_1, S_2 \) and \( \psi \):

\[ S_{12}(y_1, y_2) = \psi[\psi^{-1}[S_1(y_1)] + \psi^{-1}[S_2(y_2)]] \] (3.6)

This is the standard definition of a copula [Sklar (1959)]:

\[ S_{12}(y_1, y_2) = C[S_1(y_1), S_2(y_2)], \] (3.7)

with a survivor Archimedean copula [Genest and MacKaye (1986)]:

\[ C(u_1, u_2) = \psi[\psi^{-1}(u_1) + \psi^{-1}(u_2)], \] (3.8)

**Property 4**: Let us consider a Freund model with single proportional frailty. Under the continuity assumption, the dependence between the lifetime variables
$Y_1, Y_2$ is summarized by an Archimedean copula with the Laplace transform of the frailty as generator.

Conversely, most usual Archimedean copula admit a frailty interpretation. The actuarial literature has considered this special case with different choices of the marginal distributions of the lifetimes and of the copulas [see Tables 3.1 and 3.2, for the actuarial literature, and Nelsen (1999) for a rather extensive list of copulas].

Table 3.1

<table>
<thead>
<tr>
<th></th>
<th>Selected Marginal Distributions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weibull</td>
<td>Free et al. (1996), Youn, Shemyakin (1999), (2001), Shemyakin, Youn (2006),</td>
</tr>
</tbody>
</table>

Table 3.2

<table>
<thead>
<tr>
<th></th>
<th>Selected Copula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>Luciano et al. (2008), (2010), Spreew (2006)</td>
</tr>
<tr>
<td>Normal</td>
<td>Carriere (2000)</td>
</tr>
<tr>
<td>4.2.20 Nelsen copula</td>
<td>Spreew (2006), Luciano et al. (2008), (2010)</td>
</tr>
</tbody>
</table>

7 Indeed the Archimedean copulas that admit this representation are those whose generator is completely monotone, see McNeil and Nešlehová (2009) for a characterization of Archimedean copulas.

8 Some authors consider non Archimedean copulas, for instance normal copulas in Carriere (2000). However, these copulas are still continuous and thus do not allow for asymmetric reactions.

9 More precisely, these authors use a stochastic extension of the Gompertz law.

10 The numbers 4.2.20 indicate the copula in the list provided by Nelsen (1999).
A more recent literature [see e.g. Denuit and Cornet (1999), Spreeuw and Wang (2008), Ji et al. (2011), Spreeuw and Owadally (2013)] focuses on the broken-heart syndrome, but without introducing frailty in the specification of the intensities.

3.4 Affine intensity model

A simple extension of the bivariate survival model discussed in Section 3.2 is obtained by introducing an intercept in the basic proportional frailty model [the so-called Generalized Shared Frailty model developed in Iachine (2004) in a special case]. The specification becomes:

\[
\begin{align*}
a_1(x_1; F) &= a_1(x_1)F + b_1(x_1), \\
a_2(x_2; F) &= a_2(x_2)F + b_2(x_2), \\
a_3(x_3; z; F) &= a_3(x_3; z)F + b_3(x_3; z), \\
a_4(x_4; z; F) &= a_4(x_4; z)F + b_4(x_4; z).
\end{align*}
\]

This extended version allows for conditional directional measures of association \(\gamma_{1|2}(y; F)\) and \(\gamma_{2|1}(y; F)\) depending on frailty \(F\), and leads to non-Archimedean copulas, when considering the joint distribution of latent lifetimes \(X_1\) and \(X_2\).

The affine specification is likely the most appropriate one for representing the effect of common lifestyle \(F\) and especially the memory features. After the death of a spouse, we expect that the effect of common lifestyle will diminish and asymptotically vanish. Thus, we expect that the latent intensity \(a_3(x_3; z)\) [resp. \(a_4(x_4; z)\)] is a decreasing function of \(x_3\) (resp. \(x_4\)) tending to zero at infinity. Then functions \(b_3\) and \(b_4\) provide the limiting mortality intensity a long time after the death of the other spouse. See also Spreeuw and Wang (2008) for a detailed discussion on the long-term and short-term effect of losing his/her partner.

Finally, note that this affine intensity models assumes implicitly no remarriage or new common law of the surviving spouse. This assumption is rather realistic for our purpose, since the insurance policies of interest are generally taken by rather old couples to profit of estate tax reductions, or to provide a rent to the surviving spouse.

4 Pricing bivariate contracts

We will now derive the pricing formulas for insurance contracts written on two lives such as joint life policies, last survivor policies and policies with reversionary annuities. By considering extended Freund models (under the risk-neutral probability), we analyze the effect of jumps in intensity on prices at the contract issuing as well as on the premium updating during the life of the contract.
4.1 Prices at the inception of the contracts

The premium computations for the joint policies are based on the joint remaining lifetimes risk-neutral distribution conditional on the ages of the spouses at the beginning of their common life \( y^{*}_{10}, y^{*}_{20} \), say, and on the fact that both spouses are still alive with an age of the couple equal to \( z_0 \), say, at the inception of the contract. Thus, the joint risk-neutral density of the remaining lifetimes \( \tilde{y}_j = Y_j - z_0, j = 1, 2 \) at the inception of the contract is:

\[
\tilde{f}_0(\tilde{y}_1, \tilde{y}_2 | z_0) = \lim_{dy_1, dy_2 \to 0} \frac{1}{dy_1 dy_2} P[Y_1 \in (\tilde{y}_1 + z_0, \tilde{y}_1 + z_0 + dy_1), Y_2 \in (\tilde{y}_2 + z_0, \tilde{y}_2 + z_0 + dy_2) | Y_1 > z_0, Y_2 \geq z_0, y^{*}_{10}, y^{*}_{20}] = f_0(\tilde{y}_1 + z_0, \tilde{y}_2 + z_0)/S_0(z_0),
\] (4.1)

where the index 0 means that the distribution characteristics of Section 3 can now depend on the initial ages \( y^{*}_{10}, y^{*}_{20} \).

Let us now illustrate the premium computation in a continuous time framework with instantaneous constant interest rate \( r \). For each insurance product, we have to analyze the risk-neutral distribution of the discounted cash-flows.

i) Joint life policy

Let us denote by \( a \) the premium rate and consider a unitary insurance payoff at the first death of a spouse. The discounted sequence of cash-flows measured at the inception of the contract is:

\[
C_0^{(1)}(a, r, z_0; Y_1, Y_2) = a \int_0^{\min(Y_1, Y_2) - z_0} \exp(-rh)dh - \exp[-r(\min(Y_1, Y_2) - z_0)] = \frac{a}{r} \{1 - \exp[-r(\min(Y_1, Y_2) - z_0)]\} - \exp[-r(\min(Y_1, Y_2) - z_0)].
\] (4.2)

There exist different ways for balancing the stochastic positive and negative

\[\text{11} \text{The link between the historical and risk-neutral bivariate distributions of the lifetimes is discussed in Appendix 2. Note that the insurance literature often price the insurance contracts by means of the historical distributions to get the so called fair premium [see e.g. Ji et al. (2011), Section 5.6].}\]
cash-flows. In particular the premium rate\textsuperscript{12} can be defined by fixing equal expectations to these sequences. We get:

\[ a_0^{*(1)}(r) = r \frac{E_0\{\exp[-r(\min(Y_1, Y_2) - z_0)]|Y_1 \geq z_0, Y_2 \geq z_0}\}}{1 - E_0\{\exp[-r(\min(Y_1, Y_2) - z_0)]|Y_1 \geq z_0, Y_2 \geq z_0}\}}. \quad (4.3) \]

\[ ii) \text{ Last survivor policy} \]

Let us now assume that the death event written in the policy is the second death of a spouse. The formulas are the same as for the joint life policy above after substituting $\max(Y_1, Y_2)$ to $\min(Y_1, Y_2)$. For instance, the fair premium becomes:

\[ a_0^{*(2)}(r) = r \frac{E_0\{\exp[-r(\max(Y_1, Y_2) - z_0)]|Y_1 \geq z_0, Y_2 \geq z_0}\}}{1 - E_0\{\exp[-r(\max(Y_1, Y_2) - z_0)]|Y_1 \geq z_0, Y_2 \geq z_0}\}}. \quad (4.4) \]

\[ iii) \text{ Reversionary annuities} \]

Finally, let us consider a product in which the premium is paid when both spouses are alive and a unitary annuity is paid to the surviving spouse up to his/her death. The discounted sequence of cash-flows becomes:

\[ C^{(3)}(a, r, z_0; Y_1, Y_2) = a \int_0^{\min(Y_1, Y_2) - z_0} \exp(-rh)dh - \int_{\min(Y_1, Y_2) - z_0}^{\max(Y_1, Y_2) - z_0} \exp(-rh)dh \]

\[ = a \left\{ 1 - \exp(-[\min(Y_1, Y_2) - z_0]) \right\} \]

\[ - \frac{1}{r} \left\{ \exp(-[\min(Y_1, Y_2) - z_0]) \right\} \]

\[ - \exp(-[\max(Y_1, Y_2) - z_0]). \quad (4.5) \]

The associated premium rate is:

\[ a_0^{*(3)}(r) = \frac{E_0\{\exp(-[\min(Y_1, Y_2) - z_0]) - \exp(-[\max(Y_1, Y_2) - z_0])|Y_1 \geq z_0, Y_2 \geq z_0}\}}{1 - E_0\{\exp(-[\min(Y_1, Y_2) - z_0])|Y_1 \geq z_0, Y_2 \geq z_0\}}. \quad (4.6) \]

\[ iv) \text{ Individual products} \]

The premia for joint products have naturally to be compared with the premia of the individual life insurance products written on a single head.

\textsuperscript{12}The fair premium rate is obtained by replacing the risk-neutral distribution by the historical distribution in formula (4.3). Otherwise the premium rate accounts for a risk premium.
$j = 1, 2$. The associated fair premium is:

$$a_{j,0}^*(r) = r \frac{E_0(\exp[-r(Y_j - z_0)]|Y_j \geq z_0)}{1 - E_0(\exp[-r(Y_j - z_0)]|Y_j \geq z_0)},$$

(4.7) if only information on spouse $j$ is taken into account and

$$a_{j,0}^{**}(r) = r \frac{E_0(\exp[-r(Y_j - z_0)]|Y_1 \geq z_0, Y_2 \geq z_0)}{1 - E_0(\exp[-r(Y_j - z_0)]|Y_1 \geq z_0, Y_2 \geq z_0)},$$

(4.8) if the information on the couple is taken into account.

In the limiting case of a zero risk-free rate $r = 0$, the expressions of the premia are obtained by a Taylor expansion. We get:

$$a_{0}^{(1)}(0) = \frac{1}{E_0\{\min(Y_1, Y_2) - z_0|Y_1 \geq z_0, Y_2 \geq z_0\}},$$

$$a_{0}^{(2)}(0) = \frac{1}{E_0\{\max(Y_1, Y_2) - z_0|Y_1 \geq z_0, Y_2 \geq z_0\}},$$

$$a_{0}^{(3)}(0) = \frac{E_0\{\max(Y_1, Y_2) - \min(Y_1, Y_2)|Y_1 \geq z_0, Y_2 \geq z_0\}}{E_0\{\min(Y_1, Y_2)|Y_1 \geq z_0, Y_2 \geq z_0\}},$$

$$a_{j,0}^*(0) = \frac{1}{E_0\{Y_j - z_0|Y_j \geq z_0\}},$$

$$a_{j,0}^{**}(0) = \frac{1}{E_0\{Y_j - z_0|Y_1 \geq z_0, Y_2 \geq z_0\}}.$$

Note that the pricing of the individual contracts of two spouses cannot be done separately. The price of the contract of a widow has to account for the time elapsed since the death of her husband.

4.2 Effect of risk dependence on prices

Let us now illustrate the effect on policy prices of risk dependencies: due to the frailty and to the asymmetric jump in intensities existing in a Freund model.

We consider a model with single proportional frailty (see Section 3.2). The population of couples is such that the two spouses have the same age 30. The distribution of the heterogeneity $F$ at age 30 is assumed to be a gamma distribution. Note that when there is no jump in latent intensities, the joint distribution of the lifetimes is associated to a Clayton copula. Due to the mover-stayer phenomenon, as the population ages, the distribution given that both spouses survive up to age $z_0 > 30$, that is, the heterogeneity distribution that the insurance company applies to price a contract for a couple with an underwriting age $z_0 > 30$, will depend on
Intensities of the latent duration variables $X_1$ (female), $X_2$ (male) are of the following form:

$$a_1(x) = \exp(\alpha_1 x + \beta_1), \quad \forall x > 0,$$

and

$$a_2(x) = \exp(\alpha_2 x + \beta_2), \quad \forall x > 0.$$

For illustration purpose, we assume that the death of the spouse has a constant multiplicative effect $\gamma$ on the mortality intensity of the survivor. Thus, given $z = \min(X_1, X_2)$, the conditional intensities of $X_3, X_4$ are of the form:

$$a_3(x_3, z) = \gamma \exp\left(\alpha_1 (z + x_3) + \beta_1\right), \quad \forall x_3 > 0,$$

and

$$a_4(x_4, z) = \gamma \exp\left(\alpha_2 (z + x_4) + \beta_2\right), \quad \forall x_4 > 0,$$

where the constant $\gamma = \frac{a_3(0,z)}{a_1(z)} = \frac{a_4(0,z)}{a_2(z)}$ is larger than 1 to reflect the broken-heart syndrome. For numerical illustrations, parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ are chosen to fit the marginal intensities of American females and males at ages 31, 32, ..., 110, provided by the Human Mortality Database. Their values are reported below:

$$\alpha_1 = 0.089, \beta_1 = -7.613, \alpha_2 = 0.081, \beta_2 = -6.934.$$

The measure of association $\gamma$ is the same in both directions with values $\gamma \in \{1, 3, 5\}$. $\gamma = 5$ corresponds to a very huge impact of the death of the spouse on the survivor lifetime and $\gamma = 1$ corresponds to the case of no impact (at the individual level, indeed, even in this case there is still jump of intensity when the heterogeneity is integrated out, see e.q.(3.2)). The gamma distribution of the heterogeneity at age 30 is set to have a shape parameter $k$ and a scale parameter $1/k$. Therefore, the average mortality intensity at age 30 is the same for each value of $k$, since $E(F) = 1/k \cdot k = 1$ does not depend on $k$. The heterogeneity parameter $k$ will be set to $k \in \{2, 5, 10\}$. $k = 10$ corresponds to a low heterogeneity level and $k = 2$ corresponds to a high one. This specification of the duration distribution is the risk-neutral distribution, which can be used to price the different life insurance contracts described in Section 4.1. The risk-free interest rate is set to $r = 1\%$.

We provide in Figure 1 the evolution of the premium rates as a function of the

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13The Human Mortality Database (HMD) was created to provide detailed mortality and population data to researchers, students, journalists, policy analysts, and others interested in the history of human longevity. It is maintained by the University of California, Berkeley, and the Max Planck Institute for Demographic Research in Rostock, Germany; its official website is http://www.mortality.org.
underwriting age $z_0 \in 31, 32, \ldots, 80$, for different contracts and for $\gamma = 5, k = 2$. The contracts include a joint life policy, a last survivor policy, a contract with reversionary annuities, and the individual insurance products for female with, or without, the information on the survival of the husband up to $z_0$.

These premia are not directly comparable, since the premia paid by the insured people (resp. the payments by the insurance company) do not correspond to a same period. Nevertheless for each product, the premium rate is increasing with the age of underwriting of the couple, which is in conformity with the usual premium structure without heterogeneity.

In general, in a model with heterogeneity, the average intensity (as well as of the premium) can be or not be increasing in $z_0$. Indeed, the aging of the population has a positive impact on the premium when $z_0$ increases, while the mover-stayer phenomenon has a negative impact on the premium since couples with higher risks die out more quickly, hence the average heterogeneity is improving in time. In this example, the first effect is more important, which results in an increasing premium.

Besides, the premium rate of an individual insurance contract for a female is always lower when the insurance company know that her spouse is still alive, as shown in the lower right panel. The difference is negligible at low ages, but increases significantly with respect to $z_0$. We also observe that the curves of the premia are convex, except for reversionary annuities, where the trend is almost linear.

Let us now illustrate the effect of risk dependencies and of the heterogeneity for the different insurance contracts. We first illustrate in Tables 6.1 and 6.2 the effect of the measure of association $\gamma$ for two different ages 30 and 50. This parameter has no effect on the joint insurance policies: indeed, the contract terminates up to the first death whereas the measure of association impacts only the residual lifetime beyond the first death event. Therefore, premium rates of the joint insurance are not reported in the Tables. The two last columns correspond to the individual insurance contract for a female with and without information on the survival of her spouse. We get premia, which increase with the $\gamma$ parameter, except for the reversionary annuities. Indeed, unlike other contracts which concern death benefit, a reversionary annuity pays survival benefits; therefore its relationship with the deterioration of mortality is opposite to other products. Then we illustrate in Tables 6.3 and 6.4 the effect of heterogeneity, characterized by parameter $k$, for two different ages 30 and 50. For instance, for the joint life contract, the premium increases as the heterogeneity decreases. However, this effect is less clear for other products. Indeed, in a more heterogeneous population ($k = 2$) there are

\[ S^*(t) = \mathbb{E}[e^{-(A_1(t) + A_2(t))F}] = \frac{1}{(1 + 1/k(A_1(t) + A_2(t)))^k} \]
more couples of extremely high risk, as well as more couples of extremely low risk. The first couples contribute to increase the premium whereas the latter couples contribute to diminish the premium. For the reversionary annuity, a riskier couple is expected to trigger annuity payment earlier, which means less premium income, but the payment is also expected to terminate earlier, which spells less total payment. In our simulation studies, we observe that, for each product, the premium rate is decreasing in the heterogeneity, both for age 30 and 50. Figure 2 plots, for each \( k \), simulated lifetimes distributions for the last survivor, respectively for \( z_0 = 30 \) and 50.

Special attention should be paid when comparing premium rates at age 50 for different values of parameter \( k \). Indeed, for each value of \( k \), \( \gamma(k, 1/k) \) is the heterogeneity distribution at age 30, but the heterogeneity distribution conditional on the survival of both spouses up to age 50 is no longer the same. However, it is still a gamma distribution \( \gamma(k, 1/[k + A_1(z_1 - z_0) + A_2(z_1 - z_0)]) \), where \( z_0 = 30 \), \( z_1 = 50 \) and \( A_1, A_2 \) are the cumulative intensities (see Appendix 3). Therefore, the mean of the heterogeneity is \( k/[k + A_1(z_1 - z_0) + A_2(z_1 - z_0)] \), and quotient between the variance at age 50 and that at age 30 is \( k^2/[k + A_1(z_1 - z_0) + A_2(z_1 - z_0)]^2 \). Both quantities are decreasing functions of \( k \), that is, the mean and the variance of the heterogeneity diminish (in proportion) faster in the population with initially the highest heterogeneity (\( k = 2 \)). Figure 3 plots, for each \( k \), the probability density function of the heterogeneity both at age 30 and at age 50. The gamma distribution parameters at age 50 are reported in Table 6.5.

4.3 Evolution of the price of the contract during the life of the contract

A premium level \( a_0 \) is fixed at the inception of each contract (see Section 4.1). However, it is important to evaluate regularly the residual value of this contract during its life, for instance, to include it correctly in the balance sheet, or, if it is securitized, to evaluate the price of the corresponding component of the Insurance Linked Security.

Let us first focus on the joint life policy. The fair value of this contract at a date where both spouses are still alive and the age of the couple is \( z_1, z_1 \geq z_0 \), is given by:

\[
\lambda(t) = \frac{a_1(t) + a_2(t)}{1 + 1/k(A_1(t) + A_2(t))},
\]

and the corresponding unconditional intensity function is:

thus the premia for a joint life contract is higher for \( k = 10 \). We would like to thank the referee for pointing it out.
\[ C_{10}^{(1)}(a_0, r, z; Y_1, Y_2) = \mathbb{E}_0[C_{0}^{(1)}(a_0, r, z_1; Y_1, Y_2)|Y_1 \geq z_1, Y_2 \geq z_1]. \] (4.9)

\( a_0 \) is for instance equal to the fair premium \( a_0 = a_0^{*} \) given in (4.3) when \( z_1 = z_0 \).

The price updating is more complicated for the reversionary annuities product, since we have to distinguish the two possible regimes existing during the life of the contract. In the first regime the two spouses are both alive, with an age of the couple equal to \( z_1 \). In the second regime, there is just one surviving spouse, the available information includes the date of the first death and the fact that the surviving spouse is the husband, or the wife. In both regimes, the residual value is systematically negative. First, in the second regime the only cash flows are the payment of the annuity, which are negative. Second, in the first regime, the premium rate of the reversionary annuity is increasing in \( z_0 \) (see Figure 1), therefore, couples who entered into the contract at age \( z_0 < z_1 \) pay, at age \( z_1 \), less premium than newly underwritten couples of age \( z_1 \), while the two groups have the same heterogeneity distribution, thus the same risk profile.

For illustration, let us calculate the residual value of a reversionary annuity underwritten at the age of 30. At date \( t > 30 \), the residual value of this contract depends on the survival status of the couple. We use the same model as in the previous section and Figure [A] displays the evolution of the residual value of the contract, first when both spouses are still alive at date \( t \), then when one of the spouse died before \( t \). The parameters are \( \gamma = 5, k = 2, z_0 = 30 \). As expected we observe that in both case, the value of the contract is negative. We observe also in the second case, that the value of the contract is smaller for widows than for widowers. Indeed, at the same age and with the same marital status, women have a smaller mortality intensity than men have.

5 Statistical Inference

The aim of this section is to provide insights on the statistical inference of models introduced in the paper. We first provide a brief review of the literature of dependent competing risks models and its identification and estimation. This should clarify the confusion in the actuarial literature that competing risks models are not identifiable. As a by-product, we also discuss the potential usefulness of our framework on the joint dynamic modeling of several cause-specific death rates, a topic which has attracted much attention in recent years from longevity researchers. Then we explain how to extend this method to the estimation of a Freund model with frailty.
5.1 Competing risks models

By Bayes' formula we have

\[ p(X_1, X_2, X_3, X_4) = p(X_1, X_2)p(X_3, X_4|X_1, X_2), \]

This equation does not imply that we can model independently the distribution of \((X_1, X_2)\) and that of \((X_3, X_4|X_1, X_2)\) since the two terms on the right hand side of this equation are not independent. Nevertheless, to fix the ideas, it is useful to consider first the problem of estimating the distribution of \((X_1, X_2)\) from the observation of \((\min(X_1, X_2), 1_{X_1<X_2})\), which is the classical competing risks problem. There has been a widespread belief, following [Liatis (1975)], that the competing risks model is nonparametrically unidentified, or in other words one can freely choose the dependence structure between different latent duration variables, or the distribution of the unobserved heterogeneity in a frailty model. Following this result, Zheng and Klein (1995) propose to estimate the model with a known copula function. This methodology has been widely adopted in the insurance literature [see e.g. Carrière (1994), Kaishev et al. (2007), Dimitrova et al. (2013)]. These models are estimated in a static setting, that is, using only data from one cohort.

What is less well known is that in many cases, when covariate information becomes available, which is typically the case in Insurance and Finance, and when some rather flexible parametric assumptions are imposed, the non identifiability result actually breaks down. Roughly speaking, the presence of covariates, such as the year of birth in mortality studies, allows for cross-sectional constraints (see e.g. the Lee and Carter (1992) model) which facilitates identification. At the same time, when some functional parameters are replaced by appropriate finite dimensional parameters, the model becomes more constrained and thus we can often weaken other conditions required for identification. A typical example in the identification literature is Heckman and Singer (1984) for the univariate survival models with frailty. They show that if the statistician is willing to assume Weibull baseline hazard function, then the distribution of the frailty can be identified nonparametrically, even without covariates.

From now on let us discuss the identification of the competing risks model under the following proportional hazard specification:

\[ \lambda_1(x|X_1 \geq x, X_2 \geq x, z) = a_1(x)\phi_1(z)U \]
\[ \lambda_2(x|X_1 \geq x, X_2 \geq x, z) = a_2(x)\phi_2(z)V \]

where:

- \(z\) is the observed covariate, \(\phi_1(z), \phi_2(z)\) are regressors.
• $a_1, a_2$ are baseline hazard functions,

• $F = (U, V)$ is the bivariate frailty. This is the simplest bivariate frailty model introduced in Section 3. For instance if if $U = V$, then we have a shared frailty model. It is in general assumed that $(U, V)$ is independent of the observed covariate $x$, that is, individuals with different covariate value $x$ have the same distribution of unobserved heterogeneity at the inception of their observation.

Such a specification is called the mixed proportional hazard model (MPH) in the econometric literature.

This specification is rather general and easy to interpret. Indeed, it assumes that, conditional on the (observed and unobserved) covariate $x$ and $(U, V)$, as well as the survival up to time $t$, that is $T_1, T_2 > t$, the two duration variables $T_1$ and $T_2$ are independent. Roughly speaking this is an exogeneity condition of the covariates, or that there is no causality, between $X_1$ and $X_2$.

For instance, when the observable covariate $z$ denotes a cohort effect, the model allows us to decompose the dependence between $(X_1, X_2)$ can be into two factors:

1. The (static) dependence via the (individual) frailty.
2. The co-dependence on the (common) covariate $x$, when the value of the covariate changes.

The nonparametric identification of the previous MPH model, whose set of functional parameters is $(a_1, a_2, \phi_1, \phi_2, G)$, where $G$ is the joint p.d.f. of $(U, V)$, is well documented in the econometric literature. In a seminal paper, [Heckman and Honore (1989)] show that a more general model, which nests the MPH competing risks model, is identified so long as the couple $(\phi_1(z), \phi_2(z))$ can take values at all points of $\mathbb{R}^2$ when $z$ varies. The identification condition therein is further weakened by [Abbring and Van den Berg (2003a)], who only assume that the couple $(\phi_1(z), \phi_2(z))$ can take value at a non null open set of $\mathbb{R}^2$.

The observation of a large number of values of covariates allows us to separate these two effects and subsequently identify both effects, which is not possible without covariate. It is also logical to expect that the identification condition can be further weakened in the shared frailty framework ($U = V$).

In practice, since these identification proofs are not constructive and the covariates may only take a finite (often small) number of values, the conditions of the identification theorems may not be satisfied. Therefore it is common practice to impose further parametric conditions in order to make the model more robust. We can, for instance, assume

• a parametric shared frailty model
- a parametric functional forms for baseline hazard functions, say, power or exponential functions of age,

- a parametric form for the covariate regressor, for instance the simplest form is \( e^{\beta_i z_i} \) where \( \beta_j, j = 1, 2 \) are constants and \( z_j, j = 1, 2 \) are the value of the covariate respectively for the male and the female. In the longevity context where \( z_1, z_2 \) are the birth dates of the two lives, we would expect \( \beta_1, \beta_2 < 0 \), which reflects the decrease of baseline mortality rates across different cohorts.

Such a model is straightforward to estimate by maximizing the likelihood if the data is available.

Since [Heckman and Singer (1984)](#), it is well known that the estimation result of MPH model may depend strongly on the choice of the distributional assumption of the unobserved heterogeneity. Thus it is sometimes useful to compare the inference of the model with different parametric assumptions for the unobserved heterogeneity. An alternative way is to use, as advocated by [Heckman and Singer (1984)](#), a finite mixture of Dirac masses to approximate the distribution of the unobserved heterogeneity. This is a nonparametric specification, although in practice, the distribution is often approximated by a finite mixture with a pre-determined number of components due to parsimony concerns, see e.g. [McCall (1996)](#), [Dolton and Van der Klaauw (1999)](#), [Deng et al. (2000)](#), [Fallick and Ryu (2007)](#) and [Brown and Dinç (2009)](#) for empirical applications of this methodology.

Finally, note that the framework described above has another important application to the Insurance literature. If we take \( X_1, X_2 \) respectively the latent time of dying from two different diseases, then the same methodology can be directly applied to model the joint dynamics of cause-specific mortality rates due to different causes, say, cancer and cardiovascular diseases. As pointed out by [Andersen and Keiding (2012)](#), the literature studying the impact of eliminating a certain disease (or diminishing the mortality due to this disease) on the mortality due to other diseases dates back to the era of d’Alembert and Bernoulli and is still very active. However, up to now, this exercise is usually done in a static framework\(^{16}\), that is, without the (cohort) longevity effect. As we argued before, in a completely static case, it is not possible to identify the the dependence between the two risks. In the case of discovery of a therapy against cancer, which will impact first the mortality (or more precisely, the baseline hazard function of cancer)\(^{17}\) from cancer, and then...
gradually the mortality of CVD diseases since the change of the mortality between different cohorts implies a change of the selection process. Thus in the recent cohort who benefited from a decline of cancer mortality, the average heterogeneity of CVD diseases may either increase or decrease with respect to the older cohort. This corresponds to the analysis done in the literature, see e.g. Carriere (1994).

However, in the case of a general change of living style, the (baseline) mortality rates due to both diseases will diminish, and thus there is no meaning in measuring the impact of decrease of cancer mortality on the CVD mortality ceteris paribus in this case. Our structural model provides a more adapted framework to analyze this problem, when the cohort effect is observed. The numerical application of such a model is straightforward but omitted since it goes beyond the primary aim of the current paper.

5.2 The estimation of a Freund model with frailty

Similarly as in the competing risks model, if the covariate is available and continuously valued, and if some parametric assumptions are made on the conditional law of $X_3 | X_1$ (resp. $X_4 | X_2$), then the Freund model with frailty is in general identified. Indeed, the presence of mortality jump upon arrival of another event appears in the biostatistics and microeconometrics literature under the terminology treatment effect. The nonparametric identification of the treatment effect model with frailty is well documented, see e.g. Abbring and Van den Berg (2003b). Naturally, the proof is a natural extension of the proof for MPH competing risks models and relies on the possibility of observing a large number of cohorts.

6 Concluding remarks

The standard insurance literature for analyzing and pricing insurance contracts written on two lives are pure models. A first category assumes a continuous bivariate distribution of the spouses’ lifetimes with a continuous probability density function. This continuity assumption implies no jump in intensity when a spouse dies. A second category of models apply a pure Freund model to describe the broken-heart syndrome. These two effects impact the price of insurance contracts and of annuity values in different ways, not only the price of contracts written on

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the marginal survival function in the statistics literature, via the formula $S_1(y) = E[e^{-\Lambda_1(y)U}]$, where $U$ is the frailty associated to the event 1.

\[18\]

Intuitively, if $U$ and $V$ are positively correlated, for instance if $U = V$, then we would expect an increase of the average heterogeneity of CVD diseases. Otherwise, if $U$ and $V$ are negatively correlated, then we would also expect a decrease of the heterogeneity linked to CVD, and thus also the mortality rates due to CVD.

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two lives, but also the prices of individual contracts.\footnote{For the same reason they can impact the price of health insurance contracts or of long term care contracts, for instance since the risk of entering long term institutional care after the death of a spouse can increase [Nihiltä and Martikainen (2008)].} By considering appropriate extensions of the Freund model, we have explained how to account for both individual heterogeneity and potential jumps at the time of a spouse's death.

A similar problem arises in the credit risk literature where the death event is replaced by a default event. The standard credit risk literature prices the default intensity, not the default event itself, leading to possible mispricing of credit derivatives. The idea of introducing jumps in intensity to correct such a mispricing has been proposed in [Jarrow and Yu (2001)] for a credit derivative, written on two corporations\footnote{which is equivalent to an insurance product written on two lives.} see also the discussions in [Benzoni et al. (2012)] and [Bai et al. (2012)]. Recently [Gourieroux et al. (2014)] derived the pricing formulas for credit derivatives written on a large pool of corporations and taking into account the jumps arising when corporations in the pool default.

Finally formulas providing the prices of insurance contracts written on two lives depend on parameters explaining how the exogenous variable impact the bivariate lifetime (risk-neutral) distribution. These variables include the individual characteristics of the couple, including the information on their generation. This generation information for each given age allows for taking into account the time dependence of the mortality rate. These parameters have to be calibrated, especially the parameters measuring the magnitude of the jumps (or of the association measures), the parameters capturing the memory effect and how they depend on generation (i.e. time). Data on individual contracts considered in isolation will not be sufficient to identify these parameters and there is clearly a need of data on couples and prices of contracts written on two lives.
Appendix 1

Joint density of lifetimes

Let us assume \( y_1 < y_2 \). We have:

\[
f(y_1, y_2) = \lim_{dy_1, dy_2 \to 0} \frac{1}{dy_1 dy_2} P[Y_1 \in (y_1, y_1 + dy_1), Y_2 \in (y_2, y_2 + dy_2)]
\]

\[
= \lim_{dy_1, dy_2 \to 0} \frac{1}{dy_1 dy_2} P[X_1 < X_2, X_1 \in (y_1, y_1 + dy_1), X_1 + X_4 \in (y_2, y_2 + dy_2)]
\]

\[
= \lim_{dy_1, dy_2 \to 0} \left[ \frac{1}{dy_1} P[y_1 < X_2, X_1 \in (y_1, y_1 + dy_1)]
\right]
\]

\[
\frac{1}{dy_2} P[X_4 \in (y_2 - y_1, y_2 - y_1 + dy_2) | X_1 = \min(X_1, X_2) = y_1]
\]

\[
= \left[ -\frac{\partial S_{12}}{\partial x_1}(y_1, y_1) \right] \left[ -\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1) \right].
\]

Appendix 2

Link between the historical and risk-neutral distributions

For expository purpose we set the riskfree rate \( r = 0 \). Then we have to consider jointly the historical (or physical) distribution, with characteristics indexed by \( P \), and the risk-neutral (or adjusted for risk) distribution, with characteristics indexed by \( Q \). Since we are in an incomplete market frameworks, these two distributions can be specified independently. Let us now discuss the possible effects of the change of probability.

i) The stochastic discount factor (sdf) is the ratio between the risk-neutral and historical densities:

\[
m(y_1, y_2, F) = \frac{f^Q(y_1, y_2, F)}{f^P(y_1, y_2, F)},
\]

for a model with frailty for instance. A discontinuity of the risk-neutral density \( f^Q \) on the 45° line \( y_1 = y_2 \), that is, jumps in the risk-neutral intensities, can result from either jumps in the historical intensities, or jumps in the adjustment for risk (sdf) when a death occurs.
The standard insurance literature computing the prices from a specification of the historical distribution and the sdf has omitted the second possibility. This is typical of the practice of pricing by Esscher transforms [see e.g. Esscher (1932), Gerber and Shiu (1994)] written on factor $F$, that is choosing $m(y_1, y_2, F) = \exp(\alpha + \beta F)$, where $\alpha$ and $\beta$ are such that $E^P[\exp(\alpha + \beta F)] = 1$ to get the zero riskfree rate.

Intuitively to reintroduce the effect of death event while using the practice of Esscher transforms, we may introduce the Esscher transforms on the distributions of the latent variables, that is,

for the pair $(X_1, X_2) : \exp(\alpha_{12} + \beta_{12} F)$, say,

for the pair $X_3 : \exp(\alpha_3 + \beta_3 F)$, say,

for the pair $(X_4) : \exp(\alpha_4 + \beta_4 F)$, say.

with parameters linked by the condition of zero riskfree rate.

**Appendix 3**

**Probability distribution function of the heterogeneity given survival up to time $t$.**

We derive the probability density function of the heterogeneity of the set of couples such that both spouses survive up to age $z_0 + x$. It is denoted $g_x$. We also denote by $g_0$ the heterogeneity distribution at age $z_0 = 30$, which equals $\gamma(k, 1/k)$, therefore:

$$g_0(f) \propto f^{k-1} \exp[-kf].$$

The unconditional survival probability that both survive up to age $z_0 + x$ is:

$$S(x) = \mathbb{P}(Y_1 > z_0 + x, Y_2 > z_0 + x|Y_1 > z_0, Y_1 > z_0)$$

$$= \int \exp[-(A_1(x) + A_2(x))]g_0(f)df,$$

where $A_1$ and $A_2$ are cumulative intensities. Then the unconditional mortality...
intensity at age $z_0 + x$ is:

$$\lambda(x) = -\frac{d}{dx} \log S(x) = \int [a_1(x) + a_2(x)] f \exp[-[A_1(x) + A_2(x)]f] g_0(f) df \frac{\exp[-[A_1(x) + A_2(x)]f] g_0(f) df}{\int \exp[-[A_1(x) + A_2(x)]f] g_0(f) df}.$$  

Therefore, we deduce that the heterogeneity distribution function is:

$$g_x(f) = \frac{g_0(f) \exp[-[A_1(x) + A_2(x)]f]}{\int g_0(f) \exp[-[A_1(x) + A_2(x)]f] df} \propto f^{k-1} \exp[-[k + A_1(x) + A_2(x)]f],$$

which is a gamma distribution with shape parameter $k$ and scale parameter $1/(k + A_1(x) + A_2(x))$. 

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Figure 1: Premium rate as a function of the age of the couple at the time of underwriting. In the lower right panel for individual life insurance policies, the dashed line (respectively solid line) represents the premium rates when the information on the spouse is (respectively is not) taken into account.
Figure 2: Probability density functions of the last survivor’s lifetime upon $z_0$, for $z_0 = 30, 50$. 

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Figure 3: Probability density functions of the heterogeneity, at ages 30 and 50.
Figure 4: Evolution of the residual value of a reversionary annuity. Left panel: both spouses are still alive. Right panel: one of the spouses died before \( t \).
<table>
<thead>
<tr>
<th></th>
<th>Last survivor</th>
<th>Reversion annuity</th>
<th>Individual, female, without husband’s information</th>
<th>Individual, female, with husband’s information</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 5$</td>
<td>0.0194</td>
<td>0.134</td>
<td>0.0212</td>
<td>0.0210</td>
</tr>
<tr>
<td>$\gamma = 3$</td>
<td>0.0182</td>
<td>0.181</td>
<td>0.0203</td>
<td>0.0202</td>
</tr>
<tr>
<td>$\gamma = 1$</td>
<td>0.0153</td>
<td>0.318</td>
<td>0.0184</td>
<td>0.0183</td>
</tr>
</tbody>
</table>

Table 6.1: Effect of the broken heart syndrome on premium rates with a fixed heterogeneity distribution ($k = 6$), at age 30.

<table>
<thead>
<tr>
<th></th>
<th>Last survivor</th>
<th>Reversion annuity</th>
<th>Individual, female, with husband’s information</th>
<th>Individual, female, without husband’s information</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 5$</td>
<td>0.0279</td>
<td>0.166</td>
<td>0.0319</td>
<td>0.0303</td>
</tr>
<tr>
<td>$\gamma = 3$</td>
<td>0.0260</td>
<td>0.225</td>
<td>0.0309</td>
<td>0.0290</td>
</tr>
<tr>
<td>$\gamma = 1$</td>
<td>0.0214</td>
<td>0.404</td>
<td>0.0275</td>
<td>0.0258</td>
</tr>
</tbody>
</table>

Table 6.2: Effect of the broken heart syndrome on premium rates with a fixed heterogeneity distribution ($k = 6$), at age 50.

<table>
<thead>
<tr>
<th></th>
<th>Joint life</th>
<th>Last survivor</th>
<th>Reversion annuity</th>
<th>Individual, female, with husband’s information</th>
<th>Individual, female, without husband’s information</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 2$</td>
<td>0.0186</td>
<td>0.0153</td>
<td>0.129</td>
<td>0.0167</td>
<td>0.0167</td>
</tr>
<tr>
<td>$k = 6$</td>
<td>0.0196</td>
<td>0.0161</td>
<td>0.135</td>
<td>0.0176</td>
<td>0.0176</td>
</tr>
<tr>
<td>$k = 10$</td>
<td>0.0197</td>
<td>0.0162</td>
<td>0.136</td>
<td>0.0177</td>
<td>0.0177</td>
</tr>
</tbody>
</table>

Table 6.3: Effect of heterogeneity on premium rates with a fixed broken heart syndrome ($\gamma = 5$), at age 30.

<table>
<thead>
<tr>
<th></th>
<th>Joint life</th>
<th>Last survivor</th>
<th>Reversion annuity</th>
<th>Individual, female, with husband’s information</th>
<th>Individual, female, without husband’s information</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 2$</td>
<td>0.0334</td>
<td>0.0265</td>
<td>0.188</td>
<td>0.0299</td>
<td>0.0293</td>
</tr>
<tr>
<td>$k = 6$</td>
<td>0.0364</td>
<td>0.0287</td>
<td>0.199</td>
<td>0.0324</td>
<td>0.0318</td>
</tr>
<tr>
<td>$k = 10$</td>
<td>0.0371</td>
<td>0.0292</td>
<td>0.203</td>
<td>0.0329</td>
<td>0.0323</td>
</tr>
</tbody>
</table>

Table 6.4: Effect of heterogeneity on premium rates with a fixed broken heart syndrome ($\gamma = 5$), at age 50.

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<table>
<thead>
<tr>
<th>$k$</th>
<th>Shape parameter</th>
<th>Scale parameter</th>
<th>$\sqrt{\text{Variance at age 50}}$</th>
<th>$\sqrt{\text{Variance at age 30}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.4816</td>
<td>2</td>
<td>0.9279</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.1646</td>
<td>6</td>
<td>0.9750</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.0992</td>
<td>10</td>
<td>0.9849</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.5: Gamma distribution parameters at age 50 for different gamma distributions $\gamma(k, 1/k)$ at age 30. The scale parameter is the same as at age 30. The fourth column gives values of $k/[k + A_1(x) + A_2(x)]$, which equals also the mean of the heterogeneity distribution. It measures the reduction of the heterogeneity due to the mover-stayer phenomenon.

References


