Love and Death: A Freund Model with Frailty

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Outline

1. INTRODUCTION

2. THE MODEL
   - Jump in intensities
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3. EFFECT OF HETEROGENEITY AND JUMP PARAMETERS ON PRICING

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INTRODUCTION
The aim of the paper:
How to price insurance/annuity contracts written on two heads, both at the issuing and during its lifetime.
There exist two approaches to modeling the lifetimes of the spouses:

1. The Archimedean copula model written on the spouses’ lifetimes.
   
   see e.g. Frees, Carriere and Valdez (1996), Carriere (2000).

2. The multi-state model
   - 4 states for describing the situation of the spouses, 1 alive, 0 dead: (1, 1), (1, 0), (0, 1), (0, 0)
   - Model directly the transition intensities between different states.
   - Therefore the intensity can jump at the death of one spouse (broken-heart syndrome).
   
   see e.g. Ji, Hardy and Li (2011), Spreeuw and Wang (2012).
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The Archimedean copula models:

- \( S(y_1, y_2) = \psi \left( \psi^{-1}(S_1(y_1)) + \psi^{-1}(S_2(y_2)) \right) \)

- They are **factor copulas**, that is, the dependence structure between the two lifetimes has an interpretation in terms of unobservable static common risk factor, or **shared frailty (heterogeneity)** of the couple (see Section 2)

- The standard copulas have continuous copula densities, which implies symmetric (multiplicative) intensity jumps for the husband/wife at the death of his/her partner (see Section 2)
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The multi-state approach:

- When both spouses are alive, the two (potential) death events are usually assumed independent. Thus the heterogeneity of the couples is omitted.

![Diagram of multi-state approach](image)
The model introduced in our paper disentangles the dependence due to the common frailty and broken-heart syndrome, that are the exogenous and endogenous sources of dependence, respectively.
THE MODEL

Jump in intensities
Freund model with frailty
(Freund [1961]). We consider four latent lifetime variables:

- $X_1$ (resp. $X_2$) potential death time of spouse 1 (resp. 2) when both are alive,
- $X_3$ residual lifetime of spouse 1 after the death of spouse 2,
- $X_4$ residual lifetime of spouse 2 after the death of spouse 1,

These variables are really latent: for each couple, either $(X_2, X_3)$, or $(X_1, X_4)$ can be observed, the other pair of variables being not observable jointly.
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The observable variables are:

\[
\begin{align*}
Y_1 &= \min(X_1, X_2) + X_3 \mathbb{1}_{X_2 < X_1}, \\
Y_2 &= \min(X_1, X_2) + X_4 \mathbb{1}_{X_1 < X_2},
\end{align*}
\]

where \( \min(X_1, X_2) = \min(Y_1, Y_2) \) is the time of the first death, and \( \mathbb{1}_{X_1 < X_2} \) defines the regime:

- \( \mathbb{1}_{X_1 < X_2} = 1 \), if spouse 1 dies first,
- \( \mathbb{1}_{X_1 < X_2} = 0 \), otherwise.
The following summaries of the distribution of latent variables characterize the law of the couple \((Y_1, Y_2)\):

- the joint survivor function of \(X_1, X_2\):
  
  \[ S_{12}(x_1, x_2) = \mathbb{P}(X_1 > x_1, X_2 > x_2), \]

- the survivor function of \(X_3\) given \(X_2 = \min(X_1, X_2) = z\):
  
  \[ S_3(x_3|z) = \mathbb{P}\left[ X_3 > x_3|X_2 = \min(X_1, X_2) = z \right]. \]

- the survivor function of \(X_4\) given \(X_1 = \min(X_1, X_2) = z\):
  
  \[ S_4(x_4|z) = \mathbb{P}\left[ X_4 > x_4|X_1 = \min(X_1, X_2) = z \right]. \]
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- the survivor function of \(X_4\) given \(X_1 = \min(X_1, X_2) = z\):
  \[ S_4(x_4|z) = \Pr[X_4 > x_4|X_1 = \min(X_1, X_2) = z]. \]
Jump in intensities

- When both spouses are alive, the mortality intensity of spouse 1 is:

\[
\lambda_1(y|Y_1 > y, Y_2 > y) = -\frac{\partial}{\partial y_1} \log S_{12}(y, y),
\]

which is the crude intensity.

- When spouse 2 dies at date \(y\), the mortality intensity of 1 becomes:

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Therefore the multiplicative intensity jump (also called cross-ratio function) for spouse 1 is:

\[
\gamma_{1|2}(y) = \frac{\lambda_1(y| Y_1 > y, Y_2 = y)}{\lambda_1(y| Y_1 > y, Y_2 > y)} = \frac{\partial}{\partial y_1} \log S_3(0, y) \frac{\partial}{\partial y_1} \log S_{12}(y, y).
\]

In general \( \gamma_{1|2}(y) \neq \gamma_{2|1}(y) \).

In a standard copula-based model, we get:

\[
\gamma_{1|2}(y) = \frac{\partial}{\partial y_1} S(y, y) \frac{\partial}{\partial y_2} S(y, y) \frac{\partial^2}{\partial y_1 \partial y_2} S(y, y).\]

Therefore \( \gamma_{1|2}(y) = \gamma_{2|1}(y) \), since \( \frac{\partial^2 S}{\partial y_1 \partial y_2} = \frac{\partial^2 S}{\partial y_2 \partial y_1} \) by continuity, and we cannot have asymmetric reactions of the spouses.
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Freund model with frailty

The model extends the basic Freund model by introducing a heterogeneity variable $F$ in the intensities.

- marginal intensities of the latent variables conditional on $F$: $a_1(x_1|F), a_2(x_2|F), a_3(x_3|x_2, F), a_4(x_3|x_1, F)$.

- marginal cumulative intensities: $A_1(x_1|F), A_2(x_2|F), A_3(x_3|x_2, F), A_4(x_3|x_1, F)$.

- $X_1$ and $X_2$ are independent given $F$.

- The conditional mortality jump is defined by:

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\gamma_{1|2}(y|F) = \frac{a_3(0|y, F)}{a_1(y|F)}.
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$$\gamma_{1|2}(y|F) = \frac{a_3(0|y, F)}{a_1(y|F)}.$$
The conditional intensities are used to deduce:

- first the conditional distribution of the variables \((Y_1, Y_2)\) given frailty \(F\),
- next the distribution of \((Y_1, Y_2)\), when frailty \(F\) is integrated out.
In particular we get the following expressions of the mortality jump, when spouse 2 dies first at time $y$:

**Theorem**

$$\gamma_{1|2}(y) = \frac{\mathbb{E}^{Q_y}[a_3(0|y,F)a_2(y|F)]}{\mathbb{E}^{Q_y}[a_1(y|F)]\mathbb{E}^{Q_y}[a_2(y|F)]},$$

where $Q_y$ denotes the probability distribution of the frailty among the surviving couples at time $y$:

$$\frac{dQ_y}{dQ_0} = \frac{e^{-A_1(y|F) - A_2(y|F)}}{\mathbb{E}^{Q_0}[e^{-A_1(y|F) - A_2(y|F)}]}.$$
Corollary

By a change of measure, we have:

\[ \gamma_{1|2}(y) = \mathbb{E}^{\tilde{Q}_y}[\gamma_{1|2}(y|F)] \frac{\mathbb{E}^{Q_y}[a_1(y|F)a_2(y|F)]}{\mathbb{E}^{Q_y}[a_1(y|F)]\mathbb{E}^{Q_y}[a_2(y|F)]}, \]

where \( \tilde{Q}_y \) is the density of the frailty \( F \) among couples who die simultaneously at time \( y \):

\[ \frac{d\tilde{Q}_y}{dQ_y} = \frac{a_1(y|F)a_2(y|F)}{\mathbb{E}^{Q_y}[a_1(y|F)a_2(y|F)]}, \]
Special case: univariate shared frailty

If the frailty $F$ is univariate, and

$$a_1(x_1|F) = a_1(x_1)F, \quad a_3(x_3|x_2, F) = a_3(x_3|x_2)F,$$
$$a_2(x_2|F) = a_2(x_2)F, \quad a_4(x_4|x_1, F) = a_4(x_4|x_1)F,$$

then $\gamma_{1|2}(y|F) = \frac{a_3(0|y,F)}{a_1(y|F)} = \frac{a_3(0|y)}{a_1(y)}$ does not depend on $F$, and:

$$\gamma_{1|2}(y) = \gamma_{1|2}(y|F) \frac{\mathbb{E}^Q_y[F^2]}{\left(\mathbb{E}^Q_y[F]\right)^2} \geq \gamma_{1|2}(y|F).$$

- Failing to control for the unobserved heterogeneity creates a bias.

- $\gamma_{1|2}(y) = \gamma_{2|1}(y) \iff \gamma_{1|2}(y|F) = \gamma_{2|1}(y|F)$. 

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Interpretation of Archimedean copulas

**Theorem (Marshall, Olkin [1988])**

*In a shared frailty model with no mortality jumps, that is,*

\[
a_3(x_3|z) = a_1(x_3 + z), \quad a_4(x_4|z) = a_2(x_4 + z),
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*the survivor copula of the couple \((Y_1, Y_2)\) is Archimedean, its generator function \(\psi\) is the Laplace transform of the frailty distribution: \(\psi(u) = \mathbb{E}[e^{-uF}]\).*

- Conversely, most Archimedean copulas admit this frailty representation, e.g. Clayton [1978], Frank [1979], Gumbel [1960], Ali-Mikhail-Haq [1978].
- In other words, Archimedean copula models implicitly ignore the broken-heart syndrome.
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EFFECT OF HETEROGENEITY AND JUMP PARAMETERS ON PRICING
Characteristics of the contracts

<table>
<thead>
<tr>
<th>Type of contracts</th>
<th>Premium payment period</th>
<th>Benefit payment period/date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joint life</td>
<td>([z_0, \min(Y_1, Y_2)])</td>
<td>at (\min(Y_1, Y_2))</td>
</tr>
<tr>
<td>Last survivor</td>
<td>([z_0, \max(Y_1, Y_2)])</td>
<td>at (\max(Y_1, Y_2))</td>
</tr>
<tr>
<td>Reversionary annuity</td>
<td>([z_0, \min(Y_1, Y_2)])</td>
<td>from (\min(Y_1, Y_2)) to (\max(Y_1, Y_2))</td>
</tr>
<tr>
<td>Individual life</td>
<td>([z_0, Y_1])</td>
<td>at (Y_1)</td>
</tr>
</tbody>
</table>

\(z_0\): underwriting age.
Simulation study

- Assume $F$ is gamma distributed, $\gamma(k, 1/k)$ for couples of age 30. (But evolves as the population ages).
- Specify parametric laws for intensities of latent variables.
- Assume constant, continuous time premium rate and interest rate.
- Assume symmetric constant jumps $\gamma_{1|2} = \gamma_{2|1} = \gamma$.
- Calculate the price of insurance contracts at different underwriting ages.

For instance, for a last survivor policy, the formula of the premium is:

$$a_0(r) = r \frac{\mathbb{E}[e^{-r\max(Y_1, Y_2)-z_0}|Y_1 \geq z_0, Y_2 \geq z_0]}{1 - \mathbb{E}[e^{-r\max(Y_1, Y_2)-z_0}|Y_1 \geq z_0, Y_2 \geq z_0]}$$
Effect of jump on prices

<table>
<thead>
<tr>
<th></th>
<th>Last survivor</th>
<th>Reversion annuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 5$</td>
<td>0.0194</td>
<td>0.134</td>
</tr>
<tr>
<td>$\gamma = 3$</td>
<td>0.0182</td>
<td>0.181</td>
</tr>
<tr>
<td>$\gamma = 1$ (No jump)</td>
<td>0.0153</td>
<td>0.318</td>
</tr>
</tbody>
</table>

Table: Effect of the broken heart syndrome on premium rates with a fixed heterogeneity distribution ($k = 6$), at age 30.
Effect of heterogeneity on prices

<table>
<thead>
<tr>
<th>k</th>
<th>Joint life</th>
<th>Last survivor</th>
<th>Reversion annuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.0334</td>
<td>0.0265</td>
<td>0.188</td>
</tr>
<tr>
<td>6</td>
<td>0.0364</td>
<td>0.0287</td>
<td>0.199</td>
</tr>
<tr>
<td>10</td>
<td>0.0371</td>
<td>0.0292</td>
<td>0.203</td>
</tr>
</tbody>
</table>

Table: Effect of the heterogeneity on premium rates with a fixed jump parameter ($\gamma = 5$), at age 50.
CONCLUSION
Conclusion

- The Freund model with frailty is very flexible to introduce non symmetric jumps for males and females.
- It allows to disentangle dependence due to the frailty and the broken-heart syndrome.
- Simulation studies show that both effects are important when pricing insurance contracts.

**Outlook**
- Good joint insurance portfolio data are essential for estimation.
Thanks for your attention. Questions / Comments?