Supplementary Material on "Inference on Adverse Selection Models through Contracts Variation"

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May 2009

In this supplementary material, we present some auxiliary results on the paper "Inference on Adverse Selection Models through Contracts Variation".[1] The first section shows how to improve the bounds given in Theorem 3.2 by using the restrictions on the primitives imposed in assumptions 1 and 3. The second section presents some useful details on the application. Finally, the third section provides some technical lemmas used in the proofs of Theorems 3.5, 4.1 and 4.2.

1 Improved bounds on the primitives when $K = 2$.

The bounds on C', F_θ and S'_k presented in Theorem 3.2 do not incorporate the fact that C is convex and $S_k(.)$ is concave, by assumptions 1 and 3 respectively. We show in this section how to use this information to obtain sharper bounds.

Differentiating the first order condition of the agent yields

$$t''_1(y) = \theta'_1(y)C'(y) + \theta_1(y)C''(y) \tag{1.1}$$

$C(.)$ is convex by assumption 1. Thus,

$$\theta'_1(y)C'(y) < t''_1(y).$$

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[1] This material cannot be read independently of the paper. In particular, we adopt the same notations as in the paper, without explicit recall of them.

Besides, using the second order condition of the agent and (1.1), we get

$$\theta_1'(y)C'(y) < 0.$$

Thus, $\theta_1'(y)C'(y) < \min(0, t_1''(y))$. Hence, by the first order condition,

$$\frac{\theta_1'(y)}{\theta_1(y)} < \frac{\min(0, t_1''(y))}{t_1'(y)}.$$

Integrating between $y' \leq y$ and y yields

$$\begin{aligned}
\theta_1(y) &\leq \theta_1(y') \exp\left(\int_{y'}^{y} \frac{\min(0, t_1''(u))}{t_1'(u)} du\right) \\
&\leq \overline{\theta_1}(y') \exp\left(\int_{y'}^{y} \frac{\min(0, t_1''(u))}{t_1'(u)} du\right)
\end{aligned} \quad (1.2)$$

Let us define

$$\widetilde{\overline{\theta}}_1(y) = \inf_{y' \leq y} \left\{ \overline{\theta_1}(y') \exp\left(\int_{y'}^{y} \frac{\min(0, t_1''(u))}{t_1'(u)} du\right)\right\}.$$

By (1.2), $\theta_1(y) \leq \widetilde{\overline{\theta}}_1(y)$. Moreover, $\widetilde{\overline{\theta}}_1(y) \leq \overline{\theta_1}(y)$ by definition, so that $\widetilde{\overline{\theta}}_1(y)$ is sharper than $\overline{\theta_1}(y)$. Similarly, let

$$\widetilde{\underline{\theta}}_1(y) = \sup_{y' \leq y} \left\{ \underline{\theta_1}(y') \exp\left(\int_{y'}^{y} \frac{\min(0, t_1''(u))}{t_1'(u)} du\right)\right\}.$$

Then $\theta_1(y) \geq \widetilde{\underline{\theta}}_1(y)$ and $\widetilde{\underline{\theta}}_1(y)$ is sharper than $\underline{\theta_1}(y)$. These two bounds on $\theta_1(y)$ lead to bounds on the primitives, as the following theorem shows.

Theorem 1.1 *Suppose that $K = 2$, assumptions 1 and 4 hold and $t_1'(.) < t_2'(.)$. Then*

1. *$C'(.)$ and $F_\theta(.)$ satisfy*

$$\begin{aligned}
1 - F_{y_1}\left(\sup\{y : \theta \leq \widetilde{\overline{\theta}}_1(y)\}\right) &\leq F_\theta(\theta) \leq 1 - F_{y_1}\left(\inf\{y : \theta \geq \widetilde{\underline{\theta}}_1(y)\}\right) && (1.3) \\
\frac{t'(y)}{\widetilde{\overline{\theta}}_1(y)} &\leq C'(y) \leq \frac{t'(y)}{\widetilde{\underline{\theta}}_1(y)} && (1.4)
\end{aligned}$$

2. *If assumption 3 also holds and contracts are optimal, $S_k'(.)$ satisfies, for $k \in \{1, 2\}$,*

$$\begin{aligned}
S_k'(y) &\geq \sup_{y' > y} \left\{ \frac{\ln\left((1 - F_{y_k}(y))\widetilde{\underline{\theta}}_k(y)\right) - \ln\left((1 - F_{y_k}(y'))\widetilde{\overline{\theta}}_k(y')\right)}{\int_y^{y'} 1/q_k(u) du} \right\}, \\
S_k'(y) &\leq \inf_{y' < y} \left\{ \frac{\ln\left((1 - F_{y_k}(y'))\widetilde{\overline{\theta}}_k(y')\right) - \ln\left((1 - F_{y_k}(y))\widetilde{\underline{\theta}}_k(y)\right)}{\int_{y'}^{y} 1/q_k(u) du} \right\}.
\end{aligned}$$

Proof:

1) (1.4) follows directly from the first order condition of the agent. As for $F_\theta(\theta)$, note that $\widetilde{\underline{\theta}}_1(y_1(\theta)) \leq \theta_1(y_1(\theta)) = \theta$. Thus, $y_1(\theta) \geq \inf\{y : \widetilde{\underline{\theta}}_1(y) \leq \theta\}$. This implies that

$$F_\theta(\theta) = 1 - F_{y_1}(y_1(\theta)) \leq 1 - F_{y_1}\left(\inf\{y : \theta \geq \widetilde{\underline{\theta}}_1(y)\}\right).$$

The lower bound follows similarly.

2) By equation (3.3) in the paper p.11,

$$\frac{S_k'(y)}{q_k(y)} = \frac{f_{y_k}(y)}{1 - F_{y_k}(y)} - \frac{\theta_k'(y)}{\theta_k(y)}.$$

Thus, for all $y' < y$,

$$\int_{y'}^{y} \frac{S_k'(u)}{q_k(u)} du = \ln\left(\frac{(1 - F_{y_k}(y'))\theta_k(y')}{(1 - F_{y_k}(y))\theta_k(y)}\right).$$

Because $S_k'(.)$ is decreasing by assumption , we get

$$S_k'(y) \int_{y'}^{y} \frac{1}{q_k(u)} du \leq \ln\left(\frac{(1 - F_{y_k}(y'))\theta_k(y')}{(1 - F_{y_k}(y))\theta_k(y)}\right) \leq \ln\left(\frac{(1 - F_{y_k}(y'))\widetilde{\overline{\theta}}_k(y')}{(1 - F_{y_k}(y))\widetilde{\underline{\theta}}_k(y)}\right).$$

The upper bound follows by minimizing on $y' < y$.

The lower bound is obtained similarly. ■

2 Details on the application

2.1 Estimation of the parameters a, b, α, β

The first order condition of the agent writes as $\theta_i(y) = \delta_i / C'(y)$. Using the parametric form of the marginal cost $C'(y) = \alpha\left(\frac{y}{1-y}\right)^\beta$ yields

$$\theta_i(y) = \frac{\delta_i}{\alpha}\left(\frac{1-y}{y}\right)^\beta$$

Then, using the Weibull specification, we obtain

$$F_{y_i}(y) = 1 - F_\theta(\theta_i(y)) = \exp\left(-a\left(\frac{\delta_i}{\alpha}\right)^b \left(\frac{1-y}{y}\right)^{b\beta}\right),$$

which can also be written:

$$\ln(-\ln(F_{y_i}(y))) = \ln(a) + b\left(\ln(\delta_i) - \ln(\alpha)\right) + b\beta \ln\left(\frac{1-y}{y}\right).$$

The parameters are estimated by regressing $\ln(-\ln(\widehat{F}_j(y)))$ on $\ln(\frac{1-y}{y})$. As explained in the main text, a normalization is necessary and we impose $C'(y_0) = \delta_1/\theta_0$.

2.2 Estimation of λ_3

Recall that in subsection 6.4.3 of the main text, we suppose that Insee can only implement linear contracts $t(y, \delta) = \delta y$ and that its objective function satisfies $S_3(y) = \lambda_3 y$ for the 2003 survey. Let denote by $y(\theta, \delta)$ the response rate chosen by a θ type agent. Insee solves, for the 2003 survey,

$$\max_{\delta} \int (\lambda_3 - \delta) y(\theta, \delta) f_\theta(\theta) d\theta.$$

and the first order condition satisfies

$$-\int y(\theta, \delta_3) f_\theta(\theta) d\theta + (\lambda_3 - \delta_3) \int \frac{\partial y}{\partial \delta}(\theta, \delta_3) f_\theta(\theta) d\theta = 0. \tag{2.1}$$

By the first order condition of the agent, $y(\theta, \delta) = \frac{1}{1+\left(\frac{\alpha\theta}{\delta}\right)^{1/\beta}}$. Thus,

$$\frac{\partial y}{\partial \delta}(\theta, \delta) = \frac{1}{\beta\delta} y(\theta, \delta)(1 - y(\theta, \delta)).$$

Plugging this expression into (2.1) yields

$$-E(y(\theta, \delta_3)) + \frac{\lambda_3 - \delta_3}{\beta\delta_3} E(y(\theta, \delta_3)(1 - y(\theta, \delta_3))) = 0.$$

Hence,

$$\lambda_3 = \delta_3 \left(1 + \beta \frac{E(y(\theta, \delta_3))}{E(y(\theta, \delta_3)(1 - y(\theta, \delta_3)))}\right),$$

which is estimated by

$$\widehat{\lambda}_3 = \delta_3 \left(1 + \widehat{\beta} \frac{\overline{y}_3}{\overline{y}_3 - \overline{y^2}_3}\right),$$

where $\overline{y}_3$ (resp. $\overline{y^2}_3$) denotes the empirical mean of the response rate (resp. of the response rate's square) in 2003.

2.3 Linear contracts under incomplete information

The expected surplus of Insee, when linear contracts are used, is $\Pi = (\lambda_3 - \delta_3) E(y(\theta, \delta_3))$ which can be estimated by

$$\widehat{\Pi} = (\widehat{\lambda}_3 - \widehat{\delta}_3)\overline{y}_3$$

2.4 Optimal contracts under incomplete information

On one hand, under incomplete information, the optimal contract is defined by

$$\lambda_3 = \left[\theta + \frac{F_\theta(\theta)}{f_\theta(\theta)}\right] C'(y^I(\theta,\lambda_3)).$$

Hence, using the form of the cost function, we obtain

$$y^I(\theta,\lambda_3) = \frac{1}{1 + \left(\frac{\alpha}{\lambda_3}\right)^{1/\beta}\left[\theta + \frac{F_\theta(\theta)}{f_\theta(\theta)}\right]^{1/\beta}}.$$

On the other hand, using the first order condition of the agent, we have

$$t'(y^I(\theta,\lambda_3)) = \theta C'(y^I(\theta,\lambda_3)) = \lambda_3 \frac{\theta}{\theta + F_\theta(\theta)/f_\theta(\theta)}.$$

Hence, the surplus under incomplete information, which writes as

$$\Pi^I = \lambda_3 E(y^I(\theta,\lambda_3)) - E(t(y^I(\theta,\lambda_3)))$$

can be estimated by Monte-Carlo simulations.

More precisely, to estimate this surplus, we draw 100,000 values of θ in our estimated Weibull distribution $F_\theta(.|\widehat{a},\widehat{b})$ and compute the previous functions using the parameters $\widehat{\lambda}_3$, $\widehat{\alpha}$ and $\widehat{\beta}$.

2.5 Optimal contracts under complete information

Under complete information, the first order condition for the agent writes $\lambda_3 = \theta C'(y^C(\theta,\lambda_3))$. Hence, using the parametric form of the cost function, we obtain

$$y^C(\theta,\lambda_3) = \frac{1}{1 + \left(\frac{\alpha\theta}{\lambda_3}\right)^{1/\beta}}.$$

Furthermore, the transfer function is given by

$$t(y^C(\theta,\lambda_3),\lambda_3) = \theta C(y^C(\theta,\lambda_3)) = \alpha\theta \int_0^{y^C(\theta,\lambda_3)} \left(\frac{u}{1-u}\right)^\beta du$$

Finally, the expected surplus under complete information is given by

$$\Pi^C = \lambda_3 E(y^C(\theta,\lambda_3)) - E(t(y^C(\theta,\lambda_3))).$$

As previously, to estimate this surplus, we draw 100,000 values of θ in our estimated Weibull distribution $F_\theta(.|\widehat{a},\widehat{b})$ and compute the previous functions using the parameters $\widehat{\lambda}_3$, $\widehat{\alpha}$ and $\widehat{\beta}$.

3 Technical lemmas

In the following lemmas, the assumptions we refer to are defined in the paper.

Lemma 3.1 *Let $x, y, z > 0$ be such that $\ln(x)/\ln(y) \notin \mathbb{R}$, let C be a subset of $\mathbb{R}^+$ and define $A_m = \{x^p y^q z,\ (p, q) \in \mathbb{Z}^2 : |p| + |q| \leq m\} \cap C$, for all $m \in \mathbb{N}$. Then $\cup_{m\in\mathbb{N}} A_m$ is dense in C. Moreover, if $C = [\underline{c}, \overline{c}]$, then for all $\varepsilon > 0$, there exists m_0 such that for all $m \geq m_0$,*

$$\sup_{c\in C} \min_{\theta\in A_m} |c - \theta| < \varepsilon.$$

Proof: let us introduce

$$G = \{p \ln(x) + q \ln(y), (p, q) \in \mathbb{Z}^2\}.$$

G is an additive subgroup of $\mathbb{R}$. Because $\ln(y)/\ln(x) \notin \mathbb{R}$, G is dense in $\mathbb{R}$ (see e.g. Stillwell, 1992, p.33). By continuity of $u \mapsto z \exp(u)$, $\{x^p y^q z,\ (p, q) \in \mathbb{Z}^2\}$ is dense in $\mathbb{R}^+$. The first result follows since

$$\cup_{m\in\mathbb{N}} A_m = \{x^p y^q z,\ (p, q) \in \mathbb{Z}^2\} \cap C.$$

Now, let $\varepsilon > 0$ and suppose that $C = [\underline{c}, \overline{c}]$. Then there exists $\underline{c} = t_0 < t_1 < ... < t_{K-1} < \overline{c} = t_K$ in C such that for all $i = 1, ..., K - 1$, $t_{i+1} - t_i < \varepsilon$. Because $\cup_{m\in\mathbb{N}} A_m$ is dense in C, there exists, for all $i = 0, ..., K$, $\theta_i \in \cup_{m\in\mathbb{N}} A_m$ such that $|t_i - \theta_i| < \varepsilon/2$. Moreover, θ_i is of the form $x^{p_i} y^{q_i} z$. Let $m_i = |p_i| + |q_i|$, $m_0 = \max_{i=1,...,K} m_i$ and take $m \geq m_0$. For all $c \in C$, there exists $i \in \{0, ..., K\}$ such that $|c - t_i| < \varepsilon/2$. Then

$$|\theta_i - c| \leq |\theta_i - t_i| + |t_i - c| < \varepsilon.$$

Thus, $\min_{\theta\in A_m} |c - \theta| < \varepsilon$. The lemma follows because this holds for all $c \in C$ □

Lemma 3.2 *Suppose that assumptions 1 and 3 hold. Then, for all $k \in \{1, ..., K\}$ and all compact set $C_k \in \overset{\circ}{\mathcal{Y}}_k$, f_{y_k} is continuously differentiable on C_k and satisfies $\inf_{y\in C_k} f_{y_k}(y) > 0$. Moreover, for all $j \neq k \in \{1, ..., K\}^2$, $H_{kj}(.)$ is differentiable on C_k and $\sup_{x\in C_k} |H'_{kj}(x)| < \infty$.*

Proof: Let $\theta_k \equiv \theta_k(.)$. By equation (3.1) of the main paper, we get for all $y \in \overset{\circ}{\mathcal{Y}}_k$,

$$f_{y_k}(y) = -f_\theta(\theta_k(y))\theta'_k(y).$$

Deriving the first order condition (3.2) of the main paper, we get, because $t'_k(.) = \delta_k$,

$$\theta'_k(y) = -\frac{\delta_k}{C'^2(y)} C''(y) \tag{3.1}$$

Thus, by assumption 1, $\theta'_k(.)$ is continuously differentiable. Because f_θ is continuously differentiable by assumption 3, f_{y_k} is continuously differentiable. Besides, (3.1) together with assumption 1 imply that $\theta'_k(y) < 0$ for all $y \in \overset{\circ}{\mathcal{Y}}_k$. Moreover, because C_k is a compact subset strictly included in $\mathcal{Y}_k$, $f_\theta(\theta_k(y)) > 0$ for all $y \in C_k$. Hence, by continuity of f_θ and θ',

$$\inf_{y \in C_k} f_{y_k}(y) = \min_{y \in C_k} [-f_\theta(\theta_k(y))\theta'_k(y)] > 0.$$

Finally, because $H_{kj} = F_{y_k}^{-1} \circ F_{y_j}$, H_{kj} is differentiable on C_k. Moreover, by continuity of H_{kj}, $H_{kj}(C_k)$ is a compact set included in $\overset{\circ}{\mathcal{Y}}_j$. Thus, for all $x \in C_k$,

$$H'_{kj}(x) = \frac{f_{y_k}(x)}{f_{y_k} \circ H_{kj}(x)} < \frac{\max_{x \in C_k} f_{y_k}(x)}{\inf_{x \in H_{kj}(C_k)} f_{y_j}(x)} < \infty \, \square$$

Lemma 3.3 *For all* $j \neq k \in \{1, ..., K\}^2$, *all* $C_k = [\underline{x}, \overline{x}] \subsetneq \mathcal{Y}_k$ *and all sequences* $a_N = o(\sqrt{N})$,

$$a_N \sup_{x \in C_k} |\widehat{H}_{kj}(x) - H_{kj}(x)| \xrightarrow{\mathbb{P}} 0.$$

Proof: first, because $\sqrt{N} \sup_{x \in C_k} |\widehat{F}_{y_k}(x) - F_{y_k}(x)|$ converges in distribution (see e.g. van der Vaart, 1998, corollary 19.21), we get by Slutski's lemma

$$a_N \sup_{x \in C_k} |\widehat{F}_{y_k}(x) - F_{y_k}(x)| \xrightarrow{\mathbb{P}} 0 \tag{3.2}$$

Let $[p_1, p_2] = [F_{y_k}(\underline{x}), F_{y_k}(\overline{x})]$. Because $\inf_{y \in H_{kj}(C_k)} f_{y_j}(y) > 0$ by lemma 3.2, $\sqrt{N}(\widehat{F}_{y_j}^{-1} - F_{y_j}^{-1})$ converges in distribution in the space of real bounded functions on $[p_1, p_2]$ (see e.g. Van der Vaart, 1998, corollary 21.5). Thus, as previously,

$$a_N \sup_{x \in [p_1, p_2]} |\widehat{F}_{y_j}^{-1}(x) - F_{y_j}^{-1}(x)| \xrightarrow{\mathbb{P}} 0 \tag{3.3}$$

Now, fix $\varepsilon, \delta > 0$ and $\zeta > 0$ such that $F_{y_k}(\underline{x}) > \zeta$ and $F_{y_k}(\overline{x}) < 1 - \zeta$. For all N large enough,

$$P(\sup_{x \in C_k} |\widehat{F}_{y_k}(x) - F_{y_k}(x)| > \zeta) < \varepsilon/2 \tag{3.4}$$

If $\sup_{x \in C_k} |\widehat{F}_{y_k}(x) - F_{y_k}(x)| \leq \zeta$, we get, for all $x \in C_k$,

$$\begin{aligned} |\widehat{H}_{kj}(x) - H_{kj}(x)| &\leq |\widehat{F}_{y_j}^{-1}(\widehat{F}_{y_k}(x)) - F_{y_j}^{-1}(\widehat{F}_{y_k}(x))| + |F_{y_j}^{-1}(\widehat{F}_{y_k}(x)) - F_{y_j}^{-1}(F_{y_k}(x))| \\ &\leq \sup_{u \in [F_{y_k}(\underline{x}) - \zeta, F_{y_k}(\overline{x}) + \zeta]} |\widehat{F}_{y_j}^{-1}(u) - F_{y_j}^{-1}(u)| \\ &\quad + M \sup_{x \in C_k} |\widehat{F}_{y_k}(x) - F_{y_k}(x)| \end{aligned} \tag{3.5}$$

where $M = \sup_{u\in[F_{y_k}(\underline{x})-\zeta, F_{y_k}(\overline{x})+\zeta]} F_{y_j}^{-1\prime}(u)$. Hence, for all N large enough,

$$\begin{aligned}
& P\left(a_N \sup_{x\in C_k} |\widehat{H}_{kj}(x) - H_{kj}(x)| > \delta\right) \\
\leq\ & P\left(a_N \sup_{x\in C_k} |\widehat{H}_{kj}(x) - H_{kj}(x)| > \delta, \sup_{x\in C_k} |\widehat{F}_{y_k}(x) - F_{y_k}(x)| \leq \zeta\right) \\
& + P\left(\sup_{x\in C_k} |\widehat{F}_{y_k}(x) - F_{y_k}(x)| > \zeta\right) \\
\leq\ & P\left\{a_N \left(\sup_{u\in[F_{y_k}(\underline{x})-\zeta, F_{y_k}(\overline{x})+\zeta]} |\widehat{F}_{y_j}^{-1}(u) - F_{y_j}^{-1}(u)| + M \sup_{x\in C_k} |\widehat{F}_{y_k}(x) - F_{y_k}(x)|\right) > \delta\right\} + \frac{\varepsilon}{2} \\
\leq\ & \varepsilon
\end{aligned}$$

where the second inequality stems from (3.4) and (3.5), and the third is obtained by using (3.2) and (3.3). The result follows since ε and δ where arbitrary. $\Box$

Lemma 3.4 *Suppose that assumption 9 is satisfied, $n > 0$, $n-1 \in I_1$ and the condition (8.3) of the main paper holds. Then*

$$P(n \in \widehat{I}_1) \to \mathbb{1}_{n\in I_1}.$$

The same holds for any $n < 0$ such that $n+1 \in I_1$, replacing $n-1$ and $\widehat{H}_{12}$ by respectively $n+1$ and $\widehat{H}_{21}$ in the equation (8.3) of the main paper.

Proof: we restrict, without loss of generality, to the case where $n-1 \geq 0$. We first prove the result when $n \in I_1$. Let E_ε denote the event $\{n-1 \in \widehat{I}_1, |\widehat{H}_{12}(\widehat{y}_{n-1}) - y_n| \leq \varepsilon\}$. We have

$$P(n \notin \widehat{I}_1) \leq P(n \notin \widehat{I}_1, E_\varepsilon) + 1 - P(E_\varepsilon).$$

Thus, by the condition (8.3), it suffices to prove that for a well chosen $a > 0$,

$$P(n \notin \widehat{I}_1, E_a) \to 0 \tag{3.6}$$

By assumption 9, $y_n = H_{12}(y_{n-1}) \in \overset{\circ}{\mathcal{Y}_1}$ and thus, by lemma 3.2, $0 < F_{y_1}(H_{12}(y_{n-1})) < 1$. Hence, there exists $a > 0$ such that $F_{y_1}(H_{12}(y_{n-1}) + a) < 1$ and $F_{y_1}(H_{12}(y_{n-1}) - a) > 0$. When $|\widehat{H}_{12}(\widehat{y}_{n-1}) - H_{12}(y_{n-1})| \leq a$, $\widehat{H}_{12}(\widehat{y}_{n-1}) \notin \widehat{\mathcal{Y}_1}$ implies either $H_{12}(y_{n-1}) + a > \max_k y_{1k}$ or $H_{12}(y_{n-1}) - a < \min_k y_{1k}$. This proves (3.6), since the probability of these events is respectively $F_{y_1}(H_{12}(y_{n-1}) + a)^N$ and $(1 - F_{y_1}(H_{12}(y_{n-1}) - a))^N$.

Now let us prove the result when $n \notin I_1$. Because $\mathcal{Y}_1$ is a closed set, $\min_{x\in\mathcal{Y}_1} |H_{12}(y_{n-1}) - x| = b > 0$. Now, because $\widehat{\mathcal{Y}_1} \subset \mathcal{Y}_1$,

$$\begin{aligned}
\mathbb{P}(n \in \widehat{I}_1) &= \mathbb{P}(n-1 \in \widehat{I}_1, \widehat{H}_{12}(\widehat{y}_{n-1}) \in \widehat{\mathcal{Y}_1}) \\
&\leq \mathbb{P}(n-1 \in \widehat{I}_1, \widehat{H}_{12}(\widehat{y}_{n-1}) \in \mathcal{Y}_1) \\
&\leq \mathbb{P}(n-1 \in \widehat{I}_1, |\widehat{H}_{12}(\widehat{y}_{n-1}) - H_{12}(y_{n-1})| \geq b)
\end{aligned}$$

Thus, by the condition (8.3) of the main paper, the l.h.s. tends to zero. This shows that $\mathbb{P}(n \in \widehat{I}_1) \to 0$ □

Lemma 3.5 *Suppose that assumptions 1, 3, 8 and 10 hold and let C_k denote a compact strictly included in $\mathcal{Y}_k$ for $k \in \{1, ..., K\}$. Then, if $h_{1N} \to 0$ and $Nh_{1N}/\ln N \to \infty$,*

$$\sup_{y \in C_k} |\widehat{q}_k(y) - q_k(y)| \xrightarrow{\mathbb{P}} 0.$$

Furthermore, if $h_{2N} \to 0$ and $N^3 h_{2N}/\ln N \to \infty$,

$$\sup_{y \in C_k} |\widehat{q}'_k(y) - q'_k(y)| \xrightarrow{\mathbb{P}} 0.$$

Proof: Let $T_k = \sup_{u \in C_k} |t'_k(u)|$, it follows by the triangular inequality that

$$\begin{aligned} \sup |\widehat{q}_k(y) - q_k(y)| &\leq T_k \sup_{u \in C_k} \left| (1 - \widehat{F}_{y_k}(u)) \left(\frac{1}{\widehat{f}_{y_k}(u)} - \frac{1}{f_{y_k}(u)} \right) + \frac{1}{f_{y_k}(u)} (F_{y_k}(u) - \widehat{F}_{y_k}(u)) \right| \\ &\leq T_k \left(\frac{\sup_{u \in C_k} |\widehat{f}_{y_k}(u) - f_{y_k}(u)|}{\inf_{u \in C_k} f_{y_k}(u) \inf_{u \in C_k} |\widehat{f}_{y_k}(u)|} + \frac{\sup_{u \in C_k} |\widehat{F}_{y_k}(u) - F_{y_k}(u)|}{\inf_{u \in C_k} f_{y_k}(u)} \right) \end{aligned} \quad (3.7)$$

Because assumption 8.1-8.3 of Newey & McFadden (1994) are met here (with $d = 0$ in their notations), we get, by their lemma 8.10,

$$\sup_{u \in C_k} |\widehat{f}_{y_k}(u) - f_{y_k}(u)| = O_P \left(\sqrt{\frac{\ln N}{N h_{1N}}} + h_{1N}^2 \right).$$

Hence, by the conditions on h_{1N}, the supremum on the l.h.s. tends to zero.

Besides, by lemma 3.2, $\inf_{u \in C_k} f_{y_k}(u) > 0$. Thus, by uniform convergence, $\inf_{u \in C_k} \widehat{f}_{y_k}(u) > 0$ with an arbitrary large probability. Hence, the first term on the l.h.s. of (3.7) tends to zero in probability. The second also tends to zero by Glivenko-Cantelli Theorem. The result on $q_k(.)$ follows.

The proof of the uniform convergence of $q'_k(.)$ is similar. The key point is the uniform convergence of $\widehat{f}'_{y_k}$ on C_k. To obtain it, note that assumption 8.2 of Newey & McFadden (1994) is satisfied here with $d = 1$ because by lemma 3.2, f_{y_k} is continuously differentiable. Then, by their lemma 8.10 once more,

$$\sup_{u \in C_k} |\widehat{f}'_{y_k}(u) - f'_{y_k}(u)| = O_P \left(\sqrt{\frac{\ln N}{N^3 h_{1N}}} + h_{1N}^2 \right).$$

This and the conditions on h_{2N} ensure the uniform convergence of $\widehat{f}_{y_k}$ □

Lemma 3.6 *Let $E'_N(\varepsilon, \Delta)$ and $E_{m,N}(\varepsilon, \Delta)$ denote the events defined respectively by equations (8.6) and (8.7) of the main paper. For all (ε, Δ) such that $\varepsilon < \Delta$, we have*

$$E'_N(\varepsilon, \Delta) \subset \cup_{m=0}^{m_n} E_{m,N}(\varepsilon, \Delta).$$

Proof: Suppose that the event $E'_N(\varepsilon, \Delta)$ holds and assume, w.l.o.g., that $\varepsilon < \Delta$. Then let us show by induction on $m \in \{0, ..., m_N\}$ that $E_{m,N}(\varepsilon, \Delta)$ holds.

The result is obvious for $m = 0$. Suppose that it holds for $m - 1 \geq 0$. We first show that $I_{2,m}^{-\Delta} \subset \widehat{I}_{2,m}$. If $y_{k,l} \in Y^{-\Delta}$, then, by the proof of Theorem 3.5 applied to $Y^{-\Delta}$ instead of $\mathcal{Y}_1$, there exists $(k', l') \in I_2$ such that $|k'| + |l'| = m - 1$ and $y_{k',l'} \in Y^{-\Delta}$. Assume w.l.o.g. that $(k', l') = (k - 1, l)$. By the induction assumption, $(k - 1, l) \in \widehat{I}_{2,m-1}$. Moreover,

$$\begin{aligned} |\widehat{H}_{21}(\widehat{y}_{k-1,l}) - y_{k,l}| &\leq |\widehat{H}_{21}(\widehat{y}_{k-1,l}) - H_{21}(\widehat{y}_{k-1,l})| + |H_{21}(\widehat{y}_{k-1,l}) - H_{21}(y_{k-1,l})| \\ &\leq \sup_{x \in Y} |\widehat{H}_{21}(x) - H_{21}(x)| + M|\widehat{y}_{k-1,l} - y_{k-1,l}| \\ &\leq \varepsilon_{1,N} + M\frac{M^{m-1} - 1}{M^{m_N} - 1} \\ &\leq \varepsilon_{m,N} \end{aligned}$$

Since $\varepsilon_{m,N} \leq \Delta$, $\widehat{H}_{21}(\widehat{y}_{k-1,l}) \in Y$, which proves that $(k, l) \in \widehat{I}_2$.

Now, let us show that $\widehat{I}_{2,m} \subset I_{2,m}^{\Delta}$ and $|\widehat{y}_{k,l} - y_{k,l}| \leq \varepsilon_{m,N}$ for all $(k, l) \in \widehat{I}_{2,m}$. Let $(k, l) \in \widehat{I}_{2,m}$ and suppose for instance that $(k - 1, l) \in \widehat{I}_2$ and $\widehat{H}_{12}(\widehat{y}_{k-1,l}) \in Y$. By the induction hypothesis, $(k - 1, l) \in I_2^{\Delta}$. Moreover,

$$\begin{aligned} |H_{12}(y_{k-1,l}) - \widehat{H_{12}}(\widehat{y}_{k-1,l})| &\leq |H_{12}(\widehat{y}_{k-1,l}) - H_{12}(y_{k-1,l})| + |H_{12}(\widehat{y}_{k-1,l}) - \widehat{H}_{12}(\widehat{y}_{k-1,l})| + \\ &\leq \varepsilon_{m,N} \end{aligned} \tag{3.8}$$

Thus, because $\widehat{H_{12}}(\widehat{y}_{k-1,l}) \in Y$, $H_{12}(y_{k-1,l}) \in Y^{\Delta}$ and $(k, l) \in I_{2,m}^{\Delta}$. Moreover, by (3.8), $|y_{k,l} - \widehat{H_{12}}(\widehat{y}_{k-1,l})| < \varepsilon_{m,N}$. If $(k, l - 1) \notin \widehat{I}_2$ or $\widehat{H}_{23}(\widehat{y}_{k,l-1}) \notin Y$, this implies that $|y_{k,l} - \widehat{y}_{k,l}| \leq \varepsilon_{m,N}$. Otherwise, as previously, $|H_{23}(y_{k,l-1}) - \widehat{H_{23}}(\widehat{y}_{k,l-1})| \leq \varepsilon_{m,N}$ so that $y_{k,l} = H_{12}(y_{k-1,l}) = H_{23}(y_{k,l-1})$ and

$$\begin{aligned} |\widehat{y}_{k,l} - y_{k,l}| &\leq \frac{1}{2}\Big(|H_{23}(y_{k,l-1}) - \widehat{H_{23}}(\widehat{y}_{k,l-1})| + |H_{12}(y_{k-1,l}) - \widehat{H_{12}}(\widehat{y}_{k-1,l})|\Big) \\ &\leq \max\Big\{|H_{23}(y_{k,l-1}) - \widehat{H_{23}}(\widehat{y}_{k,l-1})|, |H_{12}(y_{k-1,l}) - \widehat{H_{12}}(\widehat{y}_{k-1,l})|\Big\} \\ &\leq \varepsilon_{m,N}. \end{aligned}$$

Thus $E_{m,N}(\varepsilon, \Delta)$ holds, and the induction is complete. □

Additional references

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