Hedging and microstructure

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Outline

1. Introduction: hedging error
2. Hedging strategies
3. Asymptotic results for the microstructural hedging error
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2. Hedging strategies
3. Asymptotic results for the microstructural hedging error
Frictionless market

Assumptions for a frictionless market

- It is possible to borrow and lend cash at a risk-free interest rate.
- The transaction price is equal to the efficient price, irrespectively of the volume of the transaction and of its sign (buy or sell).
- One can buy or sell instantaneously and continuously.
- There are no transaction costs.
- The asset is perfectly divisible (it is possible to buy or sell any fraction of a share). Moreover, short selling is authorized.
Hedging problem in a market with frictions

Failure of one of the preceding conditions

The failure of one of the preceding conditions makes the problem of hedging a derivative security more complex. The following cases are treated in the literature:

Our setting

### Microstructure noise effects

We work in the model with uncertainty zones which accommodates the stylized facts of ultra high frequency prices and durations together with a semi-martingale efficient price. In particular, transaction prices will belong to the tick grid. Consequently:

- Impossibility to buy or sell a share at the efficient price: the microstructure noise leads to a cost (possibly negative).
- Transaction price changes a finite number of times on a given time period. Therefore, it is reasonable to assume that one waits for a price change before rebalancing the hedging portfolio.
Our setting

**Studied strategies**

In our market model, we will assume an agent considers a hedging procedure derived from a theoretical (possibly misspecified) local volatility-type replicating strategy. We will study two hedging strategies, where the usual continuous time hedging portfolio is rebalanced at some random trading times:

- The hedging portfolio is rebalanced every time that the transaction price moves.
- The hedging portfolio is rebalanced only once the transaction price has varied by more than a selected value.
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Benchmark frictionless hedging strategy

The benchmark frictionless hedging strategy is those of an agent deciding (possibly wrongly) that the volatility of the efficient price at time $t$ is equal to $\sigma(t, X_t)$.

It leads to a benchmark frictionless hedging portfolio whose value $\Pi_t$ satisfies

$$\Pi_t = C(0, X_0) + \int_0^t \dot{C}_x(u, X_u) \, dX_u.$$ 

Note that, if the model is misspecified, $\Pi_t$ is different from $C(t, X_t)$. 
We naturally impose that the times when the hedging portfolio may be rebalanced are the times where the transaction price moves. Thus, the hedging portfolio can only be rebalanced at the transaction times $\tau_i$.

In this setting, we consider strategies such that, if $\tau_i$ is a rebalancing time, the number of shares in the risky asset at time $\tau_i$ is $\hat{C}_X(\tau_i, X_{\tau_i})$. 
Hedging strategies in the model with uncertainty zones

We will consider two hedging strategies:

1. The hedging portfolio is rebalanced every time that the transaction price moves.
2. The hedging portfolio is rebalanced only once the transaction price has varied by more than a selected value.
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Components of the hedging error

In our setting, the microstructural hedging error is due to:

- **Discrete trading**: the hedging portfolio is rebalanced a finite number of times.
- **Microstructure noise on the price**: between two rebalancing times, the variation of the market price (multiple of the tick size) differs from the variation of the efficient price.
Two steps analysis

We analyse this microstructure hedging error in two steps.

- First, we assume that there is no microstructure noise on the price although the trading times are endogenous (for all $i$, $P_{\tau_i} = X_{\tau_i}$).
- Second, we assume the presence of the endogenous microstructure noise and discussed the two hedging strategies.
Hedging error without microstructure noise on the price

Let $\phi(t) = \sup \{ \tau_i : \tau_i < t \}$. In the absence of microstructure noise on the price, the hedging error is given by

$$L^{(1)}_{\alpha, t} = \int_0^t [\dot{C}_x(u, X_u) - \dot{C}_x(\phi(u), X_{\phi(u)})]dX_u.$$
Hedging error without microstructure noise on the price

Theorem

As $\alpha$ tends to 0,

$$N_{\alpha,t} L_{\alpha,t} \xrightarrow{\mathbb{L}} L_t := f_t^{1/2} \int_0^t c_s^{(1)} d\mathcal{W}_s^{(1)},$$

in $\mathbb{D}[0, T]$, where $\mathcal{W}^{(1)}$ is a Brownian motion defined on an extension of the filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and independent of all the preceding quantities, and

$$(c_s^{(1)})^2 = \frac{1}{6} \tilde{C}_{xx}(\theta_s, X_{\theta_s}) \mu_4(\chi_{\theta_s}),$$

$$f_t = \int_0^t \left( \sum_{j=1}^m p_j(\chi_u) j(j - 1 + 2\eta) \right)^{-1} \sigma_u^2 X_u^2 du.$$
Hedging error without microstructure noise on the price

Comments

- The variance of the hedging error is proportional to the inverse of the number of rebalancing transactions.
- It depends on the local volatility gamma of the derivative security.
Hedging error with microstructure noise

**Total hedging error**

In the presence of microstructure noise on the price, the transaction prices differ from the efficient prices. The hedging error is now given by

\[
L^{(2)}_{\alpha, t} = \int_0^t \dot{C}_x(u, X_u) \, dX_u - \int_0^t \dot{C}_x(\phi(u), X_{\phi(u)}) \, dP_u.
\]
Hedging error with microstructure noise

Theorem

As $\alpha$ tends to 0,

$$L_{\alpha,t}^{(2)} \xrightarrow{\mathcal{I}} L_t^{(2)} := \int_0^t a_{fs}^{(2)} ds + \int_0^t b_{fs}^{(2)} dX_s + \int_0^t c_{fs}^{(2)} d\mathcal{W}_{fs}^{(2)},$$

in $\mathbb{D}[0, T]$, with

$$a_{s}^{(2)} = -(1 - 2\eta) \ddot{C}_{xx} (\theta_s, X_{\theta_s}) \mu_1^*, a (\chi_{\theta_s}) \varphi (\chi_{\theta_s})$$

$$b_{s}^{(2)} = (1 - 2\eta) \dot{C}_x (\theta_s, X_{\theta_s}) \mu_1^*, a (\chi_{\theta_s}) \varphi (\chi_{\theta_s})$$

$$(c_{s}^{(2)})^2 = (1 - 2\eta)^2 \dot{C}_x^2 (\theta_s, X_{\theta_s})$$

$$\varphi (\chi_{\theta_s}) \left( \pi_a (\chi_{\theta_s}) \varphi^{-1}(\chi_{\theta_s}) - (\mu_{1,a}^*(\chi_{\theta_s}))^2 \right).$$
Hedging error with microstructure noise

Comments (1)

- The microstructural hedging error process is not renormalized as in the previous case.
- It means that the hedging error does not vanish even if the number of rebalancing transactions goes to infinity and so is of the same order of magnitude as the usual tracking error.
- If $\eta = 1/2$, the error due to the microstructure noise on the price vanishes.
- The unusual term $\int_0^t b_{f_s}^{(2)} dX_s$ is due to the asymmetry between alternations and continuations. Indeed, when an alternation occurs, $\Delta P_{\tau_i} - \Delta X_{\tau_i} = (1 - 2\eta)\text{sign}(\Delta X_{\tau_i})$ while, when a continuation occurs, $\Delta P_{\tau_i} - \Delta X_{\tau_i} = 0$. 
Hedging error with microstructure noise

Comments (2)

- The asymptotic error is not necessarily centered.
- The quadratic variation of the asymptotic hedging error is

\[(1 - 2\eta)^2 \int_0^t \dddot{C}_x(s, X_s) \pi_a(\chi_s) \, df_s.\]

It now depends on the local volatility delta of the derivative security and on the proportion of alternations.

- Consequently, the variance of the microstructural hedging error increases with the position delta and the proportion of alternation in the price.
Optimal rebalancing

Total hedging error

The hedging portfolio is now rebalanced only once the price has varied by $l_\alpha$ ticks (we could also consider the same way that the hedging portfolio is rebalanced only once the local volatility delta has varied by a given value). The hedging error is given by

$$L_{\alpha,t}^{(3)} = \int_0^t \dot{C}_x(u, X_u) \, dX_u - \int_0^t \dot{C}_x(\phi^{(l)}(u), X_{\phi^{(l)}(u)}) \, dP_u.$$ 

with $\phi^{(l)}(t) = \sup\{\tau_i^{(l)} : \tau_i^{(l)} < t\}$ and the $\tau_i^{(l)}$ are stopping times associated to moves of $l_\alpha$ ticks.
Asymptotic results for the microstructural hedging error

Theorem

Let $l_\alpha = \alpha^{-1/2}$. As $\alpha$ tends to 0,

\[
\left( N^{(l)}_{\alpha,t} \right)^{1/4} L^{(3)}_{\alpha,t} \xrightarrow{\mathcal{I}-\mathcal{L}_s} L^{(3)}_t :
\]

\[
= (f^{(l)}_t)^{1/4} \left( \int_0^t a^{(3)}_{f^{(l)}_s} ds + \int_0^t b^{(3)}_{f^{(l)}_s} dX_s + \int_0^t c^{(3)}_{f^{(l)}_s} d\mathcal{W}^{(3)}_{f^{(l)}_s} \right)
\]

avec

\[
a^{(3)}_s = -(1 - 2\eta) \ddot{C}_x (\theta_s, X_{\theta_s}), \quad b^{(3)}_s = \frac{1 - 2\eta}{2} \dot{C}_x (\theta_s, X_{\theta_s})
\]

\[
(c^{(3)}_s)^2 = \frac{(1 - 2\eta)^2}{4} \ddot{C}_x^2 (\theta_s, X_{\theta_s}) + \frac{1}{6} \dddot{C}_{xx}^2 (\theta_s, X_{\theta_s}).
\]
Optimal rebalancing

Comments

- This optimal strategy allows to reduce significantly the hedging error in the presence of microstructure noise.
- The asymptotic variance of the hedging error now depends both on the delta and on the gamma of the derivative security.
- The optimal $l_\alpha$ is of the same order of magnitude as the square root of the number of times where the hedging portfolio is rebalanced.
Numerical study

Parameters

- European call, strike 100, maturity T=1.
- Black-Scholes model with $\sigma = 0.01$, $x_0 = 100$.
- $\alpha = 0.05$, $\eta = 0.05$.
- 1000 simulations.

<table>
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<tr>
<th></th>
<th>Avg. modifs.</th>
<th>St. Dev. modifs.</th>
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<tbody>
<tr>
<td>Modif. every chg</td>
<td>3487</td>
<td>108</td>
</tr>
<tr>
<td>Modif. after chg of 5 ticks</td>
<td>20.22</td>
<td>3.55</td>
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</tbody>
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Errors 1-2-3 (green, red, grey)