Testing the local volatility assumption: 
a statistical approach *

Mark Podolskij¹  Mathieu Rosenbaum²

¹ Fakultät für Mathematik und Informatik,
University of Heidelberg
m.podolskij@uni-heidelberg.de

² Centre de Mathématiques Appliquées,
École Polytechnique Paris
mathieu.rosenbaum@polytechnique.edu

7 May 2011

Abstract
In practice, the choice of using a local volatility model or a stochastic volatility model is made according to their respective ability to fit implied volatility surfaces. In this paper, we adopt a different point of view. Indeed, using a purely statistical methodology, we design new procedures aiming at testing the assumption of a local volatility model for the price dynamics, against the alternative of a stochastic volatility model. These test procedures are based only on historical data and do not require any calibration procedures via option prices. We also provide a convincing simulation study and an empirical analysis on future contracts on interest rates.

Key words: Local Volatility Models, Stochastic Volatility Models, Test Statistics, Semi-Martingales, Limit Theorems.

MSC2010: 60F05, 60G44, 60J60, 62M02, 62M07.

*We would like to thank Holger Dette for turning our attention to the statistical problem discussed in this paper. We are also very grateful to Jean-Marc Duprat (IRG BNP-Paribas London), Nicolas Michon (IRG BNP-Paribas London) and Sebouh Takvorian (FIRST BNP-Paribas London) for providing and discussing the data used in this work. Mark Podolskij acknowledges financial support from CREATES funded by the Danish National Research Foundation.
1 Introduction

It is well known that the Black-Scholes model does not allow for important stylized facts of asset returns such as heavy tails, gain/loss asymmetry or leverage effect. A classical way to obtain these features of financial data is to use models in which the volatility, that is the diffusion coefficient of the log price, is itself a random process. However, the success of these models with random volatility is clearly not due to their statistical properties. Indeed, practitioners essentially focus on the fact that they enable to fit implied volatility surfaces much better than the Black-Scholes model does. Two main types of models in which the volatility is a random process are particularly used: models with local volatility and models with stochastic volatility.

In local volatility models, the volatility is assumed to depend on time and of the present value of the asset only. They can be written on the form

\[ \text{d}X_t = \mu_t \text{d}t + \sigma_t \text{d}W_t, \quad \sigma_t = \sigma(X_t, t), \]

where \( X_t \) represents the price of the asset at time \( t \), \( W_t \) is a Brownian motion and \( \sigma(x, t) \) is a deterministic function. Therefore, in these models, the price \( X_t \) and the volatility \( \sigma_t/X_t \) can both be stochastic. However, there is only one factor of randomness, the Brownian motion \( W_t \). In particular, the Black-Scholes model belongs to the class of local volatility models. In that case, \( \sigma(X_t, t) = \sigma X_t \), where \( \sigma \) is a positive constant and so the volatility is constant. Among the other local volatility models, the most famous one is probably the constant elasticity of variance model (CEV model) introduced in [6]. In the CEV model, the function \( \sigma(x, t) \) is of the form \( \alpha x^{1+\beta} \), where \( \alpha > 0 \) and \( 0 \leq \beta \leq 1 \) are constants.

A first interest of local volatility models is that in spite of the random nature of the volatility, they remain arbitrage free and complete. Mostly, the popularity of these models is due to the work of Dupire. He showed in [10] that provided the market is arbitrage free, one can find a local volatility function which enables to fit exactly the implied volatility surface of the European call options, see also [7]. However, local volatility models have several drawbacks. In particular, since only a finite number of strikes and maturities are available on the market, the derivation of the local volatility function requires interpolation-type methods, leading sometimes to highly unstable results. Moreover, such models do not allow for relevant smile dynamics. In particular, it is shown in [13] that under this specification for the volatility, the spot volatility smile moves in the opposite direction as the underlying. Also, in these models, the forward smile has a flattening dynamic, see [12]. These facts are not in agreement with the behaviors observed on the market. Thus, one source of randomness is not always enough to manage the smile risk.

To remedy this, in so-called stochastic volatility models, one therefore increases
the dimension of the underlying Brownian motion. Indeed, they can be written as

\[ dX_t = \mu_t dt + \sigma_t dW_t, \]

where the process \( \sigma_t \) satisfies

\[ d\sigma_t^2 = \tau_t dt + \nu_t dW_t + \nu_t dV_t, \]

with \( W \) and \( V \) two independent Brownian motions and \( \tau_t \) a non degenerate process. Various specifications for stochastic volatility models have been proposed in the literature and are largely used in practice. Let us cite among others the works by Hull and White [15], Heston [14] and the SABR model introduced in [13]. Despite incomplete, these models are very popular among practitioners since they enable to obtain more suitable smiles and smile dynamics. For example, the SABR model has been derived in [13] in order to remedy the fact that when the price of the underlying decreases (resp. increase), local vol models predict that the smile shifts to higher (resp. lower) prices. Also, in stochastic volatility model, future implied volatility surfaces do not flatten as they do in local volatility models. Finally, note that these models are not necessarily more complex for pricing issues. Indeed, formulas can be semi explicit, as in the case of the Heston model, see [14].

Thus, it is quite clear that the practical relevance of a model is essentially assessed through the lenses of derivatives pricing and hedging. Historical data from the underlying are hardly taken into account. In this paper, we adopt a different point of view. Indeed, we are interested in what historical data have to say about volatility model selection. More precisely, we want to know if the historical data are “compatible” with a local volatility model.

This question might appear surprising. Indeed, smile dynamics already seem to give an answer. However we want to stress the following fact: option prices, and more generally implied quantities, are clearly the most important data to deal with when one wants to hedge derivatives. Nevertheless, from a statistical point of view, options data are in practice not “fully reliable”, in the sense that their link with historical data through the no free lunch assumption remains arguable. Indeed, option prices reflect the expectations of options market makers about the medium-term future of stock prices, and in this sense may contain more information than historical stock prices, a fact that can go against the practice of Markovian pricing and hedging in local volatility models. However, at the same time, the additional information cannot necessarily be equated with what may come from any unobservable portion of a stochastic volatility in stochastic volatility models. These facts illustrate the persistent complexity of the modeling problem for stock prices and their consistency with option prices. Therefore, although our question is probably not of very first interest from a classical mathematical finance perspective, it is however very natural and im-
portant if one really wants to understand the dynamics of asset returns.

Our way to partially answer the preceding question is the following. We use historical data of an asset $X$ regularly observed over a fixed time period $[0,T]$:

$$X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots$$

and work in a high frequency context, which means our asymptotic is that the time span between two observations $\Delta_n$ goes to zero. From these data, we build a test statistics $S_n$ whose behavior is approximately standard Gaussian when $X$ belongs to a given class $\Theta_0$ of processes following local volatility models. Therefore, thanks to this statistics, we can construct a statistical test with level $\alpha$ through the rejection area $\{S_n^2 > u_{1-\alpha}\}$, where $u_{1-\alpha}$ is the $1 - \alpha$ quantile of a $\chi^2(1)$ distribution. Since our aim is to compare local volatility models and stochastic volatility models, in order to get a meaningful test, we will also impose that our test statistics diverges to infinity when the data generating process belongs to a class $\Theta_1$ of stochastic volatility models. The “tricky” part of the paper will be the construction of this test statistics $S_n$. Then, deriving its asymptotic behavior when $X \in \Theta_0$ or $X \in \Theta_1$ will be essentially a direct use of recent general results about semi-martingales.

Remark that this kind of tests is in the spirit of several procedures recently developed in the literature. The papers [8], [9], for instance, deal with goodness-of-fit tests for local volatility models in the high-frequency framework. On the other hand, Aït-Sahalia and Jacod have addressed the following testing problems for Itô semi-martingales (see [1], [2] and [3]): Is the jump part of the semi-martingale equal to zero? Do the jumps have finite or infinite activity? Is the Brownian part equal to zero? Such kind of test procedures is also designed in [11] in order to assess the multifractal nature of data from a semi-martingale. Note that as in all these works, since we want to distinguish between two very large classes, we will only be able to give pointwise results and so the level of the test will not hold uniformly over $\Theta_0$ (which means that the supremum over all $X \in \Theta_0$ of the probabilities of being in the rejection area will not be controlled).

In this paper, we treat the situation where the null hypothesis is that the asset price follows a local volatility model against the alternative that it follows a stochastic volatility model. One can also ask about the case where the null and alternative hypothesis are switched. We highlight the answer to this problem, which is in fact an easier one, in Remark 3.2.

The paper is organized as follows. In Section 2, we propose a test procedure in the (unrealistic) case where both $X$ and $\sigma^2$ are observed at times $0, \Delta_n, 2\Delta_n \ldots$ In Section 3, we explain how to deduce from it a feasible test statistics when the volatility process is not observed. A simulation study is performed in Section 4 and an empirical analysis on future contracts on interest rates is provided in Section 5. The proofs are relegated to Section 6.
2 Case with observed volatility

In this section, we focus on the case where both the price \( X \) and the process \( \sigma^2 \) are observed at regular times. Remark that it implies that the volatility process \( \sigma_t/X_t \) is also observed at regular times. The results in the case where the volatility is not observed will naturally follow from those obtained in this section.

2.1 Statistical problem

On a given filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq0}, \mathbb{P})\) we consider a one dimensional continuous Itô process of the form

\[
\mathrm{d}X_t = a_t \mathrm{d}t + \sigma_t \mathrm{d}W_t, \quad t \in [0, T],
\]

where \( a \) is an adapted càdlàg drift process, \( \sigma \) is a positive adapted càdlàg process and \( W \) denotes a standard Brownian motion. The process \( X \) is assumed to be observed at equidistant time points \( t_i = i \Delta_n, i = 0, \ldots, \lfloor T/\Delta_n \rfloor \) and \( \Delta_n \to 0 \).

Our aim is to decide on the basis of observations \((X_{i\Delta_n}, \sigma^2_{i\Delta_n}), i = 0, \ldots, \lfloor T/\Delta_n \rfloor\)

whether the process \( \sigma^2 \) is a deterministic function of \( X \) (Markov diffusion case) or if it follows a stochastic volatility model. Formally, the null hypothesis is given as

\[ H_0 : \text{The process } \sigma^2_t \text{ has the form } \sigma^2_t = z(X_t) \text{ for a positive function } z \in C^2(\mathbb{R}) \text{ with } z' > 0. \]

while the alternative is given by

\[ H_1 : \text{The process } \sigma^2_t \text{ follows a stochastic volatility model } \mathrm{d}\sigma^2_t = \bar{\sigma}_t \mathrm{d}t + \sigma_t \mathrm{d}W_t + \nu_t \mathrm{d}V_t \text{ for two independent Brownian motions } W \text{ and } V \text{ and some adapted càdlàg processes } \bar{\sigma}, \sigma \text{ and } \nu \text{ with } \nu \text{ being non-vanishing on some measurable set } A \subset [0, T] \text{ with } \lambda(A) > 0 \text{ (a.s.)}. \text{ Moreover, the processes } \bar{\sigma} \text{ and } \sigma \text{ are Itô semi-martingales.} \]

Some remarks are in order.

**Remark 2.1.** By Itô’s formula we obtain under \( H_0 \) that

\[
\mathrm{d}\sigma^2_t = (a_t z'(X_t) + \sigma_t^2 z''(X_t)) \mathrm{d}t + \sigma_t z'(X_t) \mathrm{d}W_t.
\]

That is, under \( H_0 \), \( \sigma^2 \) follows a stochastic volatility model as in \( H_1 \), but with \( \nu = 0 \). Note however that the models in \( H_0 \) and \( H_1 \) do not contain all Itô semi-martingale processes for \( \sigma^2 \): the model \( \mathrm{d}\sigma^2_t = \bar{\sigma}_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \) with \( \bar{\sigma}_t \neq \sigma_t z'(X_t) \) belongs neither to \( H_0 \) nor to \( H_1 \).
Remark 2.2. The ad-hoc assumption $z' > 0$ in the null hypothesis enables us to provide a test procedure. It looks rather strange, but it is satisfied for many price models. This is in particular true in the Black-Scholes and CEV models mentioned in the introduction. Indeed, in these cases, $z(x) = x^p$ for some $p > 0$ and $X_t > 0$.

2.2 Building the test statistics

The starting idea for the construction of our test is quite simple. For any $s, t > 0$, since $z$ is increasing, we have under $H_0$

$$(X_t - X_s)(\sigma_t^2 - \sigma_s^2) = (X_t - X_s)(z(X_t) - z(X_s)) \geq 0.$$ 

Under $H_1$, the above positivity cannot hold a.s. as long as $v$ is non-vanishing (in the sense of $H_1$).

The first step for our procedure is to choose a function $g : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

(i) $g \in C^1_p(\mathbb{R})$ ($C^1$-functions of polynomial growth) with $g(x) > 0$ for $x > 0$ and $g(x) < 0$ for $x < 0$.

(ii) $g^+ = \max(g, 0) \in C^1(\mathbb{R})$.

A simple example of such a function is given by $g(x) = x^3$ (or more generally by $g(x) = x^{2k+1}$, $k \geq 1$). Our test statistic will be based on the quantity $M_{n,T}$, with for $t \leq T$,

$$M_{n,t} = \Delta_n \left( \sum_{i=2}^{[t/\Delta_n]} g^+ \left( \frac{X_{i\Delta_n} - X_{(i-2)\Delta_n}}{\sqrt{2\Delta_n}} \cdot \frac{\sigma_{i\Delta_n}^2 - \sigma_{(i-2)\Delta_n}^2}{\sqrt{2\Delta_n}} \right) 
- \sum_{i=1}^{[t/\Delta_n]} g \left( \frac{X_{i\Delta_n} - X_{(i-1)\Delta_n}}{\sqrt{\Delta_n}} \cdot \frac{\sigma_{i\Delta_n}^2 - \sigma_{(i-1)\Delta_n}^2}{\sqrt{\Delta_n}} \right) \right).$$

Thanks to the properties of $g$, the products of increments involved in $M_{n,t}$ are always positive under $H_0$. Since $g^+ = g$ on $\{g(x) > 0\} = \{x > 0\}$, this will imply that, under $H_0$, $M_{n,T}$ goes to 0 in probability (see Corollary 2.5). On the other hand, under the alternative, we will have $\lim_{n \to \infty} M_{n,t} \geq 0$.

Remark 2.3. Notice that we compare two estimators at frequencies $\Delta_n$ and $2\Delta_n$ in the definition of $M_{n,t}$. When the process $\sigma^2$ is fully observed it is more natural to use the same frequency $\Delta_n$. Indeed, in this case we would accept the null hypothesis only if $M_{n,t} = 0$ identically. However, in practice, the process $\sigma^2$ is not observed (see Section 3). Thus, we need to use an estimator $\hat{\sigma}^2$ instead of the true process $\sigma^2$ in the definition of $M_{n,t}$. Now, if we would use the same frequency in our test statistic, say $\Delta_n$, the asymptotic results would solely come
from the approximation error when replacing $\sigma^2$ by its empirical analogue $\hat{\sigma}^2$. In such situation, the obtained central limit theorems would be typically infeasible (i.e. the asymptotic results cannot be used for statistical inference). The reason is the following: the resulting centered and properly normalized statistics would essentially be an odd functional of the price process $X$ and in this case central limit theorems are known to be infeasible. We refer to [19] for asymptotic results for general functionals of continuous semi-martingales.

2.3 Law of large numbers

Before we proceed with the weak law of large numbers for $M_{n,t}$, we need to introduce some notation. For $k$ a positive integer, $\Psi$ a $2 \times 2$-matrix and $f$ a function from $(\mathbb{R}^2)^k$ to $\mathbb{R}$, we set

$$\rho^{\otimes k}_\Psi (f) = \mathbb{E}[f(\Psi U_1, \ldots, \Psi U_k)],$$

where $U_1, \ldots, U_k$ are iid bidimensional Gaussian vector with covariance matrix equal to identity. If $k = 1$, we simply write $\rho_\Psi (f)$. We also use the following representation of $(X, \sigma^2)$:

$$d\left( \frac{X^t}{\sigma^2} \right) = \left( \begin{array}{c} a_t \\ \overline{a_t} \end{array} \right) dt + \Sigma_t d\left( \begin{array}{c} W_t \\ V_t \end{array} \right), \quad \Sigma_t = \left( \begin{array}{cc} \sigma_t & 0 \\ \overline{\sigma_t} & \overline{v_t} \end{array} \right).$$

Note that this representation holds under $H_1$, and under $H_0$ with $\overline{v} = 0$ and $\overline{\sigma} = \sigma_t \overline{z}'(X_t)$. We have the following law of large numbers.

**Theorem 2.4.** Let $h(x, y) = g(xy)$. Under $H_0$ and under $H_1$, it holds that

$$M_{n,t} \rightarrow M_t = \int_0^t \rho_{\Sigma_u} (h^+ - h) du,$$

in probability, uniformly over compact sets in $[0, T]$.

Let $A = \{ s, \overline{v}_s \neq 0 \} \subset [0, t]$ (then $A = \emptyset$ under $H_0$ and $\lambda(A) > 0$ (a.s.) under $H_1$). Under $H_1$, since $\mathbb{P}(U_1 U_2 > 0) > 0$ and $\mathbb{P}(U_1 U_2 < 0) > 0$ for any normal variable $(U_1, U_2)$ with correlation $|\rho| < 1$, we have

$$\int_A \rho_{\Sigma_u} (h^+ - h) du > 0.$$

On the other hand, it holds that $M_t = 0$ under $H_0$. Therefore, $M_{n,T}$ has different behaviors under $H_0$ and under $H_1$. Thus we have the following corollary which will enable us to construct a test statistics in the remaining part of the section.

**Corollary 2.5.** We have the following convergences in probability: under $H_0$, $M_{n,T} \rightarrow 0$ and under $H_1$, $M_{n,T} \rightarrow M_T > 0$.  

2.4 Central limit theorem

Before defining the test statistics, we first give a general central limit theorem associated to the preceding law of large numbers. Let \( f_i : (\mathbb{R}^2)^2 \to \mathbb{R}, i = 1, 2, \) be defined by

\[
f_1((x_1, x_2), (y_1, y_2)) = g^+\left(\frac{(x_1 + y_1)(x_2 + y_2)}{2}\right), \quad f_2((x_1, x_2), (y_1, y_2)) = g(x_1x_2).
\]

Define also \( \tilde{f}_i : (\mathbb{R}^2)^3 \to \mathbb{R}, i = 1, 2, \) by

\[
\tilde{f}_i((x_1, x_2), (y_1, y_2), (z_1, z_2)) = f_i((x_1, x_2), (y_1, y_2)) f_i((y_1, y_2), (z_1, z_2)).
\]

Finally, we set \( \Delta_i^n X = X_i \Delta_n - X_i(\Delta_n - 1), \Delta_i^{n, 2} X = X_i \Delta_n - X_i(\Delta_n - 2), \) and we use the convention \( \Delta_i^0 X = \Delta_i^{n, 2} X = 0 \) for \( i > [t/\Delta_n] \). We have the following result which will be our basis for the test statistics.

**Theorem 2.6.** Under \( H_0 \) and under \( H_1 \), the process

\[
\Delta_n^{-1/2} \left( \Delta_1^n \sum_{i=2}^{[t/\Delta_n]} g^+\left(\frac{\Delta_i^{n, 2} X \Delta_i^{n, 2} \sigma_i^2}{2 \Delta_n}\right) - \int_0^t \rho_{\Sigma_n}(h^+)du \right)
\]

converges stably in law towards a process \( V_i \) that is, conditionally on \( F \), a centered Gaussian process with independent increments, such that \((i, j = 1, 2)\)

\[
\Theta_{ij,t} = \mathbb{E}[V_i, V_j,t|F] = \int_0^t R_{ij}^{ij} du,
\]

where

\[
R_{ij}^{11} = \rho_{ij}^2(f_1^2) + 2\rho_{ij}^3(\tilde{f}_1) - 3(\rho_{ij}^2(f_1))^2
\]

\[
R_{ij}^{12} = \rho_{ij}^2(f_1f_2) + \rho_{ij}^3(\tilde{f}_2) - 2\rho_{ij}^2(f_1)\rho_{ij}^2(f_2)
\]

\[
R_{ij}^{22} = \rho_{ij}^2(f_2^2) - (\rho_{ij}^2(f_2))^2.
\]

Consequently, we deduce that

\[
\Delta_n^{-1/2}(M_{n,t} - \int_0^t \rho_{\Sigma_n}(h^+ - h)du)
\]

converges stably in law towards a mixed normal random variable with conditional variance equal to \( \Theta_{11, t} - 2\Theta_{12, t} + \Theta_{22, t} \).

2.5 The test statistics

In order to obtain a formal test statistics, we need to estimate the conditional covariance matrix \( \Theta \). We give such estimators in the following theorem. For a function \( k \), we define

\[
k[i, n] = k\left(\frac{\Delta_i^n X \Delta_i^n \sigma_i^2}{\Delta_n}\right), \quad k[i, n, 2] = k\left(\frac{\Delta_i^{n, 2} X \Delta_i^{n, 2} \sigma_i^2}{2 \Delta_n}\right).
\]

We have the following result.
Theorem 2.7. Consistent estimators for the terms of the asymptotic covariance matrix in Theorem 2.6 are given by

\[
\hat{\Theta}_{11, t} = \Delta_n \left( \frac{t}{\Delta_n} \right)^{-3} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left\{ g^+[i + 1, n, 2] (g^+[i + 1, n, 2] + 2g^+[i + 2, n, 2] - 3g^+[i + 3, n, 2]) \right\}
\]

\[
\hat{\Theta}_{12, t} = \Delta_n \left( \frac{t}{\Delta_n} \right)^{-3} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left\{ g^+[i + 1, n, 2] (g[i, n] + g[i + 1, n] - 2g[i + 2, n]) \right\}
\]

\[
\hat{\Theta}_{22, t} = \Delta_n \left( \frac{t}{\Delta_n} \right)^{-3} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left\{ g[i, n] (g[i, n] - g[i + 1, n]) \right\}
\]

We can now define our test statistics \( S_n \) by

\[
S_n = \frac{\Delta_n^{-1/2} M_{n,T}}{\sqrt{\hat{\Theta}_{11, T} - 2\hat{\Theta}_{12, T} + \hat{\Theta}_{22, T}}}
\]

We have the following corollary.

Corollary 2.8. Under \( H_0 \), \( S_n^2 \) converges in law to a \( \chi^2(1) \) distribution. Under \( H_1 \), \( S_n^2 \) converges in probability to infinity.

Therefore, we reject the null hypothesis at level \( \alpha \in (0, 1) \) when \( S_n^2 > u_{1-\alpha} \), where \( u_{1-\alpha} \) denotes the \( 1 - \alpha \) quantile of a \( \chi^2(1) \) distribution.

3 Non observed volatility

Of course, in practice, we only observe over \([0, T]\) the sample

\[ X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots \]

The process \( \sigma^2 \) is not observed and needs to be locally estimated. To adapt the test statistics built in Section 2, the idea is to use a subsample

\[ (X_0, \hat{\sigma}_0^2), (X_{\Delta_n}, \hat{\sigma}_{\Delta_n}^2), (X_{2\Delta_n}, \hat{\sigma}_{2\Delta_n}^2), \ldots \]

with a slower frequency \( \Delta_n \) (instead of \( \Delta_n \)), where \( \hat{\sigma}_s^2 \) is a consistent estimator of the true value \( \sigma_s^2 \) based on the price observations \( (X_i, \Delta_n)_{i \geq 1} \). If the estimation accuracy of the volatility \( \sigma_s^2 \) is sufficiently good compared to \( \Delta_n \), the asymptotic results of Section 2 remain valid when \( \Delta_n \) is replaced by \( \Delta_n \) and \( \sigma_s^2 \) is replaced by \( \hat{\sigma}_s^2 \). More precisely, using classical localization procedures, we immediately deduce the following result.

Theorem 3.1. Assume that there exists an estimator \( \hat{\sigma}_s^2 \) of \( \sigma_s^2 \) such that the sequence

\[ u_n^{-1} \sup_{s \in [0, T]} |\hat{\sigma}_s^2 - \sigma_s^2| \]
is tight, for some $u_n$ tending to zero such that $(\Delta_n')^{-1}u_n \to 0$. If we replace in the definition of $M_{n,t}$ and in all the results of Section 2 the increments $\Delta_n^o \sigma^2$ by their empirical counterpart $\Delta_n \hat{\sigma}^2$, then all the results of Section 2 still hold provided we also replace $\Delta_n$ by $\Delta_n'$. In particular, the test statistic

$$
\hat{S}_n = \frac{\Delta_n^{1/2} \left( \sum_{i=2}^{[T/\Delta_n']} g^+ \left( \Delta_n^{n-2} X \Delta_n^{n-2} \hat{\sigma}^2 \right) - \sum_{i=1}^{[T/\Delta_n']} g \left( \Delta_n^o X \Delta_n^o \sigma^2 \right) \right)}{\sqrt{\hat{\Theta}_{11,T} - 2\hat{\Theta}_{12,T} + \hat{\Theta}_{22,T}}},
$$

(3.1)

where the quantities $\hat{\Theta}_{kl,T}$ are defined in the same way as $\hat{\Theta}_{kl,T}$ in Theorem 2.7 with $(\Delta_n, \sigma^2)$ replaced by $(\Delta_n', \hat{\sigma}^2)$, converges in distribution to a standard normal variable under $H_0$ and diverges to infinity in probability under $H_1$.

Consistent pointwise estimators $\sigma_{n,s}^2$ of the process $\sigma^2_t$ can be obtained using realized variance over some local window around the time $s$, that is

$$
\sigma_{n,s}^2 = \frac{1}{2k_n\Delta_n} \sum_{i=[s/\Delta_n]+k_n}^{[s/\Delta_n]+k_n} (\Delta_n^o X)^2 \to \sigma^2_s, \quad k_n \Delta_n \leq s \leq T - k_n \Delta_n,
$$

(3.2)
in probability, provided $k_n \to \infty$ with $k_n \Delta_n \to 0$; see for example [4], [20] (the estimators $\sigma_{n,s}^2$ for $s \in [0, k_n \Delta_n)$ (resp. $s \in (T - k_n \Delta_n, T]$) are obtained similarly by using $k_n$ increments of $X$ on the right hand side of $s$ (resp. on the left hand side of $s$)). The estimator $\sigma_{n,s}^2$ is probably the most intuitive one; however, one can also use any type of kernel estimators to get a proxy for $\sigma^2_t$.

The uniform bound $u_n$ for the estimator $\hat{\sigma}^2 = \sigma_{n,s}^2$ is given by $u_n = (k_n \Delta_n)^{1/2}$, where $k_n$ satisfies $k_n \leq c \Delta_n^{-1/2}$ for some constant $c > 0$ (see e.g. [4]). The restriction on the window size $k_n$ comes from the smoothness of the process $\sigma^2$ (as $\sigma^2$ is a continuous Itô semimartingale it is Hölder continuous of order $\alpha \in (0, 1/2]$). We see that the best attainable rate is essentially $u_n = \Delta_n^{1/4}$, which implies that $\Delta_n'$ has to converge to 0 slower than $\Delta_n^{1/4}$.

**Remark 3.2.** If we consider the null hypothesis of a stochastic volatility model (as described in Section 2) against the alternative of a local volatility model, the testing procedure is somewhat easier (in particular, we do not require the monotonicity assumption on the function $z \in C^2(\mathbb{R})$). Recall that the dynamics of the bivariate process $(X_t, \sigma^2_t)$ is given by

$$
d\left( \frac{X_t}{\sigma^2_t} \right) = \begin{pmatrix} a_t \\ \sigma_t \end{pmatrix} dt + \Sigma_t d\left( \begin{pmatrix} W_t \\ V_t \end{pmatrix} \right), \quad \Sigma_t = \begin{pmatrix} \sigma_t & 0 \\ \sigma_t & \nu_t \end{pmatrix},
$$

where $\nu = 0$ in the local volatility case while $\nu$ is a non-degenerate process for the stochastic volatility model. Thus, our test problem is equivalent to testing whether the bivariate process $(X_t, \sigma^2_t)$ is generated by two independent Brownian motions (null hypothesis) or not (the alternative). The latter testing problem has been discussed in [17] for a general $d$-dimensional continuous Itô
semi-martingale. Since the volatility process $\sigma^2$ is not observed, it has to be estimated by some $\hat{\sigma}^2$ (say, by $\sigma^2_n$) and then we can apply the procedure proposed in [17] for $d = 2$ (again we have to use a subsample $(X_{i\Delta_n}, \hat{\sigma}^2_{i\Delta_n})$ as above).

4 A simulation study

We give in this section some numerical results about our test procedure with $g(x) = x^3$. In the following, we work with equidistant observations

$$(X_0, X_{1/n}, \ldots, X_1),$$
in one of the two following model:

- The Black-Scholes model:

$$dX_t = \sigma_t X_t dW_t.$$  

The value of the parameters in the simulations are $X_0 = 1$ and $\sigma = 0.2$.

- The non correlated Heston model:

$$dX_t = \sigma_t X_t dW_t, \quad \sigma_t = \sigma_0 X_t,$$

$$ds_t^2 = (a - ks_t^2)dt + \varepsilon_s dV_t,$$

where $W$ et $V$ are two non correlated Brownian motions. The value of the parameters in the simulations are $X_0 = 1$, $s_0 = 0.04$, $a = 0.02$, $k = 0.5$ and $\varepsilon = 0.1$, so that the Feller condition is satisfied.

4.1 Observed volatility

We begin with the case where $\sigma_t$ is observed at the same instants as $X_t$. The obtained results for the behavior of our test over 3000 simulations with $n = 1025$ and $n = 131,073$ are given in Figure 1 and Table 1.

<table>
<thead>
<tr>
<th>Simulated Process</th>
<th>Black-Scholes</th>
<th>Heston</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number $n$ of data</td>
<td>1025</td>
<td>131 073</td>
</tr>
<tr>
<td></td>
<td>131 073</td>
<td></td>
</tr>
<tr>
<td>Level of the test</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>9.7%</td>
<td>10.4%</td>
</tr>
<tr>
<td>5%</td>
<td>3.7%</td>
<td>5.5%</td>
</tr>
<tr>
<td>1%</td>
<td>0.8%</td>
<td>1.0%</td>
</tr>
</tbody>
</table>

Table 1: Percentage of rejection of the local volatility assumption.

The results are quite satisfactory. Indeed, under $H_0$, for both values of $n$, the distributions of the test statistics are quite close to standard Gaussian and the rejection rates are not far from the theoretical ones. Also, the obtained empirical powers are quite reasonable.
4.2 Estimated volatility

We now assume that we observe \( n \) data from the price \( X \) and estimate the process \( \sigma^2 \) at \( n' < n \) equidistant points. The squared volatility at time \( t \) is estimated using the estimator (3.2) on log price data and then converting it in order to obtain estimators for \( \sigma^2 \). In the simulations, we use \( n = 131 \, 073 \), \( n' = 1025 \), \( k_n = 1000 \) and compute the test statistic \( \hat{S}_n^2 \) introduced in (3.1). The value \( k_n = 1000 \) enables to obtain satisfying estimators \( \sigma_n^2 \) in the Heston case. Remark that \( \sigma^2 \) is particularly well estimated in the Black-Scholes case since the volatility is constant. The results from 3000 simulations are given in Figure 2 and Table 2.

<table>
<thead>
<tr>
<th>Simulated Process</th>
<th>Black-Scholes</th>
<th>Heston</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number ( n ) of data</td>
<td>131 073</td>
<td>131 073</td>
</tr>
<tr>
<td>Number ( n' ) of volatility estimators</td>
<td>1025</td>
<td>1025</td>
</tr>
<tr>
<td>Level of the test</td>
<td>( 10% )</td>
<td>9.2%</td>
</tr>
<tr>
<td></td>
<td>5%</td>
<td>3.9%</td>
</tr>
<tr>
<td></td>
<td>1%</td>
<td>0.7%</td>
</tr>
</tbody>
</table>

Table 2: Percentage of rejection of the local volatility assumption.

Here again the results are fairly satisfactory. Indeed, they are of the same order of magnitude as those obtained for \( n = 1025 \) when the volatility is observed.
Figure 2: Histogram of the test statistics in the Black-Scholes case, when the volatility is not observed.

5 A numerical illustration on real data

We study here the question of the relevance of a local volatility model on real data through the lenses of our test procedure.

5.1 The assets

We consider three future contracts on interest rates, electronically traded on the EUREX market:

- The Euro-Bund future contract (FGBL),
- The Euro-Bobl future contract (FGBM),
- The Euro-Schatz future contract (FGBS).

These assets are future contracts on respectively long (10 years), medium (5 years) and short term (2 years) debt instruments issued by the Republic of Germany. The maturity of these contracts can be March, June, September or December. Therefore there is a roll period. In the whole numerical study, we consider for each day only one maturity for the assets. This maturity is given by the closest maturity except when being close to one date of maturity. When approaching this date of maturity, we change our maturity of interest according to the roll of liquidity on the market.

We choose these three future contracts for two reasons:

- They are known to be highly liquid (most liquid future contracts on interest
rates in Europe).

- They are close in term of definition (same kind of notional with different terms) and therefore they are quite correlated. Consequently, we may except a close behavior for the test on these three different assets. If it is not the case, giving an interpretation of the test is probably dubious.

5.2 The dataset

Our database is provided by the Electronic Trading Team of the Interest Rates Group from BNP Paribas London. For each of the three assets, the dataset is made of the last traded price values, every trading day, from 8.00 am (CET) to 17.00 pm (CET), every minute, between 2008, January 2 and 2010, June 30 (341 070 data for each contract). Note that we have chosen not to use a sampling frequency higher than one minute in order to avoid microstructure phenomena. Also, to treat the overnight effect, we simply translate each day the price values so that the first price data of one given day is equal to the last price data of the preceding day.

5.3 Results

We apply our test procedure on our future contracts dataset. To be consistent with the parameter used in the simulation study, for each contract, we split our dataset into two equal time periods ($n = 170 535$ data in each part). Then we estimate the process $\sigma^2$ at $n' = 1025$ equidistant points for each part, taking $k_n = 1000$. Since the price values are subsampled on the points where the volatility is estimated, we use various starting points for the procedure (the first, twentieth, fortieth and sixtieth data, corresponding to shift of 0, 20, 40, and 60 minutes). The obtained values for the test statistic $\hat{S}_n^2$ are given in Table 3.

We see that for the three assets, for both time periods and for all starting points, when considering reasonable level values, the test procedure does not reject the assumption of a local volatility type dynamics. Remark that our results are quite satisfying, in the sense that our procedure leads to the same conclusion in each case.

6 Proofs

6.1 Proof of Theorem 2.4

Let $Y = (X, \sigma^2)$. For $k = 1, 2$, we write

$$V(f_k, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} f_k \left( \frac{\Delta_n^i Y}{\sqrt{\Delta_n}}, \frac{\Delta_n^i Y}{\sqrt{\Delta_n}} \right),$$
Table 3: The test statistics $\hat{S}_n^2$ and the associated $p$-values (between parenthesis) for FGBL, FGBM and FGBS, over the two time periods, for various time shifts.

<table>
<thead>
<tr>
<th>Contract</th>
<th>Shift (min.)</th>
<th>$S_n^2$, first time period</th>
<th>$S_n^2$, second time period</th>
</tr>
</thead>
<tbody>
<tr>
<td>FGBL</td>
<td>0</td>
<td>0.98 (0.32)</td>
<td>3.34 (0.07)</td>
</tr>
<tr>
<td>FGBL</td>
<td>20</td>
<td>0.95 (0.33)</td>
<td>0.37 (0.54)</td>
</tr>
<tr>
<td>FGBL</td>
<td>40</td>
<td>1.03 (0.31)</td>
<td>0.42 (0.52)</td>
</tr>
<tr>
<td>FGBL</td>
<td>60</td>
<td>1.23 (0.27)</td>
<td>0.83 (0.36)</td>
</tr>
<tr>
<td>FGBM</td>
<td>0</td>
<td>0.36 (0.55)</td>
<td>0.33 (0.57)</td>
</tr>
<tr>
<td>FGBM</td>
<td>20</td>
<td>0.02 (0.89)</td>
<td>0.81 (0.37)</td>
</tr>
<tr>
<td>FGBM</td>
<td>40</td>
<td>1.75 (0.19)</td>
<td>0.91 (0.34)</td>
</tr>
<tr>
<td>FGBM</td>
<td>60</td>
<td>1.67 (0.20)</td>
<td>1.71 (0.19)</td>
</tr>
<tr>
<td>FGBS</td>
<td>0</td>
<td>1.81 (0.18)</td>
<td>0.97 (0.32)</td>
</tr>
<tr>
<td>FGBS</td>
<td>20</td>
<td>3.13 (0.08)</td>
<td>0.96 (0.33)</td>
</tr>
<tr>
<td>FGBS</td>
<td>40</td>
<td>3.62 (0.06)</td>
<td>0.99 (0.32)</td>
</tr>
<tr>
<td>FGBS</td>
<td>60</td>
<td>3.14 (0.08)</td>
<td>0.98 (0.32)</td>
</tr>
</tbody>
</table>

where the $f_k$ are defined before Theorem 2.6. Note that

$$
\Delta_n \sum_{i=1}^{[t/\Delta_n]} f_1 \left( \frac{\Delta^n Y_i}{\sqrt{\Delta_n}}, \frac{\Delta^n_{i+1} Y_i}{\sqrt{\Delta_n}} \right) = \Delta_n \sum_{i=2}^{[t/\Delta_n]+1} g_n^+(\frac{\Delta^n_{i+1} X \Delta_n}{2\Delta_n})
$$

$$
\Delta_n \sum_{i=1}^{[t/\Delta_n]} f_2 \left( \frac{\Delta^n Y_i}{\sqrt{\Delta_n}}, \frac{\Delta^n_{i+1} Y_i}{\sqrt{\Delta_n}} \right) = \Delta_n \sum_{i=1}^{[t/\Delta_n]} g\left( \frac{\Delta^n_{i+1} X \Delta_n}{\Delta_n} \right).
$$

From Theorem 6.1 in [16] (see also [5]), we obtain that for $k = 1, 2$,

$$
\Delta_n V(f_k, \Delta_n)_t \to \int_0^t \rho^{\otimes 2}_\Sigma(f_k) du,
$$

in probability, uniformly over compact sets in $[0, T]$. The result follows remarking that $\rho^{\otimes 2}_\Sigma(f_1) = \rho_\Sigma(h^+)$ and $\rho^{\otimes 2}_\Sigma(f_2) = \rho_\Sigma(h)$.

### 6.2 Proof of Theorem 2.6

Now consider

$$
V(f, \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]} f \left( \frac{\Delta^n Y_i}{\sqrt{\Delta_n}}, \frac{\Delta^n_{i+1} Y_i}{\sqrt{\Delta_n}} \right),
$$

where $f = (f_1, f_2)$. Formally, the result of Theorem 2.6 follows directly from Theorem 7.1 in [16], but we would like to sketch the main ideas.

First of all, we recall that the two-dimensional process $Y$ has the representation

$$
dY_t = \left( \frac{\alpha_t}{\sigma_t} \right) dt + \Sigma_t d\left( \begin{array}{c} W_t \\ V_t \end{array} \right), \quad \Sigma_t = \left( \begin{array}{cc} \sigma_t & 0 \\ \sigma_t & \nu_t \end{array} \right).
$$
As the processes \(a_-, \pi_-\) and \(\Sigma_-\) are locally bounded we may assume without loss of generality that these processes are bounded in \((\omega, t)\); this is a consequence of a standard localization technique. In a second step we observe that the quantity \(\Delta_n^{\beta}Y\) is well approximated by \(\Sigma_{(i-1)\Delta_n} \Delta_n^2 B\) with \(B = (W, V)^*\). In fact, as it was shown in [5], a more stronger statement

\[
\Delta_n^{-1/2} \left( \Delta_n V(f, \Delta_n) - \int_0^t \rho_{\Sigma_n}^2(f) \right) = \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left( f(\beta_i^n, \beta_i^n) - \rho_{\Sigma_{(i-1)\Delta_n}}^2 (f) \right) + o_p(1)
\]

with \(\beta_i^n = \frac{\Sigma_{(i-1)\Delta_n} \Delta_n B}{\sqrt{\Delta_n}}, \beta_i^n = \frac{\Sigma_{(i-1)\Delta_n} \Delta_n^2 B}{\sqrt{\Delta_n}}\) holds true. Thus, we only need to prove the central limit theorem for the right-hand side of the above decomposition. A simple computation shows that

\[
\Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left( f(\beta_i^n, \beta_i^n) - \rho_{\Sigma_{(i-1)\Delta_n}}^2 (f) \right) = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \chi_i^n + o_p(1)
\]

with

\[
\chi_i^n = \Delta_n^{1/2} \left( f(\beta_i^n, \beta_i^n) - \mathbb{E}[f(\beta_i^n, \beta_i^n)|\mathcal{F}_{(i-1)\Delta_n}] \right)
\]

A direct calculation and the fact that \(\Sigma\) is a càdlàg process imply the convergence in probability

\[
\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[\chi_i^n (\chi_i^n)^*|\mathcal{F}_{(i-1)\Delta_n}] \to \int_0^t R_{\Sigma_n}(f)\,du,
\]

(6.3)

where \(R_{\Sigma_n}^{kj}(f), k, j = 1, 2\), is equal to

\[
\sum_{l=1}^{1} (\mathbb{E}[f_k(\Sigma U_2, \Sigma U_3) f_j(\Sigma U_{l+2}, \Sigma U_{l+3}))]) - 3\mathbb{E}[f_k(\Sigma U_1, \Sigma U_2)]\mathbb{E}[f_j(\Sigma U_1, \Sigma U_2)],
\]

with \(U_1, U_2, U_3\) some iid bidimensional Gaussian vector with covariance matrix equal to identity. Since \(f\) is an even function we also obtain that

\[
\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(\chi_i^n)^k \Delta_n^2 B_j|\mathcal{F}_{(i-1)\Delta_n}] = 0
\]

(6.4)

for \(k, j = 1, 2\). Due to the boundedness of \(\Sigma\) we have for some constant \(C > 0\)

\[
\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(\chi_i^n)^k] \leq C \Delta_n \to 0.
\]

(6.5)
A martingale representation theorem (see again [5] for more details) implies the identity
\[
\sum_{i=1}^{t/\Delta_n} E[(\chi^n_i) \Delta_n^i N_i | \mathcal{F}_{(i-1)\Delta_n}] = 0
\] (6.6)
for any bounded martingale \(N\) which is orthogonal to \(B\). Finally, by the conditions (6.3)-(6.6) and Theorem IX.7.28 from [18] we deduce the stable convergence
\[
\sum_{i=1}^{t/\Delta_n} \chi^n_i \to V(f)_t,
\]
where \(V(f)\) is, conditionally on \(\mathcal{F}\), a centered Gaussian process with independent increments such that for \(k, j = 1, 2\) it holds
\[
E[V(f)_t V(f)_t | \mathcal{F}] = \int_0^t R_{\Sigma_1}^{kj}(f) du.
\]
Thus, the first part of Theorem 2.6 follows. An application of the \(\Delta\)-method for stable convergence gives the second statement.

6.3 Proof of Theorem 2.7
For the proof of Theorem 2.7, simply remark that for \(k = 1, 2\),
\[
\left(\rho_\Sigma^{\otimes 2}(f_k)\right)^2 = \rho_\Sigma^{\otimes 4}(f'_k),
\]
with \(f'_k : (\mathbb{R}^2)^4 \to \mathbb{R}\) such that
\[
f'_k((x_1, x_2), (y_1, y_2), (x'_1, x'_2), (y'_1, y'_2)) = f_k((x_1, x_2), (y_1, y_2)) f_k((x'_1, x'_2), (y'_1, y'_2))
\]
and that
\[
\rho_\Sigma^{\otimes 2}(f_1) \rho_\Sigma^{\otimes 2}(f_2) = \rho_\Sigma^{\otimes 4}(f'_{12}),
\]
with \(f'_{12} : (\mathbb{R}^2)^4 \to \mathbb{R}\) such that
\[
f'_{12}((x_1, x_2), (y_1, y_2), (x'_1, x'_2), (y'_1, y'_2)) = f_1((x_1, x_2), (y_1, y_2)) f_2((x'_1, x'_2), (y'_1, y'_2)).
\]
Then the result follows directly from Theorem 6.1 in [16].

7 Conclusion
In this paper, we study from a statistical point of view the question of the relevance of local volatility models. Therefore, we propose new statistical procedures for testing the local volatility assumption, against the alternative of a stochastic volatility model. These procedures only use historical data. In particular, they do not require any calibration step via option prices. We provide a complete mathematical analysis of our procedures, together with a convincing simulation study. We also apply our tests on some interest data, for which it is found that the assumption of a local volatility model cannot be rejected.
Acknowledgements

We are very grateful to the Associate Editor and to the three anonymous referees whose very relevant remarks and comments have significantly improved a former version of the paper.

References


