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PUBLIC ECONOMICS

Lecture Notes 3

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Chapter III

Public Goods

THE OPTIMAL PROVISION OF PUBLIC GOODS

We derive the fundamental Lindahl-Samuelson condition for the optimal production of a public good in a simple economy.

There is one private good (or "money") and one public good. There are n consumers indexed by $i = 1, \dots, n$.

Let x_i denote consumer i's consumption of the private good.

Denote by G, the production of public good.

The utility of agent i is $u_i(x_i, G)$.

Assumption 1

Utility functions are strictly quasi-concave (*i.e.*, preferences are convex) and continuously differentiable. The marginal utilities are positive (*i.e.*, the public good is not a "public bad").

Agent i has an endowment of private good (or "wealth") denoted ω_i. The aggregate endowment is denoted $\omega = \sum_i \omega_i$. The private good is used as an input to produce the public good. To produce G units of public good, $z = c(G)$ units of private good are needed at least, where c is a cost function.

Assumption 2

The cost function c is increasing, convex and continuously differentiable. The marginal cost c' is positive.

We would like to characterize the Pareto optima is this economy. We know that if we maximize a weighted sum of the utilities, *i.e.*, $\sum_i \alpha_i u_i$, subject to resource constraints, with weights $\alpha_i > 0$, we will find a Pareto optimum. (*Exercise:* prove this). Pareto optima can be characterized as solutions to the problem,

$$\max_{(x,G)} \sum_{i=1}^{n} \alpha_i u_i(x_i, G)$$

subject to

$$\sum_{i=1}^{n} x_i + c(G) \le \omega, \tag{λ}$$

We implicitly assume free disposal. Let λ be the Lagrange multiplier associated to the resource constraint. The Kuhn-Tucker conditions for a maximum under the constraint can be written as follows;

$$\alpha_i \frac{\partial u_i}{\partial x_i} = \lambda, \quad \text{for all}\, i \tag{1}$$

$$\sum_i \alpha_i \frac{\partial u_i}{\partial G} = \lambda c'(G) \tag{2}$$

$$\lambda \left(\sum_{i=1}^{n} x_i + c(G) - \omega \right) = 0. \tag{3}$$

From the first equation, we derive, first, that $\lambda > 0$, and therefore, the resource constraint holds as an equality. From (1) we also derive

$$\alpha_i = \frac{\lambda}{\frac{\partial u_i}{\partial x_i}}$$

and therefore, substituting this expression in (2), we find the well-known Lindahl-Samuelson condition for an optimal production of the public good,

$$\sum_i MRS_i(x_i, G) = c'(G) \tag{4}$$

where

$$MRS_i(x_i, G) = \frac{\left(\frac{\partial u_i}{\partial G}\right)}{\left(\frac{\partial u_i}{\partial x_i}\right)}$$

is the *marginal rate of substitution* between the public and private goods. Recall that a given indifference curve can be expressed by the equation $u_i(x_i(G), G) = \mathit{constant}$, where $x_i(G)$ is an implicit function. Differentiation yields

$$\frac{\partial u_i}{\partial x_i}\frac{dx_i}{dG} + \frac{\partial u_i}{\partial G} = 0,$$

and therefore, we have,

$$\frac{dx_i}{dG} = -MRS_i(x_i, G).$$

The function MRS_i can be interpreted as the value of agent i's marginal *willingness to pay* for the public good at point (x_i, G); it gives the maximal number of units of private good that the agent is ready to sacrifice in order to increase the production of public good by one unit. The Lindahl-Samuelson condition says that in order for G to be optimal, the total sum, over i, of willingnesses to pay must be equal to the marginal cost of producing an extra unit. If this equation did not hold, it would be possible to increase the weighted sum of utilities by increasing or decreasing production a little. Remark that (4) is only a necessary condition for optimality, and that it doesn't depend on the vector of weights $\alpha = (\alpha_1, \ldots, \alpha_n)$. But the Lindahl-Samuelson condition determines G independently of $x = (x_1, \ldots, x_n)$ only if MRS_i is independent of x. In general, the optimal production G^* is not independent of the distribution of its cost among consumers, and therefore not independent of the private good distribution x. Define the personal tax paid by i as the value t_i such that $x_i = \omega_i - t_i$. The optimal value of x itself depends on α, as shown by equation (1): the higher the weight α_i, the lower the personal tax t_i paid for the public good.

There is a well-known special case in which G^* can be chosen independently of the distribution of the private good (*i.e.*, of income): when utilities are quasi-linear, that is, if

$$u_i(x_i, G) = x_i + v_i(G)$$

for some concave function $v_i(.)$. In this particular case,

(i), the only *interior* optima, where by interior we mean $x_i > 0$ for all i, are obtained when all weights are equal, $\alpha_1 = \alpha_2 = \cdots = \alpha_n$ (*Exercise:* prove this assertion), and

(ii), $MRS_i(x_i, G) = v_i'(G)$, so that G is determined as a solution of (4) independently of x. In fact, in this quasi-linear case, the distribution of cost doesn't matter for efficiency: any $t = (t_1, \ldots, t_n)$ such that $\sum_i t_i = c(G^*)$ is efficient.

We can further specialize the model and consider the case of linear utilities and a discrete decision, $G \in \{0, 1\}$ (a public project is carried out or not). Utilities are $u_i = v_i G + x_i$, and v_i is now just a real number. Assume that the total cost is cG, where $c > 0$.

Then, the optimal decision maximizes

$$\sum_i u_i = \sum_i (v_i G - t_i + \omega_i).$$

Given that $\sum_i t_i = cG$, we see that the optimal decision maximizes $[-c + \sum_i v_i]G + \omega$. The optimal choice is clearly

$$G^* = 1 \quad \text{if and only if} \quad \sum_i v_i \geq c.$$

We have obtained a discrete version of the Lindahl-Samuelson condition.

The Private Contribution Model for Public Goods

In this section, we draw heavily from a famous paper by Theodore Bergstrom, Lawrence Blume and Hal Varian, published in *Journal of Public Econ.* (1986), vol. 29, p. 25-49, "On the Private Provision of Public Goods."

A Simple Model of Voluntary Contribution

Let $g_i \geq 0$ denote agent i's (voluntary) contribution to the public good. We assume now that $c(G) = G$ for all G: there are constant returns to scale. The production of the public good is simply $G = \sum_{i=1}^n g_i$.

To simplify notation, define $G_{-i} = \sum_{j \neq i} g_j$.

Definition. (Nash equilibrium in voluntary contributions)
A Nash equilibrium is a vector of contributions $g^* = (g_1^*, \ldots, g_n^*)$, with $g_i^* \in [0, \omega_i]$, such that for all i, (x_i^*, g_i^*) solves the problem

$$\max_{(x_i, g_i)} \quad u_i(x_i, g_i + G_{-i}^*)$$

$$\text{subject to} \begin{cases} x_i + g_i = \omega_i \\ 0 \leq g_i \leq \omega_i. \end{cases}$$

This is equivalent to the property that g_i^* solves the problem,

$$\max_{0 \leq g_i \leq \omega_i} \quad u_i(\omega_i - g_i, g_i + G_{-i}^*)$$

for all i.

The Free-Riding Problem

To see the reason for free-riding in the voluntary contribution model, assume that the Nash equilibrium is interior, *i.e.*, $0 < g_i^* < \omega_i$ for all i. The first order conditions for such a Nash equilibrium are

$$-\frac{\partial u_i}{\partial x_i}(x_i, G) + \frac{\partial u_i}{\partial G}(x_i, G) = 0, \tag{5}$$

for all i, where $x_i = \omega_i - g_i$. Note that, in this model, the marginal cost c' is equal to 1. It follows that (5) can be rewritten,

$$1 = c' = MRS_i(x_i, G)$$

for all i. Adding these equations yields

$$c' = \frac{1}{n} \sum_i MRS_i(x_i, G). \tag{6}$$

It is immediate that this condition is not compatible with the Samuelson condition (4). The Nash-equilibrium total contribution G is typically too small (under-provision), as compared with the Pareto-optimal production because marginal cost is equated with the average willingness to pay instead of being equated with the *total* willingness to pay.

The Neutrality Theorem

Implicitly, each i is in fact choosing the value of G. If i makes a zero contribution, that is, if $g_i = 0$, then $G = G_{-i}$. Agent i' optimization problem can be rewritten,

$$\max_{(x_i, G)} \quad u_i(x_i, G)$$

$$\text{subject to} \begin{cases} x_i + G = \omega_i + G^*_{-i} \\ G \geq G^*_{-i}. \end{cases}$$

This is as if i was choosing G subject to G being larger than the contributions of others, G^*_{-i}.

We can state the following result.

Neutrality Theorem

Assume that contributions g_i^* are initially a Nash equilibrium. Consider a redistribution of income among contributing consumers (*i.e.*, among contributors such that $g_i^* > 0$). The redistribution is described by transfers of private good, denoted t_i, such that,

(i) $\sum_{i=1}^n t_i = 0$;

(ii) $g_i^* = 0 \Rightarrow t_i = 0$;

(iii) $t_i < g_i^*$.

Note that t_i is a tax if $t_i > 0$, and t_i is a subsidy if $t_i < 0$.
After redistribution, there is a new Nash equilibrium, denoted $g_i^*(t)$, such that $g_i^* - g_i^*(t) = t_i$ for all i, $x_i^*(t) = x_i^*$ and $G^*(t) = G^*$. Each consumer consumes the same bundle (x_i^*, G^*) as before redistribution t.

Proof of the Neutrality Theorem

Suppose that for all $j \neq i$, the new contribution is

$$g_j^*(t) = g_j^* - t_j > 0.$$

Next,

$$\sum_{j \neq i} t_j = -t_i,$$

since

$$\sum_i t_i = 0.$$

Agent i's budget constraint becomes

$$x_i + g_i = \omega_i - t_i.$$

This is equivalent to

$$x_i + g_i + G^*_{-i}(t) = \omega_i - t_i + G^*_{-i}(t),$$

where $G^*_{-i}(t) = \sum_{j \neq i}(g^*_j - t_j)$. This is again equivalent to,

$$x + G = \omega_i + G^*_{-i}$$

where G is now defined as $G = g_i + G^*_{-i}(t)$. So, the consumer chooses $g^*_i(t)$ and $x^*_i(t)$ to solve

$$\max_{(x_i, G)} \quad u_i(x_i, G)$$

$$\text{subject to,} \quad x_i + G = \omega_i + G^*_{-i},$$
$$G \geq G^*_{-i}(t).$$

The problem is the same as above, except that the positive contribution constraint becomes $G \geq G^*_{-i}(t)$, or $G \geq G^*_{-i} + t_i$, since $-t_i = \sum_{j \neq i} t_j$.

Since, by assumption, $g^*_i \geq t_i$, the old equilibrium G^* can still be obtained, that is, $g^*_i \geq t_i$ implies

$$G^* = g^*_i + G^*_{-i} \geq t_i + G^*_{-i} = G^*_{-i}(t).$$

It is crucial that agent i can still reach the old equilibrium.

If $t_i > 0$ (a tax), the budget set is smaller than before. By the principle of revealed preference, since G^* is still feasible, and was preferred before, this implies that G^* is still preferred after the redistribution t.

If $t_i < 0$ (a subsidy), the budget set is larger than before. Since preferences are convex, if there existed a $\hat{G}$ preferred to G^*, then a convex combination of $\hat{G}$ and G^* would have been preferred to G^* under $t = 0$ and be feasible. More precisely, suppose that there exists a feasible $(\hat{x}_i, \hat{G})$ such that $u_i(\hat{x}_i, \hat{G}) > u_i(x^*_i, G^*)$. Then, for all $\alpha \in (0, 1)$, the strict quasi-concavity of u_i (strict convexity of preferences) implies

$$u_i\left(\alpha\hat{x}_i + (1-\alpha)x^*_i, \alpha\hat{G} + (1-\alpha)G^*\right) > u_i(x^*_i, G^*).$$

But if α is small enough, the convex combination $\alpha(\hat{x}_i, \hat{G}) + (1-\alpha)(x^*_i, G^*)$ would have been feasible before the redistribution, since $g^*_i > 0$. This is a contradiction.

We conclude that the new contribution is

$$g_i^*(t) = g_i^* - t_i \quad \text{and} \quad x_i^*(t) = x_i^*$$

for all i, and the total contribution doesn't change, since

$$G^*(t) = \sum_i g_i^*(t) = \sum_i g_i^* - \sum_i t_i = \sum_i g_i^* = G^*.$$

Q.E.D.

Conclusion: the optimal responses of consumers to the wealth transfer t have completely offset the effects of the redistribution.

100% Crowding-out Effect of Public Expenditure

Suppose that the government levies taxes t_i to produce an additional amount of public good T, with $T = \sum_i t_i$.

The utility of each consumer becomes $u_i(x_i, G+T)$ and the budget constraint becomes

$$x_i + g_i = \omega_i - t_i,$$

that is,

$$x_i + G + T = \omega_i - t_i + G_{-i}^*(t) + T,$$

or equivalently,

$$x_i + G + T = \omega_i + G_{-i}^*,$$

since

$$T - t_i = \sum_{j \neq i} t_j.$$

Now, if we define $Z = G + T$, agent i's problem can be rewritten,

$$\max_{(x_i, Z)} \; u_i(x_i, Z)$$

$$\text{s.t.} \quad x_i + Z = \omega_i + G_{-i}^*$$

$$\text{and} \quad Z \geq G_{-i}^*(t) + T$$

$$\Rightarrow \quad Z \geq G_{-i}^* + t_i,$$

since $T - \sum_{j \neq i} t_j = t_i$.

Again, if the set of contributors does not change and if $t_i < g_i^*$, the old equilibrium is still feasible, since $G^* \geq G^*_{-i} + t_i$.

If $t_i < 0$ (a tax) the contribution constraint is relaxed and as in the proof of the Neutrality Theorem above, by revealed preference, G^* is chosen again. If $t_i > 0$ (a subsidy) the result is true, again by convexity of preferences. If $\hat{G}$ was strictly preferred to G^*, then, for α small enough, $\alpha G^* + (1-\alpha)\hat{G}$ is both feasible and preferred in the no tax equilibrium. This is a contradiction.

We conclude that the contribution of each i will be reduced to $g_i^*(t) = g_i^* - t_i$ for all i. An the total contribution is lower. We have,

$$G^*(t) = \sum_i g_i^*(t) = G^* - \sum_i t_i = G^* - T.$$

The 100% crowding-out effect immediately follows, since we have $G^*(t) + T = G^*$. The reduction in private contributions balances T exactly. The 100%-Crowding-out Effect is a corollary to the Neutrality Theorem.

Suppose that $g_j^*(t) = g_j^* - t_j$ for all $j \neq i$, then

$$\begin{aligned} Z &= g_i^*(t) + \sum_{j \neq i} g_j^* - \sum_{j \neq i} t_j + T \\ &= g_i^*(t) + \sum_{j \neq i} g_j^* + t_i. \end{aligned}$$

So, if i chooses $g_i^*(t) = g_i^* - t_i$, we get exactly $Z = \sum_i g_i^* = G^*$, the old production.

Private contribution mechanisms for public goods with discrete decisions

We present here the simplest instance of Bagnoli and Lipman's *streetlight model.* The following elements are drawn from Mark Bagnoli and Bart Lipman (1989), "Provision of Public Goods : Fully Implementing the Core through Private contributions", in *Rev. Econ. Studies* 56, 583-601.

Are voluntary (private) provision mechanisms always failing to achieve efficient outcomes? The answer is *no* if we consider multi-stage mechanisms and (or) perfect equilibria.

The street-light model is an instance.

Bagnoli and Lipman's Game

The game is easy to describe.

Stage 1. Agents submit voluntary contributions.

Stage 2. *(a)* If the total sum of contributions is enough to fund the public good, then the public good is produced.

(b) If contributions fall short of the required sum: *money is reimbursed to contributors,* and the good is not produced.

This game has Nash equilibria that are Pareto-efficient. In fact, a much stronger result can be proved: the Nash equilibria that are *perfect* and *undominated* implement the *core* of the economy.

The undominated perfect equilibria are robust equilibria of the game (see explanations below).

Definition. The *core* is by definition the set of allocations that cannot be blocked by any coalition (*i.e.*, any subset of the set of players). An allocation in the core is necessarily efficient, because it cannot be blocked by the *grand coalition* of all players (there is no improvement that can be unanimously approved). A coalition *blocks* a proposed allocation if, using their own resources, the members of the coalition can do better for themselves than what they obtain under the proposed allocation.

There is an important difference with other games: the public decision is chosen in a *discrete set* D. For instance, example $D = \{0, 1, 2, \dots, K\}$. This happens to be a crucial difference with ordinary voluntary contribution games.

Refresher: the Perfection Criterion (Reinhard Selten (1975))

Let us consider the simplest example. Figure 1 depicts a game in extensive form (in the form of a tree).

INSERT Fig. 1 Here.

Player 1 chooses strategy A or strategy B in the first stage. If Player 1 has chosen B, the game ends and the payoffs are $(0, 3)$, where 0 is the utility of Player 1, and 3 is the utility of Player 2. If Player 1 has chosen A, a second stage begins, in which Player 2 chooses either C or D. The payoffs of path (A, D) are $(1, 1)$; the payoffs of path (A, C) are $(-1, -1)$. The strategic form (or normal form) of this game is given by the following table, in which Player 1 chooses the row and Player 2 chooses the column.

	D	C
A	(1, 1)	(-1, -1)
B	(0, 3)	(0, 3)

This game has two Nash equilibria.

1) strategies $(A, D) \Rightarrow$ payoffs $(1, 1)$;

2) strategies $(B, C) \Rightarrow$ payoffs $(0, 3)$.

Of these two equilibria, only (A, D) is *perfect*. To be precise (A, D) is the only subgame-perfect equilibrium: the only Nash equilibrium that is also a Nash equilibrium in every subgame. There is only one subgame here starting after the choice of A by player 1. We find these equilibria by means of the extensive form, applying the *backwards induction* principle to the game tree. This means that we solve the game backwards. First, we find that Player 2 would always play D (a threat of playing C is not *credible* here). Then, it is easy to see that Player 1, knowing that Player 2 is rational, will choose A to maximize his(her) payoff. But there is a more powerful definition of *equilibrium perfection*(also due to Reinhard Selten).

Trembling-hand perfection. Suppose that the other players, facing a given player, make mistakes in choosing their strategy, with small probabilities (their hands tremble). The mistakes are random and any player may choose any of his strategies. Because of *random mistakes*, a player's strategy is called *completely mixed.* This means that every pure strategy can be chosen with a small positive probability. By definition, a *perfect equilibrium* is a vector of strategies (one for each player) such that each individual strategy is a best reply to each point in a sequence of *completely mixed strategies* played by other players and converging to their equilibrium strategies.

An interpretation of this definition is that in perfect equilibrium, players choose strategies that are best responses to other players' strategies, taking mistakes made by other players with a small probability into account. This means that perfect equilibrium strategies are in a certain sense *robust* to random deviations.

In the above example, we can easily see that C is not a best response to Player 1 playing A with probability $\varepsilon > 0$ (and B with probability $1 - \varepsilon$).

		D	C
ε	A	(1, 1)	(-1,-1)
$(1-\varepsilon)$	B	(0, 3)	(0, 3)

Player 2's payoff with C is equal to $-\varepsilon + 3(1 - \varepsilon) = 3 - 4\varepsilon$.
Player 2's payoff with D is $\varepsilon + 3(1 - \varepsilon) = 3 - 2\varepsilon$.

Conclusion: D is clearly a better response to the mixed strategy $\Pr(A) = \varepsilon$, for all $\varepsilon > 0$.

Refresher 2: Elimination of Dominated Strategies

Definition: a strategy is (weakly) dominated if there exist another strategy of the player that yields a higher (or equal) payoff for every strategy of the other players (and a strictly higher payoff for at least one choice of the other players).

It is reasonable to assume that rational players will not choose weakly dominated strategies. If this is the case, weakly dominated strategies can be eliminated from the game. The elimination of a given strategy (deletion of a row or column) creates a reduced game, and may reveal the existence of weakly dominated strategies *in the reduced game*. There can be several rounds of elimination of dominated strategies. This procedure of successive elimination can be used to solve, or just to simplify, games.

The Simplest Streetlight Game

Let d denote the public decision; we assume that $d \in \{0, 1\}$ (build one streetlight, $d = 1$, or do not build and refund, $d = 0$). The total cost is cd and assume that $c = 1$. There are two players, $n = 2$.

The players have the same valuation (willingness to pay) for the public good,

$$\theta_1 = \theta_2 = \frac{2}{3}.$$

Player i contributes g_i, and his (her) payoff is

$$u_i = \begin{cases} \theta_i - g_i & \text{if} \quad d = 1 \\ 0 & \text{if} \quad d = 0 \end{cases}$$

Suppose that the public decision rule is $d = 1$ if and only if $\sum_i g_i \geq c$. Assume that the contributions are chosen in a discrete set, namely,

$$g_i \in \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}$$

The normal form of the game played by the two players can be represented by the following table.

	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
0	$0, 0$	$0, 0$	$0, 0$	$0, 0$
$\frac{1}{3}$	$0, 0$	$0, 0$	$0, 0$	$\frac{1}{3}, 0$
$\frac{1}{2}$	$0, 0$	$0, 0$	$\frac{1}{6}, \frac{1}{6}$	$\frac{1}{6}, 0$
$\frac{2}{3}$	$0, 0$	$0, \frac{1}{3}$	$0, \frac{1}{6}$	$0, 0$

To fill in the entries, we use $\frac{1}{6} = \frac{2}{3} - \frac{1}{2}$; $\frac{1}{3} + \frac{1}{2} = \frac{5}{6} < 1 = c$ and $\frac{2}{3} + \frac{1}{2} = \frac{7}{6} > 1 = c$, implying that $d = 1$ if contributions are $g = (\frac{2}{3}, \frac{1}{2})$ and $d = 0$ if $g = (\frac{1}{3}, \frac{1}{2})$. The table is obviously symmetric with respect to the diagonal.

There are many Nash equilibria in this game:

Strategies $(0, 0) \to$ payoffs $(0, 0)$ is a Nash equilibrium.

Strategies $\left(\frac{1}{3}, \frac{1}{3}\right) \to$ payoffs $(0, 0)$ is another Nash equilibrium.

Strategies $\left(\frac{1}{3}, \frac{2}{3}\right) \to$ payoffs $\left(\frac{1}{3}, 0\right)$ is yet another Nash equilibrium.

But,

$g_i = 0$ is weakly dominated by $g_i = \frac{1}{3}$,

$g_i = \frac{2}{3}$ is weakly dominated by $g_i = \frac{1}{2}$.

So, we eliminate $g_i = 0$ and $\frac{2}{3}$ and we get the following reduced normal form.

		$\frac{1}{3}$	$\frac{1}{2}$
$1-\varepsilon$	$\frac{1}{3}$	$0, 0$	$0, 0$
ε	$\frac{1}{2}$	$0, 0$	$\frac{1}{6}, \frac{1}{6}$

Strategies $\left(\frac{1}{3}, \frac{1}{3}\right)$ yielding payoffs $(0, 0)$ is a Nash equilibrium.

Strategies $\left(\frac{1}{2}, \frac{1}{2}\right)$ yielding payoffs $\left(\frac{1}{6}, \frac{1}{6}\right)$ is another Nash equilibrium.

But $x_i = \frac{1}{3}$ is not a best response to a strategy playing $\frac{1}{2}$ with probability $\varepsilon > 0$.

$$g_i = \frac{1}{3} \quad \text{yields} \quad 0\varepsilon + 0(1-\varepsilon) = 0;$$
$$g_i = \frac{1}{2} \quad \text{yields} \quad \frac{\varepsilon}{6} > 0.$$

It follows that we can eliminate $g = \left(\frac{1}{3}, \frac{1}{3}\right)$. The only surviving equilibrium is $\left(\frac{1}{2}, \frac{1}{2}\right)$: the *perfect* equilibrium, with payoffs $\left(\frac{1}{6}, \frac{1}{6}\right)$. (Note that $g_i = \frac{1}{2}$ is weakly dominant in the reduced game.)

Conclusion. If we combine the elimination of dominated strategies and the perfection criterion, we find the desired result: the remaining equilibrium is an *efficient* Nash equilibrium.

Our particular example is just an illustration for a general result, that was proved by Bagnoli and Lipman.

Implementation of the Core

The general result can be described as follows. Consider a public good economy with n agents indexed by $i = 1, \ldots, n$. There is one private good (money). Agent i's endowment is $\omega_i > 0$. The vector of endowments is now denoted $\omega = (\omega_1, ..., \omega_n)$. The public decision d si chosen in the set $D = \{0, 1\}$, *i.e.*, $d = 1$ means "build a streetlight". Let θ describe the profile of preferences in the economy; θ belongs to the set of preference profiles Θ. Each agent i has a valuation θ_i in state θ. The consumption of private good is $x_i = \omega_i - g_i$ as above. The cost of the public project is $c(d) = cd$ with $c > 0$.

Utilities are re-defined as follows,

$$u_i(d, x, \theta) = \begin{cases} \theta_i + x_i & \text{if} \quad d = 1 \\ \omega_i & \text{if} \quad d = 0. \end{cases}$$

An outcome is a point $a = (d, x)$, where $a \in D \times \mathbb{R}^N_+$, and $x = (x_1, \ldots, x_n)$ is an allocation of money. The set of feasible outcomes is denoted A, and

$$A = \left\{ (d, x) \mid \sum_i x_i \leq \sum_i \omega_i - cd,\ d \in \{0, 1\} \right\}.$$

By definition, the *core* of the economy is a correspondence,

$$C : \Theta \to P(A),$$

where $P(A)$ is the set of all subsets of A. Define a coalition T as a subset of $N = \{1, \ldots, n\}$. Then, define the set of outcomes that coalition T can achieve,

$$A_T = \left\{ (d, x) \mid \sum_{i \in T} x_i \leq \sum_{i \in T} \omega_i - cd,\ d \in \{0, 1\} \right\}.$$

Coalition T can block outcome a in state θ if there exists $a' = (d', x') \in A_T$ such that

$$u_i(d', x', \theta) \geq u_i(d, x, \theta)$$

for all $i \in T$, with a strict inequality for some $i \in T$.

Definition: An outcome $a \in A$ is in the core $C(\theta)$ if and only if there is no coalition that can block a in state θ.

Consequence: The grand coalition N is a particular coalition. Hence, if $a \in C(\theta)$, then a is Pareto-optimal in state θ.

Define the *game form* $((S_i)_{i=1,\ldots,n}, \hat{a})$, where S_i is the strategy space of i, *i.e.*, the set of possible contributions g_i. A profile of strategies is denoted $g = (g_1, \ldots, g_n)$ and belongs to the product $S = S_1 \times S_2 \times \cdots \times S_n$. The outcome, denoted $\hat{a}(g)$, belongs to A and $\hat{a}$ can be viewed as a mapping $\hat{a} : S \to A$.

A game $\Gamma(\theta)$ is generated from the *game form*, with strategy spaces S_i and payoffs

$$u_i(\hat{a}(g), \theta) = (\theta_i + \hat{x}_i(g))\hat{d}(g) + \omega_i(1 - \hat{d}(g)),$$

where $\hat{a}(g) = (\hat{d}(g), \hat{x}(g))$. Let $g^*_\Gamma(\theta)$ be the set of equilibrium strategy profiles of $\Gamma(\theta)$.

Definition. We say that Γ fully implements the core if and only if

$$C(\theta) = \hat{a}(g^*_\Gamma(\theta)),$$

for all θ. If the equilibrium concept is the perfect equilibrium of $\Gamma(\theta)$; then we say that Γ *implements the core in perfect equilibrium.*

The One-Streetlight Problem

In addition to the above assumptions, we assume that $\omega_i > \theta_i$ for all i and all θ, and that $\sum_i \omega_i > c$.

We skip the characterization of the core of the considered economy (see Bagnoli and Lipman). The Game $\Gamma(\theta)$ is formally defined as above: *(i)*, all agents i contribute a g_i; *(ii)*, if the total sum of contributions is greater than (or equal to) c, the streetlight is provided, and excess contributions (if any) are kept by the social planner (or refunded in some fashion); *(iii)*, if the total sum of contributions is strictly smaller than c, the streetlight is not built and all contributions are refunded.

Formally, we have $S_i = [0, \omega_i]$, and

$$\hat{a}(g) = \begin{cases} (1, \omega - g) & \text{if} \quad \sum_i g_i \geq c \\ (0, \omega) & \text{otherwise.} \end{cases}$$

We can now state the theorem.

Theorem (Bagnoli-Lipman, 1989)
Γ fully implements the Core $C(\theta)$ in undominated perfect equilibrium.

Remarks. Here, "undominated" means that we delete the weakly dominated strategies and apply the perfection criterion, to pick only "trembling-hand" perfect equilibria in the reduced normal form game. A consequence of the result is that all undominated perfect equilibria of the streetlight game are efficient (since they belong to the core). The extension of this result to the case of K streetlights is true, but its proof is difficult. The proof is in the work of Bagnoli and Lipman.

All equilibria which lead to *core* outcomes are *strong* Nash equilibria. This is very robust: no coalition of players can profitably deviate from the equilibrium (because the equilibrium allocation is in the core). Note, finally, that the streetlight mechanism is based on a complete information game: the introduction of private information would change the problem in a radical way.

Revelation of Preferences for Public Goods: Clarke-Groves Mechanisms

We now study the most famous example of a public good mechanism working under conditions of incomplete information: the Clarke-Groves mechanism (sometimes called the *pivotal mechanism*). These mechanisms were discovered by Edward H. Clarke (1971) and Theodore Groves (1973); they have been studied in depth by Jerry Green and Jean-Jacques Laffont in the 70s (see Green and Laffont (1979)). The elegant presentation of Moulin (1988) has been our main source of inspiration for this section.

Example. We start with an example due to Moulin (1988). A group of 4 municipalities must chose the location of a common facility. There are two possible locations: a or b. the following table displays the payoffs of the 4 towns for each of the locations. Payoffs measure the net benefits in euros.

Town:	1	2	3	4
Location a	$+20$	$+15$	-10	0
Location b	-10	-5	$+12$	$+4$

Agent 1 would be ready to pay 30 to change the decision from b to a. In this problem, utilities $u_i(z)$, where $z = a, b$, are typically comparable. We will adopt the utilitarian social welfare function. Compensations can be paid in money (transfers t_i can be made between towns). The utility of decision x and of a transfer t_i is $u_i(x) + t_i$ for agent i (we assume that utilities are quasi-linear). We know that in this framework, a different constant can be added to each utility function, without changing the optimal decision. This permits us to normalize utilities in such a way that $u_i(a) + u_i(b) = 0$, without any loss of generality. We obtain the following normalized table.

Town:	1	2	3	4
Location a	$+15$	$+10$	-11	-2
Location b	-15	-10	$+11$	$+2$

The efficient decision maximises $\sum_{i=1}^{4} u_i(z)$, choosing $z \in \{a, b\}$. It is easy to check that $\sum_i u_i(a) = 12 > -12 = \sum_i u_i(b)$, so that a is the efficient decision. Without any transfers, the efficient mechanism is highly manipulable. The “losers” in the example are towns $n^o 3$ et $n^o 4$. A loser can win by exaggerating his (her) utility losses. No agent is ready to report preferences honestly.

Consider for instance the egalitarian mechanism in which all utilities are equal at the end. With the transfers,

$$t_1 = -12, \quad t_2 = -7, \quad t_3 = +14, \quad t_4 = +5,$$

we obtain $u_i(a) = 3$ for all $i = 1, \ldots, 4$. A negative value of t_i is interpreted as a tax.

Manipulation : If 2, 3 and 4 are honestly reporting their utilities, then, agent 1 can reduce his transfer payment (his tax) by at least 9 units if he reports $u_1(a) = 3 + \varepsilon$, $u_1(b) = -3 - \varepsilon$. Decision a remains just efficient.

Transfers are used to provide incentives.

Definition of the Pivot. An agent i is called a *pivot* (or is “pivotal”) if the efficient decision for the agent set $N \setminus \{i\}$ differs from the efficient decision for $N = \{1, ..., n\}$.

In the 4 towns example, agent 1 is a *pivot.*

$\{2, 3, 4\}$ would choose b

No other agent is pivotal: $\{1, 2, 4\}$, $\{1, 2, 3\}$ and $\{1, 3, 4\}$ choose a.

The pivotal tax (*i.e.*, the tax paid by the pivot) would in this case be, for agent 1, equal to the joint loss inflicted to $\{2, 3, 4\}$ when the decision changes from b to a.

$$\begin{aligned} t_1 &= -[(-10 + 11 + 2) - (10 - 11 - 2)] \\ &= +20 - 22 - 4 = -6 \end{aligned}$$

The non-pivotal agents pay nothing: $t_2 = t_3 = t_4 = 0$. This tax system induces a truthful revelation of utilities. In the example, consider first agent 1.

Agent 1’s reports that do not change the decision a do not change the tax and are therefore not profitable.

The reports that induce a change of decision from a to b cancel the tax (since 1 is no longer a pivot), but the change of decision causes a loss of $30 = 15 - (-15)$ units of utility.

If agent 3 exaggerates his distaste for a and reports $u_3(a) = -25$, $u_3(b) = +25$, then b will become seemingly efficient, but agent 3 becomes a pivot and will be taxed by an amount equal to the loss inflicted to others, that is, to subset $\{1, 2,\ 4\}$, namely,

$$t_3 = -[(15 + 10 - 2) - (-15 - 10 + 2)] = -30 - 20 + 4 = 46.$$

This tax is greater than the benefit of a decision switching from a to b, since this benefit is only $11 - (-11) = 22$.
The reasoning would be similar for agents 2 and 4. (Exercise: Make sure that you understand how this works).

Pivotal Mechanisms

Let A denote the set of public decisions, $A = \{a, b, ..., z\}$. The set A is assumed finite (for simplicity). There are n agents with quasi-linear preferences defined on $A \times \mathbb{R}_+$. The set of individuals is $N = \{1, ..., n\}$. The choice $a \in A$ is a full description of the public decision and includes the distribution of its cost among agents. For instance, a may be of the form $a = (d, g)$ where $d \in D$ represents the choice of a public project and g is a profile of taxes covering the cost of the project $c(d) = \sum_i g_i$. The decision d itself may be multidimensional and include the decision to exclude some consumers from the consumption of the public good (if feasible). The agents' utilities may depend on the way the cost $c(d)$ is distributed. So, this model is very general. The transfers t are only used to generate revelation incentives, since they are not used to distribute the project cost.

Definition of an outcome. An outcome is a vector (a, t) where $a \in A$ and $t = (t_1, ..., t_n)$ is a vector of transfers.

Assumption. (Quasi-linear, Separable Preferences)
Agent i's utility for (a, t) is $u_i(a) + t_i$.

Since A is finite, u_i may be viewed as a vector of real numbers, $u_i \in R^{|A|}$ or as a mapping $u_i : A \longrightarrow \mathbb{R}$.

Definition of a Mechanism. A mechanism is a mapping associating an outcome $(a(u), t(u))$ with every profile of utilities $u = (u_1, \ldots, u_n)$ and with the following invariance property: if two profiles u and v differ only by a vector of constants $k = (k_1, \ldots, k_n)$, such that, $v_i(z) = u_i(z) + k_i$ for all $z \in A$ and $i \in N$, then $a(u) = a(v)$ and $t(u) = t(v)$.

Definition of an Efficient Mechanism. A mechanism $(a(.), t(.))$ is efficient if for all u, it selects an efficient decision a in A. A decision $a = a(u)$ is efficient if and only if,

$$\sum_{i=1}^{n} u_i(a) = \max_{b \in A} \left[\sum_{i=1}^{n} u_i(b) \right].$$

Definition of a Feasible Mechanism. A mechanism is feasible if $a(u) \in A$ for all u and if it generates no deficit, *i.e.*, $\sum_{i=1}^{n} t_i(u) \leq 0$ for all u.

Remark. A mechanism may be efficient and feasible and generate a budget surplus (*i.e.*, $\sum_i t_i(u) < 0$). This surplus cannot be redistributed: it is therefore a social waste. This means that an efficient and feasible mechanism may not be Pareto-optimal in this context. In other words, the efficient and feasible mechanism chooses the right decision $a(u)$, but fails to reach an optimum because it taxes the agents too much.

Definition of a Non-manipulable Mechanism. A mechanism is non-manipulable, or strategy-proof, if it is revealing in dominant strategies. Formally, mechanism $(a(.), t(.))$ is strategy-proof if for any profile u, for all agent i, we have

$$u_i[a(u)] + t_i(u) \geq u_i[a(v_i, u_{-i})] + t_i(v_i, u_{-i}),$$

for all utility function v_i.

In this framework, it must be understood that each agent i reports a preference (*i.e.*, a utility vector u_i) to the social planner. Based on the reported profile u, the social planner chooses the efficient decision $a(u)$ and implements the transfers $t(u)$. When the mechanism is strategy-proof, none of the agents has an interest in deviating from reporting her true preferences u_i, by sending a distorted report v_i. According to the above definition, revelation is in dominant strategies since an agent always prefers to say the truth, whichever the decisions of the other agents. Even if other agents lie, agent i prefers to say the truth. This notion of non-manipulability is therefore very strong. Note that this mechanism works under conditions of *incomplete information*: the social planner doesn't know the agents' preferences; a given agent doesn't know the preferences of the other agents.

To simplify notation, denote $u_N = \sum_{i=1}^{n} u_i$ and $u_T = \sum_{i \in T} u_i$ for any subset $T \subseteq N$.

Definition of a Clarke Mechanism. A Clarke Mechanism, or Pivotal Mechanism, is such that $a(u)$ is efficient for all u and the transfer functions are defined as follows,

$$
\begin{aligned}
t_i(u) &= \sum_{j \neq i} u_j(a(u)) - \max_{b \in A} \left\{ \sum_{j \neq i} u_j(b) \right\} \\
&= u_{N \setminus i}(a(u)) - \max_{A} u_{N \setminus i}
\end{aligned}
$$

Proposition. A Clarke Mechanism is efficient, feasible and strategy-proof.

If for a given profile u, there are several efficient decisions, several outcomes (a, t) satisfying the strategy-proofness property may be chosen, and they all yield the same final utility, that is,

$$
\begin{aligned}
S_i^*(u) &= \max_{A} \left\{ \sum_{j=1}^{n} u_j \right\} - \max_{A} \left\{ \sum_{j \neq i}^{n} u_j \right\} \\
&= \max_{A} u_N - \max_{A} u_{N \setminus i}
\end{aligned}
$$

Proof of the Proposition. With first check the feasibility (or budget constraint), that is, for all u, we must have $\sum_{i=1}^{n} t_i(u) \leq 0$. Using the expression for Clarke transfers above, the budget constraint can be rewritten,

$$(n-1)u_N(a(u)) - \sum_i \max_A u_{N\setminus i} \leq 0.$$

or equivalently, we must show that,

$$(n-1)\max_A u_N \leq \sum_i \max_A u_{N\setminus i}.$$

If a^* maximizes u_N, then,

$$(n-1)\max_A u_N = (n-1)u_N(a^*) = \sum_{i=1}^{n} u_{N\setminus i}(a^*) \leq \sum_{i=1}^{n} \max_A u_{N\setminus i}$$

since by definition, $u_{N\setminus i}(a^*) \leq \max_A u_{N\setminus i}$.

We now check for strategy-proofness. Fix u_i, and $v_i \in \mathbb{R}^{|A|}$ and denote $a(u) = a$, $a(v_i, u_{-i}) = b$. We have,

$$u_i(a) + t_i(u) \geq u_i(b) + t(v_i, u_{-i})$$

that is,

$$u_i(a) + u_{N\setminus i}(a) - \max_A u_{N\setminus i} \geq u_i(b) + u_{N\setminus i}(b) - \max_A u_{N\setminus i},$$

or equivalently,

$$u_N(a) \geq u_N(b).$$

But this is true since $a(u)$ is efficient.

Q.E.D.

The Class of Groves Mechanisms

The Clarke mechanism is just a member in a more general class, that we now characterize.

Definition of a Groves Mechanism. Let $h_i(u_{-i})$ be an arbitrary function of $u_j \in \mathbb{R}^{|A|}$ for all $j \neq i$ (but h_i doesn't depend on u_i).

A Groves mechanism $(a(u), t(u))$,

(i), chooses an efficient decision $a(u)$ for all u and

(ii), the transfers of a Groves mechanism are defined as follows,

$$t_i(u) = u_{N\backslash i}[a(u)] - h_i(u_{-i}).$$

Theorem 1 (Green and Laffont, 1979)
(1) If $(a(.), t(.))$ is a Groves mechanism, then $(a(.), t(.))$ is non-manipulable and efficient.
(2) Any non-manipulable and efficient mechanism is a Groves mechanism.

Proof.
Proof of (1). Same proof as the proof that Clarke mechanisms are non-manipulable (above).
Proof of (2). (The relatively more difficult part.) Let (a, t) be a non-manipulable and efficient mechanism. Define $k_i(u) = u_{N\backslash i}(a(u)) - t_i(u)$ for all i. We now show that k_i doesn't depend on u_i.

Fix u_{-i}. Then, at point (u_i, u_{-i}), non-manipulability implies

$$\begin{aligned} u_i(a(u)) + t_i(u) &= u_N(a(u)) - k_i(u) \\ &\geq u_N(a(v_i, u_{-i})) - k_i(v_i, u_{-i}) \\ &= u_i(a(v_i, u_{-i})) + t_i(v_i, u_{-i}). \end{aligned}$$

Therefore,

$$u_N(a(u_i, u_{-i})) - u_N(a(v_i, u_{-i})) \geq k_i(u_i, u_{-i}) - k_i(v_i, u_{-i}), \tag{7}$$

with a fixed u_{-i}.

Assume that $a(u_i, u_{-i}) = a(v_i, u_{-i})$, then, the roles of u_i and v_i can be exchanged in inequality (7) and we obtain $k_i(u_i, u_{-i}) = k_i(v_i, u_{-i})$. Thus, for all $b \in A$, let $\kappa_i(b)$ denote the common value of $k_i(u_i)$ for all u_i such that $a(u_i, u_{-i}) = b$.
Chose 2 decisions b, b' and construct a utility function u_i such that b and b' are efficient decisions, that is, choose for instance u_i such that, $u_N(b) = u_N(b') \geq u_N(c) + 1$ for all $c \neq b, b'$.

Define v_i^ε and w_i^ε as follows.

$$v_i^\varepsilon(b) = u_i(b) + \varepsilon; \qquad v_i^\varepsilon(c) = u_i(c) \quad \text{for all}\, c \neq b$$

$$w_i^\varepsilon(b') = u_i(b) + \varepsilon; \qquad w_i^\varepsilon(c) = u_i(c) \quad \text{for all}\, c \neq b'$$

For any $\varepsilon > 0$, the unique efficient decision at profile $(v_i^\varepsilon, u_{-i})$ is b; the only efficient decision at profile $(w_i^\varepsilon, u_{-i})$ is b'. We choose ε as small as needed. Hence, our mechanism being efficient, $a(v_i^\varepsilon, u_{-i}) = b$ and $a(w_i^\varepsilon, u_{-i}) = b'$. Applying inequality (7) to the profile $(v_i^\varepsilon, u_{-i})$ when agent i misreports w_i^ε instead of v_i^ε, we have,

$$(v_i^\varepsilon + u_{N\backslash i})(b) - (w_i^\varepsilon + u_{N\backslash i})(b') \geq \kappa_i(b) - \kappa_i(b').$$

Given our assumptions, this can be rewritten,

$$[\varepsilon + u_N(b)] - [\varepsilon + u_N(b')] \geq \kappa_i(b) - \kappa_i(b').$$

and since $u_N(b) = u_N(b')$, we derive $\kappa_i(b) - \kappa_i(b') \leq 0$. Now, if we exchange the roles of b and b' we find $\kappa_i(b) - \kappa_i(b') \geq 0$, proving that $\kappa_i(b') - \kappa_i(b) = 0$. We conclude that k_i doesn't depend on u_i.
Q.E.D.

The next question is the following, can we find a mechanism that is efficient, strategy-proof and with a balanced budget for every profile u? The answer is no.

Theorem 2 (Green and Laffont, 1979)
There doesn't exist a mechanism that is efficient, strategyproof and satisfies the budget constraint as an equality for every profile of preferences, *i.e.*, $\sum_i t_i(u) = 0$ for all u.

Remark. We may find mechanisms that are balanced if we restrict the set of possible preference profiles (the above result is valid for a universal domain of preferences). For example, see below, the exercise on the Groves and Loeb mechanism. Groves mechanisms are inefficient (they are second-best mechanisms). This is a consequence of the Gibbard-Satterthwaite theorem. Green and Laffont (1979) and others have studied the budget surplus when the number of participants n becomes large (asymptotic properties). The expected surplus of the Clarke mechanism becomes in a certain sense negligible when n is large and the distribution of preferences is well-behaved (for instance, if it is normal). There exists an essential relationship between the Groves mechanisms and the second-price sealed-bid (Vickrey) auction for the allocation of private goods. (Think about this). The Vickrey auction allocates a private good, say, a painting by Picasso, to an individual i in the set N called the winner. Efficiency requires to choose the buyer with the highest willingness to pay for the painting. Agents declare their willingness to pay, but the winner pays the second highest valuation: this is a pivotal tax, *i.e.*, the highest welfare loss caused by the mere presence of the winner in the auction mechanism. The Vickrey auction is revealing in dominant strategies. The Groves class is just a larger class of mechanisms that includes also public goods, public goods with exclusion, etc. This is why the Groves mechanism is called the VCG or Vickrey-Clarke-Groves Mechanism by many economists.

Some Exercises

Exercise 1. (Voluntary provision with Cobb-Douglas preferences)

We consider the voluntary provision model à la Bergstrom-Blume-Varian. There are n individuals with identical preferences. Using the same notation as above, assume that

$$u_i = u(x_i, G) = x_i^{1-\alpha} G^{\alpha}$$

with $\alpha \in (0, 1)$ for all $i = 1, \ldots, n$.

1. Pose the utility maximization problem of agent i. Show that we can as well suppose that agent i chooses (x_i, G) subject to constraints. Find the demand functions denoted $(x_i(.), G(.))$, for the public and the private goods, and show that $(x_i, G) = (x(R_i), G(R_i))$ where $R_i = \omega_i + G_{-i}$, if $g_i > 0$.

2. Suppose that $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$. Find conditions on ω and α for an equilibrium in which $g_2^* = \cdots = g_n^* = 0$ and agent 1 is the only contributor (only the richest contributes).

3. An interesting case is when $\omega_1 = W$ and $\omega_j \simeq 0$ for all $j \neq 1$ (the first agent has all the money). Show that agent 1's choice of G is a Pareto-optimal level of production when $W = \sum_i \omega_i = \omega_1$.

4. Let G_k denote the equilibrium production when the wealth is divided equally

among k individuals.

4(a) Suppose first that we divide the wealth W among 2 individuals. Show that $G_2 < G_1$.

4(b) More generally suppose that the wealth is divided into k equal shares W/k among k consumers. Compute the equilibrium value of G_k and show that $G_k \to 0$ when $k \to +\infty$. (The smallest amount of public production is supplied when everyone is a contributor).

Exercise 2. (Voluntary Contribution with Quasi-Linear Utilities: "Locomotive Theorem")

Suppose that $u_i(x_i, G) = x_i + v_i(G)$ and take the particular example,

$$u(x_i, G) = x_i + a_i \ln(G),$$

for all i, where $a_i > 0$. The other notations are the same as above. We assume that $\omega_i = 1$ for all i.

1) Show that if $a_i = 1$ for all $i = 1, \ldots, n$, any $g = (g_1, ..., g_n) \in [0, 1]^n$ satisfying $\sum_i g_i = 1$ is a Nash equilibrium.

2) Show that if $a_1 = 1$ and $a_i < 1$ for all $i > 1$, the unique equilibrium is $g_1 = G = 1$ and $g_k = 0$, for all $k > 1$.

Exercise 3. (Cost sharing of indivisible public good)

We use the same notation as above. Let the set of public decisions be $A = \{0, 1\}$. The cost is ca with $c > 0$ and $a \in \{0, 1\}$. Decision $a = 1$ means that the public good is produced and that its cost is shared uniformly among the n agents. Let $(\theta_1, \ldots, \theta_n)$ be the profile of benefits to agents, *i.e.*, the utilities are defined as follows $u_i(a, \theta) + t_i = a(\theta_i - (c/n)) + t_i$.

1. Describe the Pivotal mechanism (Clarke Mechanism) completely in this case.

2. If $n = 2$, $c = 4$, $\theta_1 = 1$ and $\theta_2 = 3.5$, compute the efficient decision and the budget surplus. Compare the result with the first-best optima.

Exercise 4. (the Groves and Loeb Mechanism)

Consider a divisible public production y. The cost function is linear, $c(y) = y$. There are n agents with benefit functions $v_i(y, \theta) = \theta_i \sqrt{y}$ for all $i = 1, \ldots, n$. The utilities are of the specific form $v_i(y, \theta) - t_i$ with the interpretation that $t_i > 0$ is a tax. The budget constraint now includes the cost of y, that is, $\sum_i t_i \geq c(y)$.

Show that the mechanism choosing an efficient decision and relying on the transfer functions,

$$t_i = \frac{1}{4}\theta_i^2 + \frac{1}{2(n-2)} \sum_{j,k \neq i\, j<k} \theta_j \theta_k$$

is non-manipulable and *always balances the budget.*

Solution of Exercise 1.

$$1) \qquad \max \quad u_i(x_i, g_i + G_{-i}) \quad \text{s.t.} \quad x_i + g_i = \omega_i, \quad g_i \geq 0$$

$$\Leftrightarrow \qquad \max \quad (x, G) \quad \text{s.t.} \quad x + G = \omega_i + G_{-i}, \quad G \geq G_{-i}$$

$$\Leftrightarrow \qquad \max \quad x^{1-\alpha}G^{\alpha} \quad \text{s.t.} \quad x + G = R, \quad G \geq G_{-i}$$

We easily derive the demand functions (in the textbook, Cobb-Douglas example):

(i) $$\begin{cases} x_i = (1-\alpha)R_i \\ G = \alpha R_i \quad \text{with} \quad R_i = \omega_i + G_{-i} \end{cases}$$

These expressions are valid if $g_i \geq 0$! That is, if $g_i = \alpha\omega_i + (\alpha - 1)G_{-i} \geq 0$ $\Leftrightarrow$ $\alpha\omega_i \geq (1-\alpha)G_{-i}$.

(ii) $$\begin{cases} x_i = \omega_i \\ g_i = 0 \quad \text{or} \quad G = G_{-i} \quad \textit{otherwise.} \end{cases}$$

2) Suppose that $\omega_1 \geq \omega_2 \geq ... \geq \omega_n$.

Find conditions for an equilibrium in which $g_2^* = g_3^* = ... = g_n^* = 0$ and only agent 1 contributes.

If this is true, then $g_1^* = \alpha\omega_1 = G^*$ and $g_2^* = 0$. It follows that we must have,

$$\begin{aligned} & \alpha\omega_2 - (1-\alpha)g_1^* < 0 \\ \Leftrightarrow \quad & \alpha\omega_2 < (1-\alpha)\alpha\omega_1 \\ \Leftrightarrow \quad & \omega_2 < (1-\alpha)\omega_1. \end{aligned}$$

This is possible, for instance, if $\alpha = 1/2$ and $\omega_2 < 2\omega_1$.

The same inequality holds for $i \geq 2$ since $\omega_k \leq \omega_2$ for $k \geq 2$.

3) An interesting case is when $\omega_1 = W$ and $\omega_j \simeq 0$ for all $j > 1$.

The first agent's choice $G_1 = \alpha W$ is a Pareto-optimal production. To find a Pareto optimum, we can solve the problem,

$$\max \sum_i \ln \left[x_i^{1-\alpha} G^\alpha\right]$$

subject to the ressource constraint. We can do this because $v = \ln(u)$ is an ordinally equivalent representation of the agent's preferences.

$$\Leftrightarrow \qquad \max \left[(1-\alpha)\sum_i \ln(x_i) + \alpha n \ln(G)\right]$$

$$\text{s.t.} \quad \sum_i x_i + G = W.$$

The first-order conditions are

$$\frac{(1-\alpha)}{x_i} = \lambda$$

where λ is the Lagrange multiplier of the resource constraint, and

$$\frac{(\alpha n)}{G} = \lambda.$$

We find,

$$\Rightarrow \sum_i x_i + G = W = \frac{n(1-\alpha)}{\lambda} + \frac{n\alpha}{\lambda} = \frac{n}{\lambda},$$

$$\Rightarrow \hat{x}_i = \frac{W}{n}(1-\alpha) \quad \text{and} \quad \hat{G} = \alpha W$$

is a Pareto-optimum. The richest agent, endowed with W, will therefore choose a Pareto optimum (recall that $G_{-1} = 0$).

4a) Suppose that we divide the wealth W among 2 individuals. Then, we must have

$$g_2 = \alpha \frac{W}{2} - (1-\alpha) G_{-2} \quad \text{and} \quad G_{-2} = g_1.$$

Under symmetry, we have $g_1 = g_2 = G_{-2} = G_{-1}$. So,

$$2g_2 = \alpha \frac{W}{2} + \alpha g_2,$$

or

$$G_2 = \alpha \left(\frac{W}{2} + \frac{G_2}{2} \right).$$

From this we derive,

$$G_2 = \frac{\alpha W}{2-\alpha} < G_1.$$

4b) More generally, if the wealth is divided in k equal shares $\frac{W}{k}$, we obtain,

$$\begin{aligned} & g_k = \alpha \frac{W}{k} - (1-\alpha) G_{-k} \\ \Leftrightarrow \qquad & G_k = \alpha \frac{W}{k} + \alpha (k-1) \frac{G_k}{k} \end{aligned}$$

because using equilibrium symmetry,

$$G_{-k} = \frac{G_k}{k}(k-1).$$

and $G_k = g_k + G_{-k}$.

$$\begin{aligned} \Rightarrow \qquad & G_k(1 - \alpha(1 - 1/k)) = \frac{\alpha W}{k} \\ & G_k(k - \alpha(k-1)) = \alpha W \\ & G_k = \frac{\alpha W}{k(1-\alpha) + \alpha} \xrightarrow[k \to \infty]{} 0. \end{aligned}$$

The smallest amount of public production is supplied when everyone is a contributor.

Solution of Exercise 2.

1) Show that if $a_i = 1$ for all $i = 1, \ldots, n$, any $g = (g_1, ..., g_n) \in [0,1]^n$ satisfying $\sum_i g_i = 1$ is a Nash equilibrium.

An equilibrium contribution solves,

$$\max_{g_i > 0} \left(\omega_i - g_i + \ln \left(\sum_j g_j \right) \right).$$

This implies the FOC,

$$1 + \frac{1}{G} = 0 \Rightarrow \quad G = 1.$$

It follows that if $G_{-i} < 1$ and $1 - G_{-i} \in [0,1]$, then $g_i = 1 - G_{-i}$ is a best response of agent i.

2) If $a_1 = 1$ and $a_i < 1$ for all $i > 1$. The unique equilibrium is $g_1 = G = 1$ and $g_k = 0$, for all $k > 1$.

For $i = 1$, the FOC becomes,

$$-1 + \frac{1}{G} = 0 \qquad \text{or} \qquad g_1 + G_{-1} = 1,$$

if $(1 - G_{-1}) \in [0,1]$, and,

$$g_1 = 0 \quad \text{if} \quad 1 < G_{-1},$$
$$g_1 = 1 \quad \text{if} \quad G_{-1} = 0.$$

For agent $i > 1$, the FOC is as follows:

$$-1 + \frac{a_i}{G} = 0 \qquad \text{or} \qquad g_i = a_i - G_{-i},$$
$$\text{if} \qquad (a_i - G_{-i}) \in [0,1],$$
$$\text{and} \qquad g_i = 0 \quad \text{if} \quad a_i - G_{-i} < 0.$$

Note that $a_i - G_{-i} < 1$ since $a_i < 1$. It follows that $g_1 = 1$, and $g_j = 0$, for all $j = 2, \ldots, n$ is a Nash equilibrium. $G = g_1$ is a best response to $g_j = 0$ for $j > 1$ and $g_j = 0$ is a best response to $g_1 = 1$ and $g_k = 0$, $k \neq 1$, $k \neq j$.

We now show that this is the only Nash equilibrium.

Suppose that $g' = (g'_1, ..., g'_n)$ is another equilibrium in which $i = 1$ and $i \in C$ contribute, with $|C| \geq 2$. The subset C is defined as follows,

$$C = \{i \in N \mid g'_i > 0\}.$$

We have $g'_1 + G'_{-1} = 1$ and for $k \in C \setminus \{1\}$, we have $g'_k + G'_{-k} = a_k$. Adding these equations yields

$$\sum_{k \in C} g'_k + \sum_{k \in C} G'_{-k} = 1 + \sum_{k \in C \setminus \{1\}} a_k < |C|,$$

since $k \neq 1 \Rightarrow a_k < 1$.

We also have,

$$G' = \sum_{k \in C} g'_k.$$

Hence, we obtain $G' + (|C| - 1)G' < |C|$ and $g'_1 + G'_{-1} = G' = 1$ implies $|C| < |C|$, a contradiction.

Suppose then that there is an equilibrium g' in which Player 1 doesn't contribute, $\{1\} \cap C = \emptyset$. Then, $g'_k + G'_{-k} = a_k$ for k in C implies,

$$G' + (|C| - 1)G' = \sum_C a_k < |C|$$

$$|C|.G' < |C|$$

$$\Rightarrow \qquad G' < 1.$$

But if $G' < 1$, then, the best response of Player 1 is $g'_1 = 1 - G' > 0$. This is a contradiction.

Solution of Exercise 3.

See textbooks. See Moulin (1988).

Solution of Exercise 4.

See Moulin (1988), see Laffont (1988).