Robert Gary-Bobo

PUBLIC ECONOMICS

Lecture Notes 2

2017-2018, revised in November 2017

ENSAE

Chapter II

Externalities

Some Implementation Theory

We consider, as usual, a population of agents $i = 1, ..., n$. Each i is characterized by a type θ_i drawn in a set Θ. Society must choose a public decision $x \in X$. There is a social planner, assumed benevolent. Each agent is characterized by a utility function $u_i(x, \theta_i)$ for all i. Utility depends on i and θ_i : *i.e.*, unobservable characteristics are captured by the type θ_i (private information of i) and some observable characteristics of i may be embodied in function u_i, so that u depends also on i.

A preference profile is fully determined by a vector of types $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$

A Choice Function, or Choice Rule, is by definition a mapping,

$$\begin{aligned} f: \quad & \Theta^n \longrightarrow X \\ & \theta \longmapsto f(\theta) \in X \end{aligned}$$

Definition 1: (Mechanism)

A mechanism is a pair (M, g) where M is a message space and g is an outcome function

$$g : M^n \longrightarrow X$$

$$m = (m_1, ..., m_n) \longmapsto x = g(m).$$

Interpretation: $g(m)$ is the decision made by the planner when the messages are m. The planner commits to use g as a rule of decision.

Nash, Bayesian and Dominant-Strategy Implementation

a) If the principal is not informed of θ but the agents $i = 1, \ldots, n$ know each other's types we have a *Nash implementation* problem. Agents play a complete-information game.

b) *Incomplete information.* If the principal and the agents are not informed of the other agents' types. Types are privately known (each i knows θ_i), we have either

b1) a *Bayesian Implementation* problem, or

b2) we consider the more demanding *Implementation in Dominant Strategies.*

There exist different notions of efficiency in economies with incomplete information. There are three concepts of efficiency: *ex ante* (before the drawing of types; individuals do not know their types); *ex interim* (when each individual knows only her (his) type, which is private information); and *ex post* (once types are revealed). We define only the ex post efficiency concept here.

Definition 2: (Ex-post Pareto-optimality) f is *ex post efficient* if for no θ is there an $x \in X$ such that

$$u_i(x, \theta_i) \geq u_i(f(\theta), \theta_i)$$

for all i, with a strict inequality for some i.

Nash Implementation

Let (M, g) be a mechanism. Define the payoffs $v_i(m, \theta_i) \equiv u_i(g(m), \theta_i)$. Let

$$\Gamma_\theta = ((M)_{i=1,\dots,n}, (v_i)_{i=1,\dots,n})$$

be a game in normal form. There are n players in the game. Each player i chooses a strategy m_i in the strategy space M. We must pay attention to the fact that Γ_θ may possess several Nash equilibria.

Definition 3: (Implementation in Nash Equilibrium) (M, g) implements the choice function $f(.)$ if for any Nash equilibrium of Γ_θ, denoted $(m_i^*(\theta))_{i=1,\dots,n} = m^*(\theta)$, we have

$$g(m^*(\theta)) = f(\theta) \quad \text{for all} \quad \theta \in \Theta.$$

This is called *strong implementation* because $g(\mathcal{N}(\Gamma_\theta)) = f(\theta)$ for all θ, where $\mathcal{N}(\Gamma_\theta)$ is the set of Nash equilibria of Γ_θ (*i.e.*, when preferences are determined by θ.

Definition 4: (Direct Mechanism)
A mechanism (M, g) is *direct* if $M = \Theta$.

Maskin's Theorems

These famous results involve two assumptions, *Maskin-monotonicity* and *no-veto power*. Function f "can be implemented in Nash equilibrium" more precisely means that there exists (M, g) such that f is implemented by means of the game Γ_θ.

For a better understanding of the assumptions, define the *lower contour sets*

$$L_i(x, \theta_i) = \{y \in X \mid u_i(x, \theta_i) \geq u_i(y, \theta_i)\}.$$

Assumption 1: The choice rule f is *Maskin-monotonic* if for any profiles θ and θ', we have $x = f(\theta)$ and

$$L_i(x, \theta_i) \subseteq L_i(x, \theta_i') \quad \text{for} \quad \text{all} \quad i,$$

then, we must have $x = f(\theta')$.

In words, Assumption 1 says that if a profile θ is modified to some new profile θ' in such a way that an outcome $x = f(\theta)$ does not end up being ranked below any outcome it was previously preferred or equivalent to, then x continues to be chosen, *i.e.*, $x = f(\theta')$.

Theorem A (Eric Maskin 1977-1999)
If f is implementable in Nash equilibrium, then f is Maskin-monotonic.

Thus, Maskin-monotonicity is a *necessary condition.* This condition is not far from being sufficient, but it is not sufficient, we have to add the second condition. To formulate the next assumption, define the sets of maximal elements for agent i in X,

$$M_i(X, \theta) = \{x \in X \,|\, u_i(x, \theta_i) \geq u_i(y, \theta_i) \quad \forall y \in X\}.$$

We can state the following assumption.

Assumption 2: (No veto power)
The choice rule f satisfies the *no veto power* property if for all θ and all x, whenever there exists i such that

$$x \in \bigcap_{j \neq i} M_j(X, \theta_j)$$

then $x = f(\theta)$.

No veto power says that if an alternative x is top-ranked by $n - 1$ agents, then the n-th agent cannot prevent it from being the collective choice.

In the case of many agents (at least $n \geq 3$), this is a weak assumption that may be vacuously satisfied in many cases, because it is not easy to find an agent i such that all others agents $j \neq i$ agree that x is the best decision (for instance, if there is a distribution problem and if money is used as a compensation).

Theorem B (Eric Maskin 1977-1999)
If $n \geq 3$ and f satisfies *Maskin-monotonicity* and *no veto power*, then, f is implementable in Nash equilibrium.

Proof: Raphael Repullo (1987) in *Social Choice and Welfare*, Eric Maskin (1999) in *Rev. Econ. Stud.*.

The case of two agents is more difficult : see Moore and Repullo (1990) in *Econometrica*.

Bayesian Implementation

We now consider the cases in which the principal is not informed and the types θ_i are private information.

Bayesian Games

Assume that θ is a drawing from a probability distribution on Θ^n, denoted P, *i.e.*,

$$P(\theta) = \Pr(\theta_1, ..., \theta_n).$$

Assumption (Harsanyi's Doctrine)
P is *common knowledge* of the planner and the agents.

We now define the agent's *beliefs* in probabilistic form. We will make use of the usual notation,

$$\theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n).$$

Let $P(\theta_{-i} \mid \theta_i)$ be agent $i's$ beliefs about the types of others, that is, θ_i; this is the conditional probability of θ_{-i} knowing θ_i.

A Mechanism (M, g) defines a *Bayesian Game* $G = \{P, (\Theta)_{i=1,...,n}, (M)_{i=1,...,n}, (v_i)_{i=1,...,n}\}$ where the message space M is the strategy space, the player types belong to Θ, and the payoffs are $v_i(m, \theta_i) \equiv u_i(g(m), \theta_i)$, with $m \in M^n$.

Definition 5: (Bayesian-Nash Equilibrium, Harsanyi 1967-1968)

A Bayesian-Nash equilibrium is a n-tuple of message-strategies $m_i^*(\theta)$, such that, for all i, all θ_i,

$$\mathbb{E}_P\left[v_i(m^*(\theta), \theta_i) \mid \theta_i\right] \geq \mathbb{E}_P\left[v_i((m', m_{-i}^*(\theta)), \theta_i) \mid \theta_i\right],$$

for all $m' \in M$ where $m_{-i}^*(\theta) = (m_1^*(\theta), ..., m_{i-1}^*(\theta), m_{i+1}^*(\theta), ..., m_n^*(\theta))$ is the $(n-1)$-tuple of the other players' message strategies.

Definition 6: The mechanism (M, g) strongly implements f in Bayesian equilibrium if for any Bayesian equilibrium $m^*(\theta)$ of G we have $g(m^*(\theta)) = f(\theta)$ for all $\theta \in \Theta^n$.

Results on Bayesian implementation:

For instance, we can find efficient Public-good mechanisms, see, *e.g.*, Claude d'Aspremont and Louis-André Gérard-Varet (1979). There are some general results, generalizing Maskin's results in the Bayesian game case, see Matthew Jackson (1991) in *Econometrica*, Thomas Palfrey's (1992) survey.

There are a number of problems posed by Bayesian implementation. The details of the mechanisms may depend on the agents' beliefs $P(\theta_{-i} \mid \theta_i)$. What happens then if P is not common knowledge, or if $P(. \mid \theta_i)$ is a subjective belief of i, or if the planner doesn't know the agent's beliefs?

Dominant Strategy Mechanisms

We will now focus on a more demanding notion, the implementation in *Dominant Strategies*. There are some important and well-known results on this implementation concept. In particular, characterizations of the Clarke-Groves mechanisms (Green and Laffont (1979), Laffont and Maskin (1981)) and the Gibbard-Satterthwaite theorem (Alan Gibbard (1973), Mark Satterthwaite (1975)).

Definition 7 (Equilibrium in Dominant Strategies)
An *equilibrium in dominant strategies* is an n-tuple of message-strategies $(m_i^*(\theta_i))_{i=1,\ldots,n}$, such that for all i, and all $\theta_i \in \Theta$,

$$u_i[g(m_i^*(\theta), m_{-i}), \theta_i] \geq u_i[g(m', m_{-i}), \theta_i]$$

for all $m' \in M$, for all $m_{-i} \in M^{n-1}$.

There is no profitable deviation m', whichever θ_i and m_{-i}.

Definition 8: (Implementation in Dominant Strategies)
g implements f in dominant strategies if there exist dominant strategies $(m_i^*(\theta_i))_{i=1,\ldots,n}$ such that $g(m^*(\theta)) = f(\theta)$ for all $\theta \in \Theta^n$.

Definition 9: (Direct and Revealing Mechanisms)
A Mechanism (M, g) is *direct* if $M = \Theta$. A Mechanism is *revealing in dominant strategies* (or *strategy-proof* or *non-manipulable*) if is direct and the dominant strategies m_i^* are such that $m^*(\theta_i) \equiv \theta_i$ for all $i = 1, \ldots, n$.

Revelation Principle

The Revelation principle is stated and proved in the case of dominant Strategies here, but the result can be proved in the Bayesian implementation case as well.

Theorem C (Revelation Principle)
If mechanism (M, g) implements f in dominant strategies, then, the mechanism (Θ, f) is *revealing* and implements f in dominant strategies.

Proof: We assume that there exists dominant strategies $m^*(\theta)$ such that $g[m^*(\theta)] = f(\theta)$ for all θ. Then,

$$u_i[g(m_i^*(\theta_i), m_{-i}), \theta_i] \geq u_i[g(m_i', m_{-i}), \theta_i]$$

for all $m_i' \in M$ and all $m_{-i} \in M^{n-1}$.

So, in particular, for all i and θ_i,

$$u_i[g(m_i^*(\theta_i), m_{-i}^*(\theta_{-i})), \theta_i] \geq u_i[g(m_i^*(\theta_i'), m_{-i}^*(\theta_{-i})), \theta_i]$$

for all $\theta_i' \in \Theta$ and all $\theta_{-i} \in \Theta^{n-1}$.

Now, since (M, g) implements f, by definition, we have $g(m^*(\theta)) \equiv f(\theta)$. Hence, for all i, for all θ_i,

$$u_i[f(\theta_i, \theta_{-i}), \theta_i] \geq u_i[f(\theta_i', \theta_{-i}), \theta_i]$$

for all θ_i, all θ_{-i}.

We conclude that f is implementable by the *direct and revealing* (or strategy-proof) mechanism (Θ, f), instead of (M, g).
Q.E.D.

Consequence of the Revelation Principle: There is no loss of generality in restricting the search for optimal mechanism to direct and revealing (or strategy-proof) mechanisms.

This result is also true in the context of Bayesian implementation (see, for instance, the textbook of David Kreps (1990)). See also the textbook of Andreu MasColell, Michael Whinston and Jerry Green (1995) for a presentation of these questions.

Gibbard-Satterthwaite's Theorem

Definition 12 : f is *dictatorial* if there exists an agent i such that, for all $\theta \in \Theta$, $f(\theta)$ maximizes $u_i(x, \theta_i)$ over X.

Theorem D (Gibbard-Satterthwaite)
Assume $|X| < \infty$ and $|X| \geq 3$, assume that preferences are strict for all i and $f(\Theta) = X$, and assume that all strict preferences over X are possible when θ varies in Θ. Then, f is strategy-proof *if and only if* f is *dictatorial*.

Note: the result is not true if $|X| = 2$ (since the majority rule works with two alternatives). The result holds under the *universal domain* assumption (any preference profile can happen). This means that if we consider a problem with a restricted domain of preferences (for instance quadratic utilities over an interval), we may find an efficient, revealing mechanism that is not dictatorial. There are some well-known examples.
For proofs: see Salanié (1998), Mas-Colell, Whinston and Green (1995), Moulin (1988). The result can be viewed as a corollary to Arrow's Impossibility Theorem. The result still holds

if X is infinite and preferences are not necessarily strict, but are assumed continuous (see Salvador Barbera and Bezalel Peleg (1990)).

Further remark (Corollary):
Any ex-post efficient function f must satisfy $f(\Theta) = X$ if preferences are strict and all strict preferences on X are possible for all i. Thus, if $|X| > 2$, the only ex-post efficient choice functions f that are strategy-proof are *dictatorial*!

Possible "solutions": Use weaker implementation concepts: Nash equilibrium, Bayesian-Nash equilibrium, Subgame Perfect equilibrium (multi-stage games). Study special environments (weaken the universal domain of preferences assumption)— study specific problems.

Mechanisms implementing efficient allocations in environments with externalities

Consider an economy in which agents take actions that impose *externalities* (benefits or costs) on other agents.

We consider here a situation in which agents are completely informed : the agents involved know the relevant technology and tastes of other agents. However, the "regulator" or social planner doesn't possess this information.

Problem: Can the regulator design a mechanism to implement an efficient allocation in these environments? The answer is *yes*, using two-stage games whose subgame-perfect equilibria implement Pareto optima. We study Varian's "Compensation Mechanisms".

Reference: Hal Varian (1994) "A solution to the problem of externalities when agents are well-informed," *American Econ. Rev.*, 84, 1278-1293.

Multi-stage games and subgame-perfect implementation were analyzed by Moore and Repullo (1988). See also the survey by John Moore (1992).

A Simple example of Varian's Compensation Mechanism

Suppose there are only two agents (two firms).

Firm 1 produces output $q \geq 0$ to maximize profit,

$$\pi_1 = rq - c(q),$$

where r is the competitive price of output q and $c(q)$ is a cost function assumed differentiable, strictly increasing and strictly convex.

Firm 1's choice imposes an externality on Firm 2, the profit of which is simply

$$\pi_2 = -e(q),$$

where $e(q)$ is a differentiable, strictly increasing, and strictly convex cost function.

In general, Firm 1 ignores $e(q)$ and outcomes are not efficient. Several solutions have been studied in the literature.

Ronald Coase (1960) claims that zero transactions costs, well-defined property rights and negotiation (bargaining among agents) imply efficiency. But there is no specific mechanism (the solution is incomplete).

Arrow (1970), and other authors, suggested setting up a competitive market for the externality (for instance, markets for pollution permits). But markets for particular externalities may be very thin (few participants).

Pigou (1920), in his *Economics of Welfare*, proposed that the regulator imposes taxes ("Pigovian taxes") and subsidies to correct inefficiencies. There is a difficulty: the regulator must be able to compute the correct level of the tax. This requires knowledge relative to technologies and preferences. It follows that this solution too is incomplete.

Finally, in the *Mechanism Design* literature (*e.g.*, Maskin (1977), Moore and Repullo (1988)) a game played by agents must be fully specified.

Note that if the regulator had full information, he or she could impose the costs of externality on Firm 1 by charging a "tax" equal to $e(q)$.

Firm 1 would solve,

$$Max_q[rq - c(q) - e(q)]$$

The first-best solution q^* satisfies the first-order condition (FOC)

$$r - c'(q^*) - e'(q^*) = 0.$$

The regulator could as well choose $p^* = e'(q^*)$ (the Pigovian tax) and let the firm solve

$$Max_{q\geq 0}\{(r - p^*)q - c(q)\},$$

but the regulator doesn't know the cost function $e(q)$.

Varian's Mechanism

This is a two-stage mechanism.

Announcement Stage: Firms 1 and 2 simultaneously announce the magnitude of the appropriate Pigovian tax, that is,

Firm 1 announces p_1;

Firm 2 announces p_2.

Choice Stage: The regulator makes side payments to the firms, so that the profit functions become

$$\pi_1 = rq - c(q) - p_2 q - \alpha_1(p_1 - p_2)^2$$
$$\pi_2 = p_1 q - e(q),$$

where α_1 is an arbitrary positive parameter.

Note: Firm 1 pays a tax p_2 per unit (reported by Firm 2). Firm 2 receives compensation based on p_1 (reported by Firm 1). Firm 1 pays a penalty if $p_1 \neq p_2$.

Nash equilibria and Subgame-Perfect Nash equilibrium of Varian's Compensation Mechanism

There are many Nash equilibria in this game. Any (p_1, p_2, q) such that $p_1 = p_2$ and q maximizes π_1 is a Nash equilibrium.

Proof: If p_2 fixed, $p_1 = p_2$ is a best response and q maximizing π_1 is a best response of Firm 1. In addition, p_2 is trivially a best response to (p_1, q) since π_2 doesn't depend on p_2.

Stronger Concept: Subgame-Perfect Equilibrium
We will show that there exists a unique *Subgame-Perfect Equilibrium* (hereafter SPE). The SPE is such that each agent reports $p_1 = p_2 = p^*$ and Firm 1 chooses $q = q^*$.

Solution: (Backwards Induction) We start with the last stage.

(a) Begin with the choice stage. Firm 1 chooses q to maximize profits given (p_1, p_2). It follows that q satisfies the FOC

$$r = c'(q) + p_2.$$

This determines the best response of Firm 1, $q(p_2)$, which is a function of p_2.
Note: $q'(p_2) < 0$ since, differentiating the FOC, we have,

$$c'' \frac{dq}{dp_2} + 1 = 0,$$

implying

$$\frac{dq}{dp_2} = -\frac{1}{c''} < 0.$$

(b) Solve the announcement stage. Firm 1 wants to announce $p_1 = p_2$ if Firm 2 announces p_2, since p_1 only influences Firm 1's penalty.
Firm 2's pricing decision has an *indirect* effect through $q(p_2)$.

Firm 2's choice maximizes $\pi_2(p_2)$, that is,

$$Max\ p_1 q(p_2) - e(q(p_2)).$$

The first-order condition for this problem is

$$\pi_2'(p_2) = [p_1 - e'(q)]q'(p_2) = 0.$$

But $q'(p_2) < 0$ implies

$$p_1 = e'(q(p_2)).$$

We found the equilibrium conditions,

$$\begin{aligned} r &= c'(q(p_2)) + p_2 \\ p_2 &= p_1 \\ p_1 &= e'(q(p_2)) \end{aligned}$$

Thus,

$$r = c'(q(p_2)) + e'(q(p_2))$$

and this implies $q(p_2) = q^*$, the condition for first-best optimality.

Conclusion: The only SPE of this game involves Firm 1 producing q^*.

Intuition for Varian's Mechanism

Firm 2 chooses q by setting p_2 (the price faced by Firm 1). An equilibrium can exist only if $p_1 = e'(q)$ for otherwise, Firm 2 would like to change p_2 to induce a change of q. Firm 1 wants to minimize the penalty $\alpha(p_1 - p_2)^2$. This finally implies $p_1 = p_2$.

If Firm 1 thinks that p_2 is large, Firm 1 chooses a large $p_1 = p_2$ and it follows that Firm 2 is "over compensated" and wants q to increase: this in turn implies that Firm 2 chooses a smaller p_2.

At the end (in equilibrium), Firm 2 announces $p_2 = e'(q)$, where $q = q^*$.

Remark: The budget is balanced in equilibrium only.

$$p_1 = p_2 \Rightarrow \alpha_1(p_1 - p_2)^2 = 0$$

and trivially, $p_1 q = p_2 q$.

Extensions of the basic example

With 3 agents at least, transfers can be chosen to balance the mechanism in *and out* of equilibrium (with two agents, we get budget balance in equilibrium only). To balance the budget, just distribute the surplus or deficit generated by agent 1 to agents 2 and 3.

The 3 agents case

Let q be the production of Firm 1. Let $e_2(q)$ and $e_3(q)$ measure the externality-cost of agents 2 and 3. Define,

$$\pi_1 = rq - c(q) - (p_{21}^2 + p_{31}^3)q - \alpha_2(p_{21}^1 - p_{21}^2)^2 - \alpha_3(p_{31}^1 - p_{31}^3)^2$$
$$\pi_2 = p_{21}^1 q - e_2(q)$$
$$\pi_3 = p_{31}^1 q - e_3(q),$$

where p_{ij}^k is the price announced by k that measures the marginal cost that agent $j's$ choice imposes on agent i, and α_j is a positive parameter.

This mechanism can be balanced (with slightly different penalties), as follows:

$$\pi_1 = rq - c(q) - (p_{21}^2 + p_{31}^3)q - \alpha_2(p_{21}^1 - p_{21}^2)^2 - \alpha_3(p_{31}^1 - p_{31}^3)^2$$
$$\pi_2 = p_{21}^1 q - e_2(q) + (p_{31}^3 - p_{31}^1)q + \alpha_3(p_{31}^1 - p_{31}^3)^2$$
$$\pi_3 = p_{31}^1 q - e_3(q) + (p_{21}^2 - p_{21}^1)q + \alpha_2(p_{21}^1 - p_{21}^2)^2$$

Note that the transfers t paid by Firm 1 to other firms and the regulator are as follows,

$$\begin{aligned} t &= (p_{21}^2 + p_{31}^3)q + \alpha_2(p_{21}^1 - p_{21}^2)^2 + \alpha_3(p_{31}^1 - p_{31}^3)^2 \\ &= (p_{31}^3 - p_{31}^1)q + (p_{21}^2 - p_{21}^1)q + p_{31}^1 q + p_{21}^1 q + \alpha_2(p_{31}^1 - p_{31}^3)^2 + \alpha_3(p_{21}^1 - p_{21}^2)^2 \\ &= \text{total compensation received by 2 and 3.} \end{aligned}$$

This implies that the mechanism is balanced.

Note that penalties $\alpha(p_{ij}^k - p_{ij}^j)^2$ are useless (α_j could be set equal to zero).

To derive best responses, differentiate the objective functions with respect to choices. We obtain,

$$r - c'(q) - (p_{21}^2 + p_{31}^3) = 0,$$
$$\{p_{21}^1 - e_2'(q(p_{21}^2 + p_{31}^3)) + p_{31}^3 - p_{31}^1\}q'(p_{21}^2 + p_{31}^3) = 0,$$
$$\{p_{31}^1 - e_3'(q(p_{21}^2 + p_{31}^3)) + p_{21}^2 - p_{21}^1\}q'(p_{21}^2 + p_{31}^3) = 0.$$

It is easy to check that $q'(.) < 0$. Therefore, we derive the system,

$$r - c'(q) - (p_{21}^2 + p_{31}^3) = 0,$$
$$p_{21}^1 - e_2'(q(p_{21}^2 + p_{31}^3)) + p_{31}^3 - p_{31}^1 = 0,$$
$$p_{31}^1 - e_3'(q(p_{21}^2 + p_{31}^3)) + p_{21}^2 - p_{21}^1 = 0.$$

Adding up the equations yields:

$$r - c'(q) - e_2^1(q) - e_3^1(q) = 0,$$

the necessary and sufficient condition for optimality. Hence, $q = q^*$ (production is first-best optimal).

Varian (1994) shows that a naive adjustment process will lead to equilibrium (to the SPE). With two agents,

$$\begin{aligned} p_1(t+1) &= p_2(t) \\ p_2(t+1) &= p_2(t) - \gamma[p_1(t) - e'(q(p_2(t))]. \end{aligned}$$

With $\gamma > 0$ we see that p_2 decreases between time t and $t+1$ if $p_1(t) > e'(q(p_2(t)))$. This system is locally stable (see Varian) if γ is small enough.

Varian's Compensation Mechanism works in general externality problems with utility functions of the form $u_i(q_1, ..., q_n, y_i)$, where y_i is a transferable good.

Second-Best Internalization Mechanisms under Asymmetric Information

Coase's basic argument is that inefficiency is equivalent to the existence of profitable opportunities. It follows from this that if side-payments can be arranged, all parties should benefit. But Coase doesn't provide the details of the bargaining mechanism: *efficiency is postulated as an axiom* rather than being shown to emerge as the outcome of a non-cooperative procedure (mechanism).

We will now study a Bayesian Mechanism to compensate pollution damages. The source of inspiration for this is Rafael Rob in *J. Econ. Theory*, (1989).

The inputs of the mechanism are pollution-related monetary damages suffered by victims. The outputs of the mechanism are,

a) accept or reject the construction of a pollution-generating plant;

b) compensation payments are determined (for victims of pollution).

Our goal is to design a non-cooperative game and outcomes are not assumed Pareto-optimal *a priori*. We examine the Coase argument in a non-cooperative setting; there are *asymmetries of information* relative to the cost of damages.

The results are as follows:

(1) inefficiencies do emerge as equilibrium outcomes;

(2) inefficient outcomes are more likely when the number of participants is large.

This is a fondamental difference between public and private goods mechanisms (because perfect competition is achieved in markets with many agents).

The fundamental reason for inefficiencies is that external damages are not publicly observable. Each participant has an incentive to overstate his damages, so as to receive more compensation. A project may not be undertaken, even if efficient, because of informational problems.

Formulation of a Bayesian Mechanism

The formulation is formally close to Roger Myerson's optimal auctions model, as presented in a famous paper "Optimal Auction Design", in *Mathematics of Operations Research* (1981).

A firm must decide to construct (or not) a plant. There are n individuals residing on the site, indexed by $i = 1, \dots, n$ (as usual).
The benefit of the plant is $R > 0$. This is the incremental profit over the next best alternative.
Let θ_i denote the loss to person i (private information).
Residents and the firm are risk neutral.
Each individual i is drawn from the same probability distribution with c.d.f. $F(\theta_i)$ and all types θ_i are independent. F is common knowledge (Harsanyi's doctrine).
The density of F is $f(\theta_i)$ (F is continuously differentiable).
The support of F is a non-degenerate interval $D = [\underline{\theta}, \bar{\theta}]$ with $\underline{\theta} \geq 0$.
The utility of agent i is $u_i = c_i - \theta_i$ where c_i is compensation (money).
The firm's profit (after settlement of pollution claims) is $R - \sum_i c_i$.

Notation:

$$\begin{aligned} \theta &= (\theta_1, ..., \theta_n), \\ \theta_{-i} &= (\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_n), \\ (\theta \mid t_i) &= (\theta_1, ..., \theta_{i-1}, t_i, \theta_{i+1}, ..., \theta_n). \end{aligned}$$

First-Best Decision

Operate the plant if and only if

$$\sum_{i=1}^{n} \theta_i \leq R.$$

The first-best surplus is defined as follows:

$$S(\theta) = Max\Big\{0, R - \sum_{i=1}^{n} \theta_i\Big\}.$$

The firm designs a mechanism to maximize expected profits under a number of constraints. First, the firm chooses a decision rule $p(\theta)$ with $0 \leq p(.) \leq 1$, where p is probability of choosing to operate the plant. Randomization is admissible. And $c_i(\theta)$ is the compensation to person i. We denote,

$$c(\theta) = (c_1(\theta), c_2(\theta), ..., c_n(\theta))$$

Definition: A *mechanism* is an array of functions $(p(.), c(.))$.

Individuals' strategies in the game are *reports*. Agent $i's$ report is denoted $\hat{\theta}_i \in D$. We denote a profile of reports as follows: $\hat{\theta} = (\hat{\theta}_1, ..., \hat{\theta}_n)$. In fact, p and c are functions of $\hat{\theta}$.

The firm moves first and proposes $(p, c) : D^n \longrightarrow [0, 1] \times \mathbb{R}^n$. Individual i privately knows θ_i and all agents choose reports simultaneously and non-cooperatively.

Constraints on the Mechanism: Incentive Compatibility, and Individual Rationality

We apply the Revelation Principle. We require $\hat{\theta}_i = \theta_i$ in equilibrium. Revelation must be a Bayesian-Nash equilibrium of the game.

The expected utility of agent i is defined as follows

$$U_i(\theta_i; p, c) = \mathbb{E}_\theta[c_i(\theta) - p(\theta)\theta_i \mid \theta_i].$$

This can be rewritten,

$$U_i(\theta_i) = \int_{D^{n-1}} [c_i(\theta) - p(\theta)\theta_i] f_{-i}(\theta_{-i}) d\theta_{-i}$$

where

$$f_{-i}(\theta_{-i}) = \prod_{j \neq i} f(\theta_j),$$

and to simplify notation, we denote,

$$f_{-i}(\theta_{-i}) d\theta_{-i} = \prod_{j \neq i} f(\theta_j) \prod_{j \neq i} d\theta_j.$$

We will also use the shorthand notation

$$f(\theta) d\theta = \prod_j f(\theta_j) \prod_j d\theta_j.$$

Definition 1: The Incentive Compatibility IC constraint (or revelation constraint) is by definition,

$$U_i(\theta_i) \geq \int_{D^{n-1}} [c_i(\theta \mid \hat{\theta}_i) - p(\theta \mid \hat{\theta}_i)\theta_i] f_{-i}(\theta_{-i}) d\theta_{-i},$$

for all i and *all* $\hat{\theta}_i \in D$.

Definition 2: Participation or *Individual Rationality* Constraints, denoted IR

We require

$$U_i(\theta_i) \geq 0$$

for all $\theta_i \in D$ and all $i = 1, ..., n$.

In the status quo ante: no plant is operating, and each resident enjoys “clean air”, therefore, each agent’s utility is zero. Through IR, each resident i is granted *veto power*.

Definition 3: A *feasible* mechanism satisfies IC and IR.

Analysis

Profit maximization problem

$$Max_{(p,c)} \int_{D^n} [p(\theta)R - \sum_{i=1}^{n} c_i(\theta)] f(\theta) d\theta.$$

(this is the expected profit, net of compensatory payments) subject to constraints IC and IR.

Definition (Interim expectations)

Define $C_i(t_i) = \mathbb{E}[c_i \mid t_i]$ and $Q_i(t_i) = \mathbb{E}[p \mid t_i]$, and remark that,

$$C_i(\hat{\theta}_i) = \int_{D^{n-1}} c_i(\theta \mid \hat{\theta}_i) f_{-i}(\theta_{-i}) d\theta_{-i};$$

and

$$Q_i(\hat{\theta}_i) = \int_{D^{n-1}} p(\theta \mid \hat{\theta}_i) f_{-i}(\theta_{-i}) d\theta_{-i}.$$

Lemma 1

The direct mechanism (p, c) is feasible *if and only if*,

(a) $0 \leq p(\theta) \leq 1$;

(b) $U_i(\bar{\theta}) \geq 0$ for all i;

(c) $Q_i(\hat{\theta}_i)$ is monotonically decreasing;

(d) $U_i(\theta_i) = U_i(\bar{\theta}_i) + \int_{\theta_i}^{\bar{\theta}_i} Q_i(t_i)dt_i$

Sketch of Proof: Using the interim expectations defined above, we have,

$$U_i(\theta_i) = C_i(\theta_i) - \theta_i Q(\theta_i)$$

IC implies

$$U_i(\theta_i) \geq C_i(\hat{\theta}_i) - \theta_i Q_i(\hat{\theta}_i)$$

or

$$U_i(\theta_i) \geq U_i(\hat{\theta}_i) - (\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i). \tag{1}$$

for all $(\hat{\theta}_i, \theta_i)$. And similarly, $\hat{\theta}_i$ cannot be better off mimicking θ_i, that is,

$$U_i(\hat{\theta}_i) \geq U_i(\theta_i) - (\hat{\theta}_i - \theta_i)Q_i(\theta_i) \tag{2}$$

Expressions (1) and (2) imply

$$(\theta_i - \hat{\theta}_i)Q_i(\hat{\theta}_i) \geq U_i(\hat{\theta}_i) - U_i(\theta_i) \geq (\theta_i - \hat{\theta}_i)Q_i(\theta_i).$$

We see that Q_i is a *monotonic* nonincreasing function: if $\theta_i > \hat{\theta}_i$, the above string of inequalities implies $Q_i(\hat{\theta}_i) \geq Q_i(\theta_i)$. A monotonic function defined on an interval has at most a countable number of discontinuities and it is integrable in the sense of Riemann (well-known Theorems). We will exploit this property. The monotonicity property is true for any $(\theta_i, \hat{\theta}_i)$ such that $\theta_i > \hat{\theta}_i$. Consider then a subdivision of the interval $[\theta_i, \bar{\theta}]$, that is, $(\theta_{i1}, \theta_{i2}, ..., \theta_{in})$ with $\theta_{i1} = \theta_i$ and $\theta_{in} = \bar{\theta}$, and the length of the steps is $\delta = \theta_{ik} - \theta_{i,k-1} > 0$

for all $k = 2, ..., n$. The number δ depends on n and $\delta = \delta(n)$ goes to 0 when n becomes arbitrarily large. We have

$$Q_i(\theta_{ik})\delta \geq U_i(\theta_{ik}) - U_i(\theta_{ik} + \delta) \geq Q_i(\theta_{ik} + \delta)\delta.$$

Then we can sum over k. This yields,

$$\sum_{k=1}^{n} Q_i(\theta_{ik})\delta \geq U_i(\theta_i) - U_i(\bar{\theta}) \geq \sum_{k=1}^{n} Q_i(\theta_{ik} + \delta)\delta.$$

Since Q_i is Riemann-integrable, the Darboux sums converge towards the (same) integral of Q_i,

$$\lim_{n\to\infty} \sum_{k=1}^{n} Q_i(\theta_{ik})\delta = \lim_{n\to\infty} \sum_{k=1}^{n} Q_i(\theta_{ik} + \delta)\delta = \int_{\theta_i}^{\bar{\theta}} Q_i(t_i)dt_i.$$

As a consequence, we obtain,

$$U_i(\theta_i) = U_i(\bar{\theta}) + \int_{\theta_i}^{\bar{\theta}} Q_i(t_i)dt_i.$$

Finally, since $U_i(\theta_i)$ is decreasing, the IR constraints can be rewritten,

$$U_i(\bar{\theta}) \geq 0$$

for all i. This ends the of sketch of proof. To finish the proof we need to prove that (c) and (d) imply IC (this is true).

Q.E.D.

Remark:

If there is no discontinuity in the neighborhood of point θ, we have

$$\frac{dU_i(\theta_i)}{d\theta_i} = -Q_i(\theta_i),$$

and of course $Q_i \geq 0$.

We now prove another key result, showing that the best IC mechanism maximizes a particular expression of the expected profit.

Simplification of the firm's problem

Lemma 2:

If $p : D^n \to [0,1]$ maximizes

$$\int_{D_n} p(\theta) \left\{ R - \sum_{i=1}^{n} \left[\theta_i + \frac{F(\theta_i)}{f(\theta_i)} \right] \right\} f(\theta) d\theta - \sum_{i=1}^{n} U_i(\bar{\theta}),$$

subject to Q_i being monotonically decreasing, and if

$$c_i(\theta) = p(\theta)\theta_i + \int_{\theta_i}^{\bar{\theta}} p(\theta_{-i} \mid t_i) dt_i,$$

then (p, c) maximizes the firm's expected profit.

Sketch of proof: The objective of the firm is

$$\pi = \int_{D^n} \Big[p(\theta)R - \sum_{i=1}^{n} c_i(\theta) \Big] f(\theta) d\theta = \int_{D^n} p(\theta) R f(\theta) d\theta - \sum_{i=1}^{n} \int_{\underline{\theta}}^{\bar{\theta}} C_i(\theta_i) f(\theta_i) d\theta_i.$$

Define, $\bar{R} = R \int_{D^n} p(\theta) f(\theta) d\theta$. Now, since

$$U_i(\theta_i) = C_i(\theta_i) - \theta_i Q_i(\theta_i), \qquad \text{and} \quad U_i(\theta_i) = U_i(\bar{\theta}_i) + \int_{\theta_i}^{\bar{\theta}_i} Q_i(t_i) dt_i,$$

we have,

$$\begin{aligned} \int_{\underline{\theta}}^{\bar{\theta}} C_i(\theta_i) f(\theta_i) d\theta_i &= \int_{\underline{\theta}}^{\bar{\theta}} [U_i(\theta_i) + \theta_i Q_i(\theta_i)] f(\theta_i) d\theta_i \\ &= \int_{\underline{\theta}}^{\bar{\theta}} \Big[\int_{\theta_i}^{\bar{\theta}} Q_i(t_i) dt_i + \theta_i Q_i(\theta_i) \Big] f(\theta_i) d\theta_i - U_i(\bar{\theta}). \end{aligned}$$

Now, integrating by parts, we derive

$$\int_{\underline{\theta}}^{\bar{\theta}} \Big[\int_{\theta_i}^{\bar{\theta}} Q_i(t) dt \Big] f(\theta_i) d\theta_i = \Big[\int_{\theta_i}^{\bar{\theta}} Q_i(t) dt . F(\theta_i) \Big]_{\underline{\theta}}^{\bar{\theta}} + \int_{\underline{\theta}}^{\bar{\theta}} Q_i(\theta_i) F(\theta_i) d\theta_i = \int_{\underline{\theta}}^{\bar{\theta}} Q_i(\theta_i) F(\theta_i) d\theta_i.$$

So that, using $f(\theta_i) > 0$,

$$\int_{\underline{\theta}}^{\bar{\theta}} C_i(\theta_i) f(\theta_i) d\theta_i = \int_{\underline{\theta}}^{\bar{\theta}} Q_i(\theta_i) \Big[\frac{F(\theta_i)}{f(\theta_i)} + \theta_i \Big] f(\theta_i) d\theta_i - U_i(\bar{\theta}).$$

The expected profit is

$$\pi = \bar{R} - \sum_{i=1}^{n} \int_{\underline{\theta}}^{\bar{\theta}} \Big[Q_i(\theta_i) \Big(\frac{F(\theta_i)}{f(\theta_i)} + \theta_i \Big) \Big] f(\theta_i) d\theta_i - \sum_{i} U_i(\bar{\theta}).$$

Using the above result,

$$\begin{aligned}\pi &= \bar{R} - \int_{\underline{\theta}}^{\bar{\theta}} \int_{D^{n-1}} \Big[\sum_{i=1}^{n} p(\theta)\Big(\frac{F(\theta_i)}{f(\theta_i)} + \theta_i\Big)\Big] f_{-i}(\theta_{-i}) d\theta_{-i} f(\theta_i) d\theta_i - \sum_i U_i(\bar{\theta}) \\ &= \bar{R} - \int_{D^n} p(\theta)\Big[\sum_{i=1}^{n}\Big(\frac{F(\theta_i)}{f(\theta_i)} + \theta_i\Big)\Big] f(\theta) d\theta - \sum_i U_i(\bar{\theta}).\end{aligned}$$

Finally, if

$$c_i(\theta) = p(\theta)\theta_i + \int_{\theta_i}^{\bar{\theta}} p(\theta_{-i} \mid t_i) dt_i,$$

then, taking expectations, and using the fact that utility satisfies IC, we find,

$$\begin{aligned}C_i(\theta_i) &= Q_i(\theta_i)\theta_i + \int_{\theta_i}^{\bar{\theta}} \int_{D^{n-1}} p(\theta_{-i} \mid t_i) dt_i f_{-i}(\theta_{-i}) d\theta_{-i} \\ &= Q_i(\theta_i)\theta_i + \int_{\theta_i}^{\bar{\theta}} Q_i(t_i) dt_i \\ &= Q_i(\theta_i)\theta_i + U_i(\theta_i) - U_i(\bar{\theta}),\end{aligned}$$

This shows that $U_i(\theta_i) - U_i(\bar{\theta}) = C_i(\theta_i) - Q_i(\theta_i)\theta_i$, but optimality requires $U_i(\bar{\theta}) = 0$ for all i.

This ends the sketch of proof.

Q.E.D.

Definition: Agent i's *virtual valuation* is by definition

$$\theta_i + \frac{F(\theta_i)}{f(\theta_i)}.$$

The total *virtual cost* is defined as

$$\sum_{i=1}^{n}\Big(\theta_i + \frac{F(\theta_i)}{f(\theta_i)}\Big).$$

The optimal Bayesian mechanism

Assumption A1:

$(\theta + F(\theta)/f(\theta))$ is a monotonically increasing function.

Assumption A1 is satisfied by many usual distributions.

Assumption A1 ensures that Q_i is monotonic.

We can state the main theorem.

Theorem 3

Under A1 a profit maximizing mechanism is of the form

$$p^*(\theta) = \begin{cases} 1 & \text{if} \quad \sum_{i=1}^{n} \Big(\theta_i + \dfrac{F(\theta_i)}{f(\theta_i)}\Big) \leq R \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_i^*(\theta) = p^*(\theta)\theta_i + \int_{\theta_i}^{\bar{\theta}} p^*(\theta \mid t_i) dt_i$$

for all i, all $\theta_i \in D$.

Sketch of the Proof: $p^* \in \{0,1\}$ and p^* is monotonically decreasing in each argument θ_i since $x + (F(x)/f(x))$ is increasing. Q_i is decreasing too. It is easy to check that $U_i(\bar{\theta}) = 0$ with the compensation functions c_i^*. It follows that $\sum_i U_i(\bar{\theta})$ is minimized. Finally, p^* maximizes the expression under the integral sign $\int_{D^n}$, subject to $0 \leq p \leq 1$.
Q.E.D.

Remark: The production decision is not random, *i.e.*, $p^* \in \{0,1\}$.

Interpretation

Define the virtual social loss function

$$h(\theta) = \sum_{i=1}^{n} \left(\theta_i + \frac{F(\theta_i)}{f(\theta_i)} \right).$$

The optimal mechanism is of the form:

(a) operate plant if and only if $h(\theta) \leq R$;

(b) if factory operates, pay the i-th resident the maximal amount $\tilde{\theta}_i$ for which

$$h(\theta \mid \tilde{\theta}_i) \leq R;$$

(c) pay zero if $p^* = 0$.

Note: if $p^*(\theta) = 1$, then

$$\begin{aligned} C_i^*(\theta) &= \theta_i + \int_{\theta_i}^{\bar{\theta}} p^*(\theta \mid t_i) dt_i \\ &= \theta_i + \int_{\theta_i}^{\tilde{\theta}_i} 1 dt + \int_{\tilde{\theta}_i}^{\bar{\theta}_i} 0 dt \\ &= \tilde{\theta}_i \qquad \text{where} \qquad h(\theta \mid \tilde{\theta}_i) = R \end{aligned}$$

Note: $p^*(\theta)$ is Pareto inefficient since $\theta_i + \frac{F(\theta_i)}{f(\theta_i)} > \theta_i$ if $\theta_i > \underline{\theta}$. So there are cases in which $p^*(\theta) = 0$ but $p = 1$ would have been optimal.

Asymptotic Properties of the Mechanism

Rob's Result: Welfare losses are maximal in large economies.

To see this, consider a sequence of economies indexed by n, the number of residents. Assume $R_n = rn$ (per-person profit is a constant r).

Define $z_i = F(\theta_i)/f(\theta_i)$ for every $\theta_i \in D$.

Let $\mu = E(\theta_i)$; $\lambda = E(z_i) = \int_{\underline{\theta}}^{\bar{\theta}} F(\theta) d\theta$; $\sigma^2 = Var(\theta_i)$; $\tau^2 = Var(\theta_i + z_i)$.

We define a welfare performance measure W_1, as follows,

$$W_n^1 = \Pr\left[\sum_{i=1}^{n}(\theta_i + z_i) \le R_n \,\middle|\, \sum_{i=1}^{n}\theta_i \le R_n\right].$$

Assume that,

Assumption A2:

$$\underline{\theta} < r < \bar{\theta}.$$

Assumption A3:

$$\frac{\mu - \theta}{\sigma} < \frac{\bar{\theta} - \theta}{\tau}.$$

Note: A2 is the interesting case. A3 is a technical assumption, satisfied, for instance, if $F(\theta_i) = \theta_i^{\alpha_i}$, $\theta_i \in [0, 1]$ and $\alpha_i > 0$.

We can state the following result.

Theorem 4 (Rob (1989))

Under Assumptions A2 and A3,

$$\lim_{n\to\infty} W_n^1 = 0.$$

The probability of an ex-post efficient decision goes to zero!

Consider now a second performance measure: the ratio of realized to potential welfare, that is,

$$W_n^2 = \frac{\pi_n^* + \sum_{i=1}^{n} U_i^*(n)}{V(n)}.$$

where by definition,

$$V(n) = \int_{\{\sum_i \theta_i \le R_n\}} (R_n - \sum_i \theta_i) f(\theta) d\theta,$$

and π_n^* is the firms's profit.

Remark: $V(n)$ is the expected first-best surplus and $\pi_n^* + \sum_{i=1}^{n} U_i^*(n)$ is the second-best expected surplus, achieved by (p^*, c^*).

Theorem 5 (Rob (1989))

If there exists $\beta > 0$ such that

$$\frac{F(\theta_i)}{f(\theta_i)} \geq \beta(\theta_i - \underline{\theta}),$$

when $\underline{\theta} \leq \theta_i \leq \bar{\theta}$, and if

$$\underline{\theta} < r < \mu + \beta(\mu - \underline{\theta}),$$

then,

$$\lim_n W_n^2 = 0.$$

Note: The assumption on F/f is weak since it is satisfied by many usual distributions.

However, we need $\underline{\theta} < r < E(\theta)(1+\beta) - \beta\underline{\theta}$. For instance, the technical assumptions of Theorem 5 are satisfied if $F(\theta) = \theta^\alpha$ for $\theta \in D = [0, 1]$.

In this example, we obtain,

$$\frac{F(\theta)}{f(\theta)} = \frac{\theta^\alpha}{\alpha\theta^{\alpha-1}} = \frac{\theta}{\alpha},$$

and we can choose $\beta = 1/\alpha$ since $\underline{\theta} = 0$.

Furthermore,

$$\mu = \mathbb{E}(\theta_i) = \int_0^1 \alpha\theta\theta^{\alpha-1}d\theta = \alpha\left[\frac{\theta^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{\alpha}{\alpha+1}.$$

and

$$\mu + \beta(\mu - \theta) = (1+\beta)\mu = \frac{\alpha+1}{\alpha}.\frac{\alpha}{\alpha+1} = 1.$$

So the assumption boils down to $\underline{\theta} < r < 1$ or $0 < r < 1$, and it follows that rents dissipate for all values of r in the relevant range!

Exercises

Exercise 1.

There are n agents, a finite set of collective decisions X, and each agent i has strict preferences denoted P_i on X. All the transitive, complete and strict preferences on X are admissible: the set of preference profiles is denoted $\mathcal{P}^n$. A mechanism G is a mapping $G : \mathcal{P}^n \to X$, choosing a decision in X as a function of the preference profile P. Mechanism G satisfies the *Pareto criterion* if for any profile P, and any x, y in X, if xP_iy for all i then $G(P) \neq y$. We say that G is *strongly nondictatorial* if no agent i exists such that, for all P, we have $G(P)P_iy$ for all $y \neq G(P)$, $y \in X$.

Muller and Satterthwaite (1985) state the following version of a famous result:

Theorem (Gibbard-Satterthwaite). If X has more than 3 alternatives and preferences are strict, but unrestricted, then, a mechanism G cannot simultaneously be strategy-proof and satisfy both the Pareto criterion and strong nondictatorship.

1. Give a formal definition of manipulation and of a strategy-proof mechanism G, using the above formalism.

2. Provide a proof of the theorem in the case in which there are only two agents, $n = 2$ and three alternatives, $X = \{x, y, z\}$. (This is known as Feldman's proof).

Exercise 2. (Varian (1994)) We study an application of Varian's externality compensation mechanism. The idea is to use the mechanism to regulate a duopoly. There are 3 agents: 1 consumer and 2 firms. Firm 1 chooses production x_1, Firm 2 chooses x_2. The consumer has the differentiable utility: $u(x_1, x_2) -$ expenditure. The cost functions are c_j, $j = 1, 2$.

1. The 3 payoffs are defined as follows.

$$\begin{aligned}
\pi_0 &= u(x_1, x_2) - p^1_{01}x_1 - p^2_{02}x_2, \\
\pi_1 &= p^0_{01}x_1 - c_1(x_1) - (p^1_{01} - p^0_{01})^2, \\
\pi_2 &= p^0_{02}x_2 - c_2(x_2) - (p^2_{02} - p^0_{02})^2.
\end{aligned}$$

The consumer sets the prices that the firms face; the firms set the prices that the consumer faces. More precisely, the price of Firm j, denoted p^0_{0j}, is chosen by the consumer (agent 0), the prices faced by the consumer are denoted p^j_{0j}, with $j = 1, 2$. Show that the Subgame-Perfect Equilibrium of Varian's two-stage mechanism is efficient in this model.

2. A variant of the same model. Each firm reports the price that the other firm should face. Therefore, the payoffs can be rewritten as follows:

$$\begin{aligned} \pi_0 &= u(x_1, x_2) - p^2_{01}x_1 - p^1_{02}x_2, \\ \pi_1 &= p^2_{01}x_1 - c_1(x_1), \\ \pi_2 &= p^1_{02}x_2 - c_2(x_2). \end{aligned}$$

Here, the consumer chooses x_1 and x_2 and each firm sets a price for the other firm's product. Show that the competitive allocation is the unique equilibrium of Varian's two-stage mechanism.

Exercise 3. (Rafael Rob (1989)) We consider the pollution-damages problem with asymmetric information studied above. Assume that the density of types θ_i, denoted $f(.)$, is uniform on $[0, 1]$, $n = 2$ and $R = 1$. Compute Rob's Bayesian mechanism (p^*, c^*) in this case and study its efficiency properties.

Solutions and Hints

Hint for Exercise 1.

Draw a table with the possible strict preferences of agent 1 as rows and the possible strict preferences of agent 2 as columns. Study the restrictions imposed on G by the Pareto criterion. Then look at manipulation possibilities. Eliminating the mechanisms that can be manipulated at some profile, we find a dictator. The full solution is described in E. Muller and M. Satterthwaite (1985), "Strategy-proofness: the existence of dominant-strategy mechanisms" in L. Hurwicz, D. Schmeidler and H. Sonnenschein, *Social Goals and Social Organizations*; Cambridge Univ. Press.

Hint for Exercise 2.

Not difficult. See H. Varian (1994).

Solution of Exercise 3.

Remark first that, with f uniform,

$$\theta_i + \frac{F(\theta_i)}{f(\theta_i)} = \theta_i + \frac{\theta_i}{1} = 2\theta_i$$

for all i.

The Mechanism (p^*, c^*) can be described as follows:

(a) Operate plant if and only if $\sum_{i=1}^{n} \theta_i \leq \frac{R}{2}$

(b) If $p^*(\theta) = 1$, pay $c_i = \tilde{\theta_i} = Min\left\{1, \frac{R}{2} - \sum_{j \neq i} \theta_j\right\}$.

If there are only $n = 2$ individuals, and if $R = 1$, we find

$$U_i(\theta_i) = \begin{cases} \frac{1}{2}(\frac{1}{2} - \theta_i)^2 & \text{if} \quad 0 \leq \theta_i < \frac{1}{2} \\ 0 \quad \text{if} \quad \frac{1}{2} \leq \theta_i \leq 1 \end{cases}$$

The average individual payoffs are given by the following expression:

$$U_i^* = \int_0^1 U_i(\theta_i) f_i(\theta_i) d\theta_i = \frac{1}{48}$$

for all $i = 1, 2$. The firm's expected profit is,

$$\pi^* = \int\int_{\{\theta_1 + \theta_2 \leq 1/2\}} [1 - 2(\theta_1 + \theta_2)] d\theta_1 d\theta_2 = \frac{1}{24}$$

Computation of expected utilities and profit

By IC_1,

$$U_1(\theta_1) = \int_{\theta_1}^{1} Q_1(t_1)dt_1$$
$$= \int_{\theta_1}^{1}\int_{0}^{1} p^*(t_1,\theta_2)d\theta_2 dt_1,$$

but,

$$\int_0^1 p^*(t_1,\theta_2)d\theta_2 = \begin{cases} \int_0^{\frac{R}{2}-t_1} 1.d\theta_2 & \text{if} \quad t_1 \leq \frac{R}{2} = \frac{1}{2} \\ \\ 0 & \text{if} \quad t_1 > \frac{R}{2} \end{cases}$$

Thus,

$$U_1(\theta_1) = \int_{\theta_1}^{1/2}\int_{0}^{1/2-t_1} 1.d\theta_2 dt_1$$
$$= \int_{\theta_1}^{1/2}\left(\frac{1}{2}-t_1\right)dt_1 = \left[\frac{t_1}{2}-\frac{t_1^2}{2}\right]_{\theta_1}^{1/2}$$
$$= \frac{1}{4}-\frac{1}{8}+\frac{\theta_1^2}{2}-\frac{\theta_1}{2} = \frac{1}{2}\left(\frac{1}{4}-\theta_1+\theta_1^2\right)$$
$$= \frac{1}{2}\left(\frac{1}{2}-\theta_1\right)^2 \qquad \text{if} \quad \theta_1 \leq 1/2.$$

and

$$U_1(\theta_1) = 0 \qquad \text{if} \quad \theta_1 > 1/2.$$

Then, we have,

$$U_i^* = \frac{1}{2}\int_0^{1/2}\left(\frac{1}{4}-\theta_1+\theta_1^2\right)d\theta_1$$
$$= \frac{1}{2}\left[\frac{\theta_1}{4}-\frac{\theta_1^2}{2}+\frac{\theta_1^3}{3}\right]_0^{1/2}$$
$$= \frac{1}{2}\left[\frac{1}{8}-\frac{1}{8}+\frac{1}{24}\right] = \frac{1}{48}.$$

and

$$\begin{aligned}
\pi^* &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-\theta_2} (1 - 2(\theta_1 + \theta_2)) d\theta_1 d\theta_2 \\
&= \int_0^{\frac{1}{2}} \Big[(1 - 2\theta_2)\theta_1 - \theta_1^2 \Big]_0^{1/2-\theta_2} d\theta_2 \\
&= \int_0^{\frac{1}{2}} \left[2\left(\frac{1}{2} - \theta_2\right)^2 - \left(\frac{1}{2} - \theta_2\right)^2 \right] d\theta_2 \\
&= \int_0^{\frac{1}{2}} \left(\frac{1}{2} - \theta_2\right)^2 d\theta_2 = \left[\frac{\theta_2}{4} - \frac{\theta_2^2}{2} + \frac{\theta_2^3}{3} \right]_0^{1/2} \\
&= \frac{1}{8} - \frac{1}{8} + \frac{1}{24} = \frac{1}{24}.
\end{aligned}$$

The average realized surplus is

$$\pi^* + U_1^* + U_2^* = \frac{1}{48} + \frac{1}{48} + \frac{1}{24} = \frac{1}{12}.$$

50% of the rent goes to the firm (in expected value).

Consider now the potential surplus:

$$\begin{aligned}
V = E(S(\theta)) &= E\{Max\{0, R - \Sigma_i \theta_i\}\} \\
&= E[Max(0, 1 - \theta_1 - \theta_2)] \\
&= \text{expected first-best surplus.}
\end{aligned}$$

By definition, we have

$$V = \int_A (R - \Sigma_i \theta_i) f(\theta) d\theta,$$

where $A = \{\theta \mid \sum \theta_i \leq R\}$.

If $R = 1$ and $n = 2$,

$$V = \int_{2\underline{\theta}}^{R} (R - s) g(s) ds = \int_0^1 (1 - s) g(s) ds$$

where $g(s)$ is the density of the sum $\theta_1 + \theta_2$.

But

$$g(s) = \begin{cases} s & \text{if} \quad 0 < s < 1 \\ 2 - s & \text{if} \quad 1 < s < 2. \end{cases}$$

This density is triangular.

Note: Computation of the distribution of $\theta_1 + \theta_2 = s$ if θ_i is uniform on $[0, 1]$ and *i.i.d.* Compute

$$\int_0^1 \int_0^1 (x+y) f(x) f(y) dx dy.$$

With the change of variable $s = x + y$, we derive,

$$\int_0^1 \int_0^2 s f(s-y) f(y) ds dy = \int_0^2 s \int_0^1 f(s-y) f(y) dy ds.$$

So, the density of s, that is, $g(s)$, can be expressed as follows,

$$g(s) = \int_0^1 f(s-y) f(y) dy = \begin{cases} \int_0^s 1.dy = s & \text{if} \quad 0 \leq s < 1 \\ \int_{s-1}^1 1.dy = 1 - s + 1 = 2 - s & \text{if} \quad 1 \leq s \leq 2 \end{cases}$$

using the fact that f is uniform on $[0, 1]$.

So, we obtain the following value of the expected first best surplus:

$$\begin{aligned} V &= \int_0^1 (1-s) g(s) ds \\ &= \int_0^1 (1-s) s ds = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \end{aligned}$$

We obtained $V = \frac{1}{6}$ but the realized surplus

$$\pi^* + U_1^* + U_2^* = \frac{1}{12}.$$

Conclusion: 50% of the potential surplus is realized (and there are only two agents): *the performance is not good !*

The second-best optimal decision rule p^* leads to inefficient outcomes, in this example, when

$$\frac{R}{2} < \sum_i \theta_i < R.$$

In such instances, it would be in the best interest of all parties to deviate from (p^*, c^*), arrange compensatory payments and operate the plant. But these deviations are incompatible with incentives.

Note: This negative Result depends very much on the fact that we imposed *ex-interim IR constraints*, *i.e.*, $U_i(\theta_i) \geq 0$ for all $\theta_i \in D$.