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Lecture Notes 1

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Chapter I

Social Choice and Justice

Summary: We start with elements of a theory of local justice. The term has been coined by Jon Elster (1993), but a book by Hervé Moulin (2003) *Fair Division and Collective Welfare* and the work of H. Peyton Young (1987) will be our main sources of inspiration for the first part of this chapter. In the second part, we present Arrow's classical Social Choice framework in a simple way, and state the famous impossibility Theorem. Then, we discuss the comparability of utilities (the inspiration is Amartya Sen here); the foundations of utilitarianism (presenting John Harsanyi's well-known approach); classical axiomatizations of Utilitarianism and Rawlsian egalitarianism (in fact the Leximin) following Claude d'Aspremont and Louis Gevers. A few things on voting at the end and finally, some exercises that I picked here and there. I'll try to enrich this list as time passes.

Local Justice

Local justice describes procedures to allocate scarce goods or cost sharing problems in local problems: for instance, inheritance, bankruptcy, queuing, scheduling, collective organization etc. with some fairness properties. The theory can be based on intuitive perceptions of justice. The intuition of what is fair or just has been discussed by philosophers. Moulin (2003) and Sen (2009) in his book the *Idea of Justice* (and many philosophers before them) discussed a version of the following parable of the flute.

Parable of the flute

A father dies and has four children: Ali, Bernard, Christian and David. The father was the owner of a flute. Who should inherit the flute?

Compensation principle. Ali has fewer toys than the other children: he should get the flute as a compensation.

Reward principle. Bernard worked hard to fix and clean the flute: he should get it as a reward.

Exogenous rights. Christian is the elder son and is therefore entitled to the inheritance (he has a right).

Utility or Fitness principle. David is a good flute player, so the flute must be given to him because all enjoy music and he's the best musician (hence total utility is maximized).

Many discussions about fairness involve one or a combination of these principles. To formalize these ideas we need a simple abstract model.

A model of fair distribution

The model supports many different interpretations. Firstly, there is set of n individuals $N = \{1, 2, ..., n\}$. Individuals are indexed by i or j. There is a resource or a good that must be distributed. The outcome (share) of individual i is denoted y_i (a real number). There is a budget constraint and $\Sigma_{i=1}^{n} y_i = \omega$, where ω is the total fixed amount of the resource or good. Then we rely on a notion of utility. The utility of individual i is a real-valued function u_i. For instance, y_i is food and $u_i(y_i)$ is "need" or a degree of starvation or nourishment. For instance, utility u_i could be the probability of death of i in the next 6 months. It may take more food to bring agent j to the same level of utility as i, so that $u_i(y_i) = u_j(y_j)$ implies $y_j > y_i$ (equality of utilities and equality of outcomes may be very different things). Compensation cannot be defined rigorously without a choice of the notion of utility and of the utility functions. To apply and interpret the utility/fitness principle we also need a precise definition of u_i. The utilitarian principle chooses $y = (y_j)_{j=1,...n}$ so as to maximize $\Sigma_i u_i(y_i)$ subject to constraints (we'll come back to this below of course). This type of maximization is not the same thing as equalizing utilities of course.

The next important element in this model is the notion of *claim*. Each individual i has a claim denoted x_i. Again several interpretations are possible. A claim may be a right to a share of the resource. For instance, individual i is a creditor and holds a debt of value x_i in a bankruptcy procedure, but the liquidation value of the firm is $\omega < \Sigma_i x_i$. The claim x_i can be the cost incurred by i in a joint production process. The claim x_i can also be the benefit that i derived from a public project, or the *peak* of i (*i.e.*, the value of y maximizing u_i if it exists).

We have a deficit or rationing problem if $\Sigma_i x_i > \omega$ and an excess (or surplus) problem if $\Sigma_i x_i < \omega$. These situations correspond to two important categories of problems.

In a *surplus sharing problem*, the cooperation of agents in N yields a return ω; agent i's opportunity cost is x_i and cooperation brings a surplus since $\Sigma_i x_i < \omega$. We need a rule $y = (y_1, ..., y_n)$ to share the surplus $S = \omega - \Sigma_i x_i$, so that $\Sigma_i y_i = \omega$.

In a *cost-sharing problem*, a collective project has a cost ω and brings a benefit x_i to each i. The project should be undertaken since $\Sigma_i x_i \geq \omega$. A cost-sharing problem is thus formally equivalent to a deficit or rationing problem. Individual i's cost share is then y_i. We need a rule y to allocate the cost in such a way that $\Sigma_i y_i = \omega$.

Some important rules

(a) A first important mode of division is the *proportional solution*, according to which $y_i = \alpha x_i$ where α is such that $\Sigma_i y_i = \omega$. This yields,

$$y_i = \frac{x_i}{\Sigma_j x_j}\omega.$$

This solution ensures $x_i \geq y_i \geq 0$. We also clearly have

$$\frac{y_i}{x_i} = \frac{y_j}{x_j}$$

for all (i, j).

(b) Another important rule is the *equal surplus* method. This method adds (or removes) the same number μ to x_i to all individuals, that is, $\mu = x_i - y_i = x_j - y_j$ for all (i, j). This is equivalent (prove it) to giving x_i plus an equal share S/n of the surplus to each individual i, that is,

$$y_i = x_i + \frac{1}{n}\left(\omega - \Sigma_j x_j\right).$$

When $S < 0$, a problem comes from the fact that we may have $y_i < 0$ if x_i is small or that i may be subsidized by the other agents if $x_i < (1/n)(\Sigma_j x_j - \omega)$ in a cost-sharing problem.

To make sure that $y_i \geq 0$ for all i when $S < 0$, we should reformulate the equal surplus method as follows. Define first a *common loss* μ and redefine

$$y_i = max\{x_i - \mu, 0\}$$

with μ chosen to satisfy the budget constraint $\Sigma_j max\{x_j - \mu, 0\} = \omega$. Remark that we can rewrite $max\{x_j - \mu, 0\}$ as $y_i = -\mu + max\{x_i, \mu\}$, so that, each individual i first pays the same tax μ and then gets the maximum of x_i and μ to avoid negative values of y_i. The equal tax condition becomes:

$$\text{for all } i \text{ such that} \quad x_i > \mu, \text{we have} \quad x_i - y_i = x_i - max\{x_i - \mu, 0\} = \mu$$

and if $x_i < \mu$, then $y_i = 0$. Moulin (2003) calls this the *uniform losses* method.

(c) Finally, there is another egalitarian rule, called the *uniform gains* solution. The basic idea is that with uniform gains, everybody ends up with the same allocation $y_i = \lambda$ if this does not violate the important rule that individuals should get at least their claim $y_i \geq x_i$ in case of surplus ($S > 0$) and the constraint that they cannot be subsidized, $y_i \leq x_i$, in the deficit case ($S < 0$). So we define the rule as follows

$$y_i = \begin{cases} max\{\lambda, x_i\} & \text{where } \Sigma_j max\{\lambda, x_i\} = \omega \text{ if } S > 0. \\ min\{\lambda, x_i\} & \text{where } \Sigma_j min\{\lambda, x_j\} = \omega \text{ if } S < 0. \end{cases}$$

With this rule, the individual payoffs y_i are fully equalized for large values of ω or small values of ω, and $y_i = \lambda = \omega/n$ for all i.

The three rules described above have been characterized axiomatically in various ways, for instance by Hervé Moulin, Yves Sprumont and Peyton Young. We will study a bit of that below. We now consider the utilitarian principle.

The Utilitarian Principle

This principle is very easy to define but its real meaning is more difficult to understand. Most (French) students believe that utilitarians are completely indifferent to inequality. I will show that this is a grossly mistaken interpretation. If you want to criticize Utilitarianism, you need more sophisticated or more subtle arguments (see the writings of Rawls, Sen, Roemer, etc.). The definition : an allocation y is utilitarian if it maximizes $\Sigma_i u_i(y_i)$ subject to budget and other constraints. Let us study this maximization problem in details (this is a good exercise). Our goal is to find the allocation of resources y that maximizes $\sum_i u_i(y_i)$ subject to the budget constraint, $\sum_i y_i \leq \omega$ and the sign constraints $y_i \geq 0$. To write the first-order optimality conditions, we introduce Lagrange multipliers: λ for the budget constraint and α_i for each sign constraint. We assume that u_i is strictly increasing, concave and continuously differentiable for all i. Remark that since u_i is concave and the constraints are linear, we maximize a concave function on a convex set. It follows that the Kuhn-Tucker optimality conditions are necessary and sufficient (and constraints are also *qualified*). We form the Lagrangian expression $L(y, \lambda, \alpha) = \sum_i u_i(y_i) + \lambda(\omega - \sum_i y_i) + \sum_i \alpha_i y_i$, and we write the Kuhn-Tucker conditions. The main condition is obtained by setting the partial derivatives of L equal to zero, that is, $\partial L / \partial y_i = 0$. This yields, for all i,

$$u_i'(y_i) + \alpha_i = \lambda,$$

(where u_i' denotes the derivative of u_i) with complementary slackness and dual sign conditions,

$$\lambda\,(\omega - \Sigma_i y_i) = 0, \qquad \alpha_i y_i = 0 \quad \text{for all } i, \quad \lambda \geq 0, \quad \alpha_i \geq 0 \quad \text{for all } i,$$

and of course, we must have a feasible solution satisfying

$$\Sigma_i y_i \leq \omega \quad \text{and} \quad y_i \geq 0 \quad \text{for all } i.$$

Assume first that the optimal solution is interior, that is $x_i > 0$ for all i. This will be the case if the marginal utility u_i' becomes arbitrarily large when $y_i \to 0$. Then we have $u_i'(y_i) = \lambda$ for all i, and therefore, *marginal utilities are equalized*,

$$u_1'(y_1) = u_2'(y_2) = \cdots = u_n'(y_n).$$

In a well-known paper, entitled "Equality of what?" (1979), Amartya Sen writes that theories of justice often propose to equalize something (but not the same thing). This is the case here: Utilitarianism equalizes marginal utilities. This is not the same as equalizing utilities of course, except in some important special cases.

If there is a corner solution, that is, if the solution is such that $y_j = 0$ for all j in a subset of agents $J \subset N$, then we must have $u_j'(0) + \alpha_j = \lambda$ for all $j \in J$ and $u_i'(y_i) = \lambda$ when i is such that $y_i > 0$. Since multipliers are nonnegative, and since $u_i' > 0$ by assumption, we have $\lambda > 0$ in all cases, implying that the budget constraint is binding, $\omega = \Sigma_i y_i$ and we have $u_j'(0) \leq \lambda = u_i'(y_i)$ for all $j \in J$ and all $i \in N \backslash J$. There is a corner solution if $u_j'(0)$ is too low for some j. Since u_j is concave, these people with a small marginal utility must be the already well-fed, well-to-do people.

In fact, it is not difficult to understand that the Utilitarian will always want to equalize or redistribute to a certain extent. The intuition is that if an allocation y is such that $u_1'(y_1) > u_2'(y_2)$, if agent 2 gives a little bit of his(her) share to agent 1, the total utility will increase because $u_1'(y_1 + \epsilon)$ is greater than $u_2'(y_2 - \epsilon)$ for $\epsilon > 0$ small enough (the utility of agent 2 will decrease by a smaller amount than the increase in agent 1's utility). It is clear that the Utilitarian will then want to redistribute from rich to poor and will completely equalize incomes, if that is feasible, when utility functions are all the same and strictly concave. If $u_1(.) = \cdots = u_n(.) = u(.)$, then $u'(y_1) = \cdots = u'(y_n)$ implies $y_1 = \cdots = y_n$ (because u' is a decreasing function), that is, *full equality*.

There are cases in which the differences in marginal utilities can be explained by individual characteristics that have an ethical content. Assume for instance that $u_i(x) = \theta_i v(x)$, where v is a strictly increasing and strictly concave function, and $\theta_i > 0$ is a parameter representing, say, the observable intensity of need. Then, an interior utilitarian solution is such that $\theta_i v'(y_i) = \lambda$ for all i and we find that $\theta_i > \theta_j$ implies $y_i > y_j$. The goods are allocated to the agents that are producing the greatest utility. But that may be morally justified. The meaning is not necessarily that utilitarianism is treating individuals as simple utility-production machines.

Another example is obtained when $u_i(x) = v(z_i + x)$, and z_i is a given constant (possibly representing a pre-existing allocation of goods). Then an interior utilitarian solution satisfies $v'(z_i + y_i) = \lambda$ for all i and therefore, we have $z_i - z_j = y_j - y_i$ for all (i, j), *i.e.*, the utilitarian solution compensates inequalities in the allocation z.

We conclude that when utilities are concave, Utilitarianism embodies a form of *social aversion for inequality.* We will come back to foundations of utilitarianism below. We first briefly consider the equal sacrifice approach.

We now focus on deficit or cost-sharing cases. Define $T = -S = \Sigma_i x_i - \omega$. Redefine the claim x_i as being the pretax or taxable income of i, and y_i as the after-tax income. Denote the tax of i as $t_i = x_i - y_i$. Parameter ω is now reinterpreted as the total after-tax income and T is the total tax to be levied. The budget constraint can be rewritten $\Sigma_i t_i = T$.

A tax system $t = (t_1, ..., t_n)$ is defined as *progressive* if $x_i \leq x_j$ implies $t_i/x_i \leq t_j/x_j$ for any (i, j). The tax system is *regressive* if $x_i \leq x_j$ implies $t_i/x_i \geq t_j/x_j$. Remark that the proportional solution induces a tax system that is neither progressive nor regressive (it is non-regressive). To see this, remark that under the proportional solution, we have $y_i/x_i = y_j/x_j$ and therefore $(x_i - y_i)/x_i = (x_j - y_j)/x_j$ or $t_i/x_i = t_j/x_j$.

To define equal sacrifice, we need a notion of utility. Let u be the same concave utility function for everyone.

Formally, we say that the tax system t satisfies the *u-equal sacrifice* property if there exists a constant c such that for all i, we have

$$u(x_i) - u(x_i - t_i) = c$$

and of course we want $\Sigma_i t_i = T$.

The equal sacrifice principle is not new. It has been advocated by John Stuart Mill in his *Principles of Economics* (1848) and studied by Francis Edgeworth (1898) in his *Pure Theory of Taxation*, Gustav Cassel in *Economic Journal* (1901) and others. The modern formulation is due to Peyton Young (1987) (in several papers, see *J. Public Econ.* and *Math. Operations Research*).

Example 1. Take $u(x) = \ln(x)$. Then *u-equal sacrifice* implies $x_i/y_i = x_j/y_j$ for all (i, j) (check this as an exercise). This characterizes the proportional solution.

Example 2. If we choose $u(x) = x$, then *u-equal sacrifice* implies $x_i - y_i = x_j - y_j$ for all (i, j) (check this as an exercise). This characterizes the uniform losses solution.

Clearly, we can derive a family of u-equal sacrifice tax schedules, denoted $t(x)$. For each c, a schedule is given by the expression,

$$t(x) = x - u^{-1}[u(x) - c],$$

where u^{-1} denotes the inverse of u. More generally, we could consider an *allocation problem* (x, T), where $x = (x_1, \ldots, x_n)$ and a *taxation method* $t = F_n(x, T)$ (a vector function with values in $\mathbb{R}^n$, where $t = (t_1, \ldots, t_n)$), such that $\Sigma_i t_i = T$ and $0 \leq t_i \leq x_i$ and defined for any nonempty finite set of agents N. We require taxation methods to be *anonymous* in the sense that $x_i = x_j$ implies $t_i = t_j$ (they do not depend on the individual's label) and *continuous* with respect to (x, T).

In addition, Peyton Young (1987) defines a taxation method as *consistent* if the after-tax allocation is *invariant* when viewed by any subset of individuals, that is, for all nonempty subset $J \subset N$, we have

$$t = F_n(x, T) \Rightarrow t_J = F_{|J|}\left(x_J, \sum_{j \in J} t_j\right),$$

where t_J and x_J are subvectors of x and resp. t with coordinates $j \in J$. When agents in a smaller economy J must pay the total tax amount $\Sigma_J t_j$ and keep the same claims x_j, the taxation method should yield the same individual taxes t_j as in the larger society N.

We now provide some examples of *parametric methods*.

(a) The *flat tax* is in fact the *proportional solution* presented above; it is given by the formula,

$$t_i = \gamma x_i,$$

where γ is set so that $\Sigma_i t_i = T$.

(b) The *head tax* is in fact the *equal surplus/uniform losses* solution presented above (check this), with formula

$$t_i = \min\{\gamma, x_i\},$$

where $\gamma \geq 0$ is determined by $\Sigma_i t_i = T$.

(c) The *levelling tax*, which is the taxation form of the *uniform gains* solution (check this),

$$t_i = \max\{x_i - 1/\gamma, 0\},$$

where $\gamma \geq 0$ is again determined by $\Sigma_i t_i = T$.

By definition, a general taxation method $t = F_n(x, T)$ is *parametric with representation* f if there exists a function f such that taxes are given by $t_i = f(x_i, \gamma)$ with γ in some interval $[a, b]$, and f monotone in γ, $f(x, a) = 0$, $f(x, b) = x$. The equal-sacrifice tax schedules defined above are parametric representations with the equal-sacrifice parameter c.

Every parametric method is anonymous and consistent (exercise: check this), and the converse is true, as stated by the following Theorem.

Theorem 1 (Peyton Young (1987)). A continuous taxation method $F_n(x, T)$ is anonymous and consistent *if and only if* it is parametric with some representation f.

Minimization of Total Sacrifice

Another interesting question is the possibility of representing the taxation method as the solution of a maximization problem. For instance we can define total u-sacrifice as follows,

$$s(x, t) = \sum_i (u(x_i) - u(x_i - t_i)).$$

Minimizing $s(x, t)$ with respect to t subject to the budget constraint is strictly equivalent to maximizing $\sum_i u(x_i - t_i)$ subject to $\sum_i t_i \geq T$. It immediately follows that total sacrifice minimization is utilitarian with common utility u. The first-order conditions for an interior solution $t > 0$ are $u'(x_i - t_i) = \lambda(x, T)$ for all i, where $\lambda(x, T)$ is a Lagrange multiplier that depends on all the problem parameters (x, T). The u-equal sacrifice (and utilitarian) tax schedule is $t(x, T) = x - (u')^{-1}(\lambda(x, T))$ and of course $t_i = t(x_i, T)$. Remark that the solution is extremely egalitarian, since the after-tax income $y(x, T) = x - t(x)$ is the same for everyone (assuming interiority). It is easy to check that the u-sacrifice solution that maximizes total sacrifice is the levelling tax $t_i = \max\{0, x_i - \mu\}$ with $\mu = (u')^{-1}(\lambda)$.

Exercise: (a) Show that minimizing $\Sigma_i t_i^2$ subject to $\Sigma_i t_i = T$ and $0 \leq t_i \leq x_i$ yields the head tax. (b) Show that minimizing $\Sigma_i (t_i^2 / x_i)$ under the same constraints yields the flat tax.

In fact we have a general property.

Theorem 2 (Peyton Young (1987)). A continuous, anonymous taxation method $t = F_n(x, T)$ is consistent *if and only if* t minimizes a symmetric, continuous, additively separable, strictly convex objective function of the form $\mathcal{H}(x, t) = \sum_i H(x_i, t_i)$.

Minimizing $\mathcal{H}$ is like maximizing $\mathcal{V} = -\mathcal{H}$ and therefore, all consistent taxation methods maximize a concave function $\mathcal{V}(x, t) = \sum_i V(x_i, t_i)$ (with $V = -H$) that looks like a Utilitarian objective. So you can safely conclude that the head tax, which is very regressive (everybody pays the same tax) and the levelling tax, which is very egalitarian (everybody gets the same after-tax income), are solutions of a particular kind of utilitarian program.

Equal Sacrifice, Relative Risk Aversion and Scale Invariance

Are there reasonable utility functions that could be used in practice? The answer is yes. When considering matters of distributive justice, the *relativity principle* states that the comparison of sacrifice at any two income levels x and x' depends only on the tax rates t/x and the relative income levels x_i/x_j. Some principles of experimental psychology seem to support the relativity principle as a reasonable model of human behavior. For instance, the perceived difference between two sources of noise would remain the same if the listener moved away uniformly from the noise sources. To represent the relativity principle we define the following *scale invariance* property. We say that u-equal sacrifice is *scale-invariant* if for all x, y and x', y' ,

$$u(x) - u(y) \geq u(x') - u(y') \Longleftrightarrow u(\theta x) - u(\theta y) \geq u(\theta x') - u(\theta y')$$

for all $\theta > 0$ (the ranking of sacrifices does not change if we change the scale of incomes). We have the following result.

Theorem 3 (Peyton Young (1987)). Let u be a continuous, non-decreasing utility function. Then, u-equal sacrifice is scale invariant *if and only if* $u(y)$ exhibits constant relative risk aversion; in other words, *if and only if* either $u(x) = a\ln(x) + b$ or $u(x) = (a/p)x^p + b$ with some $a > 0$, and $p \neq 0$.

Recall that the index of relative risk aversion is $-xu''(x)/u'(x)$. A consequence of Theorem 3 is that the scale-invariant, equal-sacrifice taxation methods are derived from positive linear transformations of the logarithmic utility or power functions. If we take the logarithm, we obtain the flat tax (as seen above), which is neither regressive nor progressive because $f(x)/x$ is a constant with the logarithm. If we take $u(x) = (1/p)x^p$ with $p < 0$, we find the family of schedules,

$$f(x) = x - [x^p + \gamma^p]^{1/p}.$$

This family is progressive if $p < 0$. Exercise, show this and check that $\gamma = (-cp)^{1/p}$. When $p > 0$ we find regressive and not always well-defined equal-sacrifice taxes (show this as an exercise). When $p < 0$ and $p \to 0$ the tax converges towards the flat tax. When $p \to -\infty$, the tax method converges towards the levelling tax (or uniform gains solution). Exercise: show this. In some countries and in some periods, the income-tax schedules look approximately like u-equal sacrifice methods with $0 > p > -1$ (see Young (1990)). By observing income-tax schedules, we can in principle estimate an index of (relative) "social aversion for inequality" $1 - p$, which is the coefficient of relative risk aversion of x^p/p, for a given period and society.

Aggregation of Preferences

We now turn to a much deeper and more general question: given a list of individual preference relations (that can be represented by utility functions), is there a way of aggregating these preferences to arrive at a collective preference ordering? This is Arrow's approach, in *Social Choice and Individual Values* (1951, 1963).

The population of individuals (citizens) is indexed by $i = 1, ..., n$. Let $N = \{1, ..., n\}$ be the set of individuals as above.

The set of collective decisions is X. Let $x \in X$ denote a decision. We assume that this set is finite: $|X| < +\infty$.

Each i is now endowed with a preference relation R_i.

xR_iy means that x is preferred or indifferent to y for individual i.

We define indifference as :

$$xI_iy \iff xR_iy \text{ and } yR_ix$$

Strict preference is denoted P_i. We define strict preference as follows,

$$xP_iy \Longleftrightarrow xR_iy \text{ and not } (yR_ix)$$

Remark : R_i is a subset of X^2.

Assumption 1 : R_i is complete and transitive.

Completeness : for all $(x, y) \in X^2$ either xR_iy or yR_ix.

Transitivity : for all $(x, y, z) \in X^3$ such that xR_iy and yR_iz, then xR_iz.

Utility functions

There are several definitions of utility.

The simplest is the *purely ordinal* utility.

Representation of R_i by u_i

$u_i : X \to \mathbb{R}$ represents R_i if *and only if*

$$u_i(x) \geq u_i(y) \Longleftrightarrow xR_iy$$

Preference profiles

A preference profile is a vector, $R = (R_1, R_2, ..., R_n)$.

The set of all complete and transitive preference orderings is denoted $\mathcal{R}$.

$$R \in \mathcal{R}^n$$

Let $\mathcal{P}$ denote the set of strict preferences.

Arrow's Impossibility Theorem

Domain : Let $\mathcal{D} \subseteq \mathcal{R}^n$ be a "domain" (a subset of the set of possible preference profiles).

Definition of a Social Choice Rule :

A mapping $F : \mathcal{D} \to \mathcal{R}$

$$(R_1, ..., R_n) \longmapsto F(R_1, ..., R_n)$$

and $F(R)$ is a "collective preference".

Remark : $F(R)$ is *complete* and *transitive* by definition.

Strict preference (sub-relation of $F(R)$*)*

We denote $x\varphi(R)y$ if and only if $xF(R)y$ and not $(yF(R)x)$.

Some Axioms :

Axiom 1 (Weak Pareto Principle: WP)

If $x, y \in X$ and xP_iy for all i, then x is strictly preferred to y by the collective preference : $x\varphi(R)y$.

Example : Borda's Rule (1781)

Give points to each $x \in X$.

Assume that $\mathcal{D} = \mathcal{P}^n$ (there are no ties). Points are denoted $c_i(x)$ and $c_i(x) = k$ if x has rank k in citizen i's preferences. Add up the points. Define the rule as follows,

$$xF(R)y \qquad \Longleftrightarrow \qquad \sum_{i=1}^{n} c_i(x) \leq \sum_{i=1}^{n} c_i(y)$$

Borda's Rule is complete and transitive : $F(R) \in \mathcal{R}$.

Borda's Rule satisfies Axiom 1 (the Weak Pareto Principle).

$xP_iy \Rightarrow c_i(x) < c_i(y)$ for all i and $\Sigma_i c_i(x) < \Sigma_i c_i(y) \Rightarrow xF(R)y$.

Independence of Irrelevant Alternatives

Axiom 2 (IIA)

Let R and R' be two profiles.

Let $x, y \in X$ be two decisions.

Suppose that for all $i \in N$, xR_iy precisely when xR'_iy, that is $xR_iy \Longleftrightarrow xR'_iy$.

Then, $xF(R)y \Longleftrightarrow xF(R')y$

If no individual changes his (her) ranking of x and y when we move from R to R', then, the collective ranking of x and y should not change. Collective choice on (x, y) is based on individual preferences on (x, y) only.

Remark: Borda's rule violates IIA

Example : $n = 2; X = \{x, y, z\}$; xP_1zP_1y and yP_2xP_2z. This implies :

	x	y	z
1	1	3	2
2	2	1	3
Total	3	4	5

Thus, $xF(R)yF(R)z$.

Consider now a new profile $R' = (R'_1, R'_2)$, such that $xP'_1yP'_1z$ and $yP'_2zP'_2x$.

This yields the account :

	x	y	z
1	1	2	3
2	3	1	2
Total	4	3	5

and therefore, $yF(R')xF(R')z$

Preferences between x and y have not changed when we moved from P_i to P_i' but the collective decisions have changed : this violates IIA.

Another important example

Voting and Condorcet's Paradox (discovered by Condorcet in 1785)

Consider the majority rule and $|X| \geq 3$

Agent 1 : xP_1yP_1z

Agent 2 : zP_2xP_2y

Agent 3 : yP_3zP_3x

Pairwise comparisons yield :

x beats y, $xF(R)y$

y beats z, $yF(R)z$

but z beats x, $zF(R)x$

and preferences are strict.

The majority rule doesn't define a transitive collective preference.

But majority voting satisfies IIA (Exercise: check this).

Axiom 3 (Non-dictatorship: ND)

There is no individual $k \in N$ such that, for all profiles $R \in \mathcal{D}$ and for all $(x, y) \in X^2$

$$xP_ky \Rightarrow x\varphi(R)y$$

If the negation of this held, then the society would always prefer x to y whenever k (the Dictator) prefers x to y.

Axiom 4 (Universal Domain: UD)

The domain $\mathcal{D} = \mathcal{R}^n$. The social choice rule must make a social prescription for any conceivable preference profile.

Now we can state the famous result.

Theorem 4 *(Arrow's Impossibility Theorem)*

Suppose $|X| \geq 3$. The only social choice rules $F(R)$ satisfying Axioms UD, WP and IIA are dictatorial. (Any rule F satisfying universal domain, completeness, transitivity, IIA and the weak Pareto requirement is a dictatorship).

Proof of this theorem :

Several textbooks provide a proof. See, J.J. Laffont, *Fundamentals of Public Econ.*, MIT Press.

D. Kreps, *A Course in Microeconomic Theory*, Princeton U. Press

A. Mas Colell, M. Whinston and J. Green, *Microeconomic Theory*, Oxford U.Press.

B. Salanié, *Microeconomics of Market Failures*, MIT Press.

Now, we know that to aggregate preferences, we need to weaken Arrow's axioms (a vast literature, see Amartya Sen (1986) *in Handbook of Mathematical Econ.*; see John Roemer (1996), *Theories of Distributive Justice*; Marc Fleurbaey (1996), *Théories économiques de la Justice*). We need to use more information about individuals : *utility* or *non-utility* information. Using utility information means that we make some *interpersonal* comparisons of satisfaction. Non-utility information means that we use some "principles" to judge collective choices : liberty, property rights, fairness, etc. principles of justice in general.

Reformulation of Social Choice with Utility Functions

Define a utility profile as $u = (u_1, u_2, ..., u_n)$, where u_i is a function.

$u_i : X \to \mathbb{R}$

Let $\mathcal{U}$ be the set of profiles u.

A social welfare functional is a mapping from a subset $\mathcal{D} \subseteq \mathcal{U}$ into $\mathcal{R}$.

So, $F(u)$ is a preference relation on X.

Definition : Bergson-Samuelson Welfare Function

$$\begin{aligned} W: \quad & \mathbb{R}^n \to \mathbb{R} \\ & u \mapsto W(u) \end{aligned}$$

Function W defines a social utility $\tilde{W}(x) = W(u_1(x), ..., u_n(x))$.

Three important examples:

The Utilitarian Sum

$$W(u) = \sum_{i=1}^{n} u_i.$$

The Rawlsian or Egalitarian Function

$$W(u) = Min\{u_1, ..., u_n\}.$$

The Nash Product

To define the Nash product, we need something called the *status quo*: a list of utility levels $u^0 = (u_1^0, \ldots, u_2^0)$, and we have in mind that $u_i \geq u_i^0$. Then we define,

$$W(u) = \prod_{i=1}^{n} (u_i - u_i^0).$$

Degrees of Comparability of Utilities

Remark : if u_i represents R_i and $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function then, $v_i = f \circ u_i$ also represents R_i. By definition $f \circ u_i(x) = f[u_i(x)]$

The fact that utilities are purely ordinal can be expressed as follows.

Definition: (ONC) Utility is *Ordinally Measurable* and *Noncomparable* if collective choice is invariant with respect to the transformation

$$f \circ u = (f_1 \circ u_1, f_2 \circ u_2, ..., f_n \circ u_n)$$

where f_i is *strictly increasing* for all i. We will say that F respects *ordinal noncomparable information* if $F(u) = F(f \circ u)$, that is, u and $f \circ u$ yield the same collective ordering of X.

It may be that utility *levels* are comparable, but not utility *differences*. We get *ordinal level-comparable* or *co-ordinal* utilities.

Definition: (OLC) Utilities are Ordinal Level-Comparable if and only if collective choice is invariant with respect to transformation,

$$(u_1, \ldots, u_n) \longmapsto (f(u_1), f(u_2), \ldots, f(u_n)),$$

where f is the *same* strictly increasing function.

Remark : The Rawlsian Criterion is clearly invariant with respect to OLC transformations.

$$\begin{aligned} xF(u)y &\Leftrightarrow Min_i\{u_i(x)\} \geq Min_i\{u_i(y)\} \\ &\Leftrightarrow Min_i\{f[u_i(x)]\} \geq Min_i\{f[u_i(y)]\} \\ &\Leftrightarrow xF(f(u_1), \ldots, f(u_n))y. \end{aligned}$$

We now turn to cardinal utilities.

Definition: (CNC) Utility is Cardinal Non-Comparable if and only if collective choice is invariant with respect to transformations,

$$(u_1, ..., u_n) \longmapsto (\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2, ..., \alpha_n u_n + \beta_n)$$

with $\alpha_i > 0$.

Remark: The Nash product is invariant with respect to CNC transformations, since

$$\prod_i (\alpha_i u_i + \beta_i - \alpha_i u_i^0 - \beta_i) = \prod_i \alpha_i \prod_i (u_i - u_i^0),$$

so that maximizing the left-hand side is equivalent to maximizing the right-hand side since $\alpha_i > 0$.

Definition: (CUC) Utility is Cardinal Unit-Measurable if collective choice is invariant with respect to the transformations

$$(u_1, ..., u_n) \longmapsto (\alpha u_1 + \beta_1, \alpha u_2 + \beta_2, ..., \alpha u_n + \beta_n)$$

with the *same* $\alpha > 0$, or $F(u) = F(\alpha u + \beta)$ with $\beta = (\beta_1, ..., \beta_n)$.

Remark: CUC invariance says that utility levels are meaningless but statements about differences such as

$$u_1(x) - u_1(y) > u_2(x) - u_2(y)$$

are meaningful.

Remark: Utilitarianism is clearly invariant with respect to CUC transformations

$$xF(u)y \qquad \Longleftrightarrow \qquad \sum_i u_i(x) \geq \sum_i u_i(y)$$

$$\Leftrightarrow \alpha \sum_i u_i(x) + \sum_i \beta_i \geq \alpha \sum_i u_i(y) + \sum_i \beta_i$$

$$\Leftrightarrow \sum_i (\alpha u_i(x) + \beta_i) \geq \sum_i (\alpha u_i(y) + \beta_i)$$

$$\Leftrightarrow xF(\alpha u + \beta)y.$$

It will also be useful to consider the following definitions.

Definition: (CFC) Utility is Cardinal Fully Comparable if collective choice is invariant with respect to the transformations

$$(u_1, ..., u_n) \longmapsto (\alpha u_1 + \beta, \alpha u_2 + \beta, ..., \alpha u_n + \beta)$$

with the same (α, β) and $\alpha > 0$. And finally, *Definition:* (RFC) Utility is Ratio-scale Fully Comparable if collective choice is invariant with respect to the transformations

$$(u_1, ..., u_n) \longmapsto (\alpha u_1, \alpha u_2, ..., \alpha u_n)$$

with $\alpha > 0$.

Welfarism

Condition of Irrelevance of Non-Welfare Characteristics (or Strong Neutrality)

Let $u, u' \in \mathcal{U}$ and $R = F(u), R' = F(u')$.

Let $x, y, x', y' \in X^4$.

If $u(x) = u'(x')$ and $u(y) = u'(y')$ then $xRy \Leftrightarrow x'R'y'$

Welfarism means that utility is the only thing that matters.

Representation by a Bergson-Samuelson function :

Definition: The social choice function F can be represented by W if for all x, y and u,

$$xF(u)y \Leftrightarrow W(u(x)) \geq W(u(y)).$$

[A continuous W exists under strong neutrality if $F(u)$ is a continuous preference on X. We skip this technical point. See: continuity of preferences in Micro textbooks.]

Amartya Sen's "critique" of Welfarism

Example : $X = \{x, y, z\}$ and $n = 2$.

	u_1	u_2
x	4	10
y	7	8
z	7	8

with decision x : Agent 1 is hungry and Agent 2 is eating a lot.

decision y : redistribute some food from 2 to 1.

decision z : same food allocation as x but Agent 1 is whipping Agent 2 and derives utility from it.

y and z satisfy Pareto indifference,

$u_i(y) = u_i(z)$ for all $i \Rightarrow yIz$ under $F(u_1, u_2)$.

Axiomatic Approaches to Egalitarianism and Utilitarianism

We first re-formulate a number of important axioms in terms of utility.

Axiom 1' (Weak Pareto Property : WP)

If $u_i(x) > u_i(y)$ for all $i \in N$ then x is strictly preferred to y by the collective preference : $x\varphi(u)y$.

Note : $x\varphi(u)y$ iff $xF(u)y$ and $no(yF(u)x)$.

Axiom 1" (Strong Pareto Property : SP)

If $u_i(x) \geq u_i(y)$ for all $i \in N$ then x is preferred to y by the collective preference : $xF(u)y$, and if for some $j \in N$, $u_j(x) > u_j(y)$, then x is strictly preferred: $x\varphi(u)y$.

Axiom 2' (Independence IIA)

For all $x, y \in X^2$, for all $u, u' \in \mathcal{U}$, if $u(x) = u'(x)$ and $u(y) = u'(y)$, then $xF(u)y \Leftrightarrow xF(u')y$.

Axiom 3' (Non-Dictatorship ND)

Dictatorship means, for some $k \in N$ and for all $u \in \mathcal{U}$, for all $(x, y) \in X^2$ such that,

$$u_k(x) > u_k(y) \Rightarrow x\varphi(u)y.$$

There is no such dictator.

Axiom 4' (Universal Domain : UD)

$\mathcal{D} = \mathcal{U}$: all utility profiles are possible.

It will also be useful to define the following axiom.

Axiom 5' (Pareto Indifference : PI)

If $u_i(x) = u_i(y)$ for all $i \in N$, then $xF(u)y$ and $yF(u)x$ (indifference).

We now find conditions under which welfarism holds.

Lemma : If F satisfies UD, PI and IIA, then F satisfies INW (Irrelevance of Non-Welfare Characteristics)

Proof : Let x, x', y, y' be distinct and $u, u' \in \mathcal{U}$.

	x	y	x'	y'
u	r	s		
u'			r	s
u''	r	s	r	s

We want to prove that

$$xF(u)y \iff x'F(u')y'$$

By UD, we have the existence of u. By IIA, we have $xF(u)y \Leftrightarrow xF(u'')y$, and by PI, $xF(u'')y \Leftrightarrow x'F(u'')y'$, and by IIA. Again, $x'F(u'')y' \Leftrightarrow x'F(u')y'$. Therefore, $xF(u)y \Leftrightarrow x'F(u')y'$. This proves INW. (The same type of argument works if x, x'... are not distinct). *Q.E.D.*

Characterization of Utilitarianism

Axiom 6 (Anonymity : AN)

If σ is a permutation of N, and $u'_i = u_{\sigma(i)}$ for all i, then,

$$F(u') = F(u).$$

Anonymity is clearly a stronger requirement than non-dictatorship. The next result shows that under cardinal unit-comparability, we go straight from dictatorship to utilitarianism.

Theorem 5 *(C. d'Aspremont and L. Gevers, 1977)*
F satisfies SP, IIA, AN, and CUC invariance, *if and only if* F is the utilitarian social welfare function,

$$W(u) = \sum_{i=1}^{n} u_i;$$

and

$$xF(u)y \Leftrightarrow \sum_i u_i(x) \geq \sum_i u_i(y).$$

We find a kind of "resolution" of Arrow's impossibility theorem. Arrow's Impossibility is due to the lack of information about utility that can be used by the "social planner". If we accept some *interpersonal comparisons* of utility, we find consistent ways of aggregating preferences.

Characterization of Egalitarianism — the Leximin

The social preferences represented by the Rawlsian maximin, that is, $Min_i u_i$, do not satisfy the strong Pareto property SP (Exercise: check this). In fact, under Rawlsian preferences, two decisions x and y are indifferent if they provide the same utility to the worst-off individual. We have xIy even if x and y lead to very different treatments of the relatively better off individuals. Amartya Sen (in *Collective Choice and Social Welfare*, 1970) has proposed a completion of the Maximin called the *lexicographic minimum*, or *Leximin.*

The idea is to *rank* individuals from the most disadvantaged (the worst off), with rank 1, to the highest utility level, rank n.
Let the set of ranks be denoted $\mathcal{N} = \{1, ..., n\}$.

For every u, and for every x, define the mapping i_x from $\mathcal{N}$ to N. Individual $i = i_x(h)$ is the person with rank h in $u(x)$.

The mapping i_x satisfies the following property. For all ranks h, k in $\mathcal{N}$ then

$$u_{i_x(h)}(x) < u_{i_x(k)}(x) \qquad \Longrightarrow \qquad h < k.$$

If there are ties, individuals with the same utility can be ranked in any way compatible with the above property, this doesn't matter.

Definition: (Leximin Rule)
The *Leximin* is such that or all u and all $(x, y) \in X^2$, x is strictly preferred to y if and only if there exists a rank m such that, for all ranks $h < m$,

$$u_{i_y(h)}(y) = u_{i_x(h)}(x),$$

and

$$u_{i_y(m)}(y) < u_{i_x(m)}(x).$$

Remark: The Leximin satisfies SP, IIA, AN and OLC invariance. (Exercise: check this).

To characterize the Leximin, we need two additional, but very natural axioms. Axiom 7 says that we should *eliminate the influence of indifferent individuals.*

Axiom 7 (Separability: SE)
For every u, u', if there exists $M \subset N$, such that, for all $i \in M$ and for all $x \in X$ $u_i(x) = u'_i(x)$ (individuals in M are have the same preferences with u and u'), and for all $j \in N \setminus M$, for all $(x, y) \in X^2$, $u_j(y) = u_j(x)$ and $u'_j(y) = u'_j(x)$ (the individuals not in M are completely indifferent), then $F(u) = F(u')$.

It happens that under axioms SP, AN, IIA, SE and OLC invariance, there exists a *rank dictator* (see L. Gevers (1979), A. Sen (1986)). More precisely, given that we require SP, we have either the *Leximin*, or its exact opposite, the *Leximax* (the dictatorship of the most favored individuals). But Leximax can be discarded under a minimal equity axiom. A possible formulation of the axiom is just that Leximax is not permitted.

Axiom 8 (Minimal Equity: ME)
There exists some $u \in \mathcal{U}$, $(x, y) \in X^2$ and $(i, j) \in N^2$ such that, for all $k \neq i, j$, $u_k(x) = u_k(y)$,

$$u_j(x) > u_j(y) > u_i(y) > u_i(x), \qquad \text{and} \qquad \Rightarrow yF(u)x.$$

When two individuals disagree on the ranking of x and y, there is at least one profile u where the social preference between two states x and y follows the preference of the individual who is worse off.

We can now state the characterization theorem.

Theorem 6 *(C. d'Aspremont and L. Gevers 1977)*
$F(u)$ satisfies SP, IIA, AN, SE, ME, and OLC invariance if and only if F is the *Leximin Rule.*

The Veil of Ignorance
Some theories, based on the idea of the "impartial observer" placed behind the "veil of ignorance", provide *philosophical* justifications for Utilitarianism and for Rawls's "Difference Principle" (*i.e.*, the Maximin).

Harsanyi's Approach : The Impartial Observer Theorem
Harsanyi's idea is that individual rationality is embodied in Von-Neumann Morgenstern utilities.

Quick refresher on VNM utility
We study decision under risk. Suppose that $|X| = K$ (for simplicity). $\Delta(X)$ is the set of lotteries on X representing, risk : $p \in \Delta(X), p = (p_1, ..., p_k)$, where $p_h = Pr(x_h), x_h \in X$.

A rational decision-maker should satisfy three important axioms :

Axiom A (Transitivity)

There is a *complete* and *transitive* preference ordering on $\Delta(X)$, denoted R.

Axiom B (Independence Axiom)

Let p, p', p_0 be three lotteries on X. Then, for all $\lambda \in [0,1], pR_ip'$ *if and only if*

$$[\lambda p + (1-\lambda)p_0]R_i[\lambda p' + (1-\lambda)p_0].$$

Axiom C (Continuity of Preferences)

For all $p, q, r \in \triangle(X)$ with pP_iqP_ir, there exists $\alpha, \beta \in (0,1)$ such that

$$(\alpha p + (1-\alpha)r)P_iqP_i(\beta p + (1-\beta)r)$$

Theorem 7 *(von Neumann and Morgenstern)*

The preference R_i over $\Delta(X)$ satisfies Axioms A, B and C if and only if there exists $u_i : X \to \mathbb{R}$ such that, for every $p, q \in \Delta(X)$,

$$pR_iq \Leftrightarrow \sum_{k=1}^{K} p_k u_i(x_k) \geq \sum_{k=1}^{K} q_k u_i(x_k).$$

Remark: We find a justification for expected utility under lottery p :

$$E_p u_i(x) = \sum_{k=1}^{K} u_i(x_k)p_k.$$

Remark: The VNM utility u_i is unique up to a linear-affine transformation $\alpha u_i + \beta$ with $\alpha > 0$.

Note: The VNM utility is *cardinal.*

Harsanyi's Impartial Observer

The impartial observer may become any member of society $i \in N$ with an equal probability $1/n$. He (she) considers lotteries on pairs $(x,i) \in (X) \times N$.

Under Axioms A, B and C, the observer has a VNM utility $U(x,i)$, $U : X \times N \to \mathbb{R}$ with the property that p is preferred to q if and only if

$$\sum_{(x,i)} p(x,i)U(x,i) \geq \sum_{(x,i)} q(x,i)U(x,i).$$

We *assume* that the observer's preferences, when i is fixed, are the same as individual $i's$ preferences, *i.e.*, for all i, $U(x, i) \equiv u_i(x)$ for all $x \in X$.

Note: In this identification, $(a), u_i(x) = U(x, i)$ is $i's$ VNM utility function and $(b), u_i$ measures i's cardinal welfare at the same time.

It follows that the impartial observer should choose $x \in X$ so as to maximize

$$\frac{1}{n} \sum_{i=1}^{n} u_i(x)$$

or, since the population is finite and fixed,

$$\text{Maximize } \sum_{i=1}^{n} u_i(x),$$

the Utilitarian Criterion.

For this argument to be fully valid, it seems that we should combine it with the idea of *Fundamental Preference* (developed in S. C. Kolm, *Justice et Equité*, 1972). See also J. Harsanyi (1953) ; Rawls (1971) ; J. Weymark (1991); J. Roemer (1996); M. Fleurbaey (1996).

Additional Results

Remark: If there exists a Bergson-Samuelson function $W(u_1, ..., u_n)$ that is strictly increasing with respect to each argument u_i, then W represents a collective preference F_W that satisfies Arrow's axioms. F_W is clearly transitive and complete, non-dictatorial, satisfies the (Strong) Pareto Principle and IIA in a certain sense (*i.e.*, that of Axiom 2'). Elementary Exercise: check that you can prove all these points. Thus, there is no "impossibility", but W clearly embodies value judgments and interpersonal comparisons. Armartya Sen has shown that Arrow's impossibility theorem is still true (with utility functions) if we require ONC or even CNC invariance.

Ratio-scale comparability

Kevin Roberts (1980) has proved characterization results in the case of RFC invariance, that is, $F(u) = F(\alpha u)$. Under UD, IIA, WP and RFC he showed that there exists a homothetic W function such that $W(u(x)) > W(u(y)) \Rightarrow x\varphi(u)y$. W is homothetic if $W = g \circ \omega$ and $\omega(tu) = t\omega(u)$ for all $t > 0$, with g increasing. If separability SE is imposed, one can show that W must be additively separable, *i.e.*, $W(u) = \Sigma_i \psi_i(u_i)$. If Anonymity AN is imposed, W must also be symmetric, *i.e.*,

$$W(u) = \Sigma_i \psi(u_i).$$

Now we can apply Abraham Bergson's theorem, namely, if W is additively separable, homothetic (and continuously differentiable), we have,

$$\frac{\psi'(tu_i)}{\psi'(tu_j)} = \frac{\psi'(u_i)}{\psi'(u_j)},$$

for all $t > 0$ and (i, j). If we differentiate this expression with respect to t and rearrange terms, we find, for $t = 1$,

$$\frac{u_i \psi''(u_i)}{\psi'(u_j)} = \frac{u_j \psi''(u_i)}{\psi'(u_j)} = \text{constant}.$$

This differential equation says that ψ is a utility function with a constant relative risk aversion ($CRRA$). It follows from this (Exercise: prove this statement) that W must be of the form,

$$W(u) = \sum_{i=1}^{n} \frac{(u_i)^{1-\rho}}{1-\rho},$$

if $\rho \neq 1$ and

$$W(u) = \sum_{i=1}^{n} \ln(u_i)$$

if $\rho = 1$.

Remark: (Interpretation) The coefficient ρ can be viewed as an index of *social aversion for inequality* of utility levels, as if the impartial observer, behind the veil of ignorance, decided to be more risk averse than the agents themselves, or if the social planner chose a concave transformation of the utilities to limit the inequality of welfare levels. This type of thing is routinely done in the Optimal Taxation literature.

Note that the standard Utilitarian welfare function corresponds to the case $\rho = 0$. It is also well known that if $\rho \to +\infty$, then $W \to Min_i\{u_i\}$ (Exercise: prove this). It follows that the Rawlsian Maximin is a limiting case of utilitarianism, defined as the sum of concavified utilities $\Sigma_i v_i = \Sigma_i \psi(u_i)$.

Cardinal Fully Comparable Utilities

In the CFC case, we have $F(u) = F(\alpha u + \beta e)$ with $e = (1, 1, ..., 1)$. Kevin Roberts has shown that under UD, IIA, WP and CFC invariance, there exists a function g, homogeneous of degree 1, such that W is of the form,

$$W(u(x)) = \bar{u}(x) + g[u(x) - \bar{u}(x)e],$$

where $\bar{u}(x) = (1/n)\Sigma_i^n u_i(x)$.

This is a nice result. For instance, the following mixture of Utilitarianism and Egalitarianism is justified:

$$W(u(x)) = \bar{u}(x) + \gamma Min_i\{u_i(x) - \bar{u}(x)\} = \gamma Min_i\{u_i(x))\} + (1-\gamma)\bar{u}(x),$$

with $0 \leq \gamma \leq 1$. Another possible choice is, for instance,

$$W(u) = \bar{u} - \frac{\sigma}{\sqrt{n}},$$

where

$$\sigma^2 = \frac{1}{n}\sum_i (u_i - \bar{u})^2.$$

See K. Roberts (1980) in *Rev. Econ. Stud.*; see Mas-Colell, Whinston and Green's (1995); textbook; see Hervé Moulin (1988), *Axioms of Cooperative Decision Making* for details). There is (of course) a lot more to say on the connection between social welfare functions and inequality indices.

There are other formal theories of justice that we did not study in this short introduction. For instance, the theories based on Nash bargaining and other models of axiomatic bargaining (see *e.g.*, Roemer's textbook, the work of Ken Binmore and that of William Thomson and Terje Lensberg (1989)). We also skipped the theories of fairness, based on the *no envy* criterion (see papers by W. Thomson, H. Varian) and the theory of egalitarian-equivalent allocations (see the work of Elisha Pazner and David Schmeidler, and Marc Fleurbaey's (2008) book). We did not provide a discussion of Marxian and Libertarian approaches (again see the books of John Roemer). We provided a very limited view of non-welfarist theories such as, for instance, those of Rawls and Sen.

Voting: The Case of Single-Peaked Preferences

We conclude with the famous Median Voter Theorem. This result is only valid in a restricted domain: the subset of *single-peaked* or *unimodal* preferences. We know that the Median Voter Theorem cannot be generalized. So, from now on, we no longer assume UD.

Definition: (Linear order)
A preference relation $\preceq$ on X is a *linear order* if it is *reflexive* (*i.e.*, $x \preceq x$ for all x), *transitive* and *total* (*i.e.*, $x \preceq y$ or $y \preceq x$ but *not both*).

Example: Preferences on a subset of the real line $X = \mathbb{R}$ or any set that is one-dimensional.

Definition: (Single-peaked preferences)
Preference R is single-peaked (or unimodal) with respect to the linear order $\preceq$ on X if there is an alternative x_R (*i.e.*, the *peak* of R), such that preferences "increase before the peak and decrease after the peak"; formally,

$$\text{If} \quad x_R \succeq z \succ y \quad \text{then} \quad zPy, \quad \text{and}$$

$$\text{if} \quad y \succ z \succeq x_R \quad \text{then} \quad zPy.$$

Let $\mathcal{S}$ be the set of single-peaked preferences with respect to $\preceq$ on X. Consider the domain $\mathcal{D} = \mathcal{S}^n$. Majority voting is the social preference rule F_m defined as follows. Let the profile $R \in \mathcal{S}^n$. We put $xF_m(R)y$ if

$$|\{i \in N \mid xP_iy\}| \geq |\{i \in N \mid yP_ix\}|.$$

Pairwise majority voting induces a complete preference: $F_m(R)$ is complete (Exercise: check this). Let x_i be the peak of agent i (short for x_{R_i}).

Definition: (Median voter)

Agent $m \in N$ is the median for the profile $R \in \mathcal{S}^n$ if

$$|\{i \in N \mid x_i \succeq x_m\}| \geq n/2 \quad \text{and} \quad |\{i \in N \mid x_m \succeq x_i\}| \geq n/2.$$

A median agent always exists (draw a picture). If there are no ties in peaks ($x_i \neq x_j$, all $i \neq j$) and n is odd, the definition says that $(n-1)/2$ agents have peaks smaller than x_m and $(n-1)/2$ agents have peaks greater than x_m — the median is then unique. If n is even, we have two median voters.

Definition (Condorcet winner)

A Condorcet winner is an alternative x that cannot be defeated by majority voting by any other choice x'.

Theorem 7 (Median Voter Theorem)

Let R be a profile in $\mathcal{S}^n$. Let m be a median agent. Then, a Condorcet winner exists and x_m is a Condorcet winner, *i.e.*, $x_m F_m(R) x$ for all $x \in X$. In addition, if n is odd and if preferences on X are strict, $F_m(R)$ is complete and transitive.

There are proofs of this result in many books; see *e.g.*, Mas Colell *et al.* (1995). There are many applications of this result in the economic literature, but, alas, it cannot be generalized (see Austen-Smith and Banks (1996), *Positive Political Theory*).

Some Exercises

Exercise 1. (Moulin, 1988) Two towns of equal size choose the location of a joint facility (financed exogenously). Two roads connect town A and town B. The long road is 5 km long and the short road is only 3 km long. Let C be 1 km away from A on the short road. There is mountain road between C and B that allows no feasible location of the facility.

Feasible locations are anywhere on the long road and between A and C on the short road. The disutility of a town is measured by the distance to the facility. With a location at C we have $(u_1, u_2) = (-1, -2)$, where u_1 is the utility of A, u_2 is the utility of B.

1. Draw the set of feasible disutility vectors in the (u_1, u_2)-plane.
2. Which locations are Pareto Optimal? Are equal-utility solutions optimal?
3. Determine the Rawlsian solution.

Exercise 2. (Moulin, 1988) Another facility location problem. Let $[0, 1]$ be a linear city with a continuous density of population $f(x)$, $0 \leq x \leq 1$. Total population is $\int_0^1 f(x)dx$. The disutility of an agent located at x is the distance between x and the facility located at point a, that is $|x - a|$.

1. Determine the location choice a^e of a Rawlsian-egalitarian planner.

2. Determine the choice a^u of the Utilitarian planner (show that the solution is the median location $a = x_m$).

3. Discuss the role of the density f in the two solutions above.

Exercise 3. (Roemer, 1996) A simple optimal taxation problem. There is a continuum of workers-taxpayers with productivity (wage rate) w, uniformly distributed on $[0, 1]$. Productivity is not observed by the planner. Individual income $y = wh$ is observed, where h denotes the hours worked. Individuals differ only in w and their utility function is $u(x, h) = x - h^2/2$, where x is after-tax income: $x = y - T(y)$, and $T(y)$ is the income tax schedule. We consider only linear-affine taxation of the form $T(y) = cy + d$, with two parameters (c, d). The only use of this tax is to redistribute income.

1. Given a tax *schedule* $T(y) = cy + d$, determine the labour supply $h(w)$ and the income tax $t(w)$ of an individual of type w.

2. Write the government budget constraint and derive a relationship between c and d that must be satisfied by any feasible tax schedule $T(.)$.

3. Compute the indirect utility $v(w)$ of an individual of type w under $T(.)$.

4. Determine the Rawlsian choice of parameters (c_e, d_e) (show that $c^e = 1/2$ for the Rawlsian planner).

5. Determine the choice of the utilitarian planner (c_u, d_u) (show that $c_u = 0$).

6. Compute the weighted utilitarian solution where $v(w)$ has weight $(1 - w)$.

Exercise 4. (Laffont, 1988) Consider a simple economy with one public good and one private good. Let y be the quantity of the public good. To produce y units of the public good, $z = y$ units of the private good are required. There are n consumers, indexed $i = 1, ...n$. Each i is endowed with $w_i = 1$ units of the private good. Agent i's utility function is $u_i(x_i, y) = x_i + \theta_i \ln(y)$, where x_i is the quantity of a private good consumed by i, and $\theta_i \in [0, 1]$ is a preference parameter.

1. Compute the Utilitarian optimum of this economy.

2. We now study the taxes needed to finance y under the government budget constraint. Each agent i will pay the same tax t (no discrimination between taxpayers), expressed in terms of the private good. Show that the citizens' preferences over tax levels t are single-peaked. Compute the peaks, denoted t_i.

3. Determine the tax t_m that is a Condorcet winner in this problem (apply the Median Voter Theorem).

4. Is the winning tax socially efficient?

(...)