Housing, Capital Taxation and Bequests in a Simple OLG Model*

Robert J. GARY-BOBO †, Jamil NUR‡ ,

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Abstract

We study the allocation of housing capital in an overlapping generations economy with competitive property and housing rental markets. In this economy, consumers inherit from their parents when they retire. Agents have paternalistic bequest motives. All agents are identical and there is no redistribution problem. The stationary competitive equilibrium of such a model is inefficient, since old agents consume too much perishable goods and too much housing. We then show that the golden rule stationary optimum can be achieved by means of a simple system of proportional taxes. The optimal allocation is characterized by the fact that the young agents rent their homes and that the old agents own the entire stock of housing capital. An optimal tax system has the following features: the young agents' rents must be subsidized. Housing capital and capital-income, that is, rents, are both taxed. But bequests must be subsidized. Bequest and rent subsidies are financed by labor income tax and property tax revenues. Rent subsidies are financed by the tax on rents. The government's budget is balanced. The negative tax on bequests can be interpreted as a pension benefit, paid out of a public pension fund, based on the market value of the housing-capital stock.

KEYWORDS : Housing; Real Estate; Capital Taxation; Rents; Overlapping Generations; Bequests.

*The first author would like to dedicate this article to his teacher, the late Philippe Michel.

†CREST, ENSAE, 15 boulevard Gabriel Péri, 92245 Malakoff cedex, France; E-mail:robert.gary-bobo@ensae.fr

‡Sciences Po Paris, 28 rue des Saints Pères, 75007 Paris, France. E-mail: jamil.nur@sciencespo.fr

1 Introduction

Wealth takes the form of real estate for the vast majority of the people, including the upper middle class, in most countries. Many households save to buy a home and many parents leave their property to their children. The bulk of bequests is made of more or less ordinary homes for the "99%". Land and buildings are relatively easy to tax using forms of capital taxation such as property, estate and inheritance levies. The landlords' capital income, that is, rents and imputed rents, are typically taxed. At the same time, we also observe various kinds of subsidies, mainly aiming at promoting home ownership and, simultaneously, more affordable housing. Housing policies and housing taxation vary from one country to the next, but there are strong similarities. Home-ownership is typically unbalanced in favor of the oldest households. Big cities like New York, Tokyo and Paris are expensive places and young urban households live in small (and sometimes cramped) apartments. Longer life expectancy changes the timing of personal investments, insofar as bequests arrive later in life, and as many households use inheritance (or inter vivos gifts) as a downpayment to buy their homes.

In what follows, we put these facts together as ingredients of a theory of housing taxation. We study the allocation of real estate capital in an overlapping generations model with competitive markets for property and renting. Housing is simultaneously a durable good, producing a flow of services, and a form of investment, yielding rents and possible capital gains. Agents choose how much surface area to rent, to occupy as owners or to buy as an investment. We assume that agents live for two periods. They work in the first period of life and retire in the second. Rents are the main form of income for the old agents. The lifespan of agents is long and parents pass away when their children retire. It follows that young workers wait until they themselves become seniors to inherit property (and thus money) from their parents. In the model, each agent has only one child and parents have bequest motives. We assume that parents are "impure altruists" who derive utility from the transmission of wealth to their children. Individuals are assumed identical and there is no risk. As a consequence, there is no need for insurance and redistribution. There is no agent heterogeneity and no asymmetric information and it follows that we do not need to study incentive-compatible second-best allocations.

In such an economy, the competitive equilibrium is inefficient. As compared to the golden-rule stationary optimum, young agents consume too little perishable goods and too little housing, while the old agents consume too much and occupy large houses in stationary competitive equilibrium. These inefficiencies have two independent sources: on the one hand, the bequest motive is one source, and on the other hand, the double nature of housing (as consumption and capital good) is in itself a second source. The stationary equilibrium would be inefficient even if there was no bequest motive, and it would be inefficient if housing had no utility, apart from a possible use as a store of value. This is why the two sources of inefficiency are independent. We find that golden-rule optimality can be restored by means of a simple and familiar system of tax rates, à la Ramsey.

We study an economy with Cobb-Douglas preferences. It happens that the equilibrium with taxes can be computed and that simulations are not needed to study our simple model. Optimum and equilibrium allocations can be compared and we completely characterize the subset of optimal tax systems. Our results can be summarized as follows. It is optimal that the old agents (*i.e.*, the retirees) own the entire stock of housing capital. The young workers should rent their homes and not invest in real estate; they will inherit the entire capital stock when they retire. Rents are a source of income for the old who do not work. A set of corrective taxes can ensure that the young and the old have the appropriate incentives. Firstly, rents should be subsidized. This makes larger homes affordable to the young agents. Secondly, real estate capital is taxed by means of a property tax, making sure that the young do not buy their homes or invest savings on the property market. At the same time, the property tax increases the cost of owner-occupancy for the old, so that they reduce the size of their houses and put up the rest of the capital on the rental market. Third, inheritance is not taxed. On the contrary, bequests must be subsidized. The bequest subsidy is like a pension benefit, based on the market value of the housing stock. In addition, it is optimal to tax rents, *i.e.*, capital income is taxed. Bequest subsidies are financed by means of the labor income and capital (*i.e.*, property) taxes. Rental subsidies are paid out of the capital-income (*i.e.*, rent) tax revenues: money is taken out of the pocket of old landlords by means of this tax, and given back to young renters. Finally, the government's budget is balanced. We also discuss the possible role of imputed rent taxation for owner-occupiers, but this latter form of taxation is not useful if capital can be taxed. We propose an interpretation of the optimal tax system as equivalent to a public pension fund based on the housing capital stock, with competitive rental and property markets for housing.

Let us now briefly examine relationships with the literature. There exists an important literature on intergenerational transfers, estate (or inheritance) taxation. Recent surveys are provided by Cremer and Pestieau (2006), Lafferère and Wolff (2006), Boadway, Chamberlain and Emmerson (2010) and Kopczuk (2013). On the theory of estate an inter vivos gift taxation, see also, *e.g.*, Kaplow (2001), Farhi and Werning (2010), Piketty and Saez (2013).

The theory of taxation in Diamond-Samuelson[1], overlapping generations models is of course not new; it is viewed as an important alternative to taxation à la Ramsey in models with infinitely lived agents or dynasties. Pioneering work is due, among other contributions, to Pestieau (1974), Ordover and Phelps (1979), Atkinson and Sandmo (1980). Some recent contributions have studied the role of bequest motives in the overlapping generations model, see, *e.g.*, Michel and Pestieau (2004), Lambrecht, Michel and Thibault (2006), Bossmann, Kleiber and Wälde (2007), Piketty and Saez (2012). These papers do not consider housing or real estate capital specifically.

In the literature, there is a debate on the nature of bequest motives and on the most appropriate models. Theories relying on pure altruism, on the one hand, lead to infinite horizon models in which each generation discounts the utility of future generations in their own utility function. Ricardian equivalence à la Barro typically holds in these models, because dynasties, linked by intergenerational transfers, behave like single infinitely-lived agents subjected to an inter-temporal

[1]See Samuelson (1958), Diamond (1965).

budget constraint (see Barro (1974), Kopczuk (2013)). But Ricardian equivalence is rejected by a number of tests (see, *e.g.*, Altonji, Hayashi and Kotlikoff (1997)). The real world seems to be characterized by a mix of bequest motives (see, *e.g.*, Kopczuk and Lupton (2007), Ameriks *et al.* (2011)). Overlapping generations models with bequests and impure altruism, that is, in which parents value the very fact of giving or leaving bequests to their children constitute an interesting alternative and may fit the data better than some dynastic models.

The idea of modelling gifts as a consumption good for the donor is very old, as recalled by Becker (1974). Many facts can be explained by the joy-of-giving, or "warm glow" generated by gifts (see the classic discussion of these points in Andreoni (1989); see also Diamond (2006)). The idea can be immediately applied to model bequest motives. This approach is sometimes called bequest-as-consumption or impure altruism, or paternalistic bequest approach. There are several variants of this idea in the literature, yielding different properties of taxation. A slightly more general specification is to add the individual's wealth directly as an argument of the utility function. Wealth may be valued because of a bequest motive, or for more general reasons. The donor may care (or not) about the value effectively received by the recipient; in such a case, the nature of bequest or gift-based taxes may matter. For instance, an estate tax reduces the gift but an inheritance tax doesn't. Under some assumptions, the joy-of-giving formulation may be analytically close to a reduced form for a purely altruistic formulation (see Abel and Warshawsky (1988)). In our model, the parents care about the after tax bequest, expressed in real housing units, *i.e.*, in square feet.

There is a literature on land or housing in overlapping generations models (for instance, see Deaton and Laroque (2001)). There also exists a burgeoning literature on taxation and housing policies in applied macroeconomics, using simulations of OLG models with real estate capital (see, among other contributions, Gervais (2002); Chambers, Garriga and Schlagenhauf (2009), Nakajima (2010)). The political economy of urban growth provides a complementary view of the problems posed in the present paper, namely, that there is an intergenerational problem in the market allocation of housing. It is impossible to survey this important literature here (see however, for instance, Gleaser, Gyourko and Saks (2005), Ortalo-Magné and Prat (2014)).

In the following, Section 2 describes the model, computes the stationary equilibrium and the golden rule allocation. Section 3 presents the economy with taxes and solves the model for its equilibrium with taxation. It is then possible to find the optimal tax system, and to provide interpretations. Concluding remarks are in section 4. A few proofs are given in an Appendix.

2 A simple overlapping generations model with housing

A population of identical agents live for two periods. A new generation is born in every period. Young and old agents consume a (nondurable) consumption good; they can also rent housing, invest in real property and borrow (or lend) on competitive markets. The young agents work and earn labor income while young. The old agents do not work and leave a bequest, in the form of real property, for the next generation. Housing capital and bequests are measured in units of surface area (square meters or square feet) and assumed perfectly divisible. Units of area produce a flow of housing services. Area can be bought or sold on the property market and rented on a rental market. The agents may be renters (*i.e.*, tenants) or owner-occupiers, or both. They can also buy to let the owned area on the rental market. Housing capital is therefore simultaneously a consumption good, a store of value and an asset yielding income in the form of rents. There is no housing depreciation and no building, for simplicity. Real estate is the only form of capital in the economy.[2]

We consider a stationary economy with a fixed amount of perfectly divisible inhabitable area denoted H. Let c_{1t}, resp. $c_{2,t+1}$ be the nondurable consumption of the young, resp. old, agent of generation t. Let q_{1t}, resp. $q_{2,t+1}$, denote the owner-occupied property of a young, resp. old, agent of generation t. Let z_{1t}, resp. $z_{2,t+1}$, denote the rented area of a young, resp. old, agent of generation t, and denote k_{1t}, resp. $k_{2,t+1}$, denote the buy-to-let property of a young, resp. old, agent of generation t. The area consumed by agent j in period t is

$$s_{jt} = q_{jt} + z_{jt},$$

where $j = 1$ is a young agent, $j = 2$ is an old agent. The housing property of jt is denoted h_{jt}, and we pose,

$$h_{jt} = q_{jt} + k_{jt}.$$

Finally, the agents can borrow from other agents on a credit market. Loans are denominated in units of the nondurable consumption good. Let M_{jt} denote the amount borrowed by agent jt. Negative borrowing represents lending. A unit borrowed in period t costs an interest i_t in the next period. There are competitive prices in this economy. The rent per unit of area is denoted r_t in period t. The (spot market) price of a square foot is denoted p_t. The interest rate on loans is denoted i_t and labour income is denoted w_t. By definition, the bequest of generation t is the net estate of the old agent at the end of the second period of life. More precisely, let b_{t+1} denote this bequest, expressed in square feet. We have,

$$b_{t+1} = h_{2,t+1} - \frac{(1 + i_{t+1})}{p_{t+2}} M_{2,t+1}. \tag{1}$$

The agent's debt $(1 + i_{t+1})M_{2,t+1}$ is divided by p_{t+2} to be converted in square feet available in period $t+2$. The young agent active in period t is endowed with a Cobb-Douglas utility function, denoted U. This utility depends on consumption, housing and bequests. We have,

$$U(c_{1t}, c_{2,t+1}, s_{1t}, s_{2,t+1}, b_{t+1}) = \ln(c_{1t}) + \alpha \ln(s_{1t}) + \beta\{\ln(c_{2,t+1}) + \alpha \ln(s_{2,t+1}) + \delta \ln(b_{t+1})\}.$$

Parameter β measures time preference, α measures the relative intensity of preference for housing services and δ measures the intensity of the bequest motive. This type of formulation of the bequest motive is called "joy-of-giving" or paternalistic in the literature. Parents like to transmit property to their children, but they do not take their children's real welfare as an argument of

[2]We could introduce productive industrial capital and would obtain closely related results. This form of capital is set equal to zero for the sake of simplicity.

their own utility. Parents have their own appreciation of what's good for their heirs, or they simply derive satisfaction from the donation of their property.[3] Note that, in this model, parents pass away when their children retire. As a consequence, old agents can control the lifetime indirect utility of their heirs, since they can change their wealth, namely, the right-hand side of their inter-temporal budget constraint, and to the extent that labor income and house prices are predictable, as will be shown below.

2.1 Demand functions

To derive the demand functions (for the consumption good, housing and bequests), we maximize U subject to the following budget constraints. During the first period of life, we have,

$$c_{1t} + r_t z_{1t} + p_t q_{1t} + (p_t - r_t)k_{1t} = w_t + M_{1t}. \tag{2}$$

Remark that the price of buy-to-let investment k_{1t} is in fact $p_t - r_t$ since the square feet are rented at price r_t. During the the second period of life, we have,

$$c_{2,t+1} + r_{t+1}z_{2,t+1} + p_{t+1}q_{2,t+1} + (p_{t+1} - r_{t+1})k_{2,t+1} = p_{t+1}[q_{1t} + k_{1t} + b_t] + M_{2,t+1} - (1+i_t)M_{1t}. \tag{3}$$

The right-hand side of equation (3) shows that the old agent can sell up to $q_{1t} + k_{1t} + b_t$ units of area on the housing market, can borrow $M_{2,t+1}$ and must reimburse $(1+i_t)M_{1t}$. The no-arbitrage conditions provide us with important relationships between prices and rates. A loan $M_{1t} > 0$, expressed in units of consumption buys

$$m_{1t} = \frac{M_{1t}}{p_t - r_t}$$

square feet on the property market. These units can be resold at date $t+1$, at price p_{t+1}. This yields a profit,

$$\frac{p_{t+1}M_{1t}}{p_t - r_t} - (1 + i_t)M_{1t}.$$

The profit is positive if and only if $p_{t+1} > (p_t - r_t)(1 + i_t)$. Define the discount factor as $\rho_{t+1} = (1 + i_t)^{-1}$. In competitive equilibrium, we must have

$$\rho_{t+1} = \frac{1}{1 + i_t} = \frac{p_t - r_t}{p_{t+1}}. \tag{4}$$

With the help of equation (4), the above definitions and notation, equation (1) and the budget constraints (2) and (3) can be equivalently rewritten as follows,

$$\begin{aligned} b_{t+1} &= h_{2,t+1} - m_{2,t+1}; & (5)\\ c_{1t} + (p_t - r_t)(h_{1t} - m_{1t}) + r_t s_{1t} &= w_t; & (6)\\ c_{2,t+1} + (p_{t+1} - r_{t+1})(h_{2,t+1} - m_{2,t+1}) + r_{t+1}s_{2,t+1} &= p_{t+1}(h_{1t} - m_{1t} + b_t). & (7) \end{aligned}$$

Using equation 4 we can now write the inter-temporal budget constraint. The wealth of an agent of generation t can be defined as follows,

$$y_t = w_t + \rho_{t+1}p_{t+1}b_t.$$

[3]This can also be called imperfect altruism.

If we multiply the second budget constraint by ρ_{t+1} and add the two budget constraints, we obtain,

$$c_{1t} + \rho_{t+1}c_{2,t+1} + r_t s_{1t} + \rho_{t+1}r_{t+1}s_{2,t+1} + \rho_{t+1}(p_{t+1} - r_{t+1})b_{t+1} = y_t. \tag{8}$$

To obtain the demand functions, we now maximize U with respect to (c, s, b), subject to the inter-temporal budget constraint (8). This is a standard microeconomics exercise. The following proposition summarizes the results.

Proposition 1. *Define the sum of the Cobb-Douglas coefficients*

$$\kappa = 1 + \alpha + \beta(1 + \alpha + \delta).$$

Let c^*_{jt}, s^*_{jt} and b^*_t denote the demand functions. We have,

$$c^*_{1t} = \frac{y_t}{\kappa}; \qquad s^*_{1t} = \frac{\alpha y_t}{\kappa r_t};$$

$$c^*_{2,t+1} = \frac{\beta y_t}{\kappa \rho_{t+1}}; \qquad s^*_{2,t+1} = \frac{\beta \alpha y_t}{\kappa \rho_{t+1} r_{t+1}};$$

$$b^*_{t+1} = \frac{\beta \delta y_t}{\kappa \rho_{t+1}(p_{t+1} - r_{t+1})}; \qquad h^*_{1t} - m^*_{1t} + b_t = \frac{\beta(1 + \alpha + \delta)y_t}{\kappa(p_t - r_t)}.$$

All demands are well-defined, provided that $p_t > r_t > 0$. In addition, inherited wealth is never negative, since $b^*_t > 0$ for all t (as soon as $y_t > 0$).

Note that the difference $h^*_{1t} - m^*_{1t}$ can be derived from the budget constraint (6), or from (7), but that the model does not determine h^*_{1t} and m^*_{1t} separately.

2.2 Equilibrium conditions

The competitive equilibrium conditions are the following. First, the rental market must clear. All the area to let is rented, that is,

$$z_{1t} + z_{2t} = k_{1t} + k_{2t}.$$

Second, the property market clears, *i.e.*, each unit of available housing has an owner at each date t,

$$H = h_{1t} + h_{2t}. \tag{9}$$

The market for loans clears, that is,

$$0 = m_{1t} + m_{2t}. \tag{10}$$

Recall that $q_{1t} + q_{2t}$ is the surface area used by owner-occupiers at date t. From (9) we derive,

$$H - q_{1t} - q_{2t} = k_{1t} + k_{2t}.$$

Hence, the rental market equilibrium condition above can be rewritten

$$H = s_{1t} + s_{2t}, \tag{11}$$

using $s_{jt} = q_{jt} + z_{jt}$. The stock of available housing is entirely consumed at date t. By Walras's law, the market for the nondurable consumption good is then balanced at each t. Remark, in addition, that in equilibrium, using (10), we must have,

$$H = h^*_{1t} + h^*_{2t} = (h^*_{1t} - m^*_{1t}) + (h^*_{2t} - m^*_{2t}),$$

and therefore, we have the key relationship,

$$H = h^*_{1t} - m^*_{1t} + b^*_t. \tag{12}$$

An equilibrium allocation satisfies (11) and (12) for all t. Using Proposition 1, the competitive equilibrium can be written as the solution of a system of three equations in three unknowns, (p^*_t, r^*_t, y^*_t). The three equations are (11), (12) and the definition of equilibrium wealth y^*_t, that is,

$$H = \frac{\alpha y^*_t}{\kappa r^*_t} + \frac{\beta \alpha y^*_{t-1}}{\kappa \rho^*_t r^*_t}; \tag{13}$$

$$H = \frac{\beta(1+\alpha+\delta) y^*_t}{\kappa(p^*_t - r^*_t)}; \tag{14}$$

$$y^*_t = w_t + \frac{\beta\delta y^*_{t-1}}{\kappa \rho^*_t}. \tag{15}$$

If we assume that labor income is a constant, *i.e.*, $w_t = w$ for all t, this system can easily be solved for its stationary equilibrium (p^*, r^*, y^*) where $r^* = r^*_t = r^*_{t-1}$, $p^* = p^*_t = p^*_{t-1}$, etc. To solve (13-15), we first use a change of variables

$$(p, r, y) \mapsto \left(\frac{p}{p-r}, \frac{y}{p-r}, y \right).$$

This change of variables is one-to-one and well-defined, provided that $p \neq r$. Define

$$\gamma = 1 + i = \frac{p}{p-r}.$$

Define

$$\chi = \frac{y}{p-r}.$$

With the above definitions, the first variable is just the interest rate (or the inverse of the discount rate ρ) and the second variable is the value of wealth, expressed in square feet. Remark that the equilibrium rent satisfies

$$\frac{r}{p-r} = \gamma - 1.$$

It is easy to check that our stationary equilibrium conditions, (13-15), can be rewritten with the new variables, as follows,

$$H = \frac{\alpha \chi^*(1+\beta\gamma^*)}{\kappa(\gamma^* - 1)} \tag{16}$$

$$\chi^* = \frac{H\kappa}{\beta(1+\alpha+\delta)} \tag{17}$$

$$y^* = \frac{w}{1 - (\beta\delta\gamma^*/\kappa)}. \tag{18}$$

Equation (17) immediately gives χ^*. Substituting this result in (16), we easily find the equilibrium value of γ, that is,

$$\gamma^* = 1 + i^* = \frac{\kappa - 1}{\beta(1+\delta)}. \tag{19}$$

The result is positive since $\kappa > 1$. finally, with the help of (18) we find y^* and we can solve for the rest of the unknowns. Simple algebra yields the following proposition.

Proposition 2. *The stationary competitive equilibrium is unique and given by the following expressions,*

$$r^* = \frac{\alpha w}{H}; \tag{20}$$

$$p^* = \frac{(\kappa - 1)w}{(1+\beta)H}; \tag{21}$$

$$y^* = \frac{(1+\delta)\kappa w}{(1+\beta)(1+\alpha+\delta)}. \tag{22}$$

From these values, we derive the equilibrium allocation,

$$c_1^* = \frac{(1+\delta)w}{\kappa+\delta}, \qquad c_2^* = \frac{(\kappa-1)w}{\kappa+\delta}; \tag{23}$$

$$s_1^* = \frac{(1+\delta)H}{\kappa+\delta}, \qquad s_2^* = \frac{(\kappa-1)H}{\kappa+\delta}; \tag{24}$$

$$b^* = \frac{\delta(\kappa-1)H}{\beta(1+\delta)(1+\alpha+\delta)}. \tag{25}$$

It is easy to check that $c_1^* + c_2^* = w$. Remark that $b^* < H$ if and only if $h_1^* - m_1^* > 0$. This is true if and only if $\delta\alpha < \beta(1+\alpha+\delta)$. The young agent will therefore save during the first period of her life if and only if

$$\beta > \frac{\delta\alpha}{1+\alpha+\delta}. \tag{26}$$

The above inequality will be satisfied if agents are sufficiently patient. If the bequest motive is strong, *i.e.*, if δ is large, then $\beta > \alpha$ is a sufficient condition. If individuals like large homes, *i.e.*, if α is large, then, $\beta > \delta$ is a sufficient condition for positive savings. If the condition does not hold, the agents will borrow to consume and buy large houses and will repay the loans while old with the help of their heritage.

2.3 The golden rule: optimality of stationary allocations

The Golden Rule allocation is the stationary allocation that maximizes the welfare of a generation. By definition, this allocation maximizes,

$$\ln(c_1) + \alpha\ln(s_1) + \beta\{\ln(c_2) + \alpha\ln(s_2) + \delta\ln(b)\},$$

subject to the constraints

$$c_1 + c_2 = w, \qquad s_1 + s_2 = H; \tag{27}$$

$$b \le H. \tag{28}$$

Inequality (28) says that the old agent cannot bequeath more than the entire housing stock. The solution is obtained by maximizing $\ln(c_1) + \alpha \ln(s_1) + \beta \ln(w - c_1) + \beta\alpha \ln(H - s_1)$, and $b = H$ is obviously the optimal choice: if agents derive satisfaction from the transmission of housing to their children, there is no reason not to let them bequeath the largest possible amount to the next generation. Easy computations yield the solution.

Proposition 3. *The optimal golden rule allocation is given by the following expressions.*

$$\hat{c}_1 = \frac{w}{1+\beta}, \qquad \hat{c}_2 = \frac{\beta w}{1+\beta}; \tag{29}$$

$$\hat{s}_1 = \frac{H}{1+\beta}, \qquad \hat{s}_2 = \frac{\beta H}{1+\beta}; \tag{30}$$

$$\hat{b} = H. \tag{31}$$

Proposition 3 provides us with the optimal way of sharing labor income w and the housing stock H between generations in a stationary allocation. The golden rule allocation says that all agents should wait until they retire to inherit their house from their parents. Agents should rent their home while young and become owners only when old. We can now compare the golden rule allocation of surface with the stationary competitive equilibrium allocation. It is easy to check that

$$c_1^* < \hat{c}_1$$

if $\alpha > 0$. Hence, the young do not consume enough (perishable goods) in equilibrium. It follows that $c_2^* > \hat{c}_2$, *i.e.*, the old consume too much. In addition, if $\alpha > 0$, we have

$$s_1^* < \hat{s}_1.$$

The young use too little space in competitive equilibrium (their homes are too small). Of course, as a counterpart, we have $s_2^* > \hat{s}_2$, *i.e.*, the old live in houses that are too spacious. Remark that c^* and s^* are inefficient, even if $\delta = 0$, that is, even if there is no bequest motive. Inefficiencies in terms of perishable good and housing consumption exist as soon as $\alpha > 0$, that is, as soon as agents have a demand for housing services. To see this, it is enough to check that $s_1^* < \hat{s}_1$ (or $c_1^* < \hat{c}_1$) if and only if $(1+\delta)(1+\beta) < (1+\beta)(1+\alpha+\delta)$ which boils down to $\alpha > 0$. We also have an inefficiency of the equilibrium allocation insofar as $h_1^* - m_1^* > 0$, since this is equivalent to $b^* < H$. The young should not invest in real property. We know that bequests are too small if and only if inequality (26) holds.

Case 1. If $\alpha = \delta = 0$, the equilibrium and the golden rule allocations are well-defined and coincide. We have $r^* = 0$, $p^* > 0$ and $y^* = w$. We find that $b^* = 0$, but in this case, $\hat{b}$ may optimally be set equal to zero since there is no bequest motive. Finally, since $c_1^* = \hat{c}_1$ and $s_1^* = \hat{s}_1$, the equilibrium is equal to the golden rule. Equilibrium inefficiencies are driven by the bequest motive, that is $\delta > 0$, and by the fact that housing services (*i.e.*, surface) are a valuable consumption good, or $\alpha > 0$. The bequest motive and the double nature of housing are independent sources of inefficiency.

Case 2. When $\alpha = 0$ and $\delta > 0$, we have $r^* = 0$, $p^* > 0$, $y^* > w$: the equilibrium is well-defined. Housing is useless as a form of consumption, but property can still play the role of a store of value since we have $p^* > 0$ but the interest rate is zero: $\gamma^* = p^*/p^* = 1 + i^* = 1$. The levels of consumption are optimal since we find $s_1^* = \hat{s}_1$ and $c_1^* = \hat{c}_1$, but the allocation is inefficient since $b^* = H\delta/(1+\delta) < H$.

Case 3. When $\alpha > 0$, housing surface is both a capital good and a consumption good, and this alone is a source of inefficiency. To see this, suppose that $\alpha > 0$ and $\delta = 0$. It is then easy to check that $y^* = w$, $b^* = 0$, $r^* = \alpha(w/H) > 0$ and $p^* = r^* + (\beta w)/((1+\beta)H) > 0$. Hence, $\gamma^* = 1 + \alpha(1+\beta)/\beta$, so that $i^* = \alpha(1+\beta)/\beta > 0$. There exists a well-defined stationary equilibrium with a positive price of property, a positive rent per square foot and a positive interest rate, but the equilibrium is inefficient since $s_1^* < \hat{s}_1$ and $c_1^* < \hat{c}_1$.

3 Equilibrium with Taxation

We now study a system of taxes, to restore efficiency in the economy.

1. We first introduce a renters tax (or residence tax). This tax is paid by the agents who rent their home. It is proportional to the rent. To add flexibility we assume that the young pay a tax $\tau_z r_t z_{1t}$ added to their rent $r_t z_{1t}$. The old pay a tax $\tilde{\tau}_z r_{t+1} z_{2,t+1}$. We can choose different rates for the young and the old, *i.e.*, $\tilde{\tau}_z \neq \tau_z$, if this happens to be useful.

2. We also create a tax on property, with a rate denoted τ_h. The young pay $\tau_h p_t h_{1t}$. The old pay $\tau_h p_{t+1} h_{2,t+1}$. This is clearly a tax on housing capital.

3. We also introduce an income tax with a taxation of imputed rents. Let τ_w denote the flat rate on labor income. Let τ_r be the housing-capital income tax rate. Let finally $\tilde{\tau}_r$ denote the rate of taxation of imputed rents. Labor income is w. Housing-capital income is $r_t k_{1t}$ for the young and $r_{t+1} k_{2,t+1}$ for the old. The imputed rent of the young is $r_t q_{1t}$. The imputed rent of the old is $r_{t+1} q_{2,t+1}$.

4. The inheritance tax is a specific, proportional tax on inheritance, with rate denoted τ_b. The old pay $\tau_b p_{t+1} b_t$ when they receive their heritage.

5. Finally, we consider the possibility of lump sum transfers to the young, denoted μ_t.

3.1 Demand functions in the economy with taxes

With these taxes, the after-tax income of a young agent is by definition,

$$W_{1t} = (1 - \tau_w)w + (1 - \tau_r) r_t k_{1t} - \tilde{\tau}_r r_t q_{1t} + \mu_t. \tag{32}$$

The next to last term in expression (32) is the tax on the imputed rent, since since q_{1t} is the area occupied and owned by the young agent. The after-tax wealth of the old agent can be expressed as follows,

$$W_{2,t+1} = p_{t+1}(k_{1t} + q_{1t}) + (1 - \tau_r) r_{t+1} k_{2,t+1} - \tilde{\tau}_r r_{t+1} q_{2,t+1} + (1 - \tau_b) p_{t+1} b_t. \tag{33}$$

Expressions (32) and (33) appear as the right hand side of the agents budget constraints. More precisely, with the same notations as above we now have,

$$c_{1t} + (1+\tau_z)r_t z_{1t} + (1+\tau_h)p_t h_{1t} = W_{1t} + M_{1t}; \tag{34}$$

$$c_{2,t+1} + (1+\tilde{\tau}_z)r_{t+1}z_{2,t+1} + (1+\tau_h)p_{t+1}h_{2,t+1} = W_{2,t+1} + M_{2,t+1} - (1+i_t)M_{1t}. \tag{35}$$

The no-arbitrage conditions now depend on tax rates. Borrowing M_{1t} at date t yields m_{1t} square feet, namely,

$$m_{1t} = \frac{M_{1t}}{(1+\tau_h)p_t - r_t(1-\tau_r)}. \tag{36}$$

The appropriate denominator in (36) is $(1+\tau_h)p_t - r_t(1-\tau_r)$ because each square foot of buy-to-let investment costs $(1+\tau_h)p_t$ and at the same time yields $r_t(1-\tau_r)$. The square feet can be resold on the spot market in period $t+1$ to reimburse the loan. The benefit of this investment is

$$\frac{p_{t+1}M_{1t}}{(1+\tau_h)p_t - r_t(1-\tau_r)} - (1+i_t)M_{1t}.$$

Therefore, in a competitive equilibrium with taxes, we must have,

$$1 + i_t = \frac{1}{\rho_{t+1}} = \frac{p_{t+1}}{(1+\tau_h)p_t - r_t(1-\tau_r)}. \tag{37}$$

The appropriate expression for bequests becomes,

$$b_t = h_{2t} - \frac{M_{2t}}{(1+\tau_h)p_t - r_t(1-\tau_r)} = h_{2t} - m_{2t}. \tag{38}$$

We now introduce an array of tax-adjusted prices, for each of the decisions made by the agent on the housing markets, namely z_{jt}, q_{jt} and k_{jt}. Define

$$\pi_{1t} = (1+\tau_z)r_t, \qquad \tilde{\pi}_{1t} = (1+\tilde{\tau}_z)r_t; \tag{39}$$

$$\pi_{2t} = (1+\tau_h)p_t + \tilde{\tau}_r r_t; \tag{40}$$

$$\pi_{3t} = (1+\tau_h)p_t - (1-\tau_r)r_t. \tag{41}$$

In the above definitions, expressions (39) give the prices per rented square foot for the young and the old, respectively; (40) gives the price of an owner-occupied square foot and (41) gives the price of a square foot bought to let. With the help of these prices, the budget constraints (34)-(35) can be rewritten in the following convenient form,

$$c_{1t} + \pi_{1t}z_{1t} + \pi_{2t}q_{1t} + \pi_{3t}(k_{1t} - m_{1t}) = (1-\tau_w)w + \mu_t; \tag{42}$$

$$c_{2,t+1} + \tilde{\pi}_{1,t+1}z_{2,t+1} + \pi_{2,t+1}q_{2,t+1} + \pi_{3,t+1}(k_{2,t+1} - m_{2,t+1}) = p_{t+1}[q_{1t} + k_{1t} - m_{1t} + (1-\tau_b)b_t]. \tag{43}$$

To simplify notation, define the variables x_{jt} as follows,

$$x_{jt} = k_{jt} - m_{jt},$$

where $j = 1, 2$. To obtain the intertemporal budget constraint, we multiply (43) by $\rho_{t+1} = \pi_{3t}/p_{t+1}$ and add the two budget constraints. We obtain the following expression.

$$\begin{aligned} c_{1t} + \pi_{1t}z_{1t} + (\pi_{2t} - \pi_{3t})q_{1t} \\ + \rho_{t+1}[c_{2,t+1} + \tilde{\pi}_{1,t+1}z_{2,t+1} + \pi_{2,t+1}q_{2,t+1} + \pi_{3,t+1}x_{2,t+1}] = y_t(\tau), \end{aligned} \tag{44}$$

where, by definition,

$$y_t(\tau) = (1-\tau_w)w + (1-\tau_b)\rho_{t+1}p_{t+1}b_t + \mu_t. \tag{45}$$

Remark that $\pi_{2t} - \pi_{3t}$ appears in (44). We have

$$\pi_{2t} - \pi_{3t} = r_t(1 - \tau_r + \tilde{\tau}_r).$$

We assume that $1 - \tau_r + \tilde{\tau}_r > 0$, *i.e.*, rates τ_r and $\tilde{\tau}_r$ should not be too different. To compute the demand functions, we now maximize the utility,

$$\ln(c_{1t}) + \alpha \ln(z_{1t} + q_{1t}) + \beta\{\ln(c_{2,t+1}) + \alpha \ln(z_{2,t+1} + q_{2,t+1}) + \delta \ln(q_{2,t+1} + x_{2,t+1})\},$$

with respect to (c_{1t}, z_{1t}, q_{1t}), and $(c_{2,t+1}, z_{2,t+1}, q_{2,t+1}, x_{2,t+1})$, subject to (44) and sign constraints on (c, z, q). In fact we focus on a particular form of a corner solution in which young agents are not owner-occupiers, *i.e.*, $q_{1t} = 0$ and old agents do not rent their homes, *i.e.*, $z_{2,t+1} = 0$. Note in passing that to compute demands, it does not matter if we specify the utility of bequests as $\delta \ln(b)$ or $\delta \ln((1-\tau_b)b)$, since we assumed that utilities are logarithmic and τ_b is a given parameter. Standard microeconomics lead to the following result.

Proposition 4. *Assume that* $1 - \tau_r + \tilde{\tau}_r > 0$, *assume in addition that,*

$$\tau_z < \tilde{\tau}_r - \tau_r < \tilde{\tau}_z.$$

and finally assume that $\pi_{jt} > 0$ *for* $j = 1, 2, 3$ *and for all* t. *Then, the demand functions are as follows,*

$$c^*_{1t} = \frac{y_t(\tau)}{\kappa}, \qquad c^*_{2,t+1} = \frac{\beta y_t(\tau)}{\kappa \rho_{t+1}}; \tag{46}$$

$$z^*_{1t} = \frac{\alpha y_t(\tau)}{\kappa \pi_{1t}}, \qquad z^*_{2,t+1} = 0; \tag{47}$$

$$q^*_{1t} = 0, \qquad q^*_{2,t+1} = \frac{\beta \alpha y_t(\tau)}{\kappa \rho_{t+1}(\pi_{2,t+1} - \pi_{3,t+1})}; \tag{48}$$

$$b^*_{t+1} = q^*_{2,t+1} + x^*_{2,t+1} = \frac{\beta \delta y_t(\tau)}{\kappa \rho_{t+1} \pi_{3,t+1}}. \tag{49}$$

The interpretation of this result is simple. If τ_z is sufficiently small, and possibly negative, the young will have an incentive to rent their homes in the first period and will not invest in real estate, except possibly to let. A negative value of τ_z would be a rent subsidy. If $\tilde{\tau}_z$ is larger than $\tilde{\tau}_r - \tau_r$, then, the old will choose to buy their homes and avoid renting. Note that this is what we want, to maximize welfare. For a proof of this result, see the appendix.

3.2 Competitive equilibrium in the economy with taxes

The equilibrium conditions are the same as before, but we add the government budget constraint, and we use the fact that, under the assumptions of Proposition 4, $q^*_{1t} = z^*_{2,t+1} = 0$. First, the rental market clears,

$$H - q^*_{2t} = z^*_{1t}.$$

The property market is balanced,

$$H = q_{2t}^* + k_{1t}^* + k_{2t}^*.$$

The loans market clears,

$$0 = m_{1t}^* + m_{2t}^*.$$

Substituting the latter equation in the property market equation, we find a convenient form of the latter, namely,

$$H = x_{1t}^* + b_t^*.$$

Finally the government budget constraint gives the lump-sum transfer μ_t as a function of equilibrium tax revenues, that is,

$$\mu_t = \tau_z r_t z_{1t}^* + \tau_h p_t H + \tau_w w + \tau_r r_t (H - q_{2t}^*) + \tilde{\tau}_r r_t q_{2t}^* + \tau_b p_t b_{t-1}^*. \tag{50}$$

On the right-hand side of (50), we find the revenues of the tax on young renters (that may be negative if we in fact subsidize renters) plus the revenue of the property tax $\tau_h p_t H$, plus the income tax $\tau_w w$, plus the tax on capital income $\tau_r r_t (H - q_{2t}^*)$, plus the tax on imputed rents $\tilde{\tau}_r r_t q_{2t}^*$ paid by owner-occupiers, and finally, the revenue of the inheritance tax $\tau_b p_t b_{t-1}^*$. It happens that we can solve the model analytically for a stationary equilibrium. We fix the tax system, *i.e.*, we fix the vector $\tau = (\tau_z, \tau_r, \tilde{\tau}_r, \tau_w, \tau_b)$ and solve the following system of equations. There are four equations with four unknowns (p, r, y, μ). The first equation is the rental market equation, that is, using the demand functions,

$$H = \frac{\alpha y}{\kappa \pi_1} + \frac{\beta \alpha y}{\kappa \rho (\pi_2 - \pi_3)}. \tag{$E1$}$$

The second equation is the property market equation. Using the first-period budget constraint to express x_1^* and the demand functions, we obtain,

$$H = \frac{1}{\pi_3} \left[(1 - \tau_w) w + \mu - \frac{y}{\kappa} - \frac{\alpha y}{\kappa} \right] + \frac{\beta \delta y}{\kappa \rho \pi_3}. \tag{$E2$}$$

The third equation is just an expression of equilibrium wealth y, that is,

$$y = (1 - \tau_w) w + \mu + (1 - \tau_b) \frac{\beta \delta y}{\kappa \rho}. \tag{$E3$}$$

Finally, the fourth equation is the government's budget constraint. Using the demand functions again, we have,

$$\mu = \tau_z r \frac{\alpha y}{\kappa \pi_1} + \tau_w w + \tau_h p H + \tau_r r \left[H - \frac{\beta \alpha y}{\kappa \rho (\pi_2 - \pi_3)} \right] + \tilde{\tau}_r r \frac{\beta \alpha y}{\kappa \rho (\pi_2 - \pi_3)} + \tau_b \frac{\beta \delta y}{\kappa \rho}. \tag{$E4$}$$

To study the system ($E1$-$E4$), we use a change of variables: $(p, r, y, \mu) \to (\gamma, \chi, \phi, \lambda)$, with the following definitions. As in section 2 above,

$$\gamma = \frac{1}{\rho} = \frac{p}{\pi_3}, \qquad \chi = \frac{y}{\pi_3}.$$

We also define,

$$\phi = \frac{w}{\pi_3}, \qquad \lambda = \frac{\mu}{\pi_3}.$$

This change of variable is of course well defined if $\pi_3 > 0$. For the sake of notational elegance, we introduce notation for a number of key parameters.

$$\begin{aligned} R &= 1 - \tau_r; & S &= (1 - \tau_r + \tilde{\tau}_r); \\ Q &= 1 + \tau_h; & T &= 1 - \tau_w; \\ Z &= 1 + \tau_z; & B &= 1 - \tau_b; \\ \xi &= 1 + \alpha + \delta. \end{aligned}$$

Note that

$$\frac{r}{\pi_3} = \frac{Q\gamma - 1}{R}.$$

With these notational changes, and rearranging terms, we find equivalent forms for the equilibrium equations.

$$\frac{H(Q\gamma - 1)}{R} = \frac{\alpha\chi}{\kappa}\left[\frac{1}{Z} + \frac{\beta\gamma}{S}\right]; \tag{$E1a$}$$

$$H = \lambda + T\phi - \frac{\chi}{\kappa}\left[1 + \alpha - \beta\gamma\delta\right]; \tag{$E2a$}$$

$$\lambda + T\phi = \frac{\chi}{\kappa}\left[\kappa - B\beta\gamma\delta\right]; \tag{$E3a$}$$

and finally, the government budget constraint becomes,

$$\begin{aligned} &\lambda + (T-1)\phi = \\ &\quad \frac{\chi}{\kappa}\left[\frac{\alpha(Z-1)}{Z} + \frac{(S-1)\alpha\beta\gamma}{S} + (1-B)\beta\gamma^2\delta\right] + (Q-1)\gamma H + \frac{(1-R)(Q\gamma-1)H}{R}. \end{aligned} \tag{$E4a$}$$

The above system is non-linear and can be solved with the help of simple but somewhat lengthy calculations. Parameter γ can be obtained as the solution of a quadratic equation. The next proposition gives a convenient expression of the solution, in which the formula for γ^* is not explicit. The proof is in the appendix.

Proposition 5. *A solution of system ($E1a$-$E4a$) is a stationary competitive equilibrium with taxes τ.* Define

$$\Omega^* = \xi + (1-B)\delta\gamma^*.$$

We have,

$$\gamma^* = \frac{[\beta Z\Omega^* + \alpha R]S}{[SQ\Omega^* - \alpha R]\beta Z}. \tag{$E1b$}$$

Equation ($E1b$) is quadratic in γ^*. *Thus, γ^* is the real positive solution of this quadratic equation, if it exists. Assuming the existence of such a γ^*, we have,*

$$\chi^* = \frac{\kappa H}{\beta\Omega^*}; \tag{$E2b$}$$

$$\phi^* = \frac{H(1+\beta\gamma^*)}{\beta\Omega^*}; \tag{$E3b$}$$

$$\lambda^* = \frac{H}{\beta\Omega^*}\left[\kappa - B\beta\delta\gamma^* - T(1+\beta\gamma^*)\right]; \tag{E4b}$$

From these results we derive the stationary equilibrium values of (p^*, r^*, y^*, μ^*). Inverting the change of variables, the unknowns can be expressed as a function of γ^* (and Ω^*) as follows,

$$p^* = \frac{w\beta\Omega^*\gamma^*}{H(1+\beta\gamma^*)}; \tag{S1}$$

$$r^* = \frac{w(Q\gamma^*-1)\beta\Omega^*}{HR(1+\beta\gamma^*)}; \tag{S2}$$

$$y^* = \frac{w\kappa}{1+\beta\gamma^*}; \tag{S3}$$

$$\mu^* = \frac{w(\kappa - B\beta\delta\gamma^*)}{1+\beta\gamma^*} - wT; \tag{S4}$$

and

$$Q\gamma^* > 1.$$

3.3 Optimal tax system

We now study the possibility of implementing the golden rule allocation with the help of a tax system τ. In fact we will find a manifold of optimal tax systems. To implement the golden rule, we need to find a τ such that the following three equations are simultaneously satisfied,

$$s_1^*(\tau) = \hat{s}_1. \tag{A}$$

$$b^*(\tau) = H. \tag{B}$$

$$c_1^*(\tau) = \hat{c}_1. \tag{C}$$

Using (29) and $(S3)$, we find that (C) is equivalent to,

$$c_1^*(\tau) = \frac{y^*}{\kappa} = \frac{w}{1+\beta\gamma^*} = \frac{w}{1+\beta} = \hat{c}_1.$$

This immediately yields a key result,

$$\gamma^* = 1.$$

And as a consequence, the after-tax interest rate i^* is equal to zero. Using (30) and (49), we find that (B) can be rewritten as follows,

$$H = b^*(\tau) = \frac{\beta\delta y^*}{\kappa\rho^*\pi_3^*} = \beta\delta\gamma^*\frac{\chi^*}{\kappa} = \beta\delta\gamma^*\frac{H}{\beta\Omega^*}.$$

This yields

$$\delta\gamma^* = \Omega^* = \xi + (1-B)\delta\gamma^*.$$

Exploiting the fact that $\gamma^* = 1$ and simplifying the above expression yields another key result,

$$B^* = \frac{1+\alpha+\delta}{\delta}.$$

Since $1-\tau_b^* = B^* > 1$ we conclude that

$$\tau_b^* = -\frac{(1+\alpha)}{\delta} < 0,$$

that is, we found that bequests must be subsidized! We provide an interpretation for this surprising result below. From $\gamma^* = 1$ and $\Omega^* = \delta$, we derive the following: if the tax system is optimal, then,

$$\gamma^* = \frac{(\beta\delta Z + \alpha R)S}{(\delta SQ - \alpha R)\beta Z} = 1; \tag{C'}$$

$$p^* = \frac{\beta\delta w}{(1+\beta)H}; \tag{51}$$

$$r^* = \frac{(Q-1)}{R} p^*. \tag{52}$$

Now, condition (A) implies

$$s_1^*(\tau) = z_1^*(\tau) = \frac{\alpha y^*}{\kappa Z r^*} = \frac{H}{1+\beta} = \hat{s}_1.$$

Using ($S3$), this can be rearranged to yield $r^* H/(1+\beta) = \alpha y^*/(\kappa Z) = \alpha w/((1+\beta)Z)$. And finally, using ($S2$), we obtain an equation relating Z to the ratio of $Q-1$ to R,

$$\frac{Q-1}{R} = \frac{\alpha(1+\beta)}{\beta\delta Z}. \tag{A'}$$

From (C') above, we derive another equation, namely,

$$\frac{Q-1}{R} = \frac{\alpha(S+\beta Z)}{\beta\delta S Z}; \tag{C''}$$

Combining (A') and (C''), we find that the two relations imply

$$Z^* = S^*,$$

and going back to the definition of these variables, we see that at the optimum, the latter equality implies

$$\tau_z^* = \tilde{\tau}_r^* - \tau_r^*. \tag{53}$$

As a consequence, if $\tilde{\tau}_r^* < \tau_r^*$ it may be optimal to subsidize the young renters, *i.e.*, set $\tau_z^* < 0$. This result yields

$$\frac{Q^*-1}{R^*} = \frac{\alpha(1+\beta)}{\beta\delta S^*}. \tag{A''}$$

It follows that we have a degree of freedom in the choice of the property tax rate $Q - 1 = \tau_h$ and the choice of the capital income tax rate τ_r since $R = 1 - \tau_r$. The value of S can be chosen by the government in various ways. We will discuss the possibilities below. The labour income tax can be chosen to balance the government's budget, in the particular sense that the lump-sum transfer to the young is zero, *i.e.*, we can set $\mu^* = 0$. If we do this, ($S4$) shows that

$$T^* = \frac{\kappa - \beta\xi}{1+\beta} = \frac{1+\alpha}{1+\beta},$$

since $\kappa - \beta\xi = 1 + \alpha$. We can now summarize these findings in the next proposition.

Proposition 6. *The golden rule allocation can be implemented as an equilibrium with taxes. The array of tax rates must be chosen as follows.*

1. *Bequests must be subsidized.*
$$1-\tau_b^* = \frac{1+\alpha+\delta}{\delta} > 1. \tag{TB}$$
2. *The government can choose the difference between the tax rate on imputed rents and the tax on capital income. Let $\sigma = \tau_r^* - \tilde{\tau}_r$. We require $\sigma < 1$. Then, the tax on young renters must be chosen as follows*
$$\tau_z^* = -\sigma. \tag{TZ}$$
3. *The property tax and the capital-income tax rates are constrained by a linear-affine relation, given σ. We must have,*
$$\frac{\tau_h^*}{1-\tau_r^*} = \frac{\alpha(1+\beta)}{\beta\delta(1-\sigma)}. \tag{THR}$$
4. *The labor-income tax rate can be chosen to balance the budget. We have $\mu^* = 0$ and the following,*
$$1-\tau_w^* = \frac{1+\alpha}{1+\beta}. \tag{TF}$$
5. *Finally, the tax on old renters must be chosen so that $\tilde{\tau}_z^* > -\sigma$.*

We now study some important special cases. Assume first that imputed rents are not taxed, *i.e.*, $\tilde{\tau}_r = 0$. Then, $S^* = (1-\tau_r^*) = R^*$ and, using (A''), we find,

$$\tau_h^* = \frac{\alpha(1+\beta)}{\beta\delta} > 0.$$

In this case, it is clear that the tax on capital has a positive rate. Condition (TZ) becomes

$$\tau_z^* = -\tau_r^*,$$

meaning that renters are subsidized. The government gives money back to the renters by means of the capital income tax. In other words, to restore efficiency, the government redistributes from the old landlords to the young tenants.

Assume now that $\tilde{\tau}_r = \tau_r^*$. Then $S^* = 1$. This implies $Z^* = 1$, thus, $\tau_z^* = 0$, and

$$\tau_h^* = \frac{\alpha(1+\beta)}{\beta\delta}(1-\tau_r^*).$$

Finally, it is not possible to implement the golden rule without a bequest subsidy. If $\tau_b = 0$, then $\Omega = \xi$. Condition (C) still implies $\gamma^* = 1$, but condition (B), *i.e.*, $b^* = H$, yields $H\delta\gamma^*/\xi = H$, that is, $\delta = \xi$, which is impossible. It is not possible to satisfy conditions (C) and (B) simultaneously if $\tau_b = 0$.

3.4 Interpretation

Intuitively, the golden rule allocation can be interpreted as a form of "housing socialism" in which the entire capital stock belongs to the government. Houses are rented to the young in exchange for public rent payments. The rest of the available surface is used to house the seniors. The rents paid by the young are used to finance pensions for the old, so that they can consume (because they do not work). This "socialist" allocation can be implemented in a private property economy with competitive real-estate and housing rental markets, and corrective taxes. Recall that agents inherit property from their parents when they retire. Parents pass away when their children retire. In the "market socialism" interpretation, the entire housing capital stock belongs to the old, who themselves inherited it entirely from their parents. Part of the stock is kept by the seniors for their own accommodation needs; the seniors are owner-occupiers. The rest of the stock is rented on a competitive rental market. The rental revenue provides an income for the old agents, but they also receive a *pension benefit*, *i.e.*, the *negative* bequest tax, that is indexed to the value of the housing stock. The rental and property markets are used to decentralize the allocation of housing surface and the allocation of ownership. But we have shown that the stationary competitive equilibrium does not support optimal shares of the available surface, in part because the young agents' consumption and homes are too small. The tax system is designed to make sure that the rates of consumption of the young are appropriate. It also implements the appropriate allocation of ownership by making sure that the young are not owner-occupiers. They must wait for retirement age and will then inherit the entire stock of housing. This is typically realized by means of rent subsidies. At the same time, it can be shown that the capital stock must be taxed, using a property tax. Property-tax and income-tax revenues will be used to finance bequest subsidies and rent subsidies.

Another formulation of the result is that we provide the theory of an optimal public pension fund based on the housing stock. The allocation under study is decentralized by means of the rental and property markets, but contributions based on housing and labor income fuel a pension fund investing in real property. Pension benefits are the sum of two components: they take the form of rents related to the property to let, plus a pension expressed as a fraction of the housing stock's market value.

To see this more precisely, remark that, if we have $b = H$, in stationary equilibrium, the old agent's budget constraint can be written,

$$c_2 + \pi_2 q_2 + \pi_3 x_2 = p[q_1 + x_1 + (1 - \tau_b)H]. \tag{54}$$

where $x_2 = k_2 - m_2$ is the net capital investment of the old agent. Consider now the simple case in which imputed rents are not taxed, *i.e.*, $\tilde{\tau}_r = 0$. Then, from Proposition 6, condition (TZ), we know that $\tau_z = -\tau_r$ (rents are subsidized). This implies $\pi_2 = (1 + \tau_h)p$, $\pi_1 = (1 - \tau_r)r$ and $q_1 = 0$. From this, we derive $\pi_3 = \pi_2 - \pi_1$, and (54) above can be rewritten,

$$c_2 + \pi_2(q_2 + x_2) = \pi_1 x_2 + p[x_1 + (1 - \tau_b)H]. \tag{55}$$

Using the expressions for π_2 and π_1 and rearranging terms in (55), we derive,

$$c_2 + \tau_h p(q_2 + x_2) + \tau_r r x_2 = r x_2 + p x_1 + p(H - q_2 - x_2) - \tau_b p H. \tag{56}$$

But in stationary equilibrium, we have $H = q_1 + x_1 + q_2 + x_2$ and $H = x_1 + b$. Since the golden rule allocation entails $b = H$, we have $x_1 = 0$ (the young's net capital investment is zero). Finally, since $q_1 = 0$, we have $H = q_2 + x_2$. Therefore, (56) implies

$$c_2 + \tau_h p H + \tau_r r x_2 = r x_2 - \tau_b p H. \tag{57}$$

Since $\tau_b < 0$, the old agent's budget constraint (57) can be interpreted as follows,

$$\text{consumption} + \text{taxes} = \text{rents} + \text{pension benefit}.$$

Given that $\tau_r = -\tau_z$, the government budget constraint imposes

$$\tau_w w + \tau_h p^* H = -\tau_b p^* H.$$

This is true since, using (TF), we have

$$\tau_w w = \frac{(\beta - \alpha) w}{(1 + \beta)},$$

(TB) and (THR) yield

$$(\tau_h + \tau_b) p^* H = \frac{(\alpha - \beta)}{\beta\delta} p^* H,$$

and (51) gives $p^* H = \beta\delta w/(1 + \beta)$. This confirms that the labor-income tax and the property tax revenues balance the bequest subsidy, *i.e.*, both taxes finance the pension benefits.

4 Conclusion

We studied the allocation of housing capital in an overlapping generations economy with competitive property and housing rental markets. In this economy, consumers inherit from their parents when they retire. All agents are identical and there is no redistribution problem. We have shown that the stationary competitive equilibrium of such a model is inefficient, since old agents consume too much perishable goods and too much housing. We have then shown that the golden rule stationary optimum can be achieved by means of a simple system of proportional taxes. The optimal allocation is characterized by the fact that the young agents rent their homes and that the old agents own the entire stock of housing capital. An optimal tax system has the following features: the young agents' rents must be subsidized. Housing capital is taxed and capital-income, that is, rents, are taxed. But bequests must be subsidized. Bequest and rent subsidies are financed by labor income tax and property tax revenues. Rent subsidies are financed by the tax on rents. The government's budget is balanced. The negative tax on bequests can be interpreted as a pension benefit, paid out of a public pension fund based on the market value of the housing-capital stock.

5 References

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6 Appendix: proofs

Proof of Proposition 4. We maximize U subject to the intertemporal budget constraint (44) and two sign constraints, namely, $q_{1t} \geq 0$ and $z_{2,t+1} \geq 0$. Let λ denote the Lagrange multiplier of the budget constraint. Let η and ν be the Lagrange multipliers of $q_{1t} \geq 0$, and $z_{2,t+1} \geq 0$, respectively. We assume that the solution satisfies $c_{jt} > 0$, $q_{2,t+1} > 0$ and $z_{1t} > 0$. The first-order conditions can be written as follows

$$\frac{1}{c_{1t}} = \lambda; \qquad \frac{\beta}{c_{2,t+1}} = \lambda\rho_{t+1}; \tag{58}$$

$$\frac{\alpha}{z_{1t} + q_{1t}} = \lambda\pi_{1t}; \qquad \frac{\alpha}{z_{1t} + q_{1t}} = \lambda(\pi_{2t} - \pi_{3t}) - \eta; \tag{59}$$

$$\frac{\beta\alpha}{z_{2,t+1} + q_{2,t+1}} = \lambda\rho_{t+1}\tilde{\pi}_{1,t+1} - \nu; \qquad \frac{\beta\delta}{q_{2,t+1} + x_{2,t+1}} = \lambda\rho_{t+1}\pi_{3,t+1}; \tag{60}$$

$$\frac{\beta\alpha}{z_{2,t+1} + q_{2,t+1}} + \frac{\beta\delta}{q_{2,t+1} + x_{2,t+1}} = \lambda\rho_{t+1}\pi_{2,t+1}; \tag{61}$$

$$\eta q_{1t} = 0, \quad \eta \geq 0; \qquad \nu z_{2,t+1} = 0, \quad \nu \geq 0. \tag{62}$$

Clearly, we must have $\lambda > 0$. From the third and fourth conditions above (in (58) and (59)), we then derive

$$\frac{\eta}{\lambda} = \pi_{2t} - \pi_{3t} - \pi_{1t} \geq 0.$$

If $r_t > 0$, this yields,

$$\tau_z \leq \tilde{\tau}_r - \tau_r.$$

From (60) and (61), we now derive,

$$\frac{\beta\alpha}{z_{2,t+1} + q_{2,t+1}} + \lambda\rho_{t+1}\pi_{3,t+1} = \lambda\rho_{t+1}\pi_{2,t+1},$$

and given that

$$\frac{\beta\alpha}{z_{2,t+1} + q_{2,t+1}} = \lambda\rho_{t+1}\tilde{\pi}_{1,t+1} - \nu,$$

if $\rho_{t+1} > 0$, we find,

$$\frac{\nu}{\lambda\rho_{t+1}} = \tilde{\pi}_{1,t+1} - \pi_{2,t+1} + \pi_{3,t+1} \geq 0.$$

Hence,

$$\tilde{\tau}_z \geq \tilde{\tau}_r - \tau_r.$$

When the inequalities are strict, *i.e.*, $\tau_z < \tilde{\tau}_r - \tau_r < \tilde{\tau}_z$, the complementary slackness relations (62) imply $z^*_{2,t+1} = 0 = q^*_{1t}$. To find the result, we solve the first-order conditions above in this case. This is a standard exercise. Using (44), we first easily find that $\lambda y_t(\tau) = \kappa$. With the help of this result, we then find the solutions stated as (46)-(49) above. *Q.E.D.*

Proof of Proposition 5. Substituting $(E3a)$ in $(E2a)$ yields

$$H = \frac{\chi}{\kappa}\left[\beta\xi + (1 - B)\beta\gamma\delta\right], \tag{$E2a'$}$$

if we use the fact that $\kappa - 1 - \alpha = \beta\xi$. This immediately yields $(E2b)$ since $\Omega = \xi + (1-B)\gamma\delta$. Now, if we substitute $(E1a)$ in $(E2a')$, we find

$$\beta\Omega\frac{(Q\gamma - 1)}{R} = \frac{\alpha}{Z} + \frac{\alpha\beta\gamma}{S}, \tag{$E\Gamma$}$$

a quadratic equation in γ. We do not solve this equation. We rearrange the terms in $(E\Gamma)$ and easily find expression $(E1b)$. It is then easy to check that $Q\gamma^* > 1$ is equivalent to $SQ > -Z\beta$. The latter inequality will be true if S, Q, $Z > 0$. To find the other two variables, first substitute $(E1a)$ in $(E4a)$. This yields,

$$\lambda + (T-1)\phi = \frac{\chi}{\kappa}\left[\alpha(1+\beta\gamma^*) + (1-B)\gamma^{*2}\beta\delta\right] + H(1-\gamma^*). \tag{$E4a'$}$$

We then substract $(E4a')$ from $(E3a)$. This immediately yields ϕ^*, that is,

$$\phi^* = \frac{\chi^*}{\kappa}[\kappa - B\beta\gamma^*\delta - \alpha(1+\beta\gamma) - (1-B)\gamma^{*2}\beta\delta] + H(\gamma^* - 1).$$

exploiting the fact that $H = \beta\Omega^*\chi^*/\kappa$ and simplifying, the above expression boils down to

$$\phi^* = \frac{\chi^*}{\kappa}[1 + \beta\gamma^*],$$

and therefore, substituting the expression for χ^*, we find expression $(E3b)$. The value of λ^* given by $(E4b)$ is immediately derived with the help of $(E3a)$, combined with $(E2b)$ and $(E3b)$. To find $(S1)$ we just rewrite $p^* = \gamma^*\pi_3^* = \gamma^* w/\phi^*$. To find $(S2)$ we use $r^* = (Q\gamma^* - 1)w/(R\phi^*)$. Expression $(S3)$ is derived from $y^* = \chi^*\pi_3^* = \chi^* w/\phi^*$. Finally, we derive μ^* from $\mu^* = \lambda^*\pi_3^* = \lambda^* w/\phi^*$. *Q.E.D.*