Econometric Asset Pricing Modelling

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Abstract
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The purpose of this paper is to propose a general econometric approach to no-arbitrage asset pricing modelling based on three main ingredients: (i) the historical discrete-time dynamics of the factor representing the information, (ii) the Stochastic Discount Factor (SDF), and (iii) the discrete-time risk-neutral (R.N.) factor dynamics. Retaining an exponential-affine specification of the SDF, its modelling is equivalent to the specification of the risk sensitivity vector and of the short rate, if the latter is neither exogenous nor a known function of the factor. In this general framework, we distinguish three modelling strategies: the Direct Modelling, the Risk-Neutral Constrained Direct Modelling and the Back Modelling. In all the approaches we study the Internal Consistency Conditions (ICCs), implied by the absence of arbitrage opportunity assumption, and the identification problem. The general modelling strategies are applied to two important domains: security market models and term structure of interest rates models. In these contexts we stress the usefulness (and we suggest the use) of the Risk-Neutral Constrained Direct Modelling and of the Back Modelling approaches, both allowing to conciliate a flexible (non-Car) historical dynamics and a Car R.N. dynamics leading to explicit or quasi explicit pricing formulas for various derivative products. Moreover, we highlight the possibility to specify asset pricing models able to accommodate non-Car historical and non-Car R.N. factor dynamics with tractable pricing formulas. This result is based on the notion of (Risk-Neutral) Extended Car process that we introduce in the paper, and which allows to deal with sophisticated models like Gaussian and Inverse Gaussian GARCH-type models with regime-switching, or Wishart Quadratic Term Structure models.

Keywords : Direct Modelling, Risk-Neutral Constrained Direct Modelling, Back Modelling, Internal Consistency Conditions (ICCs), identification problem, Car and Extended Car processes, Laplace Transform.
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1 Introduction

Financial econometrics and no-arbitrage asset pricing remain rather disconnected fields mainly because the former is essentially based on discrete-time processes (like, for instance, VAR, GARCH and stochastic volatility models or switching regime models) and the latter is in general based on continuous-time diffusion processes, jump-diffusion processes and Lévy processes. Recently, a few papers have tried to build a bridge between these two literatures [see Heston and Nandi (2000), Garcia, Ghysels and Renault (2003), and Christoffersen, Heston and Jacobs (2006) for the econometrics of option pricing, Gourieroux, Monfort and Polimenis (2003), Dai, Le and Singleton (2006), Dai, Singleton and Yang (2007), Monfort and Pegoraro (2007) for interest rates models, Gourieroux, Monfort and Sufana (2005) for exchange rates models, Gourieroux, Monfort and Polimenis (2006) for credit risk models], and the aim of the present work is in the same spirit. More precisely, the general objective of our paper is organized in the following four steps.

First, we propose a general and flexible pricing framework based on three main ingredients: i) the discrete-time historical ($\mathbb{P}$) dynamics of the factor (w_t, say) representing the information (in the economy) used by the investor to price assets; ii) the (one-period) Stochastic Discount Factor (SDF) $M_{t,t+1}$, defining the change of probability measure between the historical and risk-neutral world; iii) the discrete-time risk-neutral (R.N. or $\mathbb{Q}$) factor dynamics. The central mathematical tool used in the description of the historical and R.N. dynamics of the factor is the conditional Log-Laplace transform (or cumulant generating function). The SDF is assumed to be exponential-affine [see Gourieroux and Monfort (2007)], and its specification is equivalent to the specification of a risk sensitivity vector (α_t, say) and of the short rate r_t, if the latter is neither exogenous nor a known function of the factor. Moreover, the notion of risk sensitivity is linked to the usual notion of Market Price of Risk in a way which depends on the financial context (security markets or interest rates).

Second, we focus on the tractability of this general framework, in terms of explicit or quasi explicit derivative pricing formulas, by defining the notion of Extended Car (ECar) process, based on the fundamental concept of Car (Compound Autoregressive, or discrete-time affine) process introduced by Darolles, Gourieroux and Jasiak (2006). More precisely, we first recall that the discrete-time Car approach is much more flexible than the corresponding continuous-time affine one, since, although every discretized continuous-time affine model is Car, the converse is not true. In other words, the Car family of processes is much wider than the discretized affine family, mainly because of the time consistency constraints (embedding condition) applying to the latter [see Darolles, Gourieroux and Jasiak (2006), Gourieroux, Monfort and Polimenis (2003, 2006), Monfort and Pegoraro (2006a, 2006b, 2007)]. Then, thanks to the concept of ECar process we define, we show that, even if the starting factor in our pricing model ($w_{1,t}$, say) is not Car in the R.N. world (implying, in principle, pricing difficulties), there is the possibility to find a second factor ($w_{2,t}$, say), possibly function of the first one, such that the extended process $w_t = (w'_{1,t}, w'_{2,t})'$ turns out to be R.N. Car. The process $\{w_{1,t}\}$ is called (Risk-Neutral) Extended Car. If the R.N. dynamics is Extended Car, the whole machinery of multi-horizon complex Laplace transform, truncated real Laplace transform and inverse Fourier transform of Car-based pricing procedures [see Bakshi and Madan (2000), and Duffie, Pan and Singleton (2000)] becomes available.

Third, in this general asset pricing setting we formalize three modelling strategies: the Direct Modelling strategy, the Risk-Neutral Constrained Direct Modelling strategy, and the Back Modelling strategy. Since the three elements of the general framework, namely the $\mathbb{P}$-dynamics, the SDF $M_{t,t+1}$ and the $\mathbb{Q}$-dynamics, are linked together (through the SDF change of probability measure), each strategy proposes a parametric modelling of two elements, the third one being a by-product. In the Direct Modelling strategy, we specify the historical dynamics and the SDF, that is to say, the risk sensitivity vector and the short rate and, thus, the R.N. dynamics is obtained as a by-product. In the second strategy, the Risk-Neutral Constrained Direct Modelling strategy, we specify the $\mathbb{P}$-dynamics and we constrain the R.N. dynamics to belong to a given family, typically the family of Car or ECar processes. In this case, the risk sensitivity vector characterizing the SDF is obtained as a by-product. Finally, in the Back Modelling strategy (the third strategy), we specify the $\mathbb{Q}$-dynamics, the short rate process r_t, as well as the risk sensitivity vector α_t and, consequently, the historical dynamics is obtained as by-product. Thus, we get three kinds of Econometric Asset Pricing Models (EAPMs). In these strategies we carefully take into account the following important points: a) the status of the short rate; b) the internal consistency conditions (ICCs) ensuring the compatibility of the pricing model with the absence of arbitrage opportunity principle [the ICCs are conveniently (explicitly) imposed through the Log-Laplace transform]; c) the identification problem; d) the possibility to have a $\mathbb{Q}$-dynamics of Car or Extended Car type. In this respect, two of the proposed strategies, the Back Modelling and the Risk-Neutral Constrained Direct Modelling strategies, are particularly attractive since they control for the R.N. dynamics and they allow for a rich class of nonlinear historical dynamics (non-Car, in general). Moreover, these two approaches may be very useful for the computation of the (exact) likelihood function. For instance, in the Back modelling approach, the nonlinear historical conditional density function is easily deduced from the, generally tractable (known in closed form), p.d.f. in the R.N. world and from the possibly complex, but explicitly specified, risk sensitivity vector.

Fourth, we apply these strategies to two important domains: security market models and interest rate models[5]. In the first domain, we show how the Back Modelling strategy provides quasi explicit derivative pricing formulas even in sophisticated models like the R.N. switching regimes GARCH models generalizing those proposed by Heston and Nandi (2000) and Christoffersen, Heston and Jacobs (2006). In the second domain, we show how both the Back Modelling and the R.N. Constrained Direct Modelling strategies provide models able to generate, at the same time, nonlinear historical dynamics and tractable pricing procedures. In particular, we show how the introduction of lags and switching regimes lead to a rich and tractable modelling of the term structure of interest rates [see Monfort and Pegoraro (2007)].

The strategies formalized in this paper have been already used, more or less explicitly, in the continuous-time literature. However, it is worth noting that, very often, rather specific Direct Modelling or Back Modelling strategies are used: the dynamics of the factor is assumed to be affine under the historical (the R.N., respectively) probability, the risk sensitivity vector (and the short rate) is specified as affine function of the factor, and the R.N. (historical, respectively) dynamics is found to be also affine once the Girsanov change of probability measure is applied[6]. These strategies could be called "basic" Direct and Back Modelling strategies.

If we consider the option pricing literature, the stochastic volatility (SV) diffusion models [based on Heston(1993)] with jumps [in the return and/or volatility dynamics] of Bates (2000), Pan (2002) and Eraker (2004) are derived following this basic Direct Modelling strategy. The pricing models proposed by Bakshi, Cao and Chen (1997, 2000) can be seen as an application of the basic Back

[5]See also, in the (security market and) option pricing literature, among the others, Christoffersen, Elkamhi and Jacobs (2005), Christoffersen, Jacobs and Wang (2006), Duan (1995), Duan, Ritchken and Sun (2005), and Leon, Mencia and Sentana (2007). With regard to the term structure modelling, see Ang and Piazzesi (2003), Ang, Piazzesi and Wei (2006), Ang, Bekaert and Wei (2008), Backus, Foresi and Telmer (1998), Gourieroux and Sufana (2003) and Monfort and Pegoraro (2006a). In the general equilibrium setting see, among the others, Garcia and Renault (1998), Garcia, Luger and Renault (2003), and Eraker (2007).

[6]See Duffie, Filipovic and Schachermeyer (2003) for a general mathematical characterization of continuous time affine processes with jumps.

Modelling approach, given that they work directly under the R.N. (pricing) probability measure. Even if these (affine) parametric specifications are able to explain relevant empirical features of asset price dynamics, the introduction of nonlinearities in the $\mathbb{P}$-dynamics of the factor seems to be very important, as suggested by Chernov, Gallant, Ghysels and Tauchen (2003) and Garcia, Ghysels and Renault (2003).

In the continuous-time term structure literature, for instance, Duffie and Kan (1996) and Cheridito, Filipovic and Kimmel (2007) follow the basic Direct Modelling strategy, while Dai and Singleton (2000, 2002) and Duffee (2002) use the basic Back Modelling counterpart. In other words, the classes of completely and essentially affine term structure models are derived following a basic Direct or Back Modelling strategy. A notable exception, however, is given by the semi-affine model of Duarte (2004). Following a Back Modelling approach, he proposes a square-root bond pricing model which is affine under the R.N. probability, but not under the historical, given the non-affine specification of the market price of risk. This nonlinearity improves the model's ability to match the time variability of the term premium, but it is not able to solve the tension between the matching of the first and the second conditional moments of yields [see Dai and Singleton (2002) and Duffee (2002)], and it makes the estimation more difficult (less precise) given that the likelihood function of yield data becomes intractable.

The last example highlights the kind of limits typically affecting the continuous-time setting: the affine specification is necessary (under both $\mathbb{P}$ and $\mathbb{Q}$ measures) to make the econometric analysis of the model tractable, and, therefore, certain relevant nonlinearities are missed. As indicated above, we can overcome these limits in our discrete-time asset pricing setting if the right strategy is followed. For instance, Dai, Le and Singleton (2006), following a well chosen Back modelling strategy, propose a nonlinear discrete-time term structure model which nests (the discrete-time equivalent of) the specifications adopted in Duffee (2002), Duarte (2004) and Cheridito, Filipovic and Kimmel (2007). In their work, the $\mathbb{Q}$-dynamics of the factor is Car, the market price of risk is assumed to be a nonlinear (polynomial) function of the factor, the $\mathbb{P}$-dynamics is not Car, and the likelihood function of the bond yield data is known in closed form. This nonlinearity is shown to significantly improve the statistical fit and the out-of-sample forecasting performance of the nested models.

The paper is organized as follows. In Section 2 we define the historical and risk-neutral dynamics of the factor, and the SDF. In Section 3 we briefly review Car processes and their main properties, we introduce the important notion of (Internally and Externally) Extended Car (ECar) process, we provide several examples and we briefly describe the pricing of derivative products when the underlying asset is Car (or ECar) in the Risk-Neutral world. In Section 4 we discuss the status of the short rate, we describe the various modelling strategies for the specification of an EAPM, and we present the associated inference problem. Sections 5 and 6 consider, respectively, applications to Econometric Security Market Models and to Econometric Term Structure Models, while, in Section 7 we present an example of Security Market Model with stochastic dividends and short rate. Section 8 concludes, and the proofs are gathered in the appendices.

2 Historical and Risk-Neutral Dynamics

2.1 Information and Historical Dynamics

We consider an economy between dates 0 and T. The new information in the economy at date t is denoted by w_t, the overall information at date t is $\underline{w_t} = (w_t, w_{t-1}, ..., w_0)$, and the σ-algebra generated by $\underline{w_t}$ is denoted $\sigma(\underline{w_t})$. The random variable w_t is called a factor or a state vector, and it may be observable, partially observable or unobservable by the econometrician. The size of w_t

is K.

The historical dynamics of w_t is defined by the joint distribution of $\underline{w}_T$, denoted by $\mathbb{P}$, or by the conditional p.d.f. (with respect to some measure):

$$f_t(w_{t+1}|\underline{w}_t)\,,$$

or by the conditional Laplace transform (L.T.):

$$\varphi_t(u|\underline{w}_t) = E[\exp(u'w_{t+1})|\underline{w}_t]\,,$$

which is assumed to be defined in an open convex set of $\mathbb{R}^K$ (containing zero). We also introduce the conditional Log-Laplace transform :

$$\psi_t(u|\underline{w}_t) = \mathrm{Log}[\varphi_t(u|\underline{w}_t)]\,.$$

The conditional expectation operator, given $\underline{w}_t$, is denoted by E_t. $\varphi_t(u|\underline{w}_t)$ and $\psi_t(u|\underline{w}_t)$ will be also denoted by $\varphi_t(u)$ and $\psi_t(u)$.

2.2 The Stochastic Discount Factor (SDF)

Let us denote by L_{2t} the (Hilbert) space of square integrable functions[7] $g(\underline{w}_t)$. Following Hansen and Richard (1987) we consider the following assumptions :

A1 (Existence and uniqueness of a price): Any payoff $g(\underline{w}_s)$ of L_{2s}, delivered at s, has a unique price at any $t < s$, for any $\underline{w}_t$, denoted by $p_t[g(\underline{w}_s)]$, function of $\underline{w}_t$.

A2 (Linearity and continuity):

- $p_t\left[\lambda_1 g_1(\underline{w}_s) + \lambda_2 g_2(\underline{w}_s)\right] = \lambda_1 p_t\left[g_1(\underline{w}_s)\right] + \lambda_2 p_t\left[g_2(\underline{w}_s)\right]$ (law of one price)
- if $g_n(\underline{w}_s) \xrightarrow[n\to\infty]{L_{2s}} 0$, $p_t\left[g_n(\underline{w}_s)\right] \underset{n\to\infty}{\longrightarrow} 0$.

A3 (Absence of Arbitrage Opportunity): At any $t \in \{0,\dots,T\}$ it is impossible to constitute a portfolio (of future payoffs), possibly modified at subsequent dates, such that: *i*) its price at t is non positive; *ii*) its payoffs at subsequent dates are non negative; *iii*) there exists at least one date $s > t$ such that the net payoff, at s, is strictly positive with a strictly positive conditional probability at t.

Under **A1**, **A2** and **A3**, a conditional version of the Riesz representation theorem implies, for each $t \in \{0,\dots,T-1\}$, the existence and uniqueness of the stochastic discount factor $M_{t,t+1}(\underline{w}_{t+1})$, belonging to $L_{2,t+1}$, such that the price at date t of the payoff $g(\underline{w}_s)$ delivered at $s > t$ is given by [see Appendix 1] :

$$p_t\left[g(\underline{w}_s)\right] = E_t\left[M_{t,t+1}\ldots M_{s-1,s}\,g(\underline{w}_s)\right]\,. \tag{1}$$

Moreover, under **A3**, $M_{t,t+1}$ is positive for each $t \in \{0,\dots,T-1\}$. The process $M_{0t} = \prod_{j=0}^{t-1} M_{j,j+1}$ is called the state price deflator over the period $\{0,\dots,t\}$.

Since $L_{2,t+1}$ contains 1, the price at t of a zero-coupon bond maturing at $t+1$ is :

$$B(t,1) = \exp(-r_{t+1}) = E_t(M_{t,t+1}),$$

where r_{t+1} is the (geometric) short rate, between t and $t+1$, known at t.

[7] We do not distinguish functions which are equal almost surely.

2.3 Exponential-affine SDF

We assume that $M_{t,t+1}(\underline{w}_{t+1})$ has an exponential-affine form :

$$M_{t,t+1} = \exp\left[\alpha_t(\underline{w}_t)' w_{t+1} + \beta_t(\underline{w}_t)\right] ,$$

where α_t is the "factor loading" or "risk sensitivity" vector[8]. Since $\exp(-r_{t+1}) = E_t(M_{t,t+1}) = \exp\left[\psi_t(\alpha_t \,|\, \underline{w}_t) + \beta_t\right]$, the SDF can also be written :

$$M_{t,t+1} = \exp\left[-r_{t+1}(\underline{w}_t) + \alpha_t'(\underline{w}_t) w_{t+1} - \psi_t(\alpha_t|\underline{w}_t)\right] . \tag{2}$$

In the case where w_{t+1} is a vector of geometric returns of basic assets or a vector of yields, the risk sensitivity vector $\alpha_t(\underline{w}_t)$ can be seen, respectively, as the opposite of a market price of risk vector, or as a market price of risk vector [see Appendix 2 for a complete proof]. More precisely, if we consider the vector of arithmetic returns $\rho_{A,t+1}$ of the basic assets in the first case, and of zero-coupon bonds in the second case, the arithmetic risk premia $\pi_{At} = E_t(\rho_{A,t+1}) - r_{A,t+1}e$ (where $r_{A,t+1}$ is the arithmetic risk-free rate, and where e denotes the unitary vector) is given by $\pi_{At} = -\exp(r_{t+1})\Sigma_t\alpha_t$ in the first case, and it is $\pi_{At} = \exp(r_{t+1})\Sigma_t\alpha_t$ in the second case (Σ_t is the conditional variance-covariance matrix of w_{t+1} given $\underline{w}_t$).

2.4 Risk-Neutral Dynamics

The joint historical distribution of $\underline{w}_T$, denoted by $\mathbb{P}$, is defined by the conditional distribution of w_{t+1} given $\underline{w}_t$, characterized either by the p.d.f. $f_t(w_{t+1}|\underline{w}_t)$ or the Laplace transform $\varphi_t(u|\underline{w}_t)$, or the Log-Laplace transform $\psi_t(u|\underline{w}_t)$.

The Risk-Neutral (R.N.) dynamics is another joint distribution of $\underline{w}_T$, denoted by $\mathbb{Q}$, defined by the conditional p.d.f., with respect to the corresponding conditional historical probability, given by :

$$\begin{aligned} d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) &= \frac{M_{t,t+1}(\underline{w}_{t+1})}{E_t\left[M_{t,t+1}(\underline{w}_{t+1})\right]} \\ &= \exp(r_{t+1})M_{t,t+1}(\underline{w}_{t+1}). \end{aligned}$$

So, the R.N. conditional p.d.f. (with respect to the same measure as the corresponding conditional historical probability) is :

$$f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t) = f_t(w_{t+1}|\underline{w}_t) d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t),$$

and the conditional p.d.f. of the conditional historical distribution with respect to the R.N. one is given by :

$$d_t^{\mathbb{P}}(w_{t+1}|\underline{w}_t) = \frac{1}{d_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t)}.$$

[8] The justification of this exponential-affine specification is now well documented in the asset pricing literature. First, this form naturally appears in equilibrium models like CCAPM [see e.g. Cochrane (2005)], Consumption-based asset pricing models with habit formation or with Epstein-Zin preferences [see, among the others, Bansal and Yaron (2004), Campbell and Cochrane (1999), Eraker (2007), Garcia, Meddahi and Tedongap (2006), Garcia, Renault and Semenov (2006)]. Second, in general continuous time security market models the discretized version of the SDF is exponential-affine [see Gourieroux and Monfort (2007)]. Third, the exponential-affine specification is particularly well adapted to the Laplace Transform which is a central tool in discrete-time asset pricing theory [see e.g. Bertholon, Monfort and Pegoraro (2006), Darolles, Gourieroux and Jasiak (2006), Gourieroux, Jasiak and Sufana (2004), Gourieroux, Monfort and Polimenis (2003, 2006), Monfort and Pegoraro (2006a, 2006b, 2007), Pegoraro (2006), Polimenis (2001)].

When the SDF is exponential-affine, we have the convenient additional result:

$$\begin{aligned} d_t^{\mathbb{Q}}(w_{t+1}|\underline{w_t}) &= \frac{\exp(\alpha_t' w_{t+1} + \beta_t)}{E_t \exp(\alpha_t' w_{t+1} + \beta_t)} \\ &= \exp\left[\alpha_t' w_{t+1} - \psi_t(\alpha_t)\right], \end{aligned}$$

so $d_t^{\mathbb{Q}}$ is also exponential-affine. It is readily seen that the conditional R.N. Laplace transform of the factor w_{t+1}, given $\underline{w_t}$, is [see Gourieroux and Monfort (2007)]:

$$\varphi_t^{\mathbb{Q}}(u|\underline{w_t}) = \frac{\varphi_t(u+\alpha_t)}{\varphi_t(\alpha_t)}$$

and, consequently, the associated conditional R.N. Log-Laplace transform is :

$$\psi_t^{\mathbb{Q}}(u) = \psi_t(u+\alpha_t) - \psi_t(\alpha_t)\,. \tag{3}$$

Conversely, we get :

$$d_t^{\mathbb{P}}(w_{t+1}|\underline{w_t}) = \exp\left[-\alpha_t' w_{t+1} + \psi_t(\alpha_t)\right]$$

and, taking $u = -\alpha_t$ in $\psi_t^{\mathbb{Q}}(u)$, we can write :

$$\psi_t^{\mathbb{Q}}(-\alpha_t) = -\psi_t(\alpha_t) \tag{4}$$

and, replacing u by $u - \alpha_t$, we obtain :

$$\psi_t(u) = \psi_t^{\mathbb{Q}}(u-\alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t). \tag{5}$$

We also have :

$$d_t^{\mathbb{P}}(w_{t+1}|\underline{w_t}) = \exp\left[-\alpha_t' w_{t+1} - \psi_t^{\mathbb{Q}}(-\alpha_t)\right],$$

$$d_t^{\mathbb{Q}}(w_{t+1}|\underline{w_t}) = \exp\left[\alpha_t' w_{t+1} + \psi_t^{\mathbb{Q}}(-\alpha_t)\right].$$

3 Car (Compound Autoregressive) and Extended Car (ECar) Processes

For sake of completeness we give, in this section, a brief review of Car (or discrete time affine) processes and of their main properties [for more details, see Darolles, Gourieroux and Jasiak (2006), Gourieroux and Jasiak (2006), and Gourieroux, Monfort and Polimenis (2006)]. We will also introduce the notion of Extended Car process, which will be very useful in the rest of the paper. All the processes $\{y_t\}$ considered will be such that y_t is a function of the information at time t: $\underline{w_t}$.

3.1 Car(1) Processes

A n-dimensional process $\{y_t\}$ is called Car(1) if its conditional Laplace transform $\varphi_t(u\,|\,\underline{y_t}) = E[\exp(u' y_{t+1})\,|\,\underline{y_t}]$ is of the form:

$$\varphi_t(u\,|\,\underline{y_t}) = \exp[a_t(u)' y_t + b_t(u)]\,, \quad u \in \mathbb{R}^n\,, \tag{6}$$

where a_t and b_t may depend on t in a deterministic way. The Log-Laplace transform $\psi_t(u\,|\,\underline{y_t}) = \operatorname{Log}\varphi_t(u\,|\,\underline{y_t})$ is therefore affine in y_t, which implies that all the conditional cumulants, and in particular the conditional mean and the conditional variance-covariance matrix, are affine in y_t. Let us consider some examples of Car(1) processes.

i) Gaussian AR(1) processes

If y_{t+1} is a Gaussian AR(1) process defined by:

$$y_{t+1} = \mu + \rho y_t + \varepsilon_{t+1}$$

where ε_{t+1} is a gaussian white noise distributed as $\mathcal{N}(0, \sigma^2)$, then the process is Car(1) with $a(u) = u\rho$ and $b(u) = u\mu + \frac{\sigma^2}{2}u^2$.

ii) Compound Poisson processes (or integer valued AR(1) processes)

If y_{t+1} is defined by:

$$y_{t+1} = \sum_{i=1}^{y_t} z_{it} + \varepsilon_{t+1}$$

where the z_{it}'s follow independently the Bernoulli distribution $\mathcal{B}(\rho)$ of parameter $\rho \in]0,1[$, and the ε_{t+1}'s follow independently (and independently from the z_{it}'s) a Poisson distribution $\mathcal{P}(\lambda)$ of parameter $\lambda > 0$. It is easily seen that $\{y_t\}$ is Car(1) with $a(u) = \text{Log}[\rho \exp(u) + 1 - \rho]$ and $b(u) = -\lambda[1 - \exp(u)]$.

In particular, the correlation between y_{t+1} and y_t is given by ρ, and we can write $y_{t+1} = \lambda + \rho y_t + \eta_{t+1}$, where η_{t+1} is a martingale difference and, therefore, $\{y_t\}$ is an integer valued weak AR(1) process.

iii) Autoregressive Gamma processes (ARG(1) or positive AR(1) processes)

The ARG(1) process y_{t+1} is the exact discrete-time equivalent of the square-root (CIR) diffusion process, and it can be defined in the following way:

$$\frac{y_{t+1}}{\mu} | z_{t+1} \sim \gamma(\nu + z_{t+1})\,, \;\; \nu > 0\,,$$

$$z_{t+1} | y_t \sim \mathcal{P}(\rho y_t / \mu)\,, \;\; \rho > 0\,, \mu > 0\,,$$

where γ denotes a Gamma distribution, μ is the scale parameter, ν is the degree of freedom, ρ is the correlation parameter, and z_t is the mixing variable. The conditional probability density function $f(y_{t+1} \,|\, y_t; \mu, \nu, \rho)$ (say) of the ARG(1) process is a mixture of Gamma densities with Poisson weights. It is easy to verify that $\{y_t\}$ is Car(1) with $a(u) = \frac{\rho u}{1 - u\mu}$ and $b(u) = -\nu \text{Log}(1 - u\mu)$. Moreover, we have:

$$y_{t+1} = \nu\mu + \rho y_t + \eta_{t+1}\,,$$

where η_{t+1} is a martingale difference sequence, so $\{y_t\}$ is a positive weak AR(1) process with $E[y_{t+1} \,|\, y_t] = \nu\mu + \rho y_t$ and $V[y_{t+1} \,|\, y_t] = \nu\mu^2 + 2\rho\mu y_t$.

It is also possible, thanks to the recursive methodology followed by Monfort and Pegoraro (2006b), to build discrete-time multivariate autoregressive gamma processes. A notable advantage of the vector ARG(1) process, with respect to the continuous time analogue, is given by its conditional probability density (and likelihood) function known in closed-form even in the case of conditionally correlated scalar components. Indeed, the multivariate CIR process has a known discrete transition density only in the case of uncorrelated components and, therefore, in continuous-time this particular case, only, opens the possibility for an exact maximum likelihood estimation approach.

iv) Wishart Autoregressive processes (or positive definite matrix valued AR(1) processes)

The Wishart Autoregressive (WAR) process y_{t+1} is a process valued in the space of $(n \times n)$ symmetric positive definite matrices, such that its conditional historical Log-Laplace transform is given by:

$$\begin{aligned} \psi_t(\Gamma) &= \text{Log}\{E_t \exp(Tr\Gamma y_{t+1})\} \\ &= Tr\left[M'\Gamma(I_n - 2\Sigma\Gamma)^{-1}My_t\right] - \tfrac{K}{2}\text{Log}\ \det[(I_n - 2\Sigma\Gamma)]\,, \end{aligned} \tag{7}$$

where Γ is a $(n \times n)$ matrix of coefficients, which can be chosen symmetric [since, with obvious notations, $Tr(\Gamma y_{t+1}) = \sum_{i\,j} \Gamma_{ij} y_{ij,t+1} = \sum_{i \leq j} (\Gamma_{ij} + \Gamma_{ji}) y_{ij,t+1}$]. This dynamics is Car(1) and, if K is integer, it can be defined as:

$$\begin{aligned} & y_t = \sum_{k=1}^{K} x_{k,t} x'_{k,t}\,, \quad (K \geq n) \\ & x_{k,t+1} = M x_{k,t} + \varepsilon_{k,t+1}\,,\ k \in \{1, \ldots, K\} \\ & \varepsilon_{k,t+1} \overset{\mathbb{P}}{\sim} IIN(0, \Sigma)\,,\ k \in \{1, \ldots, K\}\,,\ \text{independent.} \end{aligned} \tag{8}$$

Moreover, we have:

$$y_{t+1} = M y_t M' + k\Omega + \eta_{t+1}\,,$$

where η_{t+1} is a matrix martingale difference. So, $\{y_t\}$ is a positive definite matrix valued AR(1) process. Note that, if $n = 1$, $\Gamma = u$, $M = m$ and $\Omega = \sigma^2$, relation (7) reduces to $\psi_t^{\mathbb{P}}(u) = [\frac{m^2}{1-2\sigma^2 u} - \frac{k}{2}\text{Log}(1 - 2\sigma^2 u)]$, and $\{y_t\}$ is found to be an ARG(1) process with $\rho = m^2$, $\nu = k/2$, $\mu = 2\sigma^2$. This means that the Wishart Autoregressive process is a multivariate (matrix) generalization of the ARG(1) process.

v) Markov Chains

Let us consider a J-state homogeneous Markov Chain y_{t+1}, which can take the values $e_j \in \mathbb{R}^J$, $j \in \{1, \ldots, J\}$, where e_j is the j^{th} column of the $(J \times J)$ identity matrix I_J. The transition probability, from state e_i to state e_j is $\pi(e_i, e_j) = Pr(y_{t+1} = e_j \,|\, y_t = e_i)$. The process $\{y_t\}$ is a Car(1) process with:

$$\begin{aligned} a(u) &= \left[\log\left(\textstyle\sum_{j=1}^{J} \exp(u'e_j)\pi(e_1, e_j)\right), \ldots, \log\left(\textstyle\sum_{j=1}^{J} \exp(u'e_j)\pi(e_J, e_j)\right)\right]'\,, \\ b(u) &= 0\,. \end{aligned}$$

3.2 Extended Car(1) (or ECar(1)) processes

An important generalization of the Car(1) family is given by the family of Extended Car(1) [ECar(1)] processes.

Definition: A process $\{y_{1,t}\}$ is said to be ECar(1) if there exists a process $\{y_{2,t}\}$ such that $y_t = (y'_{1,t}, y'_{2,t})'$ is Car(1). Moreover, if the σ-algebra $\sigma(\underline{y}_{1t})$ spanned by $\underline{y}_{1t}$ is equal to $\sigma(\underline{y}_t)$, $\{y_{1,t}\}$ will be called Internally Extended Car(1) process. Otherwise, if $\sigma(\underline{y}_{1t}) \subset \sigma(\underline{y}_t)$, $\{y_{1,t}\}$ will be called Externally Extended Car(1) process.

3.2.1 Internally Extended Car(1) Processes

Car(p) processes

The process $y_{1,t+1}$ is Car(p) if its conditional Log-Laplace transform satisfies:

$$\psi_t(u \,|\, \underline{y}_t) = \sum_{i=1}^{p} a_{i,t}(u)' y_{t+1-i} + b_t(u)\,, \quad u \in \mathbb{R}^n\,. \tag{9}$$

It easily seen that the process $y_t = (y'_{1,t}, y'_{2,t})'$, with $y_{2,t} = (y'_{1,t-1}, \ldots, y'_{1,t-p+1})'$, is Car(1) [see Darolles, Gourieroux and Jasiak (2006)], and that $\sigma(\underline{y}_{1t}) = \sigma(\underline{y}_t)$. Moreover, starting from a Car(1) process, we can easily construct Index-Car(p) processes like ARG(p) and Gaussian AR(p) processes [see Monfort and Pegoraro (2007)].

ARMA processes

If we consider an ARMA$(1,1)$ process $\{y_{1,t}\}$ defined by:

$$y_{1,t+1} - \varphi y_{1,t} = \varepsilon_{t+1} - \theta \varepsilon_t\,,$$

where $\varepsilon_{t+1} \sim IIN(0, \sigma^2)$, it is well known that y_{t+1} is not Markovian and, consequently, it is not Car(1), or even Car(p). However, using the state-space representation of ARMA processes, we have that the process $y_t = (y_{1,t}, \varepsilon_t)'$ satisfies:

$$y_{t+1} = \begin{bmatrix} \varphi & -\theta \\ 0 & 0 \end{bmatrix} y_t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \varepsilon_{t+1}\,. \tag{10}$$

This means that $\{y_t\}$ is Car(1) since it is a Gaussian bivariate AR(1) process, and that $\{y_{1,t}\}$ is a ECar(1) process. Clearly, $\sigma(\underline{y}_{1t}) = \sigma(\underline{y}_t)$, so $\{y_{1,t}\}$ is an Internally ECar(1). It is important to observe that, in the bivariate AR(1) representation (10), one eigenvalue of the autoregressive matrix is equal to zero and, therefore, this process has no continuous time bivariate Ornstein-Uhlenbeck analogue, since in this kind of process the autoregressive matrix Φ (say) is of the form $\Phi = \exp(A)$. This result is also a consequence of the fact that a discrete-time ARMA(p,q), with $q \geq p$, cannot be embedded in a continuous-time ARMA (CARMA) process [see Brockwell (1995), Huzii (2007)]. This example of Extended Car process can obviously be generalized to ARMA(p,q) and VARMA(p,q) processes. The VARMA model belongs also to the class of generalized affine models proposed in finance by Feunou and Meddahi (2007) to provide tractable derivative prices.

GARCH-Type processes

Let us consider the process $\{y_{1,t}\}$ defined by:

$$\begin{cases} y_{1,t+1} &= \mu + \varphi y_{1,t} + \sigma_{t+1}\varepsilon_{t+1}\,, \\ \sigma^2_{t+1} &= \omega + \alpha\varepsilon^2_t + \beta\sigma^2_t\,, \end{cases}$$

where $\varepsilon_{t+1} \sim IIN(0,1)$. $\{y_{1,t}\}$ is not Car(1), but the (extended) process $y_t = (y_{1,t}, \sigma^2_{t+1})'$ is Car(1). Indeed, we have that:

$$\begin{aligned} &E[\exp(u y_{1,t+1} + v\sigma^2_{t+2}) \,|\, y_{1,t}, \sigma^2_{t+1}] \\ &= \exp\left[\left(u\mu + v\omega - \tfrac{1}{2}\mathrm{Log}(1 - 2v\alpha)\right) + u\varphi y_{1,t} + \left(v\beta + \frac{u^2}{2(1-2v\alpha)}\right)\sigma^2_{t+1}\right]. \end{aligned}$$

Therefore, $\{y_{1,t}\}$ is ECar(1) and $\sigma(\underline{y}_{1t}) = \sigma(\underline{y}_t)$. Section 5.6 shows that this result still applies when switching regimes are introduced. Observe that this model [called also Heston and Nandi (2000) model] is not a generalized affine one, and it belongs to the class of generalized non-affine models mentioned in Feunou and Meddahi (2007).

3.2.2 Externally Extended Car(1) Processes

Quadratic transformation of Gaussian AR(1) processes

Let us consider the following Gaussian AR(1) process:

$$x_{t+1} = \mu + \rho x_t + \varepsilon_{t+1}\,, \quad \varepsilon_{t+1} \sim IIN(0, \sigma^2)\,.$$

If $\mu = 0$ the process $y_{1,t} = x_t^2$ is Car(1). If $\mu \neq 0$, the process $y_{1,t} = x_t^2$ is not Car, however it can be shown that $y_t = (y_{1,t}, x_t)'$ is Car(1) [see Gourieroux and Sufana (2003)] and, thus, $y_{1,t}$ is ECar(1) [see Section 6.4 for a proof in a multivariate context]. Obviously, we have $\sigma(\underline{y}_{1t}) \subset \sigma(\underline{y}_t)$.

Switching regimes Gaussian AR(1) processes

In the classical Gaussian AR(1) model defined in Section 3.1.i), the conditional distribution of y_{t+1}, given $\underline{y}_t$, has a skewness $\tilde{\mu}_3 = 0$ and a kurtosis $\tilde{\mu}_4 = 3$. If we want to introduce a more flexible specification for $\tilde{\mu}_3$ and $\tilde{\mu}_4$, a first possibility is to assume that ε_{t+1} is still a zero mean, unit variance white noise, but with a distribution belonging to some parametric family [like, for instance, the truncated Gram-Charlier expansion used by Jondeau and Rockinger (2001) to price foreign exchange options, or the semi-nonparametric (SNP) distribution employed by Léon, Mencia and Sentana (2006) for European-type option pricing]. However, this approach has some drawbacks: the set of possible pair of conditional skewness-kurtosis of y_{t+1} (i.e., the set of skewness and kurtosis generated by ε_{t+1}) is not the maximal set $\mathcal{D} = \{(\tilde{\mu}_3, \tilde{\mu}_4) \in \mathbb{R} \times \mathbb{R}_+^* : \tilde{\mu}_4 \geq \tilde{\mu}_3^2 + 1\}$ and, moreover, $\tilde{\mu}_3$ and $\tilde{\mu}_4$ do not depend on $\underline{y}_t$. One way to solve these problems is to consider a 2-state switching regimes Gaussian AR(1) process $\{y_{1,t}\}$ given by:

$$y_{1,t+1} = \mu' y_{2,t+1} + \rho y_{1,t} + (\sigma' y_{2,t+1})\varepsilon_{t+1}\,,$$

where $\varepsilon_{t+1} \sim IIN(0,1)$, $\mu' = (\mu_1, \mu_2)$, $\sigma' = (\sigma_1, \sigma_2)$, and where $\{y_{2,t}\}$ is a 2-state homogeneous Markov chain [as defined in Section 3.1.v)] with $\pi(e_1, e_1) = p$ and $\pi(e_2, e_2) = q$, independent of $\{\varepsilon_t\}$. The Laplace transform of $y_{1,t+1}$, conditionally to $\underline{y}_{1,t}$, is not exponential-affine, but it is easy to verify that the bivariate process $y_t = (y_{1,t}, y_{2,t})'$ is Car(1) [see Monfort and Pegoraro (2007)]. In other words, $y_{1,t}$ is an Externally ECar(1) process, given the additional information introduced by the Markov chain.

Given that the probability density function of $y_{1,t+1}$, conditionally to $\underline{y}_t$, is a mixture of the Gaussian densities $n(y_{1,t+1}; \mu_j + \rho y_{1,t}, \sigma_j^2)$, with $j \in \{1, 2\}$, this kind of Car process is able to generate (conditionally to $\underline{y}_t$) stochastic skewness [$\tilde{\mu}_3(\underline{y}_t)$, say] and kurtosis [$\tilde{\mu}_4(\underline{y}_t)$, say] and, moreover, it is able to reach, for each time t, any possible pair of skewness-kurtosis in the domain of maximal size $\mathcal{D}_t = \{(\tilde{\mu}_3(\underline{y}_t), \tilde{\mu}_4(\underline{y}_t)) \in \mathbb{R} \times \mathbb{R}_+^* : \tilde{\mu}_4(\underline{y}_t) \geq \tilde{\mu}_3(\underline{y}_t)^2 + 1\}$ [see Bertholon, Monfort and Pegoraro (2006) for a formal proof].

It is important to highlight that these features do not characterize just the distribution of $y_{1,t+1}$ conditionally to both its own past ($\underline{y}_{1,t}$) and the past of the latent variable ($\underline{z}_t$). Indeed, the distribution of $y_{1,t+1}$, conditionally only to its own past $\underline{y}_{1,t}$, is still a mixture of Gaussian distributions with probability density function given by:

$$f(y_{1,t+1}\,|\,\underline{y}_{1,t}) = n(y_{1,t+1};\mu_1+\rho y_{1,t},\sigma_1^2)[pP(y_{2,t}=e_1\,|\,\underline{y}_{1,t})+(1-q)P(y_{2,t}=e_2\,|\,\underline{y}_{1,t})]$$
$$+n(y_{1,t+1};\mu_2+\rho y_{1,t},\sigma_2^2)[(1-p)P(y_{2,t}=e_1\,|\,\underline{y}_{1,t})+qP(y_{2,t}=e_2\,|\,\underline{y}_{1,t})]\,.$$

In Section 5, we will see that, thanks to the exponential-affine specification (2) of the SDF, these statistical properties (used to describe the dynamics of geometric returns) are transferred from the historical to the risk-neutral distribution, with important pricing implications.

Stochastic volatility in mean processes

We can specify also a stochastic volatility in mean AR(1) process defined by:

$$y_{1,t+1}=\mu_1+\mu_2 y_{2,t+1}+\rho y_{1,t}+y_{2,t+1}^{1/2}\varepsilon_{t+1}\,,$$

where $\varepsilon_{t+1}\sim IIN(0,1)$, and where $\{y_{2,t}\}$ is an ARG(1) process, as defined in Section 3.1.*iii*), independent of $\{\varepsilon_t\}$. The process $\{y_{1,t}\}$ is an Externally ECar(1) since $y_t=(y_{1,t},y_{2,t})$ is Car(1) and $\sigma(\underline{y}_{1t})\subset\sigma(\underline{y}_t)$. We can also consider a n-variate stochastic volatility in mean AR(1) process defined by:

$$y_{1,t+1}=\mu+Ry_{1,t}+\begin{bmatrix}TrS_1y_{2,t+1}\\ \vdots\\ TrS_ny_{2,t+1}\end{bmatrix}+y_{2,t+1}^{1/2}\varepsilon_{t+1}\,,$$

where $\varepsilon_{t+1}\sim IIN(0,I)$, of size n, R is a $(n\times n)$ matrix, the S_i's are $(n\times n)$ symmetric matrices, and $\{y_{2,t}\}$ is an n-dimensional Wishart Autoregressive process independent of $\{\varepsilon_t\}$. In this multivariate setting, $\{y_{1,t}\}$ is an n-dimensional ECar(1) process, because $y_t=(y'_{1,t},vech(y_{2,t})')'$ is Car(1) [see Gourieroux, Jasiak and Sufana (2004) and, in continuous time, Buraschi, Porchia and Trojani (2007), Da Fonseca, Grasselli and Tebaldi (2007a, 2007b), Da Fonseca, Grasselli and Ielpo (2008)].

3.3 Pricing with R.N. Car(1) or ECar(1) processes

It is well known that if $\{y_t\}$ is Car(1) with conditional Laplace transform $\varphi_t(u\,|\,\underline{y}_t)=\exp[a(u)'y_t+b(u)]$, the multi-horizon (conditional) Laplace transform takes the following exponential-affine form:

$$E_t\left[\exp(u'_{t+1}y_{t+1}+\ldots+u'_Ty_T)\right]=\exp\left[A_T(t)'y_t+B_T(t)\right]\,,$$

where the functions A_T and B_T are easily computed recursively, for $j\in\{T-t-1,\ldots,0\}$, by:

$$A_T(t+j) = a_{t+j+1}[u_{t+j+1}+A_T(t+j+1)]\,,$$
$$B_T(t+j) = b_{t+j+1}[u_{t+j+1}+A_T(t+j+1)]+B_T(t+j+1)\,,$$

starting from the terminal conditions $A_T(T)=0$, $B_T(T)=0$.

If we want to determine the price at t of a payoff $g(\underline{y}_T)$ at T, we have to compute a conditional expectation under the risk-neutral probability, namely $E_t^{\mathbb{Q}}[\exp(r_{t+1}+\ldots+r_T)\,g(\underline{y}_T)]$. If $\{y_t\}$ is Car(1) or ECar(1) in the risk-neutral world, this computation leads to explicit or quasi explicit pricing formulas for several derivative products. For instance, if the one-period risk-free rate r_{t+1} is exogenous or affine in $\underline{y}_t$ and if $g(\underline{y}_T)=[\exp(\mu'_1y_{t,T})-\exp(\mu'_2y_{t,T})]^+$, where $y_{t,T}=(y'_t,\ldots,y'_T)'$, the computation reduces to two truncated multi-horizon Laplace transforms which, in turn, are obtained by simple integrals based on the untruncated complex Laplace transform easily deduced from the recursive equations given above [see Bakshi and Madan (2000) and Duffie, Pan and Singleton (2000), Gourieroux, Monfort and Polimenis (2003), Monfort and Pegoraro (2007)].

4 Econometric Asset Pricing Models (EAPMs)

The true value of the various mathematical tools introduced in Section 2, for instance $\psi_t, M_{t,t+1}$ or $\psi_t^{\mathbb{Q}}$, are unknown by the econometrician and, therefore, they have to be specified and parameterized. In other words, we have to specify an Econometric Asset Pricing Model (EAPM). What we really need, in order to derive explicit or quasi explicit pricing formulas, is a factor w_{t+1} which is Car or ECar under the risk-neutral probability, while its historical dynamics does not necessarily belong to this family of processes [see also Duarte (2004) and Dai, Le and Singleton (2006)]. In other words, the tractability of the asset pricing model is associated to a conditional Log-Laplace transform $\psi_t^{\mathbb{Q}}$ which is affine in w_t, while the specification and parameterization of ψ_t can be more general.

We are going to present three ways of specifying an EAPM: the Direct Modelling, the R.N. Constrained Direct Modelling and the Back Modelling. In all approaches, we first need to make more precise the status of the short rate r_{t+1}.

4.1 The status of the short rate

The short rate r_{t+1} is a function of $\underline{w_t}$. This function may be known or unknown by the econometrician. It is known in two main cases :

i) r_{t+1} is exogenous, i.e. $r_{t+1}(\underline{w_t})$ does not depend on $\underline{w_t}$, and, therefore, $r_{t+1}(.)$ is a known constant function of $\underline{w_t}$;

ii) r_{t+1} is an endogenous factor , i.e. r_{t+1} is a component of w_t.

If the function $r_{t+1}(\underline{w_t})$ is unknown, it has to be specified parametrically. So we assume that the unknown function belongs to a family :

$$\left\{ r_{t+1}(\underline{w_t}, \tilde{\theta}), \tilde{\theta} \in \tilde{\Theta} \right\} ,$$

where $r_{t+1}(.,.)$ is a known function.

4.2 Direct Modelling

In the Direct Modelling approach we first specify the historical dynamics, i.e. we choose a parametric family for the conditional Log-Laplace transform $\psi_t(u \,|\, \underline{w_t})$:

$$\{\psi_t(u \,|\, \underline{w_t}, \theta_1), \theta_1 \in \Theta_1\} . \tag{11}$$

Then, we have to specify the SDF

$$\begin{aligned} M_{t,t+1} &= \exp\left[\alpha_t(\underline{w_t})' w_{t+1} + \beta_t(\underline{w_t})\right] \\ &= \exp\left[-r_{t+1}(\underline{w_t}) + \alpha_t'(\underline{w_t}) w_{t+1} - \psi_t(\alpha_t | \underline{w_t})\right] . \end{aligned}$$

Once r_{t+1} has been specified, according to its status described in Section 4.1, as well as ψ_t, the remaining function to be specified is $\alpha_t(\underline{w_t})$. We assume that $\alpha_t(\underline{w_t})$ belongs to a parametric family :

$$\{\alpha_t(\underline{w_t}, \theta_2), \theta_2 \in \Theta_2\} .$$

Finally, $M_{t,t+1}$ is specified as :

$$M_{t,t+1}(\underline{w_{t+1}}, \theta) = \exp\left\{ -r_{t+1}(\underline{w_t}, \tilde{\theta}) + \alpha_t'(\underline{w_t}, \theta_2) w_{t+1} - \psi_t\left[\alpha_t(\underline{w_t}, \theta_2) | \underline{w_t}, \theta_1\right] \right\} , \tag{12}$$

where $\theta = (\tilde{\theta}', \theta_1', \theta_2')' \in \tilde{\Theta} \times \Theta_1 \times \Theta_2 = \Theta$; note that $\tilde{\Theta}$ may be reduced to one point.

This kind of modelling may have to satisfy some Internal Consistency Conditions (ICCs). Indeed, for any payoff $g(\underline{w}_s)$ delivered at $s > t$, that has a price $p(\underline{w}_t)$ at t which is a known function of $\underline{w}_t$, we must have :

$$p(\underline{w}_t) = E\left\{M_{t,t+1}(\theta)...M_{s-1,s}(\theta)\, g(\underline{w}_s) \,|\, \underline{w}_t, \theta_1\right\} \quad \forall\, \underline{w}_t, \theta\,. \tag{13}$$

These AAO pricing conditions may imply strong constraints on the parameter θ, for instance when components of w_t are returns of some assets or interest rates with various maturities [see Sections 5 and 6].

The specification of the historical dynamics (11) and of the SDF (12) obviously implies the specification of the R.N. dynamics :

$$\psi_t^{\mathbb{Q}}(u|\underline{w}_t, \theta_1, \theta_2) = \psi_t\left[u + \alpha_t(\underline{w}_t, \theta_2)|\underline{w}_t, \theta_1\right] - \psi_t\left[\alpha_t(\underline{w}_t, \theta_2)|\underline{w}_t, \theta_1\right] .$$

The particular case in which the historical dynamics is Car, $\alpha_t(\underline{w}_t, \theta_2)$ is an affine function of the factor, along with the short rate $r_{t+1}(\underline{w}_t, \tilde{\theta})$, is the (discrete-time) counterpart of the basic Direct Modelling strategy frequently followed in continuous time.

4.3 R.N. Constrained Direct Modelling

In the previous kind of modelling, the family of R.N. dynamics $\psi_t^{\mathbb{Q}}(u|\underline{w}_t)$ is obtained as a by-product and therefore is, in general, not controlled.

In some cases it may be important to control the family of R.N. dynamics and, possibly, the specification of the short rate, if we want to have explicit or quasi-explicit formulas for the price of some derivatives. For instance, it is often convenient to impose that the R.N. dynamics be described by a Car (Compound Autoregressive) process. If we want, at the same time, to control the historical dynamics, for instance to have good fitting when w_t is observable, the by-product of the modelling becomes the factor loading vector $\alpha_t(\underline{w}_t)$. More precisely, we may wish to choose a family $\{\psi_t(u|\underline{w}_t, \theta_1), \theta_1 \in \Theta_1\}$ and a family $\{\psi_t^{\mathbb{Q}}(u|\underline{w}_t, \theta^*), \theta^* \in \Theta^*\}$ such that, for any pair $(\psi_t^{\mathbb{Q}}, \psi_t)$ belonging to these families, there exists a unique function $\alpha_t(\underline{w}_t)$ denoted by $\alpha_t(\underline{w}_t, \theta_1, \theta^*)$ satisfying :

$$\psi_t^{\mathbb{Q}}(u|\underline{w}_t) = \psi_t\left[u + \alpha_t(\underline{w}_t)|\underline{w}_t\right] - \psi_t\left[\alpha_t(\underline{w}_t)|\underline{w}_t\right] .$$

In fact, this condition may be satisfied only for a subset of pairs (θ_1, θ^*). In other words (θ_1, θ^*) belongs to Θ_1^* strictly included in $\Theta_1 \times \Theta^*$, but such that any $\theta_1 \in \Theta_1$ and any $\theta^* \in \Theta^*$ can be reached [see Section 5]. Once the parameterization $(\tilde{\theta}, \theta_1, \theta^*) \in \tilde{\Theta} \times \Theta_1^*$ is defined, internal consistency conditions similar to (13) may be imposed.

4.4 Back Modelling

The final possibility is to parameterize first the R.N. dynamics $\psi_t^{\mathbb{Q}}(u|\underline{w}_t, \theta_1^*)$, and the short rate process $r_{t+1}(\underline{w}_t)$, taking into account, if relevant, internal consistency conditions of the form:

$$p(\underline{w}_t) = E_t^{\mathbb{Q}}\left[\exp(-r_{t+1}(\underline{w}_t, \tilde{\theta}) - ... - r_s(\underline{w}_s, \tilde{\theta}))g(\underline{w}_s)|\underline{w}_t, \theta_1^*\right] \;,\; \forall \underline{w}_t\,, \tilde{\theta}\,, \theta_1^*. \tag{14}$$

Once this is done, the specification of $\alpha_t(\underline{w}_t)$ is chosen, *without any constraint*, providing the family $\{\alpha_t(\underline{w}_t, \theta_2^*), \theta_2^* \in \Theta_2^*\}$, and the historical dynamics is a by-product :

$$\psi_t(u|\underline{w}_t, \theta_1^*, \theta_2^*) = \psi_t^{\mathbb{Q}}\left[u - \alpha_t(\underline{w}_t, \theta_2^*)|\underline{w}_t, \theta_1^*\right] - \psi_t^{\mathbb{Q}}\left[-\alpha_t(\underline{w}_t, \theta_2^*)|\underline{w}_t, \theta_1^*\right] .$$

The basic Back Modelling approach (frequently adopted in continuous time) is given by the particular case in which $\psi_t^{\mathbb{Q}}(u|\underline{w}_t, \theta_1^*)$, the short rate $r_{t+1}(\underline{w}_t, \tilde{\theta})$ and the risk sensitivity vector $\alpha_t(\underline{w}_t, \theta_2^*)$ are assumed to be affine functions of the factor.

Also note that, if the R.N. conditional p.d.f. $f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t, \theta_1^*)$ is known in (quasi) closed form, the same is true for the historical conditional p.d.f.:

$$f_t(w_{t+1}|\underline{w}_t, \theta_1^*, \theta_2^*) = f_t^{\mathbb{Q}}(w_{t+1}|\underline{w}_t, \theta_1^*) \exp\left\{-\alpha_t'(\underline{w}_t, \theta_2^*)w_{t+1} - \psi_t^{\mathbb{Q}}\left[-\alpha_t(\underline{w}_t, \theta_2^*)|\underline{w}_t, \theta_1^*\right]\right\} . \qquad (15)$$

In particular, if w_t is observable we can compute the likelihood function. However the identification of the parameters (θ_1^*, θ_2^*), from the dynamics of the observable components of w_t must be carefully studied (see examples in Sections 5 and 6) and observations of derivative prices may be necessary to reach identifiability.

4.5 Inference in an Econometric Asset Pricing Model

In order to estimate an EAPM, we assume that the econometrician observes, at dates $t \in \{0, \ldots, T\}$, a set of prices x_{ti} corresponding to payoffs $g_i(\underline{w}_s), i \in \{1, ..., J_t\}$, $s > t$, given by (using the parameter notations of Direct Modelling) :

$$q_{ti}(\underline{w}_t, \theta) = E\left[g_i(\underline{w}_s)M_{t,s}(\underline{w}_s, \theta)|\underline{w}_t, \theta_1\right] , \ i \in \{1, ..., J_t\} .$$

Therefore, we have two kinds of equations representing respectively the historical dynamics of the factors and the observations:

$$w_t = \tilde{q}_t(\underline{w}_{t-1}, \varepsilon_{1t}, \theta_1) \, , \, (\text{say}) \, , \qquad (16)$$

$$x_t = q_t(\underline{w}_t, \theta) \, , \qquad (17)$$

where the first equation is a rewriting of the conditional historical distribution of w_t given $\underline{w}_{t-1}$, ε_{1t} is a white noise (which can be chosen Gaussian without loss of generality), $x_t = (x_{t1}, \ldots, x_{tJ_t})'$ and $q_t(\underline{w}_t, \theta) = [q_{t1}(\underline{w}_t, \theta), \ldots, q_{tJ_t}(\underline{w}_t, \theta)]'$.

Note that, if r_{t+1} is not a known function of $\underline{w}_t$, we must have $r_{t+1} = r_{t+1}(\underline{w}_t, \tilde{\theta})$ among equations (17), and that if some components of w_t are observed they should appear also in (17) without parameters.

System (16)-(17) is a nonlinear state space model and appropriate econometric methods may be used for inference in this system (in particular, Maximum Likelihood methods possibly based on Kalman filter, Kitagawa-Hamilton filter, Simulations-based methods or Indirect Inference).

For given x_t's, equations (17) may have no solutions in w_t's and, in this case, an additional white noise is often introduced leading to

$$x_t = q_t(\underline{w}_t, \theta) + \varepsilon_{2t} \, . \qquad (18)$$

Moreover, when w_t is (partially) observable, θ_1 may be identifiable from (16) and in this case a two step estimation method is available : i) ML estimation of θ_1 from (16); ii) estimation of θ_2, and possibly of $\tilde{\theta}$, by Nonlinear Least Square using (18) in which θ_1 is replaced by its ML estimator (and, possibly, the unobserved components of w_t are replaced by their smoothed values).

5 Applications to Econometric Security Market Modelling

5.1 General Setting

In an Econometric Security Market Model we assume that the short rate r_{t+1} is exogenous and that the first K_1 components of w_t, denoted by y_t, are observable geometric returns of K_1 basic assets. The remaining $K_2 = K - K_1$ components of w_t, denoted by z_t, are factors not observed by the econometrician. Since the payoffs $\exp(y_{j,t+1})$ delivered at $t+1$, for each $j \in \{1, ..., K_1\}$, have a price at t which are known function of $\underline{w}_t$, namely 1, we have to guarantee internal consistency conditions. In the Direct Modelling approach, and in the Risk-Neutral Constrained Direct Modelling one, these conditions are [using the notation of the (unconstrained) direct approach] :

$$1 = E_t \left\{ \exp(y_{j,t+1} - r_{t+1} + \alpha_t(\underline{w}_t, \theta_2)' w_{t+1} - \psi_t \left[\alpha_t(\underline{w}_t, \theta_2) | \underline{w}_t, \theta_1 \right] \right\} \;,\; j \in \{1, ..., K_1\}$$

or :

$$r_{t+1} = \psi_t \left[\alpha_t(\underline{w}_t, \theta_2) + e_j | \underline{w}_t, \theta_1 \right] - \psi_t \left[\alpha_t(\underline{w}_t, \theta_2) | \underline{w}_t, \theta_1 \right] \;,\; \forall \; \underline{w}_t, \theta_1, \theta_2 \,;\, j \in \{1, ..., K_1\} \,. \tag{19}$$

In the Back Modelling approach, these conditions are :

$$r_{t+1} = \psi_t^{\mathbb{Q}}(e_j | \underline{w}_t, \theta_1^*) \,,\;\; \forall \underline{w}_t, \theta_1^* \,;\, j \in \{1, ..., K_1\} \,. \tag{20}$$

If we consider the case where the factor w_{t+1} is a R.N. Car(1) process (the generalization to the case of a Car(p) process is straightforward), with conditional R.N. Log-Laplace transform $\psi_t^{\mathbb{Q}}(u \,|\, \underline{w}_t) = a^{\mathbb{Q}}(u)' w_t + b_t^{\mathbb{Q}}(u)$, the internal consistency conditions (19) or (20) are given by (using the Back Modelling notation):

$$\left\{ \begin{array}{lcl} a^{\mathbb{Q}}(e_j, \theta_1^*) & = & 0 \,, \\ & & \\ b_t^{\mathbb{Q}}(e_j, \theta_1^*) & = & r_{t+1} \,,\;\; \forall \theta_1^* \,;\, j \in \{1, ..., K_1\} \,. \end{array} \right. \tag{21}$$

5.2 Back Modelling for Nonlinear Conditionally Gaussian Models

Let us consider a conditionally Gaussian setting, and let us assume that all the components of w_t are geometric returns ($K_1 = K$), that is, we consider $w_t = y_t$. If we follow the Back Modelling approach, we specify, first, the risk-neutral Car(1) dynamics:

$$y_{t+1} | \, \underline{y}_t \overset{\mathbb{Q}}{\sim} N \left[m_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*), \Sigma^{\mathbb{Q}}(\theta_1^*) \right] \,,$$

$$\text{or} \qquad \psi_t^{\mathbb{Q}}(u | \underline{y}_t, \theta_1^*) = u' m_t^{\mathbb{Q}}(\underline{y}_t, \theta_1^*) + \tfrac{1}{2} u' \Sigma^{\mathbb{Q}}(\theta_1^*) u \;.$$

Then, we impose the internal consistency conditions, which are (with obvious notations) given by:

$$r_{t+1} = m_{jt}^{\mathbb{Q}}(\underline{y}_t, \theta_1^*) + \frac{1}{2} \Sigma_{jj}^{\mathbb{Q}} \,,\; j \in \{1, \ldots, K\} \,, \tag{22}$$

and, consequently, the conditional R.N. distribution compatible with arbitrage restrictions is:

$$N \left[r_{t+1} e - \frac{1}{2} \; vdiag \; \Sigma^{\mathbb{Q}}(\theta_1^*), \Sigma^{\mathbb{Q}}(\theta_1^*) \right]$$

i.e.

$$\psi_t^{\mathbb{Q}}(u | \underline{y}_t, \theta_1^*) = u' r_{t+1} e - \frac{1}{2} u' \; vdiag \; \Sigma^{\mathbb{Q}} + \frac{1}{2} u' \Sigma^{\mathbb{Q}} u \,.$$

Finally, choosing *any* $\alpha_t(\underline{y}_t, \theta_2^*)$, we deduce the historical dynamics :

$$\begin{aligned}\psi_t(u|\underline{y}_t, \theta_1^*, \theta_2^*) &= u' \left[r_{t+1}e - \tfrac{1}{2}\, vdiag\, \Sigma^{\mathbb{Q}}(\theta_1^*) \right. \\ &\qquad \left. - \Sigma^{\mathbb{Q}}(\theta_1^*)\alpha_t(\underline{y}_t, \theta_2^*) \right] + \tfrac{1}{2} u'\Sigma^{\mathbb{Q}}(\theta_1^*)u \,,\end{aligned}$$

which is not Car, in general, and therefore the process $\{y_t\}$ is not Gaussian. In other words, we have:

$$y_{t+1} \,|\, \underline{y}_t \overset{\mathbb{P}}{\sim} N \left[r_{t+1}e - \frac{1}{2}\, vdiag\, \Sigma^{\mathbb{Q}}(\theta_1^*) - \Sigma^{\mathbb{Q}}(\theta_1^*)\alpha_t(\underline{y}_t, \theta_2^*),\ \Sigma^{\mathbb{Q}}(\theta_1^*) \right].$$

Thus, for a given R.N. dynamics, we can reach *any* conditional historical mean of the factor, whereas the historical conditional variance-covariance matrix is the same as the R.N. one. Moreover θ_1^* and θ_2^* can be identified from the dynamics of y_t only [see Gourieroux and Monfort (2007), for a derivation of conditionally Gaussian models using the Direct Modelling approach].

This modelling generalizes the basic Black-Scholes framework to the multivariate case, with arbitrary (nonlinear) historical conditional mean. Therefore, options with any maturity have standard Black-Scholes prices, but their future values are predicted using the joint non-Gaussian historical dynamics of the factor y_t.

5.3 Back Modelling of Switching Regime Models

The class of conditionally Mixed-Normal models contains many static, dynamic, parametric, semi-parametric or nonparametric models [see Bertholon, Monfort, Pegoraro (2006), and Garcia, Ghysels and Renault (2003)]. Let us consider, for instance, the switching regime models. The factor w_t is equal to $(y_t, z_t')'$, where y_t is an observable geometric return and z_t is a J-state homogeneous Markov chain, valued in $(e_1, ..., e_J)$, and unobservable by the econometrician.

The Direct Modelling approach, described in Bertholon, Monfort, Pegoraro (2006), has two main drawbacks. First, the ICC associated to the risky asset must be solved numerically for any t. Second, the R.N. dynamics is not Car in general, and the pricing of derivatives needs simulations which, in turn, imply to solve the ICC for any t and any path.

Let us consider now the Back Modelling approach, starting from a Car R.N. dynamics defined by :

$$y_{t+1} = \nu_t + \rho y_t + \nu_1' z_t + \nu_2' z_{t+1} + (\nu_3' z_{t+1})\xi_{t+1} \,,$$

where ν_t is a deterministic function of t and where:

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j \,|\, z_t = e_i) = \pi_{ij}^* \,.$$

In other words, z_t is an exogenous Markov chain in the risk-neutral world. The conditional R.N. Laplace transform is given by :

$$\begin{aligned}\varphi_t^{\mathbb{Q}}(u, \nu) &= E_t^{\mathbb{Q}} \exp(u y_{t+1} + v' z_{t+1}) \\ &= \exp\left[u(\nu_t + \rho y_t + \nu_1' z_t)\right] E_t^{\mathbb{Q}} \exp\left[\left(u\nu_2 + \tfrac{1}{2}u^2\nu_3^2 + v \right)' z_{t+1} \right],\end{aligned} \tag{23}$$

[ν_3^2 is the vector containing the square of the components in ν_3] and we get :

$$\begin{aligned}\psi_t^{\mathbb{Q}}(u,v) &= \text{Log } \varphi_t^{\mathbb{Q}}(u,v) \\ &= u(\nu_t + \rho y_t + \nu_1' z_t) + \Lambda'(u,v,\nu_2,\nu_3,\pi^*) z_t \, ,\end{aligned}$$

where the i^{th} component of $\Lambda(u,v,\nu_2,\nu_3,\pi^*)$ is :

$$\Lambda_i(u,v,\nu_2,\nu_3,\pi^*) = \text{ Log } \sum_{j=1}^J \pi_{ij}^* \exp\left(u\nu_{2j} + \tfrac{1}{2}u^2\nu_{3j}^2 + v_j\right) \, .$$

So, as announced, the joint R.N. dynamics of the process $(y_t, z_t')'$ is Car since :

$$\psi_t^{\mathbb{Q}}(u,v) = a^{\mathbb{Q}}(u,v)' w_t + b_t^{\mathbb{Q}}(u,v)$$

with

$$\begin{aligned}a^{\mathbb{Q}}(u,v)' &= [u\rho, u\nu_1' + \Lambda'(u,v,\nu_2,\nu_3,\pi^*)] \, , \\ b_t^{\mathbb{Q}}(u,v) &= u\nu_t \, .\end{aligned}$$

The internal consistency condition is :

$$\psi_t^{\mathbb{Q}}(1,0) = r_{t+1}$$

that is :

$$-r_{t+1} + \nu_t + \rho y_t + \nu_1' z_t + \lambda'(\nu_2,\nu_3,\pi^*) z_t = 0 \qquad \forall\, y_t,\, z_t \, , \tag{24}$$

and where the i^{th} component of $\lambda(\nu_2,\nu_3,\pi^*)$ is

$$\lambda_i(\nu_2,\nu_3,\pi^*) = \text{ Log } \sum_{j=1}^J \pi_{ij}^* \exp\left(\nu_{2j} + \frac{1}{2}\nu_{3j}^2\right) \, .$$

Condition (24) implies, since r_{t+1} and ν_t are deterministic functions of time :

$$\begin{cases} \rho = 0 \, , \\ \nu_1 = -\lambda(\nu_2,\nu_3,\pi^*) \, , \\ \nu_t = r_{t+1} \, . \end{cases} \tag{25}$$

Finally, the R.N. dynamics compatible with the AAO conditions is :

$$y_{t+1} = r_{t+1} - \lambda'(\nu_2,\nu_3,\pi^*) z_t + \nu_2' z_{t+1} + (\nu_3' z_{t+1}) \xi_{t+1} \, , \tag{26}$$

where

$$\begin{aligned}&\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1) \\ &\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j \,|\, z_t = e_i) = \pi_{ij}^* \, .\end{aligned} \tag{27}$$

Note that, if ν_2 is replaced by $\nu_2 + c\,e$, $\nu_2' z_{t+1}$ is replaced by $\nu_2' z_{t+1} + c$ and $-\lambda' z_t$ by $-\lambda' z_t - c$, so the RHS of (26) is unchanged and therefore we can impose, for instance, $\nu_{2J} = 0$.

The SDF is specified as :

$$M_{t,t+1} = \exp\left[-r_{t+1} + \gamma_t(\underline{w}_t, \theta_2^*)\, y_{t+1} + \delta_t(\underline{w}_t, \theta_2^*)' z_{t+1} - \psi_t(\gamma_t, \delta_t)\right] \, ,$$

and the historical dynamics can then be deduced by specifying $\gamma_t(\underline{w}_t, \theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$ without any constraints (and assuming, for instance, $\delta_{Jt} = 0$). We get the Log-Laplace transform :

$$\psi_t(u, v) = \psi_t^{\mathbb{Q}}(u - \gamma_t, v - \delta_t) - \psi_t^{\mathbb{Q}}(-\gamma_t, -\delta_t)\,,$$

where

$$\psi_t^{\mathbb{Q}}(u, v) = u(r_{t+1} - \lambda' z_t) + \Lambda'(u, v) z_t\,,$$

and thus

$$\psi_t(u, v) = u(r_{t+1} - \lambda' z_t) + [\Lambda(u - \gamma_t, v - \delta_t) - \Lambda(-\gamma_t, -\delta_t)]' \, z_t\,, \tag{28}$$

with

$$\begin{aligned}
&\Lambda_i(u - \gamma_t, v - \delta_t) - \Lambda_i(-\gamma_t, -\delta_t) = \\
= \;& \mathrm{Log}\, \frac{\sum_{j=1}^{J} \pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_t^2 \nu_{3j}^2 - \delta_{jt}\right) \exp\left[u(\nu_{2j} - \gamma_t \nu_{3j}^2) + \frac{1}{2} u^2 \nu_{3j}^2 + v_j\right]}{\sum_{j=1}^{J} \pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_j^2 \nu_{3j}^2 - \delta_{jt}\right)} \\
= \;& \mathrm{Log} \sum_{j=1}^{J} \pi_{ij,t} \exp\left[u(\nu_{2j} - \gamma_t \nu_{3j}^2) + \frac{1}{2} u^2 \nu_{3j}^2 + v_j\right]
\end{aligned}$$

and

$$\pi_{ij,t} = \frac{\pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_t^2 \nu_{3j}^2 - \delta_{jt}\right)}{\sum_{j=1}^{J} \pi_{ij}^* \exp\left(-\gamma_t \nu_{2j} + \frac{1}{2}\gamma_t^2 \nu_{3j}^2 - \delta_{jt}\right)}\,.$$

Therefore, the historical dynamics is :

$$y_{t+1} = r_{t+1} - \lambda'(\nu_2, \nu_3, \pi^*) z_t + (\nu_2 - \gamma_t \nu_3^2)' z_{t+1} + (\nu_3' z_{t+1}) \varepsilon_{t+1} \tag{29}$$

where

$$\begin{aligned}
&\varepsilon_{t+1} \,|\, \underline{\varepsilon}_t, \underline{z}_{t+1} \overset{\mathbb{P}}{\sim} N(0, 1) \\
&\mathbb{P}(z_{t+1} = e_j \,|\, \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \pi_{ij,t} \\
&\lambda_i(\nu_2, \nu_3, \pi^*) = \mathrm{Log} \sum_{j=1}^{J} \pi_{ij}^* \exp\left(\nu_{2j} + \frac{1}{2}\nu_{3j}^2\right)\,,
\end{aligned}$$

and

$$\varepsilon_{t+1} = \xi_{t+1} + \gamma_t \, (\nu_3' z_{t+1})\,. \tag{30}$$

Conditionally to $\underline{w}_t$, the historical distribution of y_{t+1} is a mixture of J Gaussian distributions with means $(r_{t+1} - \lambda' z_t + \nu_{2j} - \gamma_t \nu_{3j}^2)$ and variances ν_{3j}^2, and with weights given by $\pi_{ij,t}$, $j \in \{1, \ldots, J\}$, when $z_t = e_i$.

Since γ_t and δ_t are arbitrary functions of $\underline{w}_t$ (assuming, for instance, $\delta_{Jt} = 0$), we obtain a large class of historical (non-Car) switching regime dynamics which can be matched with a Car switching regime R.N. dynamics. These features give the possibility to specify a tractable option pricing model able, at the same time, to provide historical and risk-neutral stochastic skewness and kurtosis which are determinant to fit stock return and implied volatility surface dynamics [see the survey on econometrics of option pricing proposed by Garcia, Ghysels and Renault (2003), where mixture models are studied, and the works of Bakshi, Carr and Wu (2008) and Carr and Wu (2007), where the important role of stochastic skewness in currency options is analyzed].

As mentioned in Section 4.4, the identification problem must be discussed. Let us consider the case where γ and δ are constant. In this case, the parameters π_{ij} are constant and the identifiable parameters are the π_{ij}, ν_3, the vector of the J coefficients of z_{t+1} in (29), and $(J-1)$ coefficients of z_t [assuming, for instance, $\lambda_J = 0$], i.e. $J(J-1)+3J-1 = J(J+2)-1$ parameters, whereas the parameters to be estimated are the π^*_{ij}, ν_2 (with $\nu_{2J}=0$), ν_3, γ, δ (with $\delta_J = 0$) i.e. $J(J+2)-1$ parameters also. So all the parameters might be estimated from the observations of the $y_t's$.

5.4 Back Modelling of Stochastic Volatility Models

We focus on the Back Modelling, starting from a Car representation of the R.N. dynamics of the factor $w_t = (y_t, \sigma_t^2)$, where y_t is an observable geometric return, whereas σ_t^2 is an unobservable stochastic variance. More precisely the R.N. dynamics is assumed to satisfy :

$$y_{t+1} = \lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2 + (\lambda_3 \sigma_t)\xi_{t+1}\,, \tag{31}$$

where λ_t is a deterministic function of t and

$$\begin{aligned} \xi_{t+1} \,|\, \underline{\xi_t}, \underline{\sigma^2_{t+1}} &\overset{\mathbb{Q}}{\sim} N(0,1) \\ \sigma^2_{t+1} \,|\, \underline{\xi_t},\, \underline{\sigma_t^2} &\overset{\mathbb{Q}}{\sim} ARG(1,\nu,\rho) \end{aligned} \tag{32}$$

and where the conditional $ARG(1,\nu,\rho)$ distribution [characterizing an Autoregressive Gamma process of order one (ARG(1)) with unit scale parameter[9]] is defined by the affine conditional R.N. Log-Laplace transform :

$$\psi_t^{\mathbb{Q}}(v) = a^{\mathbb{Q}}(v)\sigma_t^2 + b^{\mathbb{Q}}(v)\,,$$

where $a^{\mathbb{Q}}(v) = \frac{\rho v}{1-v}$, $b^{\mathbb{Q}}(v) = -\nu \text{Log}(1-v)$, $v<1$, $\rho>0$, $\nu>0$. The conditional R.N. Log-Laplace transform of $(y_{t+1}, \sigma^2_{t+1})$ is :

$$\psi_t^{\mathbb{Q}}(u,v) = (\lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2)u + \frac{1}{2}\lambda_3^2\sigma_t^2 u^2 + a^{\mathbb{Q}}(v)\sigma_t^2 + b^{\mathbb{Q}}(v)\,. \tag{33}$$

The internal consistency condition is :

$$\psi_t^{\mathbb{Q}}(1,0) = r_{t+1}$$

or

$$r_{t+1} = \lambda_t + \lambda_1 y_t + \lambda_2 \sigma_t^2 + \frac{1}{2}\lambda_3^2\sigma_t^2\,,$$

which implies :

$$\lambda_t = r_{t+1},\ \lambda_1 = 0,\ \lambda_2 = -\frac{1}{2}\lambda_3^2\,. \tag{34}$$

[9]See Darolles, Gourieroux and Jasiak (2006), Gourieroux and Jasiak (2006) and Monfort and Pegoraro (2006b) for a presentation of single regime and regime-switching (scalar and vector) Autoregressive Gamma processes.

So, the R.N. dynamics compatible with the AAO restriction is given by (32) and :

$$y_{t+1} = r_{t+1} - \frac{1}{2}\lambda_3^2\sigma_t^2 + \lambda_3\sigma_t\xi_{t+1}\,,$$

that is

$$\psi_t^{\mathbb{Q}}(u,v) = \left(r_{t+1} - \frac{1}{2}\lambda_3^2\sigma_t^2\right)u + \frac{1}{2}\lambda_3^2\sigma_t^2u^2 + a^{\mathbb{Q}}(v)\sigma_t^2 + b^{\mathbb{Q}}(v)\,. \tag{35}$$

The historical dynamics is defined by specifying $\gamma_t(\underline{w}_t\theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$, and we get :

$$\begin{aligned}
\psi_t(u,v) &= \psi_t^{\mathbb{Q}}(u-\gamma_t, v-\delta_t) - \psi_t^{\mathbb{Q}}(-\gamma_t, -\delta_t) \\
&= \left(r_{t+1} - \tfrac{1}{2}\lambda_3^2\sigma_t^2\right)u - \lambda_3^2\sigma_t^2\gamma_t u + \tfrac{1}{2}\lambda_3^2\sigma_t^2u^2 \\
&\qquad + \left[a^{\mathbb{Q}}(v-\delta_t) - a^{\mathbb{Q}}(-\delta_t)\right]\sigma_t^2 + b^{\mathbb{Q}}(v-\delta_t) - b^{\mathbb{Q}}(-\delta_t) \\
&= \left(r_{t+1} - \tfrac{1}{2}\lambda_3^2\sigma_t^2 - \lambda_3^2\sigma_t^2\gamma_t\right)u + \tfrac{1}{2}\lambda_3^2\sigma_t^2u^2 + a_t(v)\sigma_t^2 + b_t(v)\,,
\end{aligned}$$

with

$$\begin{aligned}
a_t(v) &= \frac{\rho_t v}{1 - v\mu_t}\,,\ b_t(v) = -\nu\mathrm{Log}(1 - v\mu_t)\,, \\
\rho_t &= \frac{\rho}{(1+\delta_t)^2}\,,\ \mu_t = \frac{1}{1+\delta_t}\,.
\end{aligned}$$

So, the only conditions, when we define the historical dynamics, are $\mu_t > 0$, i.e. $\delta_t > -1$, and $v < 1/\mu_t$. The historical dynamics can be written:

$$y_{t+1} = r_{t+1} - \frac{1}{2}\lambda_3^2\sigma_t^2 - \lambda_3^2\sigma_t^2\gamma_t + \lambda_3\sigma_t\varepsilon_{t+1} \tag{36}$$

where

$$\begin{aligned}
&\varepsilon_{t+1}\,|\,\underline{\varepsilon_t}, \underline{\sigma_{t+1}^2} \overset{\mathbb{P}}{\sim} N(0,1) \\
&\sigma_{t+1}^2\,|\,\underline{\varepsilon_t}, \underline{\sigma_t^2} \overset{\mathbb{P}}{\sim} ARG(\mu_t, \nu, \rho_t)\,.
\end{aligned} \tag{37}$$

Note that, the conditional historical distribution of σ_{t+1}^2, given $(\underline{y_t}, \underline{\sigma_t^2})$, is given by the Log-Laplace Transform

$$\psi_t(v) = \frac{\rho_t v}{1 - v\mu_t}\,\sigma_t^2 - \nu\mathrm{Log}(1 - v\mu_t)$$

which is not affine in σ_t^2, except in the case where δ_t is constant (or a deterministic function of t). Moreover we have :

$$\varepsilon_{t+1} = \xi_{t+1} + (\lambda_3\sigma_t)\gamma_t\,. \tag{38}$$

If γ_t and δ_t are constant, the identifiable parameters are the coefficients of σ_t^2 and $\sigma_t\varepsilon_{t+1}$ in (36) as well as the two parameters of the ARG dynamics (with unit scale). So, we have four identifiable parameters. The parameters to be estimated are λ_3, ν, ρ, γ, δ, i.e. five parameters. So these parameters are not identifiable from the dynamics of the $y_t' s$. Observations of derivative prices must be added.

In this example we have assumed $\sigma_{t+1}^2 \sim ARG(1)$ and absence of instantaneous causality between y_{t+1} and σ_{t+1}^2 just for ease of exposition. It is possible to specify a stochastic volatility model in which $\sigma^2_{t+1} \sim ARG(p)$, with an instantaneous correlation between the stock return and the stochastic volatility. For instance, we can consider:

$$\begin{cases} y_{t+1} &= \lambda_t + \lambda_1 y_t + \lambda_2 \sigma^2_{t+1} + \lambda_3 \sigma^2_t + (\lambda_4 \sigma_{t+1})\xi_{t+1} \\ \sigma^2_{t+1} &= \nu + \varphi_1 \sigma^2_t + \ldots + \varphi_p \sigma^2_{t-p+1} + \eta_{t+1} \end{cases}$$

where η_{t+1} is an heteroscedastic martingale difference sequence. This specification generalizes the exact discrete-time equivalent of the SV diffusion model typically used in continuous-time (and based on the CIR process). It has the potential features to explain not only the volatility smile in option data, but also to improve the fitting of the observed time varying persistence in stock return volatility [see Garcia, Ghysels and Renault (2003), end the references therein]. Indeed, the conditional mean and variance of σ^2_{t+1} show the following specifications: $E[\sigma^2_{t+1} \,|\, \underline{\sigma_t}^2] = \nu + \varphi_1 \sigma^2_t + \ldots + \varphi_p \sigma^2_{t-p+1}$ and $V[\sigma^2_{t+1} \,|\, \underline{\sigma_t}^2] = \nu + 2(\varphi_1 \sigma^2_t + \ldots + \varphi_p \sigma^2_{t-p+1})$.

5.5 Back Modelling of Switching GARCH Models with leverage effect: a first application of Extended Car Processes

In this section, following a Back Modelling approach, we consider specifications generalizing those proposed by Heston and Nandi (2000) to the case where switching regimes are introduced in the conditional mean and conditional (GARCH-type) variance of the geometric return [see also Elliot, Siu and Chan (2006)].

Like in Section 5.4, we assume $w_t = (y_t, z_t')'$, where y_t is an observable geometric return and z_t an unobservable J-state homogeneous Markov chain valued in $\{e_1, ..., e_J\}$. The new feature is the introduction of a GARCH effect (with leverage). More precisely, the R.N. dynamics is assumed to be of the following type :

$$y_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \nu_3' z_{t+1} + \nu_4 \sigma^2_{t+1} + \sigma_{t+1}\xi_{t+1} \tag{39}$$

where ν_t is a deterministic function of t and

$$\xi_{t+1} \,|\, \underline{\xi_t}, \underline{z_{t+1}} \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1(\xi_t - \alpha_2 \sigma_t)^2 + \alpha_3 \sigma^2_t \,,$$

and

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y_t}, \underline{z_{t-1}}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi^*_{ij} \,.$$

Note that σ^2_{t+1} is a deterministic function of $(\underline{\xi_t}, \underline{z_t})$, and therefore of $\underline{w_t} = (\underline{y_t}, \underline{z_t})$. Also note that, following Heston and Nandi (2000), in this switching GARCH(1,1) model, ξ_t replaces the usual term $\sigma_t\, \xi_t$ in the R.H.S. of the equation giving σ^2_{t+1} and the term $\alpha_2 \sigma_t$ captures an asymmetric or "leverage" effect.

It is easily seen that the R.N. conditional Log-Laplace transform of (y_{t+1}, z_{t+1}) is :

$$\begin{aligned} \psi^{\mathbb{Q}}_t(u,v) &= \operatorname{Log} E^{\mathbb{Q}}_t \exp(u y_{t+1} + v' z_{t+1}) \\ &= (\nu_t + \nu_1 y_t + \nu_2' z_t + \nu_4 \sigma^2_{t+1})u + \tfrac{1}{2}\sigma^2_{t+1} u^2 + \Lambda'(u, v, \nu_3, \pi^*) z_t \,, \end{aligned} \tag{40}$$

where the i^{th} component of $\Lambda(u, v, \nu_3, \pi^*)$ is :

$$\Lambda_i(u, v, \nu_3, \pi^*) = \operatorname{Log} \sum_{j=1}^{J} \pi^*_{ij} \exp(u \nu_{3j} + v_j) \,. \tag{41}$$

The internal consistency condition, or AAO constraint, is :

$$\psi_t^{\mathbb{Q}}(1,0) = r_{t+1} \quad \forall \underline{w}_t \, ,$$

implying

$$r_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \nu_4 \sigma_{t+1}^2 + \frac{1}{2}\sigma_{t+1}^2 + \lambda'(\nu_3, \pi^*) z_t \, ,$$

where the i^{th} component of $\lambda(\nu_3, \pi^*)$ is given by:

$$\lambda_i(\nu_3, \pi^*) = \mathrm{Log} \sum_{j=1}^{J} \pi_{ij}^* \exp(\nu_{3j}) \tag{42}$$

and, therefore, the arbitrage restriction implies:

$$\begin{cases} \nu_1 = 0 \, , \\ \nu_2 = -\lambda(\nu_3, \pi^*) \, , \\ \nu_4 = -\frac{1}{2} \, , \\ \nu_t = r_{t+1} \, . \end{cases}$$

Thus, equation (39) becomes:

$$y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t - \frac{1}{2}\sigma_{t+1}^2 + \nu_3' z_{t+1} + \sigma_{t+1}\xi_{t+1} \tag{43}$$

with

$$\sigma_{t+1}^2 = \omega' z_t + \alpha_1(\xi_t - \alpha_2 \sigma_t)^2 + \alpha_3 \sigma_t^2$$

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^* \, ,$$

(again, we can take $\nu_{3J} = 0$) which gives the R.N. dynamics compatible with the AAO restriction. The corresponding Log-Laplace transform is :

$$\psi_t^{\mathbb{Q}}(u,v) = \left(r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma_{t+1}^2 \right) u + \frac{1}{2}\sigma_{t+1}^2 u^2 + \Lambda'(u, v, \nu_3, \pi^*) z_t \tag{44}$$

The historical dynamics is obtained by specifying $\gamma_t(\underline{w}_t, \theta_2^*)$ and $\delta_t(\underline{w}_t, \theta_2^*)$, with, for instance $\delta_{Jt} = 0$, and in particular we have :

$$\psi_t(u,v) = \psi_t^{\mathbb{Q}}(u - \gamma_t, v - \delta_t) - \psi_t^{\mathbb{Q}}(-\gamma_t, -\delta_t) \, .$$

We obtain :

$$\begin{aligned} \psi_t(u,v) \quad &= \quad \left(r_{t+1} - \lambda' z_t - \tfrac{1}{2}\sigma_{t+1}^2 - \gamma_t \sigma_{t+1}^2 \right) u + \tfrac{1}{2}\sigma_{t+1}^2 u^2 \\ & \qquad + \left[\Lambda(u - \gamma_t, v - \delta_t, \nu_3, \pi^*) - \Lambda(-\gamma_t, -\delta_t, \nu_3, \pi^*) \right]' z_t \end{aligned}$$

where

$$\begin{aligned} &\Lambda_i(u - \gamma_t, v - \delta_t, \nu_3, \pi^*) - \Lambda_i(-\gamma_t, -\delta_t, \nu_3, \pi^*) = \mathrm{Log} \Sigma_{j=1}^J \pi_{ij,t} \exp(u\nu_{3j} + v_j) \\ &\text{with} \quad \pi_{ij,t} = \frac{\pi_{ij}^* \exp(-\gamma_t \nu_{3j} - \delta_{jt})}{\Sigma_{j=1}^J \pi_{ij}^* \exp(-\gamma_t \nu_{3j} - \delta_{jt})} \, . \end{aligned} \tag{45}$$

So the non-affine historical dynamics is given by :

$$
\begin{aligned}
& y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t - \tfrac{1}{2}\sigma_{t+1}^2 - \gamma_t(\underline{w}_t, \theta_2^*)\sigma_{t+1}^2 + \nu_3' z_{t+1} + \sigma_{t+1}\varepsilon_{t+1} \\
& \varepsilon_{t+1} \,|\, \underline{\varepsilon}_t, \underline{z}_{t+1} \overset{\mathbb{P}}{\sim} N(0,1)\,,
\end{aligned}
\tag{46}
$$

with

$$
\sigma_{t+1}^2 = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma_t^2
$$

$$
\mathbb{P}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \pi_{ij,t}\,.
$$

Comparing (43) and (46) we get :

$$
\xi_{t+1} = \varepsilon_{t+1} - \gamma_t\sigma_{t+1}\,,
$$

and, therefore, the equation giving σ_{t+1}^2 can be rewritten :

$$
\sigma_{t+1}^2 = \omega' z_t + \alpha_1\left[\varepsilon_t - (\alpha_2 + \gamma_t)\sigma_t\right]^2 + \alpha_3\sigma_t^2\,.
$$

One may observe, from (44), that $w_{t+1} = (y_{t+1}, z_{t+1}')$ does not have a Car R.N. dynamics. So, the pricing seems a priori difficult. Fortunately, it can be shown [see Appendix 3] that the (extended) factor $w_{t+1}^e := (y_{t+1}, z_{t+1}', \sigma_{t+2}^2)'$ is R.N. Car, that is, w_{t+1} is an Internally Extended Car(1), and therefore the pricing methods based on Car dynamics apply. In particular, the R.N. conditional Log-Laplace transform of w_{t+1}^e, given $\underline{w}_t^e$, is:

$$
\psi_t^{\mathbb{Q}}(u, v, \tilde{v}) = a_1^{\mathbb{Q}}(u, v, \tilde{v})' z_t + a_2^{\mathbb{Q}}(u, \tilde{v})\sigma_{t+1}^2 + b_t^{\mathbb{Q}}(u, \tilde{v})\,,
\tag{47}
$$

where

$$
\begin{aligned}
a_1^{\mathbb{Q}}(u, v, \tilde{v}) \quad &= \quad \tilde{\Lambda}(u, v, \tilde{v}, \nu_3, \omega, \pi^*) - \lambda(\nu_3, \pi^*)\, u \\
\text{with} \qquad & \tilde{\Lambda}_i(u, v, \tilde{v}, \nu_3, \omega, \pi^*) = \mathrm{Log}\sum_{j=1}^{J} \pi_{ij}^* \exp(u\nu_{3j} + v_j + \tilde{v}\omega_j)\,,\ i \in \{1, \ldots, J\}\,, \\
a_2^{\mathbb{Q}}(u, \tilde{v}) \quad &= \quad -\frac{1}{2}u + \tilde{v}(\alpha_1\alpha_2^2 + \alpha_3) + \frac{(u - 2\alpha_1\alpha_2\tilde{v})^2}{2(1 - 2\alpha_1\tilde{v})} \\
b_t^{\mathbb{Q}}(u, \tilde{v}) \quad &= \quad u r_{t+1} - \frac{1}{2}\mathrm{Log}(1 - 2\alpha_1\tilde{v})\,,
\end{aligned}
$$

which is affine in $(z_t', \sigma_{t+1}^2)'$, with an intercept deterministic function of time.

Finally, let us consider the identification problem from the historical dynamics when functions γ and δ are constant. In this case, we can identify from (46) J coefficients of z_{t+1}, $(J-1)$ coefficients of z_t, the coefficient of σ_{t+1}^2, ω, α_1, $(\alpha_2 + \gamma)$, α_3, and π_{ij}, i.e. $3J + 3 + J(J-1) = J(J+2) + 3$ parameters. The parameters to be estimated are ν_3 (with $\nu_{3J} = 0$), $\omega, \alpha_1, \alpha_2, \alpha_3$, π_{ij}^*, γ, δ (with $\delta_J = 0$), that is, $2(J-1) + J + 4 + J(J-1) = J(J+2) + 2$ parameters. Therefore, the historical model is over identified.

5.6 Back Modelling of Switching IG GARCH Models : a second application of Extended Car Processes

The purpose of this section is to introduce, following the Back Modelling approach, several generalizations of the Inverse Gaussian[10] (IG) GARCH model proposed by Christoffersen, Heston and Jacobs (2006). First, we consider switching regimes in the (historical and risk-neutral) dynamics of the geometric return y_t and in the GARCH variance σ^2_{t+1}. Second, we price not only the factor risk but also the regime-shift risk and, third, risk correction coefficients are in general time-varying. The factor is given by $w_t = (y_t, z_t')'$, where z_t is the unobservable J-state homogeneous Markov chain valued in $\{e_1, ..., e_J\}$. The R.N. dynamics is given by:

$$y_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \nu_3' z_{t+1} + \nu_4 \sigma^2_{t+1} + \eta \xi_{t+1} \tag{48}$$

where ν_t is a deterministic function of t and

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} IG\left(\frac{\sigma^2_{t+1}}{\eta^2}\right)$$

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1 \sigma^2_t + \alpha_2 \xi_t + \alpha_3 \frac{\sigma^4_t}{\xi_t}\,,$$

with

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi^*_{ij}\,.$$

The R.N. conditional Log-Laplace transform of (y_{t+1}, z_{t+1}) is :

$$\begin{aligned}
\psi^{\mathbb{Q}}_t(u,v) &= \text{Log}\, E^{\mathbb{Q}}_t \exp(u y_{t+1} + v' z_{t+1}) \\
&= (\nu_t + \nu_1 y_t + \nu_2' z_t + \nu_4 \sigma^2_{t+1}) u + \Lambda'(u, v, \nu_3, \pi^*) z_t + \frac{\sigma^2_{t+1}}{\eta^2}\left[1 - (1-2u\eta)^{1/2}\right]\,,
\end{aligned}$$

where the i^{th} component of $\Lambda(u, v, \nu_3, \pi^*)$ is given by (41). The absence of arbitrage constraint is $\psi^{\mathbb{Q}}_t(1,0) = r_{t+1}\,,\ \forall \underline{w}_t$, implying

$$r_{t+1} = \nu_t + \nu_1 y_t + \nu_2' z_t + \lambda'(\nu_3, \pi^*) z_t + \sigma^2_{t+1}\left(\nu_4 + \frac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\right)\,,$$

with the i^{th} component of $\lambda(\nu_3, \pi^*)$ given by (42). Therefore, the arbitrage restriction implies:

$$\begin{cases}
\nu_1 = 0\,, \\
\nu_2 = -\lambda(\nu_3, \pi^*)\,, \\
\nu_4 = -\frac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\,, \\
\nu_t = r_{t+1}\,.
\end{cases}$$

Thus, equation (48) becomes:

$$y_{t+1} = r_{t+1} - \lambda(\nu_3, \pi^*)' z_t - \frac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right] \sigma^2_{t+1} + \nu_3' z_{t+1} + \eta \xi_{t+1} \tag{49}$$

[10] The strictly positive random variable y has an Inverse Gaussian distribution with parameter $\delta > 0$ [denoted IG(δ)] if and only if its distribution function is given by $F(y;\delta) = \int_0^y \frac{\delta}{\sqrt{2\pi z^3}} e^{-(\sqrt{z} - \delta/\sqrt{z})^2/2} dz$. The generalized Laplace transform is $E[\exp(\varphi y + \theta/y)] = \frac{\delta}{\sqrt{\delta^2 - 2\theta}} \exp\left(\delta - \sqrt{(\delta^2 - 2\theta)(1-2\varphi)}\right)$ and $E(y) = V(y) = \delta$ [see Christoffersen, Heston and Jacobs (2006) for further details].

with

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1 \sigma^2_t + \alpha_2 \xi_t + \alpha_3 \frac{\sigma^4_t}{\xi_t}$$

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} IG\left(\frac{\sigma^2_{t+1}}{\eta^2}\right) ,$$

$$\mathbb{Q}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi^*_{ij} ,$$

(again, we can take $\nu_{3J} = 0$) which gives the R.N. dynamics compatible with the AAO restriction. The corresponding Log-Laplace transform is :

$$\begin{aligned} \psi^{\mathbb{Q}}_t(u,v) &= \left(r_{t+1} - \lambda' z_t - \tfrac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\sigma^2_{t+1}\right) u + \\ &\qquad \Lambda'(u,v,\nu_3,\pi^*) z_t + \tfrac{\sigma^2_{t+1}}{\eta^2}\left[1 - (1-2u\eta)^{1/2}\right] . \end{aligned} \tag{50}$$

Given the specification of $\gamma_t(\underline{w}_t, \theta^*_2)$ and $\delta_t(\underline{w}_t, \theta^*_2)$ (with, for instance, $\delta_{Jt} = 0$), the conditional historical Log-Laplace transform of the factor is given by:

$$\begin{aligned} \psi_t(u,v) &= \left(r_{t+1} - \lambda' z_t - \tfrac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\sigma^2_{t+1}\right) u \\ &\quad + \left[\Lambda(u-\gamma_t, v-\delta_t, \nu_3, \pi^*) - \Lambda(-\gamma_t, -\delta_t, \nu_3, \pi^*)\right]' z_t \\ &\quad + \tfrac{\sigma^2_{t+1}}{\eta^2}\left[(1+2\gamma_t\eta)^{1/2} - \left[1 - 2(u-\gamma_t)\eta\right]^{1/2}\right] \\ &= \left(r_{t+1} - \lambda' z_t - \tilde{\eta}_t^{-3/2}\eta^{-1/2}\left[1 - (1-2\eta)^{1/2}\right]\tilde{\sigma}^2_{t+1}\right) u \\ &\quad + \left[\Lambda(u-\gamma_t, v-\delta_t, \nu_3, \pi^*) - \Lambda(-\gamma_t, -\delta_t, \nu_3, \pi^*)\right]' z_t \\ &\quad + \tfrac{\tilde{\sigma}^2_{t+1}}{\tilde{\eta}^2_t}\left[1 - (1-2u\tilde{\eta}_t)^{1/2}\right] , \end{aligned}$$

with $\Lambda_i(u-\gamma_t, v-\delta_t) - \Lambda_i(-\gamma_t, -\delta_t)$ specified by (45), and where $\tilde{\eta}_t = \frac{\eta}{1+2\gamma_t\eta}$ and $\tilde{\sigma}^2_{t+1} = \sigma^2_{t+1}\left(\frac{\tilde{\eta}_t}{\eta}\right)^{3/2}$. So, the non-affine historical dynamics is given by :

$$\begin{aligned} & y_{t+1} = r_{t+1} - \lambda(\nu_3,\pi^*)' z_t + \nu_3' z_{t+1} - \tilde{\eta}_t^{-3/2}\eta^{-1/2}\left[1 - (1-2\eta)^{1/2}\right]\tilde{\sigma}^2_{t+1} + \tilde{\eta}_t \varepsilon_{t+1} \\ & \varepsilon_{t+1} \,|\, \underline{\varepsilon}_t, \underline{z}_{t+1} \sim IG\left(\frac{\tilde{\sigma}^2_{t+1}}{\tilde{\eta}^2_t}\right) , \end{aligned} \tag{51}$$

with, using (49) and (51), $\eta\xi_{t+1} = \tilde{\eta}_t \varepsilon_{t+1}$ and

$$\tilde{\sigma}^2_{t+1} = \tilde{\omega}'_t z_t + \tilde{\alpha}_{1,t}\tilde{\sigma}^2_t + \tilde{\alpha}_{2,t}\varepsilon_t + \tilde{\alpha}_{3,t}\frac{\tilde{\sigma}^4_t}{\varepsilon_t}$$

$$\mathbb{P}(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \pi_{ij,t} ,$$

where $\tilde{\omega}_t = \omega(\tilde{\eta}_t/\eta)^{3/2}$, $\tilde{\alpha}_{1,t} = \alpha_1(\tilde{\eta}_t/\tilde{\eta}_{t-1})^{3/2}$, $\tilde{\alpha}_{2,t} = \alpha_2(\tilde{\eta}_t^{3/2}\tilde{\eta}_{t-1}/\eta^{5/2})$ and $\tilde{\alpha}_{3,t} = \alpha_3\tilde{\eta}_t^{3/2}/(\tilde{\eta}^4_{t-1}\eta^{-5/2})$.

As in the previous section, the factor $w_{t+1} = (y_{t+1}, z'_{t+1})'$ is not a R.N. Car process, but it can be verified that the factor $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is R.N. Car [see Appendix 4], and that w_{t+1}

is an Internally Extended Car(1) process. Indeed, the R.N. conditional Log-Laplace transform of w^e_{t+1}, given $\underline{w}^e_t$, is :

$$\psi^{\mathbb{Q}}_t(u,v,\tilde{v}) = a^{\mathbb{Q}}_1(u,v,\tilde{v})'z_t + a^{\mathbb{Q}}_2(u,\tilde{v})\sigma^2_{t+1} + b^{\mathbb{Q}}_t(u,\tilde{v})\,, \tag{52}$$

where

$$\begin{aligned}
a^{\mathbb{Q}}_1(u,v,\tilde{v}) &= \tilde{\Lambda}(u,v,\tilde{v},\nu_3,\omega,\pi^*) - \lambda(\nu_3,\pi^*)\,u \\
\text{with} \qquad & \tilde{\Lambda}_i(u,v,\tilde{v},\nu_3,\omega,\pi^*) = \mathrm{Log}\sum_{j=1}^{J}\pi^*_{ij}\exp(u\nu_{3j}+v_j+\tilde{v}\omega_j)\,,\ i\in\{1,\ldots,J\}\,, \\
a^{\mathbb{Q}}_2(u,\tilde{v}) &= \tilde{v}\alpha_1 - \frac{1}{\eta^2}\left(u\left[1-(1-2\eta)^{1/2}\right] + 1 - \sqrt{(1-2\tilde{v}\alpha_3\eta^4)\,(1-2(u\eta+\tilde{v}\alpha_2))}\right)\,, \\
b^{\mathbb{Q}}_t(u,\tilde{v}) &= ur_{t+1} - \frac{1}{2}\mathrm{Log}(1-2\tilde{v}\alpha_3\eta^4)\,,
\end{aligned}$$

which is affine in $(z'_t, \sigma^2_{t+1})'$, with an intercept deterministic function of time.

As far as the identification problem is concerned, with functions γ and δ constant, we can identify, from the historical dynamics (51), $3J + J(J-1) + 4$ coefficients, while the parameters to be estimated are ν_3 (with $\nu_{3J} = 0$), $\omega, \alpha_1, \alpha_2, \alpha_3$, π^*_{ij}, γ, δ (with $\delta_J = 0$), and η, that is, $2(J-1) + J + 5 + J(J-1) = 3J + J(J-1) + 3$ parameters. Thus, as in the previous section, the historical model is over identified.

6 Applications to Econometric Term Structure Modelling

It is well known that, if the R.N. dynamics of w_t is Car and if r_{t+1} is an affine function of w_t, the term structure of interest rates $[r(t,h), h \in \{1,...,H\}]$ is easily determined recursively and is affine in w_t [see Gourieroux, Monfort and Polimenis (2003), or Monfort and Pegoraro (2007)]. Indeed, if :

$$\psi^{\mathbb{Q}}_t(u|w_t;\theta^*_1) = a^{\mathbb{Q}}(u,\theta^*_1)'w_t + b^{\mathbb{Q}}(u,\theta^*_1)$$

and $r_{t+1} = \tilde{\theta}_1 + \tilde{\theta}'_2 w_t$, then

$$r(t,h) = -\frac{c'_h}{h}w_t - \frac{d_h}{h}\,, \tag{53}$$

where

$$\left\{\begin{aligned}
c_h &= -\tilde{\theta}_2 + a^{\mathbb{Q}}(c_{h-1}) \\
d_h &= d_{h-1} - \tilde{\theta}_1 + b^{\mathbb{Q}}(c_{h-1}) \\
c_0 &= 0,\ d_0 = 0\,.
\end{aligned}\right. \tag{54}$$

Moreover, applying the transform analysis, various interest rates derivatives have quasi explicit pricing formulas. Note that if the i^{th} component of w_t is a rate $r(t,h_i)$, $i \in \{1,...,K_1\}$, we must satisfy the internal consistency conditions:

$$c_{h_i} = -h_i e_i,\ d_{h_i} = 0, \quad i \in \{1,...,K_1\}\,.$$

Therefore, it is highly desirable to have a Car R.N. dynamics and this specification is obtained by one of the three modelling strategies described in Section 4. Let us consider some examples.

6.1 Direct Modelling of VAR(p) Factor-Based Term Structure Models

For sake of notational simplicity we consider the one factor case, but the results can be extended to the multivariate case [see Monfort and Pegoraro (2006a)]. We assume, for instance, that the factor w_t is unobservable, and has a historical dynamics given by a Gaussian AR(p) model:

$$\begin{aligned} w_{t+1} &= \nu + \varphi_1 w_t + ... + \varphi_p w_{t+1-p} + \sigma\varepsilon_{t+1} \\ &= \nu + \varphi' W_t + \sigma\varepsilon_{t+1} \end{aligned} \tag{55}$$

where $\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,1)$, $\varphi = (\varphi_1, ..., \varphi_p)'$ and $W_t = (w_t, ..., w_{t+1-p})'$. This dynamics can also be written :

$$W_{t+1} = \tilde{\nu} + \Phi W_t + \sigma\tilde{\varepsilon}_{t+1}$$

where $\tilde{\nu} = \nu e_1$, $\tilde{\varepsilon}_{t+1} = \varepsilon_{t+1} e_1$ [e_1 denotes the first column of the identity matrix I_p] and

$$\Phi = \begin{bmatrix} \varphi_1 & \dots & \dots & \varphi_{p-1} & \varphi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} \text{ is a } (p \times p) \text{ matrix}.$$

The SDF takes the following exponential-affine form :

$$M_{t,t+1} = \exp\left[-r_{t+1} + \alpha_t w_{t+1} - \psi_t(\alpha_t)\right], \tag{56}$$

with

$$\begin{aligned} \psi_t(u) &= (\nu + \varphi' W_t)u + \tfrac{1}{2}\sigma^2 u^2, \\ \alpha_t &= \alpha_0 + \alpha' W_t \\ &= \alpha_0 + \alpha_1 w_t + ... + \alpha_p w_{t+1-p}, \end{aligned}$$

and the short rate is given by :

$$r_{t+1} = \tilde{\theta}_1 + \tilde{\theta}_2' W_t.$$

If $r_{t+1} = w_t$, we have $\tilde{\theta}_2 = e_1$ and $\tilde{\theta}_1 = 0$.

The conditional R.N. Log-Laplace transform is given by :

$$\begin{aligned} \psi_t^{\mathbb{Q}}(u) &= \psi_t(u + \alpha_t) - \psi_t(\alpha_t) \\ &= (\nu + \varphi' W_t)u + \sigma^2 \alpha_t u + \tfrac{1}{2}\sigma^2 u^2 \\ &= [\nu + \sigma^2\alpha_0 + (\varphi + \sigma^2\alpha)' W_t]u + \tfrac{1}{2}\sigma^2 u^2. \end{aligned}$$

Therefore, the R.N. dynamics of the factor is given by :

$$w_{t+1} = (\nu + \sigma^2\alpha_0) + (\varphi + \sigma^2\alpha)' W_t + \sigma\xi_{t+1} \tag{57}$$

where $\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,1)$. Moreover, we have $\varepsilon_{t+1} = \xi_{t+1} + \sigma(\alpha_0 + \alpha' W_t)$.

The yield-to-maturity formula at date t is given by [see Monfort and Pegoraro (2006a) for the proof] :

$$r(t,h) = -\frac{c_h'}{h}W_t - \frac{d_h}{h}\,, \quad h \geq 1\,, \tag{58}$$

with

$$\begin{cases} c_h = -\tilde{\theta}_2 + \Phi' c_{h-1} + c_{1,h}\sigma^2\alpha \\ \\ d_h = -\tilde{\theta}_1 + c_{1,h-1}(\nu + \sigma^2\alpha_0) + \frac{1}{2}c_{1,h-1}^2\sigma^2 + d_{h-1} \\ \\ c_0 = 0 \ \ d_0 = 0\,. \end{cases}$$

The pricing model presented in this section is derived following the discrete-time equivalent of the basic Direct Modelling strategy typically used in continuous-time [see Duffie and Kan (1996), and Cheridito, Filipovic and Kimmel (2007)]. If we specify α_t as a nonlinear function of W_t, $\psi_t^{\mathbb{Q}}(u)$ turns out to be non-affine (in W_t) and, therefore, we loose the explicit representation of the yield formula. We will see in Section 6.3 that we can go beyond this limit following the Back Modelling approach.

6.2 R.N. Constrained Direct Modelling of Switching VAR(p) Factor-Based Term Structure Models

Again for sake of simplicity we consider the univariate case [see Monfort and Pegoraro (2007) for extensions] where the factor is given by $w_t = (x_t, z_t')'$, with z_t a J-state non-homogeneous Markov chain valued in $\{e_1, ..., e_J\}$. The first component x_t is observable or unobservable, z_t is unobservable and the historical dynamics is given by :

$$x_{t+1} = \nu(Z_t) + \varphi_1(Z_t)x_t + ... + \varphi_p(Z_t)x_{t+1-p} + \sigma(Z_t)\varepsilon_{t+1} \tag{59}$$

where

$$\begin{aligned} &\varepsilon_{t+1}\,|\,\underline{\varepsilon}_t, \underline{z}_{t+1} \overset{\mathbb{P}}{\sim} N(0,1) \\ &\mathbb{P}(z_{t+1} = e_j | \underline{x}_t \ \underline{z}_{t-1}, z_t = e_i) = \pi(e_i, e_j; X_t) \\ &Z_t = (z_t', ..., z_{t-p}') \\ &X_t = (x_t, ..., x_{t+1-p})'\,. \end{aligned}$$

Observe that the joint historical dynamics of $(x_t, z_t')'$ is not Car. Functions ν, $\varphi_1, \ldots, \varphi_p$, σ and π are parameterized using a parameter θ_1.

We specify the SDF in the following way :

$$M_{t,t+1} = \exp\left[-r_{t+1} + \Gamma(Z_t, X_t)\varepsilon_{t+1} - \frac{1}{2}\Gamma(Z_t, X_t)^2 - \delta(Z_t, X_t)' z_{t+1}\right]\,, \tag{60}$$

with $\Gamma(Z_t, X_t) = \gamma(Z_t) + \tilde{\gamma}(Z_t)' X_t$ and, in order to ensure that $E_t M_{t,t+1} = \exp(-r_{t+1})$, we add the condition :

$$\sum_{j=1}^{J} \pi(e_i, e_j, X_i) \exp[-\delta(Z_t, X_t)' e_j] = 1, \ \ \forall Z_t, X_t\,.$$

The short rate is given by :

$$r_{t+1} = \tilde{\theta}_1' X_t + \tilde{\theta}_2' Z_t\,,$$

and, in the observable factor case ($x_t = r_{t+1}$), we have $\tilde{\theta}_1 = e_1$ and $\tilde{\theta}_2 = 0$.

It is easily seen that the R.N. dynamics is given by :

$$
\begin{aligned}
&x_{t+1} = \nu(Z_t) + \gamma(Z_t)\sigma(Z_t) + [\varphi(Z_t) + \tilde{\gamma}(Z_t)\sigma(Z_t)]' X_t + \sigma(Z_t)\xi_{t+1} \\
&\xi_{t+1} \,|\, \underline{\xi_t}, \underline{z_{t+1}} \overset{\mathbb{Q}}{\sim} N(0,1) \qquad (61)\\
&\mathbb{Q}(z_{t+1} = e_j | \underline{x_t}, \underline{z_{t-1}}, z_t = e_i) = \pi(e_i, e_j, ; X_t) \exp[-\delta(Z_t, X_t)' e_j] \,.
\end{aligned}
$$

So, if we want the R.N. dynamics of w_t to be Car, we have to impose :

$$
\begin{aligned}
&\text{i) } \sigma(Z_t) = \sigma^{*'} Z_t \;\; (\text{linearity in } z_t, ..., z_{t-p}) \\
&\text{ii) } \gamma(Z_t) = \tfrac{\nu^{*'} Z_t - \nu(Z_t)}{\sigma^{*'} Z_t} \\
&\text{iii) } \tilde{\gamma}(Z_t) = \tfrac{\varphi^{*} - \varphi(Z_t)}{\sigma^{*'} Z_t} \qquad (62)\\
&\text{iv) } \delta_j(Z_t, X_t) = \text{Log}\left[\frac{\pi(z_t, e_j, X_t)}{\pi^*(z_t, e_j)}\right] ,
\end{aligned}
$$

where σ^*, ν^*, φ^* are free parameters, $\pi^*(e_i, e_j)$ are the entries of an homogeneous transition matrix. All of these parameters constitute the parameter $\theta^* \in \Theta^*$ introduced in Section 4.3. Also note that, because of constraints $(62 - \text{i})$ above, θ and θ^* do not vary independently.
So the R.N. dynamics is :

$$
\begin{aligned}
&X_{t+1} = \Phi^* X_t + [\nu^{*'} Z_t + (\sigma^{*'} Z_t)\xi_{t+1}] e_1 \,, \\
&\Phi^* = \begin{bmatrix} \varphi_1^* & \dots & \dots & \varphi_{p-1}^* & \varphi_p^* \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} \text{ is a } (p \times p) \text{ matrix} \,, \qquad (63)\\
&\xi_{t+1} \,|\, \underline{\xi_t}, \underline{z_{t+1}} \overset{\mathbb{Q}}{\sim} N(0,1) \,, \\
&\mathbb{Q}(z_{t+1} = e_j | \underline{x_t}, \underline{z_{t-1}}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j | z_t = e_i) = \pi_{ij}^*
\end{aligned}
$$

and the affine (in X_t and Z_t) term structure of interest rates is easily derived [see Monfort and Pegoraro (2007) for the proof, and Dai, Singleton and Yang (2006) for the case $p = 1$]. The empirical study proposed in Monfort and Pegoraro (2007), shows that the introduction of multiple lags and switching regimes, in the historical and risk-neutral dynamics of the observable factor (short rate and spread between the long and the short rate), leads to term structure models which are able to fit the yield curve and to explain the violation of the Expectation Hypothesis Theory, over both the short and long horizon, as well as or better than competing models like 2-Factor CIR, 3-Factor CIR, 3-Factor $\mathbb{A}_1(3)$ (using the Dai and Singleton (2000) notation) and the 2-Factor regime switching CIR term structure model proposed by Bansal and Zhou (2002). Dai, Singleton and Yang (2007) show the determinant role of priced, state-dependent regime-shift risks in capturing the dynamics of expected excess bond returns. Moreover, they show that the well-known hump shaped term structure of volatility of bond yield changes is a low-volatility phenomenon.

6.3 Back Modelling of VAR(p) Factor-Based Term Structure Models

Let us consider the (bivariate) case where w_t is given by $[r(t,1), r(t,2)]'$. We want to impose the following Gaussian VAR(1) R.N. dynamics:

$$w_{t+1} = \nu + \Phi w_t + \xi_{t+1}\,, \tag{64}$$

where $\xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,\Sigma)$. In this case, the internal consistency conditions are satisfied if we impose, in (53) and (54), $\tilde{\theta}_1 = 0$, $\tilde{\theta}_2' = (1,0)$, $c_2 = -2e_2$ and $d_2 = 0$, or :

$$\begin{cases} -2e_2 = a^{\mathbb{Q}}\begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\,, \\ \\ 0 = b^{\mathbb{Q}}\begin{pmatrix} -1 \\ 0 \end{pmatrix}\,, \end{cases} \tag{65}$$

where $a^{\mathbb{Q}}(u) = \Phi' u$ and $b^{\mathbb{Q}}(u) = u'\nu + \frac{1}{2}u'\Sigma u$. So, relation (65) becomes, with obvious notations:

$$\begin{bmatrix} \varphi_{11} \\ \varphi_{12} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\,,$$

$$\nu_1 = \tfrac{1}{2}\sigma_1^2\,,$$

and (64) must be written:

$$\begin{cases} r(t+1,1) = \frac{1}{2}\sigma_1^2 - r(t,1) + 2r(t,2) + \xi_{1,t+1} \\ \\ r(t+1,2) = \nu_2 + \varphi_{21} r(t,1) + \varphi_{22} r(t,2) + \xi_{2,t+1}\,, \end{cases} \tag{66}$$

with $\xi_t \overset{\mathbb{Q}}{\sim} IIN(0,\Sigma)$. Consequently, the R.N. conditional Log-Laplace transform of w_{t+1}, compatible with the AAO restrictions is:

$$\psi_t^{\mathbb{Q}}(u) = u' \left[\begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} w_t \right] + \frac{1}{2} u' \Sigma u\,,$$

the yield-to-maturity formula will be affine in w_t, as indicated by (53), and, moreover, independent of the specification of the factor loading α_t. Now, if we move back to the historical conditional Log-Laplace transform, we get:

$$\begin{aligned} \psi_t(u) &= \psi_t^{\mathbb{Q}}(u - \alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t) \\ &= u' \left[\begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} w_t \right] - u'\Sigma\alpha_t + \tfrac{1}{2}u'\Sigma u\,. \end{aligned}$$

If we assume $\alpha_t = \gamma + \Gamma w_t$, we get:

$$\psi_t(u) = u' \left\{ \begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} - \Sigma\gamma + \left[\begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} - \Sigma\Gamma \right] w_t \right\} + \frac{1}{2}u'\Sigma u\,,$$

or, equivalently, we have the following Car $\mathbb{P}$-dynamics:

$$w_{t+1} = \begin{pmatrix} \frac{1}{2}\sigma_1^2 \\ \nu_2 \end{pmatrix} - \Sigma\gamma + \left[\begin{pmatrix} -1 & 2 \\ \varphi_{21} & \varphi_{22} \end{pmatrix} - \Sigma\Gamma \right] w_t + \varepsilon_{t+1}\,, \tag{67}$$

where $\varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0, \Sigma)$ and $\varepsilon_t = \xi_t + \Sigma(\gamma + \Gamma w_t)$. If $\Gamma = 0$, the historical dynamics of w_t is constrained, the parameters Σ, φ_{12} and φ_{22} are identifiable from the observations on w_t, whereas γ and ν_2 are not. If $\Gamma \neq 0$, the historical dynamics of w_t is not constrained and only Σ is identifiable from the observations on w_t.

Observe that, even if we assume α_t to be nonlinear in w_t, the interest rate formula is still affine (contrary to the Direct Modelling case of Section 6.1), and the historical conditional p.d.f. of non-Car factor w_t remains known in closed form [see relation (15)]. This means that, at the same time, we have a tractable pricing model, we can introduce (non-Car) nonlinearities in the interest rate historical dynamics [as suggested, for instance, by Ait-Sahalia (1996)], and we maintain the possibility to estimate the parameters by exact maximum likelihood. Following the Back Modelling strategy, Dai, Le and Singleton (2006) develop a family of discrete-time nonlinear term structure models [exact discrete-time counterpart of the models in Dai and Singleton(2000)] characterized by these three important features.

6.4 Direct Modelling of Wishart Term Structure Models and Quadratic Term Structure Models: a third application of Extended Car Processes

The Wishart Quadratic Term Structure model, proposed by Gourieroux and Sufana (2003), is characterized by an unobservable factor W_t which follows (under the historical probability) the Wishart autoregressive (WAR) process introduced in Section 3.1. The SDF is defined by:

$$M_{t,t+1} = \exp\left[Tr(CW_{t+1}) + d\right] , \tag{68}$$

where C is a $(n \times n)$ symmetric matrix and d is a scalar. The associated R.N. dynamics is defined by:

$$\begin{aligned}\psi_t^{\mathbb{Q}}(\Gamma) &= Tr\left[M'\left\{(C+\Gamma)[I_n - 2(C+\Gamma)]^{-1} - C(I_n - 2C)^{-1}\right\}MW_t\right] \\ &\qquad -\frac{K}{2}\mathrm{Log}\det[(I_n - 2(I_n - 2C)^{-1}\Gamma)] ,\end{aligned}$$

which is also Car(1). The term structure of interest rates at date t is affine in W_t and given by:

$$\begin{aligned}r(t,h) &= -\frac{1}{h}Tr[A(h)W_t] - \frac{1}{h}b(h) , \ h \geq 1 \\ A(h) &= M'[C + A(h-1)]\left\{I_n - 2[C + A(h-1)]\right\}^{-1} M \\ b(h) &= d + b(h-1) - \frac{K}{2}\mathrm{Log}\det[I_n - 2(C + A(h-1))] \\ A(0) &= 0 , \ b(0) = 0 .\end{aligned}$$

In particular, if K is integer, we get:

$$\begin{aligned}r(t,h) &= -\frac{1}{h}Tr[\sum_{k=1}^{K} A(h)x_{k,t}x'_{k,t}] - \frac{1}{h}b(h) , \ h \geq 1 \\ &= -\frac{1}{h}\sum_{k=1}^{K} x'_{k,t}A(h)x_{k,t} - \frac{1}{h}b(h) , \ h \geq 1 ,\end{aligned} \tag{69}$$

which is a sum of quadratic forms in $x_{k,t}$. If $K = 1$, we get the standard Quadratic Term Structure Model which is, therefore, a special affine model [see Beaglehole and Tenney (1991), Ahn, Dittmar and Gallant (2002), Leippold and Wu (2002), Cheng and Scaillet (2006), and Buraschi, Cieslak and Trojani (2008) for a generalization in the continuous-time general equilibrium setting].

We can also define a quadratic term structure model with a linear term, if the historical dynamics of x_{t+1} is given by the following Gaussian VAR(1) process:

$$\begin{aligned} x_{t+1} &= m + Mx_t + \varepsilon_{t+1}\,, \\ \varepsilon_{t+1} &\overset{\mathbb{P}}{\sim} IIN(0, \Sigma)\,. \end{aligned} \tag{70}$$

Indeed (as suggested by example c) in Section 3.3), the factor $w_t = [x_t', vech(x_t x_t')']'$ is Car(1), that is, w_t is an Extended Car process in the historical world [see Appendix 5 for the proof]. Moreover, choosing:

$$\begin{aligned} M_{t,t+1} &= \exp\left[C' x_{t+1} + Tr(C x_{t+1} x_{t+1}') + d\right] \\ &= \exp(C' x_{t+1} + x_{t+1}' C x_{t+1} + d)\,,\ (C \text{ is a symmetric } (n \times n) \text{ matrix})\,, \end{aligned} \tag{71}$$

the process w_t is also Extended Car in the risk-neutral world. The term structure at date t is affine in w_t, that is, of the form:

$$r(t,h) \;=\; x_t' \Lambda(h) x_t + \mu(h)' x_t + \nu(h)\,,\ h \geq 1\,, \tag{72}$$

where $\Lambda(h), \mu(h)$ and $\nu(h)$ follow recursive equations [see also Gourieroux and Sufana (2003), Cheng and Scaillet (2006) and Jiang and Yan (2006)].

7 An Example of Back Modelling for a Security Market Model with Stochastic Dividends and Short Rate

The purpose of this section is to consider an Econometric Security Market Model where the risky assets are dividend-paying assets and the short rate is endogenous. More precisely, the factor is given by $w_t = (y_t, \delta_t, r_{t+1})'$, where:

- $y_t = (y_{1,t}, \ldots, y_{K_1,t})'$ denotes, for each date t, the K_1-dimensional vector of geometric returns associated to cum dividend prices $S_{j,t}$, $j \in \{1, \ldots, K_1\}$;
- $\delta_t = (\delta_{1,t}, \ldots, \delta_{K_1,t})$ is the associated K_1-dimensional vector of (geometric) dividend yields and, denoting $\tilde{S}_{j,t}$ as the ex dividend price of the j^{th} risky asset, we have $S_{j,t} = \tilde{S}_{j,t}\ \exp(\delta_{j,t})$;
- r_{t+1} denotes the (predetermined) stochastic short rate for the period $[t, t+1]$;

Observe that, compared to the setting of Section 5.1 (where r_{t+1} was exogenous), this model proposes a more general K-dimensional factor w_t (with $K = (2K_1 + 1)$), where we jointly specify y_t, δ_t (which is considered as an observable factor), and the short rate r_{t+1}. It would be straightforward to add an unobservable factor z_t.

Following the Back Modelling approach, we propose a R.N. Gaussian VAR(1) dynamics for the factor and the conditional distribution of w_{t+1}, given $\underline{w_t}$, is assumed to be Gaussian with mean vector $(A_0 + A_1\, w_t)$ and variance-covariance matrix Σ. The process w_{t+1} is, therefore, a Car(1) process with a conditional R.N. Laplace Transform given by :

$$\varphi_t^{\mathbb{Q}}(u \,|\, \underline{w_t}) = E_t^{\mathbb{Q}}\left[\exp\left(u'\, w_{t+1}\right)\right] = \exp\left[a^{\mathbb{Q}}(u)'\, w_t \,+\, b^{\mathbb{Q}}(u)\right]$$

where the functions $a^{\mathbb{Q}}$ and $b^{\mathbb{Q}}$ are the following :

$$\begin{cases} a^{\mathbb{Q}}(u) &=& A_1{}'\, u \\ \\ b^{\mathbb{Q}}(u) &=& A_0{}'\, u + \frac{1}{2}u'\Sigma u\,. \end{cases}$$

The R.N. dynamics can also be written:

$$\begin{aligned} & w_{t+1} = A_0 + A_1\, w_t + \xi_{t+1} \\ & \\ & \xi_{t+1} \overset{\mathbb{Q}}{\sim} IIN(0,\Sigma)\,. \end{aligned} \tag{73}$$

The AAO restrictions, applied to the K_1-dimensional vector y_{t+1}, are given by :

$$\begin{aligned} & E_t^{\mathbb{Q}}\left[\exp[\mathrm{Log}\left(\tfrac{S_{j,t+1}}{\tilde{S}_{j,t}}\right)\right] = \exp(r_{t+1})\,,\;\; j\in\{1,\ldots,K_1\}\,, \\ & \\ \iff \quad & E_t^{\mathbb{Q}}\left[\exp(y_{j,t+1})\right] = \exp(r_{t+1}-\delta_{j,t})\,,\;\; j\in\{1,\ldots,K_1\}\,, \\ & \\ \iff \quad & \begin{cases} a^{\mathbb{Q}}(e_j) &=& A_1{}'\, e_j &=& e_K - e_{j+K_1}\,,\;\; j\in\{1,\ldots,K_1\}\,, \\ \\ b^{\mathbb{Q}}(e_j) &=& A_0{}'\, e_j + \frac{1}{2}e_j'\Sigma e_j \;=\; 0\,,\;\; j\in\{1,\ldots,K_1\}. \end{cases} \end{aligned}$$

This means that the first K_1 rows of A_1 and the first K_1 components of A_0 are, for $j\in\{1,\ldots,K_1\}$, respectively given by $(e_K - e_{j+K_1})'$ and $-\frac{1}{2}\sigma_j^2$ [where e_K and e_{j+K_1} denote, respectively, the K^{th} and the $(j+K_1)^{th}$ column of the Identity matrix I_K, while σ_j^2 is the (j,j)-term of Σ]. In other words, the K_1 first equations of (73) are:

$$y_{j,t+1} = -\tfrac{1}{2}\sigma_j^2 + r_{t+1} - \delta_{j,t} + \xi_{j,t+1}\,,\;\; j\in\{1,\ldots,K_1\}\,.$$

Then, coming back to the historical dynamics of w_t, we get :

$$\begin{aligned} \psi_t(u) &= \psi_t^{\mathbb{Q}}(u-\alpha_t) - \psi_t^{\mathbb{Q}}(-\alpha_t) \\ &= \left(a^{\mathbb{Q}}(u-\alpha_t) - a^{\mathbb{Q}}(-\alpha_t)\right)'\, w_t + b^{\mathbb{Q}}(u-\alpha_t) - b^{\mathbb{Q}}(-\alpha_t) \\ &= u'A_1 w_t + u'A_0 + \tfrac{1}{2}(u-\alpha_t)'\Sigma(u-\alpha_t) - \tfrac{1}{2}\alpha_t'\Sigma\alpha_t \\ &= u'\left(A_0 + A_1 w_t - \Sigma\alpha_t\right) + \tfrac{1}{2}u'\Sigma u\,. \end{aligned} \tag{74}$$

So, if we impose $\alpha_t = (\alpha_0 + \alpha\, w_t)$, the historical dynamics of the factor is also Gaussian VAR(1) with a modified conditional mean vector equal to $[A_0 - \Sigma\alpha_0 + (A_1 - \Sigma\alpha)w_t]$ and the same variance-covariance matrix Σ, that is:

$$\begin{aligned} & w_{t+1} = A_0 - \Sigma\alpha_0 + (A_1 - \Sigma\alpha)w_t + \varepsilon_{t+1}\,,\;\; \varepsilon_{t+1} \overset{\mathbb{P}}{\sim} IIN(0,\Sigma)\,, \\ & \\ & \text{and } \varepsilon_{t+1} = \xi_{t+1} + \Sigma(\alpha_0 + \alpha\, w_t)\,. \end{aligned}$$

We notice that, under the historical probability, any VAR(1) distribution can be reached, but only Σ is identifiable. If we add the constraint $\alpha = 0$, then the historical dynamics of w_t is constrained, and A_0 and α_0 are not identifiable.

8 Conclusions

In this paper we have proposed a general econometric approach to no-arbitrage asset pricing modelling based on three main elements : (i) the historical discrete-time dynamics of the factor representing the information, (ii) the Stochastic Discount Factor (SDF), and (iii) the risk-neutral (R.N.) discrete-time factor dynamics. We have presented three modelling strategies : the Direct Modelling, the R.N. Constrained Direct Modelling and the Back Modelling. In all the approaches we have considered the internal consistency conditions, induced by the AAO restrictions, and the identification problem. These three approaches have been explicited for several discrete time security market models and affine term structure models. In all cases, we have indicated the important role played by the R.N. Constrained Direct Modelling and the Back Modelling strategies in determining, at the same time, flexible historical dynamics and Car R.N. dynamics leading to explicit or quasi explicit pricing formulas for various contingent claims. Moreover, we have shown the possibility to derive asset pricing models able to accommodate non-Car historical and risk-neutral factor dynamics with tractable pricing formulas. This result is achieved when the starting R.N. non-Car factor turns out to be a R.N. Extended Car process. These strategies, already implicitly adopted in several papers, clearly could be the basis for the specification of new asset pricing models leading to promising empirical analysis.

Appendix 1
Proof of the existence and uniqueness of $M_{t,t+1}$ and of the pricing formula (1)

Using A1 and A2, the Riesz representation theorem implies :

$$\forall s > t,\ \forall \underline{w_t}, \exists\ M_{t,s}(\underline{w_s}), \text{unique, such that } \forall g(\underline{w_s}) \in L_{2s}$$

$$p_t[g(\underline{w_s})] = E[M_{t,s}(\underline{w_s})\, g(\underline{w_s}) \,|\, \underline{w_t}].$$

In particular, the price at t of a zero-coupon bond with maturity s is $E[M_{t,s}(\underline{w_s}) \,|\, \underline{w_t}]$. A3 implies that $\mathbb{P}[M_{t,s} > 0 \,|\, \underline{w_t}] = 1$, $\forall t, s \in \{0, \ldots, T\}$, since otherwise the payoff $\mathbb{1}_{(M_{t,s} \leq 0)}$ at s, would be such that $\mathbb{P}[\mathbb{1}_{(M_{t,s} \leq 0)} > 0 \,|\, \underline{w_t}] > 0$ and $p_t\left[\mathbb{1}_{(M_{t,s} \leq 0)}\right] = E_t[M_{t,s} \mathbb{1}_{(M_{t,s} \leq 0)}] \leq 0$, contradicting A3.

Relation (1) will be shown if we prove that, $\forall\ t < r < s$, $\underline{w_t}$, $g(\underline{w_s}) \in L_{2s}$ we have:

$$p_t[g(\underline{w_s})] = p_t\{p_r[g(\underline{w_s})]\}.$$

Let us show, for instance, that if (with obvious notations) $p_t(g_s) > p_t[p_r(g_s)]$, we can construct over the time interval $[t,\ s\,]$ a sequence of portfolios with strictly positive payoff at s, with zero payoffs at any date $r \in]t,\ s\,[$, and with price zero at t, contradicting A3. The sequence of portfolios is defined by the following trading strategy:

at t: buy $p_r(g_s)$, (short) sell g_s, buy $\dfrac{p_t(g_s) - p_t[p_r(g_s)]}{E[M_{t,s} \,|\, \underline{w_t}]}$ zero-coupon bonds with maturity s, generating a zero payoff;

at r: buy g_s and sell $p_r(g_s)$, generating a zero payoff;

at s: the net payoff is $g_s - g_s + \dfrac{p_t(g_s) - p_t[p_r(g_s)]}{E[M_{t,s} \,|\, \underline{w_t}]} > 0$.

A similar argument shows that $p_t(g_s) < p_t[p_r(g_s)]$ contradicts A3 and, therefore, relation (1) is proved.

Appendix 2
Risk Premia and Market Price of Risk

Notation

In this appendix $[f_t(e_i)]$ will denote, for given scalar or row K-vectors $f_t(e_i)$, $i \in \{1, \ldots, K\}$, the K-vector or the $K \times K$ matrix $(f_t(e_1)', \ldots, f_t(e_K)')'$ with rows $f_t(e_i)$, $i \in \{1, \ldots, K\}$; e will denote the K-dimensional unitary vector.

Geometric and Arithmetic Risk Premia

Let p_t be the price at t of any given asset. The geometric return between t and $t+1$ is

$$\rho_{G,t+1} = \text{Log}\left(\frac{p_{t+1}}{p_t}\right),$$

whereas the arithmetic return is :

$$\rho_{A,t+1} = \frac{p_{t+1}}{p_t} - 1 = \exp(\rho_{G,t+1}) - 1\,.$$

In particular, for the risk-free asset we have :

$$\begin{aligned} \rho^f_{G,t+1} &= r_{t+1}\,, \\ \rho^f_{A,t+1} &= \exp(r_{t+1}) - 1 = r_{A,t+1}\,. \end{aligned}$$

So, we can define two risk premia of the given asset :

$$\begin{aligned} \pi_{Gt} &= E_t(\rho_{G,t+1}) - r_{t+1}\,, \\ \pi_{At} &= E_t(\rho_{A,t+1}) - r_{A,t+1} = E_t\left[\exp(\rho_{G,t+1})\right] - \exp(r_{t+1})\,. \end{aligned} \tag{A.1}$$

Note that the arithmetic risk premia have the advantage to satisfy $\pi_{At}(\lambda) = \Sigma_{j=1}^J \lambda_j \pi_{At,j}$, if $\pi_{At}(\lambda)$ is the risk premium of the portfolio defined by the shares in value λ_j for the asset j. Let us now consider two important particular cases in order to have more explicit forms of these risk premia and to obtain intuitive interpretations of the factor loading vector α_t [see also Dai, Le and Singleton (2006) for a similar analysis].

The factor is a vector of geometric returns

If w_{t+1} is a K-vector of geometric returns, the vectors of risk premia π_{Gt} and π_{At} whose entries are:

$$\pi_{Gt,i} = e_i' \psi_t^{(1)}(0) - r_{t+1}, \quad i \in \{1, ..., K\},$$

(where $\psi_t^{(1)}$ is the gradient of ψ_t and e_i is the i^{th} column of the identity matrix I_K),

$$\pi_{At,i} = \varphi_t(e_i) - \exp(r_{t+1}), \quad i \in \{1, ..., K\}.$$

Moreover, we have the pricing identities :

$$1 = E_t\left\{\exp\left[e_i' w_{t+1} + \alpha_t' w_{t+1} - r_{t+1} - \psi_t(\alpha_t)\right]\right\}, \;\; i \in \{1, ..., K\}, \tag{A.2}$$

that is

$$\exp(r_{t+1}) = \frac{\varphi_t(\alpha_t + e_i)}{\varphi_t(\alpha_t)} = \varphi_t^{\mathbb{Q}}(e_i),$$

or

$$r_{t+1} = \psi_t(\alpha_t + e_i) - \psi_t(\alpha_t) = \psi_t^{\mathbb{Q}}(e_i).$$

So, for each $i \in \{1, ..., K\}$, the risk premia can be written :

$$\begin{aligned} \pi_{Gt,i} &= e_i' \psi_t^{(1)}(0) - \psi_t(\alpha_t + e_i) + \psi_t(\alpha_t) \\ \pi_{At,i} &= \varphi_t(e_i) - \frac{\varphi_t(\alpha_t + e_i)}{\varphi_t(\alpha_t)}. \end{aligned}$$

Note that, for $\alpha_t = 0$, i.e. when the historical and the R.N. dynamics are identical, we have :

$$\pi_{Gt,i} = m_{it} - \psi_t(e_i) \neq 0, \quad i \in \{1, ..., K\},$$

(m_{it} denotes the conditional mean of $w_{i,t+1}$ given $\underline{w_t}$) and

$$\pi_{At,i} = 0, \qquad i \in \{1, ..., K\}.$$

So the arithmetic risk premia seem to have more natural properties. Moreover, considering first order expansions around $\alpha_t = 0$ and neglecting conditional cumulants of order strictly larger than 2 (which are zero in the conditionally gaussian case), we get:

$$\begin{aligned} \pi_{Gt} &\simeq -\frac{1}{2} vdiag(\Sigma_t) - \Sigma_t \alpha_t && \text{(A.3)} \\ \pi_{At} &\simeq -\exp(r_{t+1}) \Sigma_t \, \alpha_t \, . && \text{(A.4)} \end{aligned}$$

where $vdiag(\Sigma_t)$ is the vector whose entries are the diagonal terms of Σ_t, and Σ_t is the conditional variance-covariance matrix of w_{t+1} given $\underline{w_t}$. So, α_t can be viewed as the opposite of a market price of risk vector. We will see in the proof below that the expression of π_{Gt} is exact in the conditionally Gaussian case.

<u>*Proof of relations (A.3) and (A.4)*</u>

We have seen above that the geometric risk premium can be written as:

$$\pi_{Gt} = \psi_t^{(1)}(0) - [\psi_t(\alpha_t + e_i)] + \psi_t(\alpha_t) e \, .$$

Using a first order expansion of $\pi_{Gt} = \pi_{Gt}(\alpha_t)$ around $\alpha_t = 0$ we obtain :

$$\pi_{Gt} \simeq \psi_t^{(1)}(0) - [\psi_t(e_i)] - [\psi_t^{(1)}(e_i)'] \alpha_t + (\psi_t^{(1)}(0)' \alpha_t) e \, ,$$

and neglecting conditional cumulants of order ≥ 3 we can write :

$$\begin{aligned} \pi_{Gt} &\simeq m_t - m_t - \tfrac{1}{2} vdiag \Sigma_t - (m_t' \alpha_t) e - \Sigma_t \alpha_t + (m_t' \alpha_t) e \\ &\simeq -\tfrac{1}{2} v \, diag \, \Sigma_t - \Sigma_t \alpha_t \, . \end{aligned}$$

If we consider now the arithmetic risk premium, and we apply the same procedure, we get:

$$\begin{aligned} \pi_{At} &= [\varphi_t(e_i)] - \left[\tfrac{\varphi_t(\alpha_t + e_i)}{\varphi_t(\alpha_t)} \right] \\ &\simeq \left[\varphi_t(e_i) - \varphi_t(e_i) \left(1 + \tfrac{\varphi_t^{(1)}(e_i)' \alpha_t}{\varphi_t(e_i)} - \varphi_t^{(1)}(0)' \alpha_t \right) \right] \\ &\simeq [-\varphi_t(e_i)(\psi_t^{(1)}(e_i)' \alpha_t - \varphi^{(1)}(0)' \alpha_t)] \\ &\simeq -diag[\varphi_t(e_i)]((m_t' \alpha_t) e + \Sigma_t \alpha_t - (m_t' \alpha_t) e) \\ &\simeq -diag[\varphi_t(e_i)] \Sigma_t \alpha_t \\ &\simeq -\exp(r_{t+1}) \Sigma_t \alpha_t \, , \end{aligned}$$

since $\varphi_t(e_i) = E_t \exp(w_{i,t+1}) \simeq E_t^{\mathbb{Q}} \exp(w_{i,t+1}) = \exp(r_{t+1})$.
In the conditionally Gaussian case, where

$$\varphi_t(u) = \exp\left(m_t' u + \frac{1}{2} u' \Sigma_t u \right), \;\; \psi_t(u) = m_t' u + \frac{1}{2} u' \Sigma_t u \, ,$$

the geometric risk premium becomes

$$
\begin{aligned}
\pi_{Gt} &= \psi_t^{(1)}(0) - [\psi_t(\alpha_t + e_i)] + \psi_t(\alpha_t)e \\
&= m_t - \left[m_t'(\alpha_t + e_i) + \tfrac{1}{2}(\alpha_t + e_i)'\Sigma_t(\alpha_t + e_i) - m_t'\alpha_t - \tfrac{1}{2}\alpha_t'\Sigma\alpha_t\right] \\
&= -\frac{1}{2} vdiag\, \Sigma_t - \Sigma_t\alpha_t \, ,
\end{aligned}
$$

while, the arithmetic risk premium is

$$
\begin{aligned}
\pi_{At} &= [\varphi_t(e_i)] - \left[\tfrac{\varphi_t(\alpha+e_i)}{\varphi_t(\alpha_t)}\right] \\
&= \left[\exp\left(m_{it} + \frac{1}{2}\Sigma_{ii,t}\right) - \exp\left(m_{it} + \frac{1}{2}\Sigma_{ii,t} + e_i'\Sigma_t\alpha_t\right)\right] \\
&= \left[\exp\left(m_{it} + \frac{1}{2}\Sigma_{ii,t}\right)(1 - \exp(e_i'\Sigma_t\alpha_t))\right] \\
&\simeq -diag\left[\exp\left(m_{it} + \frac{1}{2}\Sigma_{ii,t}\right)\right]\Sigma_t\alpha_t = -\exp(r_{t+1})\,\Sigma_t\alpha_t \, ,
\end{aligned}
$$

since $\varphi_t(e_i) = \exp(m_{it} + \frac{1}{2}\Sigma_{ii,t})$.

The factor is a vector of yields

Let us denote by $r(t,h)$ the yield at t with residual maturity h; if $B(t,h)$ denotes the price at t of the zero coupon bond with time to maturity h, we have :

$$r(t,h) = -\frac{1}{h}\text{ Log }[B(t,h)] \, .$$

We assume that the components of w_{t+1} are :

$$w_{t+1,i} = h_i\, r(t+1, h_i) \, , \qquad i \in \{1, ..., K\} \, ,$$

where h_i are various integer residual maturities; this definition of $w_{t+1,i}$ leads to simpler notations than the equivalent definition $w_{t+1,i} = r(t+1, h_i)$. The payoffs $B(t+1, h_i) = \exp(-w_{t+1,i})$ have price at t equal to

$$B(t, h_i + 1) = \exp\left[-(h_i + 1) r(t, h_i + 1)\right] .$$

So, we have

$$1 = E_t\left\{\exp\left[-w_{t+1,i} + (h_i + 1)r(t, h_i + 1) + \alpha_t' w_{t+1} - r_{t+1} - \psi_t(\alpha_t)\right]\right\} \, , \; i \in \{1, ..., K\}, \qquad \text{(A.5)}$$

that is :

$$r_{t+1} = \psi_t(\alpha_t - e_i) - \psi_t(\alpha_t) + (h_i + 1)r(t, h_i + 1) \, ,$$

or :

$$\exp(r_{t+1}) = \frac{\varphi_t(\alpha_t - e_i)}{\varphi_t(\alpha_t)}\exp\left[(h_i + 1)r(t, h_i + 1)\right] \, .$$

The risk premia associated to the geometric returns :

$$\text{Log}\left[\frac{B(t+1,h_i)}{B(t,h_i+1)}\right] = -w_{t+1,i} + (h_i+1)r(t,h_i+1)$$

are the vectors with components :

$$\begin{aligned} \pi_{Gt,i} &= -E_t(w_{t+1,i}) + (h_i+1)r(t,h_i+1) - r_{t+1} \\ &= -e_i'\psi_t^{(1)}(0) - \psi_t(\alpha_t - e_i) + \psi_t(\alpha_t)\,, \end{aligned} \tag{A.6}$$

and :

$$\begin{aligned} \pi_{At,i} &= \exp\left[(h_i+1)r(t,h_i+1)\right]\varphi_t(-e_i) - \exp(r_{t+1}) \\ &= \exp\left[(h_i+1)r(t,h_i+1)\right]\left[\varphi_t(-e_i) - \tfrac{\varphi_t(\alpha_t-e_i)}{\varphi_t(\alpha_t)}\right]\,. \end{aligned} \tag{A.7}$$

Expanding relations (A.6) and (A.7) around $\alpha_t = 0$, and neglecting conditional cumulants of order strictly larger than 2, we get :

$$\pi_{Gt} \simeq -\frac{1}{2}\,vdiag(\Sigma_t) + \Sigma_t\alpha_t \tag{A.8}$$

$$\pi_{At} \simeq \exp(r_{t+1})\Sigma_t\alpha_t\,, \tag{A.9}$$

where Σ_t is the conditional variance-covariance matrix of w_{t+1} given $\underline{w_t}$. So, α_t can be viewed as a market price of risk vector. Moreover, the formula for π_{Gt} is exact in the conditionally gaussian case.

Proof of relations (A.8) and (A.9)

Following the same procedure presented above, the geometric risk premium associated to $w_{t+1} = (h_1 r(t+1,h_1),\ldots,h_K r(t+1,h_K))'$ can be written as

$$\begin{aligned} \pi_{Gt} &= -\psi_t^{(1)}(0) - [\psi_t(\alpha_t - e_i)] + \psi_t(\alpha_t)e \\ &\simeq -\psi_t^{(1)}(0) - [\psi_t(-e_i) + \psi_t^{(1)}(-e_i)'\alpha_t] + (\psi_t^{(1)}(0)'\alpha_t)e \\ &\simeq -m_t - \left(-m_t + \tfrac{1}{2}\,vdiag\,\Sigma_t + (m_t'\alpha_t)e - \Sigma_t\alpha_t\right) + (m_t'\alpha_t)e \\ &\simeq -\tfrac{1}{2}\,vdiag\,\Sigma_t + \Sigma_t\alpha_t\,, \end{aligned}$$

while, the arithmetic risk premium is

$$\begin{aligned} \pi_{At} &= \left[\exp((h_i+1)r(t,h_i+1))\left(\varphi_t(-e_i) - \tfrac{\varphi_t(\alpha_t-e_i)}{\varphi_t(\alpha_t)}\right)\right] \\ &\simeq [\exp((h_i+1)r(t,h_i+1))(\varphi_t(-e_i) - \varphi_t(-e_i)(1+\psi_t^{(1)}(-e_i)'\alpha_t - \varphi_t^{(1)}(0)'\alpha_t))] \\ &\simeq -diag[\varphi_t(-e_i)\exp((h_i+1)r(t,h_i+1))][\psi_t^{(1)}(-e_i)'\alpha_t - \varphi_t^{(1)}(0)'\alpha_t] \\ &\simeq diag[\varphi_t(-e_i)\exp((h_i+1)r(t,h_i+1))]\Sigma_t\alpha_t \\ &\simeq \exp(r_{t+1})\,\Sigma_t\alpha_t\,, \end{aligned}$$

since $\varphi_t(-e_i) = E_t \exp[-h_i r(t+1,h_i)] = E_t B(t+1,h_i) \simeq E_t^{\mathbb{Q}} B(t+1,h_i) = \exp(r_{t+1}) B(t,h_i+1)$.

In the conditionally Gaussian case, we have:

$$
\begin{aligned}
\pi_{Gt} &= -\frac{1}{2} vdiag\, \Sigma_t + \Sigma_t \alpha_t\,, \text{ and} \\
\pi_{At} &\simeq diag \left[\exp\left(-m_{it} + \frac{1}{2}\Sigma_{ii,t} \right) \exp((h_i+1) r(t,h_i+1)) \right] \Sigma_t \alpha_t = \exp(r_{t+1})\, \Sigma_t \alpha_t\,,
\end{aligned}
$$

given that $\varphi_t(-e_i) = \exp(-m_{it} + \frac{1}{2}\Sigma_{ii,t})$.

Appendix 3
Switching GARCH Models and Extended Car processes

The purpose of this appendix is to show, in the context of Section 5.6, that under the R.N. probability, even if $w_{t+1} = (y_{t+1}, z'_{t+1})'$ is not a Car process, the extended factor $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is Car. The proof of this result is based on the following two lemmas.

Lemma 1: For any vector $\mu \in \mathbb{R}^n$ and any symmetric positive definite $(n \times n)$ matrix Q, the following relation holds:

$$
\int_{\mathbb{R}^n} \exp(-u'Qu + \mu' u) du = \frac{\pi^{n/2}}{(\det\, Q)^{1/2}} \exp\left(\frac{1}{4} \mu' Q^{-1} \mu \right) .
$$

Proof: The LHS of the previous relation can be written as

$$
\begin{aligned}
& \int_{\mathbb{R}^n} \exp\left[-\left(u - \frac{1}{2} Q^{-1}\mu \right)' Q \left(u - \frac{1}{2} Q^{-1}\mu \right) \right] \exp\left(\frac{1}{4} \mu' Q^{-1} \mu \right) du \\
= & \frac{\pi^{n/2}}{(\det Q)^{1/2}} \exp\left(\frac{1}{4} \mu' Q^{-1} \mu \right)
\end{aligned}
$$

given that the n-dimensional Gaussian distribution $N\left(\frac{1}{2}Q^{-1}\mu, (2Q)^{-1}\right)$ admits unit mass.

Lemma 2: If $\varepsilon_{t+1} \sim N(0, I_n)$, we have

$$
\begin{aligned}
& E_t \left\{ \exp[\lambda' \varepsilon_{t+1} + \varepsilon'_{t+1} V \varepsilon_{t+1}] \right\} \\
= & \frac{1}{[\det(I-2V)]^{1/2}} \exp\left[\frac{1}{2} \lambda' (I-2V)^{-1} \lambda \right] .
\end{aligned}
$$

Proof: From Lemma 1, we have:

$$
\begin{aligned}
& E_t \left\{ \exp(\lambda' \varepsilon_{t+1} + \varepsilon'_{t+1} V \varepsilon_{t+1}) \right\} \\
= & \tfrac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp\left[-u' \left(\frac{1}{2} I - V \right) u + \lambda' u \right] du \\
= & \tfrac{1}{2^{n/2}\left[\det\left(\frac{1}{2}I - V\right)\right]^{1/2}} \exp\left[\tfrac{1}{4} \lambda' \left(\tfrac{1}{2} I - V \right)^{-1} \lambda \right] \\
= & \tfrac{1}{[\det(I-2V)]^{1/2}} \exp\left[\tfrac{1}{2} \lambda' (I-2V)^{-1} \lambda \right] .
\end{aligned}
$$

Proposition : In the context of Section 5.6, the process $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is Car(1) under the R.N. probability.

Proof : We have:

$$y_{t+1} = r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma^2_{t+1} + \nu'_3 z_{t+1} + \sigma_{t+1}\xi_{t+1}$$

$$\xi_{t+1} \,|\, \underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1(\xi_t - \alpha_2\sigma_t)^2 + \alpha_3\sigma^2_t$$

$$\mathbb{Q}\left(z_{t+1} = e_j | \underline{y}_t, \underline{z}_{t-1}, z_t = e_i\right) = \pi^*_{ij}\,.$$

So, the conditional R.N. Laplace transform of $(y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is:

$$\begin{aligned}
\varphi^{\mathbb{Q}}_t(u,v,\tilde{v}) &= E^{\mathbb{Q}}_t \exp\left(u y_{t+1} + v' z_{t+1} + \tilde{v}\sigma^2_{t+2}\right) \\
&= E^{\mathbb{Q}}_t \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma^2_{t+1} + \nu'_3 z_{t+1} + \sigma_{t+1}\xi_{t+1}\right)\right. \\
&\qquad \left. + v' z_{t+1} + \tilde{v}[\omega' z_{t+1} + \alpha_1(\xi_{t+1} - \alpha_2\sigma_{t+1})^2 + \alpha_3\sigma^2_{t+1}]\right\} \\
&= \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma^2_{t+1}\right) + \tilde{v}\alpha_1\alpha_2^2\sigma^2_{t+1} + \tilde{v}\alpha_3\sigma^2_{t+1}\right\} \\
&\qquad E^{\mathbb{Q}}_t \exp[\xi_{t+1}\sigma_{t+1}(u - 2\alpha_1\alpha_2\tilde{v}) + \tilde{v}\alpha_1\xi^2_{t+1} + (u\nu_3 + v + \tilde{v}\omega)' z_{t+1}]\,.
\end{aligned}$$

Using Lemma 2:

$$\begin{aligned}
\varphi^{\mathbb{Q}}_t(u,v,\tilde{v}) &= \exp[u(r_{t+1} - \lambda' z_t - \frac{1}{2}\sigma^2_{t+1}) + \tilde{v}\alpha_1\alpha_2^2\sigma^2_{t+1} + \tilde{v}\alpha_3\sigma^2_{t+1}] \\
&\quad \times \exp\left[-\frac{1}{2}\mathrm{Log}(1 - 2\alpha_1\tilde{v}) + \frac{(u - 2\alpha_1\alpha_2\tilde{v})^2}{2(1 - 2\alpha_1\tilde{v})}\sigma^2_{t+1} + \tilde{\Lambda}'(u,v,\tilde{v},\omega,\nu_3,\pi^*) z_t\right],
\end{aligned}$$

where the i^{th} component of $\tilde{\Lambda}(u,v,\tilde{v},\omega,\nu_3,\pi^*)$ is given by:

$$\tilde{\Lambda}_i(u,v,\tilde{v},\omega,\nu_3,\pi^*) = \mathrm{Log}\sum_{j=1}^{J}\pi^*_{ij}\exp(u\nu_{3j} + v_j + \tilde{v}\omega_j)\,,$$

and relation (47) is proved.

Appendix 4
Switching IG GARCH Models and Extended Car processes

In this appendix we show, in the context of Section 5.7, that under the R.N. probability, even if $w_{t+1} = (y_{t+1}, z'_{t+1})'$ is not a Car process, the extended factor $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is Car.

Proposition : In the context of Section 5.7, the process $w^e_{t+1} = (y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is Car(1) under the R.N. probability.

Proof : Let us recall equation (49) :

$$y_{t+1} = r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\sigma^2_{t+1} + \nu'_3 z_{t+1} + \eta\xi_{t+1}$$

with

$$\sigma^2_{t+1} = \omega' z_t + \alpha_1\sigma^2_t + \alpha_2\xi_t + \alpha_3\frac{\sigma^4_t}{\xi_t}$$

$$\xi_{t+1}|\,\underline{\xi}_t, \underline{z}_{t+1} \overset{\mathbb{Q}}{\sim} IG\left(\frac{\sigma^2_{t+1}}{\eta^2}\right),$$

$$\mathbb{Q}(z_{t+1} = e_j|\underline{y}_t, \underline{z}_{t-1}, z_t = e_i) = \mathbb{Q}(z_{t+1} = e_j|z_t = e_i) = \pi^*_{ij},$$

So, the conditional R.N. Laplace transform of $(y_{t+1}, z'_{t+1}, \sigma^2_{t+2})'$ is:

$$\begin{aligned}
\varphi^{\mathbb{Q}}_t(u, v, \tilde{v}) &= E^{\mathbb{Q}}_t \exp\left(uy_{t+1} + v'z_{t+1} + \tilde{v}\sigma^2_{t+2}\right) \\
&= E^{\mathbb{Q}}_t \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\sigma^2_{t+1} + \nu'_3 z_{t+1} + \eta\xi_{t+1}\right)\right. \\
&\qquad \left. + v'z_{t+1} + \tilde{v}\left[\omega' z_{t+1} + \alpha_1\sigma^2_{t+1} + \alpha_2\xi_{t+1} + \alpha_3\frac{\sigma^4_{t+1}}{\xi_{t+1}}\right]\right\} \\
&= \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\sigma^2_{t+1}\right) + \tilde{v}\alpha_1\sigma^2_{t+1}\right\} \\
&\qquad E^{\mathbb{Q}}_t \exp[(u\eta + \tilde{v}\alpha_2)\,\xi_{t+1} + \frac{\tilde{v}\alpha_3\sigma^4_{t+1}}{\xi_{t+1}} + (u\nu_3 + v + \tilde{v}\omega)' z_{t+1}]\,.
\end{aligned}$$

Using the formula of the generalized Laplace transform of an Inverse Gaussian distribution given in footnote 8 (section 5.7):

$$\begin{aligned}
\varphi^{\mathbb{Q}}_t(u, v, \tilde{v}) &= \exp\left\{u\left(r_{t+1} - \lambda' z_t - \frac{1}{\eta^2}\left[1 - (1-2\eta)^{1/2}\right]\sigma^2_{t+1}\right) + \tilde{v}\alpha_1\sigma^2_{t+1}\right\} \\
&\times \exp\left[-\frac{1}{2}\text{Log}(1 - 2\tilde{v}\alpha_3\eta^4) + \frac{1}{\eta^2}\left(1 - \sqrt{(1 - 2\tilde{v}\alpha_3\eta^4)\,(1 - 2(u\eta + \tilde{v}\alpha_2))}\right)\sigma^2_{t+1}\right. \\
&\left. + \tilde{\Lambda}'(u, v, \tilde{v}, \nu_3, \omega, \pi^*) z_t\right],
\end{aligned}$$

where the i^{th} component of $\tilde{\Lambda}(u, v, \tilde{v}, \nu_3, \omega, \pi^*)$ is given by:

$$\tilde{\Lambda}_i(u, v, \tilde{v}, \nu_3, \omega, \pi^*) = \text{Log}\sum_{j=1}^{J}\pi^*_{ij}\exp(u\nu_{3j} + v_j + \tilde{v}\omega_j)\,,$$

and relation (52) is proved.

Appendix 5
Quadratic Term Structure Models and Extended Car processes

Given the Gaussian VAR(1) process defined by relation (70), we have that, for any real symmetric matrix V, the conditional historical Laplace transform of $(x_{t+1}, x_{t+1}\, x'_{t+1})$ is given by:

$$
\begin{aligned}
& E_t \exp[u'x_{t+1} + Tr(Vx_{t+1}x'_{t+1})] \\
= \;& \exp\{u'm + u'Mx_t + TrV[mm' + Mx_tx'_tM' + mx'_tM' + Mx_tm']\} \\
& \quad E_t \exp\{u'\varepsilon_{t+1} + TrV[\varepsilon_{t+1}\varepsilon'_{t+1} + m\varepsilon'_{t+1} + \varepsilon_{t+1}m' + Mx_t\varepsilon'_{t+1} + \varepsilon_{t+1}x'_tM']\} \\
= \;& \exp\{u'm + u'Mx_t + m'Vm + 2m'VMx_t + Tr(M'VMx_tx'_t)\} \\
& \quad E_t \exp\{[u' + 2(m + Mx_t)'V]\varepsilon_{t+1} + \varepsilon'_{t+1}V\varepsilon_{t+1}\}
\end{aligned}
$$

and, using Lemma 2 in Appendix 3, we can write:

$$
\begin{aligned}
& E_t \exp[u'x_{t+1} + Tr(Vx_{t+1}x'_{t+1})] \\
= \;& \exp\{u'm + m'Vm + (M'u + 2M'Vm)'x_t + x'_tM'VMx_t \\
& \quad + \tfrac{1}{2}[u' + 2(m + Mx_t)'V](I - 2V)[u + 2V(m + Mx_t)] - \tfrac{1}{2}\,\mathrm{Log}\, det(I - 2V)\}\,,
\end{aligned}
$$

which is exponential-affine in $[x'_t, vech(x_tx'_t)']'$.

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