Asset pricing with second-order Esscher transforms

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Abstract

The purpose of the paper is to introduce, in a discrete-time no-arbitrage pricing context, a bridge between the historical and the risk-neutral state vector dynamics which is wider than the one implied by a classical exponential-affine stochastic discount factor (SDF) and to preserve, at the same time, the tractability and flexibility of the associated asset pricing model. This goal is achieved by introducing the notion of Exponential-Quadratic SDF or, equivalently, the notion of Second-Order Esscher Transform. The log-pricing kernel is specified as a quadratic function of the factor and the associated sources of risk are priced by means of possibly non-linear stochastic first-order and second-order risk-correction coefficients. Focusing on security market models, this approach is developed in the multivariate conditionally Gaussian framework and its usefulness is testified by the specification and calibration of what we name the Second-Order GARCH Option Pricing Model. The associated European Call option pricing formula generates a rich family of implied volatility smiles and skews able to match the typically observed ones.

JEL classification: G12; G13.

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1. Introduction

Discrete-time asset pricing models are now widespread in the economic and financial literature and they are successfully used in many research fields, like bond and option pricing, longevity risk, liquidity and credit risk modelling, as well as exchange rate and macro-finance modelling. This large class of models contains two important families following two different asset pricing modelling principles : the first one is built on the notion of stochastic discount factor (SDF), while the second one is based on the concept of (local) risk-neutral valuation relationship (RNVR or LRNVR).

The first set of models invokes the absence of arbitrage opportunity in order to typically introduce an exponential-affine (in the factor) SDF which provides a bridge between the historical world and the risk-neutral one [see Gourieroux and Monfort (2007)]. Since the three mathematical objects specifying the models, namely the historical dynamics of the state vector, its risk-neutral (R.N.) dynamics and the one-period SDF are linked together, three modelling strategies naturally appear (the so-called Direct Modelling, Risk-Neutral Constrained Direct Modelling and Back Modelling strategies). In each of them two objects are specified (and, possibly, the short rate if it is not assumed to be exogenous or a known function of the state vector) and the third one is obtained as a byproduct. This general discrete-time no-arbitrage asset pricing setting, formalized by Bertholon, Monfort and Pegoraro (2008) [BMP (2008), hereafter], has shown its large flexibility in various contexts (finance and macro-finance yield curve models, credit risk analysis, longevity risk, exchange rate risk).

In the second set of models the vector of state variables is made only of asset returns and a RNVR or LRNVR is introduced imposing that: *i*) the historical and risk-neutral dynamics belong to the same parametric families; *ii*) the R.N. expectation of the (arithmetic) returns of the basic assets are equal to the riskless (arithmetic) returns; *iii*) the historical and risk-neutral variance-covariance matrix of the state-vector, conditional to the past, are the same functions of the past. These RNVR or LRNVR are usually justified by a combination of assumptions on agents preferences and on probability distributions [see Rubinstein (1976), Brennan (1979), Duan (1995), Camara (2003)].

The assumptions made in both approaches obviously reduce the set of possible admissible pairs of historical and risk-neutral dynamics. For instance, in the first approach, even if the assumption of an exponential-affine SDF is well justified in the literature, in particular in consumption-based asset pricing models, in terms of minimal entropy martingale measure, in terms of discretization of continuous time security market models and in terms of tractability of the pricing formula[1], it is not imposed by the absence of arbitrage opportunity principle which only requires the positivity of the pricing kernel and, possibly, internal consistency conditions. Among the consequences of this assumption let us mention the fact that, in conditionally Gaussian models, the historical and risk-neutral conditional variance-covariance matrices of the state vector are the same function of the past, like in the LRNVR approach.

In this paper we adopt the first kind of approach and we introduce a wider bridge between the historical and the risk-neutral probability. More precisely, we introduce the notion of Exponential-Quadratic SDF or, equivalently, the notion of Second-Order Esscher Transform generalizing the classical Esscher Transform introduced by Gerber and Shiu (1994) [see also Buhlmann, Delbaen, Embrechts and Shiryaev (1998) and Gourieroux and Monfort (2007)]. The log-pricing kernel is specified as a quadratic function of the factor and the associated sources of risk are priced by means of first-order and second-order risk-correction coefficients[2].

Focusing on security market models, this approach, developed in the multivariate condi-

[1]See, among others, Bansal and Yaron (2004), Bertholon, Monfort and Pegoraro (2006), Campbell and Cochrane (1999), Darolles, Gourieroux and Jasiak (2006), Bonomo, Garcia, Meddahi and Tedongap (2011), Gourieroux, Jasiak and Sufana (2009), Gourieroux, Monfort and Polimenis (2006), Monfort and Pegoraro (2007), Stutzer (1995).

[2]Empirical evidence of a second-order term in the pricing kernel and in the log-pricing kernel can be found in Chapman (1997) and Engle and Rosenberg (2002), respectively.

tionally Gaussian framework, makes possible to price *mean-based* and *variance-covariance-based* sources of risk regardless the presence of homoscedasticity or conditional heteroscedasticity in the factor's dynamics. More precisely, our methodology implies that the factor's risk-neutral mean is a function of the short rate and of the risk-neutral conditional variance-covariance matrix which is different from the historical one because of the second-order risk-correction coefficients. Therefore, our exponential-quadratic change of probability measure involves a factor's risk-neutral conditional variance-covariance matrix different from the historical one, while keeping the risk-neutral probability equivalent to the historical one. This result is a relevant generalization of the continuous-time (Girsanov-based) change of probability measure where a risk-neutral diffusion term different from the historical one would imply mutually singular historical and risk-neutral probabilities. In this way we also extend to a general multivariate no-arbitrage asset pricing (SDF-based) framework the results provided by Bakshi and Madan (2007) and Hansen, Heaton and Li (2008) in structural frameworks, and those of Christoffersen, Elkhami, Feunou and Jacobs (2010) proposed in a no-arbitrage scalar setting. Moreover, if the Back Modelling strategy is adopted, the (stochastic) first-order and second-order risk-correction coefficients can be specified as *any non-linear* function of the *present and past values* of the factor, while keeping the historical factor dynamics computationally tractable (i.e. with a likelihood function known in closed form or by standard filtering techniques). This specification also provides a generalization of the exponential-quadratic changes of probability measures proposed by the continuous-time and discrete-time literature on Quadratic-Gaussian (QG) or on Wishart Autoregressive (WAR) asset pricing models where, for tractability reasons, the risk-correction coefficients are constant or specified as deterministic functions of time or as affine functions of the factor [see Buraschi, Porchia and Trojani (2010), Cheng and Scaillet (2007), Gourieroux, Jasiak and Sufana (2009), Gourieroux and Sufana (2011) and Leippold and Wu (2002)]. In order to present a more precise interpretation of the first-order and second-order stochastic risk-sensitivity functions, we calculate (in our conditionally Gaussian framework) the one-period risk premium and we compare it with the first-order risk premium generated by the exponential-affine SDF. We also calculate the Second-Order Black and Scholes (1973) pricing formula for European Call options and we find that it explicitly depends on the second-order risk-sensitivity parameter.

In addition, in order to provide further insight into the usefulness of our approach, we specify and calibrate what we name the *Second-Order GARCH Option Pricing Model*. First, we assume that the historical dynamics of the index return is described by an AR(1)-GARCH(1,1) process with leverage [see, among the others, Amin and Ng (1993), Engle and Mustafa (1992), Duan (1995) and Heston and Nandi (2000)] and we estimate, by the maximum likelihood method, historical parameters using S&P500 daily stock returns observed from July 1, 1962 to December 31, 2001. Second, we adopt our exponential-quadratic SDF-based change of probability measure generating a risk-neutral dynamics characterized by a market price of variance risk (i.e., the second-order risk-correction coefficient) that we specify as a function of not only the present return but also of the lagged variance risk term. Third, we calibrate risk-neutral parameters in order to study the larger set of implied volatility shapes that our model is able to provide, compared to the one obtained in the exponential-affine SDF setting [see Christoffersen and Jacobs (2004)]. Indeed, it is important to highlight that, in the latter case, once historical parameters are estimated using the time series of stock returns and once the no-arbitrage principle is imposed, the risk-neutral dynamics does not deliver any free parameter and therefore the associated implied volatility curve (smile or smirk) turns out to be indifferent to the cross-section of option prices. The calibration exercise shows the ability of our option pricing model to generate a rich family of implied volatility smiles and skews which can match the typically observed ones [see, for instance, Pan (2002)].

It is worth noting that, even if the paper focuses on security market models, we do not make any particular assumption about the state vector and therefore this SDF-based approach (contrary to the RNVR and LRNVR ones) could be used not only in option pricing models, but also for instance in interest rate and credit risk models.

The paper is organized as follows. In Section 2 we define the Second-Order Esscher Transform of a probability density function and we show, thanks to some examples, how it generalizes the family of probability distributions generated by the classical (First-Order) Esscher Transform (associated proofs are provided in the Appendix). Section 3 presents the Exponential-Quadratic Stochastic Discount Factor modelling principle in a multivariate setting, and shows how the associated change of probability measure is given by a conditional Second-Order Esscher Transform. Sections 4.1 to 4.3 deal with multivariate conditionally Gaussian economies specified by following either the Direct or the Back Modelling strategy defined in BMP (2008). In Section 4.4 we present and calibrate the Second-Order GARCH Option Pricing Model, Section 4.5 mentions the generalization to Conditionally Gaussian Switching Regimes ($CGSR$) economies, while Section 5 concludes.

2. The second-order Esscher transform

Let us consider a probability $\mathbb{P}$ defined on $\mathbb{R}^n$, and f its probability density function (p.d.f.) with respect to some measure ν. The purpose of this section is to introduce a new family of probability distributions, associated with the p.d.f. f, having the classical Esscher Transform as a subset [see Gerber and Shiu (1994)]. This new family, that we call Second-Order Esscher Transforms and which is built upon the concept of Second-Order Laplace Transform, gives the possibility, for instance, to modify not only the mean but also the variance-covariance matrix of a multivariate Gaussian distribution or the mean and the variance-covariance matrix of the components of a mixture of multivariate Gaussian distributions (see examples below).

Definition 1 (Second-Order Laplace Transform) : *The Second-Order Laplace Transform of the p.d.f. $f(y)$ is :*

$$\varphi_S(\theta_1, \theta_2) = \int_{\mathbb{R}^n} f(y) \exp(\theta_1' y + y'\theta_2 y) d\nu(y) \tag{1}$$

with $\theta_1 \in \mathbb{R}^n$, $\theta_2 \in \mathcal{S}_n(\mathbb{R})$ an $(n \times n)$ real symmetric matrix[3] *and $\theta = (\theta_1, \theta_2) \in \Theta$, Θ being the definition set $\{(\theta_1, \theta_2) \in \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R}) : \int_{\mathbb{R}^n} f(y) \exp(\theta_1' y + y'\theta_2 y) d\nu(y) < \infty\}$.*

Definition 2 (Second-Order Esscher Transform) : *The Second-Order Esscher Transform of $\mathbb{P}$ associated with (θ_1, θ_2), denoted by $S_{(\theta_1,\theta_2)}(\mathbb{P})$, is given by the family of probability distributions defined by the p.d.f.:*

$$g(y; \theta_1, \theta_2) = \frac{f(y) \exp(\theta_1' y + y'\theta_2 y)}{\varphi_S(\theta_1, \theta_2)} . \tag{2}$$

If we assume $\theta_2 = 0$ in (2), we find the classical (First-Order, say) Esscher Transform that we may denote $F_{(\theta_1)}(\mathbb{P})$. Let us now present examples of Second-Order Esscher Transforms [the proofs are given in the Appendix].

2.1. Discrete distribution

Let us assume that ν is a counting measure on a (possibly infinite) discrete space $D \subset \mathbb{R}^n$ defined by the point masses $\{p_d, d \in D\}$. The associated Second-Order Esscher transform is the family of probability distributions on D with probability masses:

$$\frac{p_d \exp(\theta_1' d + d'\theta_2 d)}{\sum_{d \in D} p_d \exp(\theta_1' d + d'\theta_2 d)} , \quad d \in D , \tag{3}$$

[3] Observe that the assumption $\theta_2 \in \mathcal{S}_n(\mathbb{R})$ is not a restriction since any square matrix A (say) is the sum of a symmetric matrix $(A + A')/2$ and of an antisymmetric matrix $(A - A')/2$, and since a quadratic form associated to an antisymmetric matrix is equal to zero.

assuming $\sum_{d \in D} p_d \exp(\theta_1' d + d'\theta_2 d) < \infty$.

2.2. *Univariate Gaussian distribution*

The Second-Order Esscher transform of the p.d.f. of a univariate ($n = 1$) Gaussian random variable $N(\mu, \sigma^2)$ is given by:

$$g(y; \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{1-2\theta_2\sigma^2}}} \exp\left[-\left(\frac{1-2\theta_2\sigma^2}{2\sigma^2} \right) \left(y - \frac{(\mu + \theta_1\sigma^2)}{1-2\theta_2\sigma^2} \right)^2 \right], \tag{4}$$

which is, under the condition $\theta_2 < \frac{1}{2\sigma^2}$, the p.d.f. of the family of the Gaussian random variables $N\left(\frac{(\mu+\theta_1\sigma^2)}{1-2\theta_2\sigma^2}, \frac{\sigma^2}{1-2\theta_2\sigma^2} \right)$. Compared with $N(\mu, \sigma^2)$, this family has not only different means (driven by the two parameters (θ_1, θ_2)) but also different variances (driven by θ_2). Observe that any Gaussian distribution can be reached when $\theta = (\theta_1, \theta_2)$ varies in $\Theta = \mathbb{R} \times \left]-\infty, \frac{1}{2\sigma^2}\right[$.

2.3. *Multivariate Gaussian distribution*

The Second-Order Esscher transform of the p.d.f. of a n-dimensional Gaussian random variable $N(\mu, \Sigma)$ is:

$$\begin{aligned} g(y; \theta_1, \theta_2) &= \frac{1}{(2\pi)^{n/2}\sqrt{\det\left[(\Sigma^{-1} - 2\theta_2)^{-1}\right]}} \times \\ &\exp\left[-\frac{1}{2}(y - (I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1))'(\Sigma^{-1} - 2\theta_2)(y - (I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1)) \right], \end{aligned} \tag{5}$$

that is the p.d.f. of the family of the n-dimensional Gaussian random variable $N((I - 2\Sigma\theta_2)^{-1}(\mu + \Sigma\theta_1), (\Sigma^{-1} - 2\theta_2)^{-1})$ if $(\Sigma^{-1} - 2\theta_2)$ is assumed to be a symmetric positive definite matrix, that is if $(\Sigma^{-1} - 2\theta_2) \in \mathcal{S}_n^+(\mathbb{R})$ or, equivalently, if the eigenvalues of $\theta_2\Sigma$ are smaller than $\frac{1}{2}$ that is, if $\theta_2 = \Sigma^{-1/2}ADA'\Sigma^{-1/2}$, where D is a diagonal matrix with diagonal terms smaller than $\frac{1}{2}$ and A is an orthogonal matrix[4]. Like in the previous example, for any given (θ_1, θ_2), the Gaussian random variable generated by (2) has a different mean as well as a different variance-covariance matrix and any n-dimensional Gaussian distribution can be reached. When we assume $\theta_2 = 0$, we degenerate to the First-Order Esscher transform $F_{(\theta_1)}(\mathbb{P})$ inducing a different conditional mean $(\mu + \Sigma\theta_1)$ but maintaining the same variance-covariance matrix Σ.

2.4. Finite mixture of multivariate Gaussian distributions

Given a finite mixture of n-dimensional Gaussian random variables with p.d.f. $f(y) = \sum_{j=1}^{J} \lambda_j \, n(y; \mu_j, \Sigma_j)$, the associated family of probability density functions generated by the Second-Order Esscher Transform is:

$$g(y; \theta_1, \theta_2) = \sum_{j=1}^{J} \lambda_j^* \, n\left(y; (I - 2\Sigma_j\theta_2)^{-1}(\mu_j + \Sigma_j\theta_1), (\Sigma_j^{-1} - 2\theta_2)^{-1}\right), \qquad (6)$$

$$\text{with } \lambda_j^* = \frac{\lambda_j \varphi_{S,j}(\theta_1, \theta_2)}{\sum_{j=1}^{J} \lambda_j \varphi_{S,j}(\theta_1, \theta_2)},$$

$$\varphi_{S,j}(\theta_1, \theta_2) = \int_{\mathbb{R}^n} \exp(\theta_1 y + y'\theta_2 y) n(y; \mu_j, \Sigma_j) dy$$

$$= \exp\left[-\frac{1}{2}\log\det(I - 2\Sigma_j\theta_2) - \frac{1}{2}\mu_j'\Sigma_j^{-1}\mu_j + \frac{1}{2}(\Sigma_j^{-1}\mu_j + \theta_1)'(\Sigma_j^{-1} - 2\theta_2)^{-1}(\Sigma_j^{-1}\mu_j + \theta_1)\right],$$

$$\text{and } 0 \leq \lambda_j^* \leq 1, \; \sum_{j=1}^{J} \lambda_j^* = 1. \qquad (7)$$

This is the family of p.d.f. of a n-dimensional Finite Mixture of J Gaussian random variables $N((I - 2\Sigma_j\theta_2)^{-1}(\mu_j + \Sigma_j\theta_1), (\Sigma_j^{-1} - 2\theta_2)^{-1})$, $j \in \{1, \ldots, J\}$, each component having a different

[4]Indeed, $\theta_2\Sigma$ has the same eigenvalues as the symmetric matrix $\Sigma^{1/2}\,\theta_2\,\Sigma^{1/2}$.

mean and a different variance-covariance matrix, as well as different mixing weights.

3. The exponential-quadratic stochastic discount factor modelling principle

3.1. General information and historical distribution

In what follows, we consider an economy between dates 0 and T. The new information in the economy at date t is denoted by w_t, while $\underline{w}_t = (w_t, w_{t-1}, ..., w_0)$ is the entire information between 0 and t. The random vector w_t is called a factor or a state vector, its dimension is n and it can be made up of *latent or observable* variables like asset prices or macro variables.

The historical dynamics of w_t is defined by the conditional distribution of w_{t+1} given $\underline{w}_t$, denoted by $\mathbb{P}_{t+1}$ (say) and characterized either by the p.d.f. $f_t(w_{t+1}|\underline{w}_t)$ or the Laplace transform $\varphi_t(u|\underline{w}_t)$, or the Log-Laplace transform $\psi_t(u|\underline{w}_t) = \log[\varphi_t(u|\underline{w}_t)]$.

3.2. The exponential-quadratic stochastic discount factor

3.2.1. The general case with non-linear stochastic path-dependent risk-correction coefficients

The purpose of this section is to generalize the classical exponential-affine SDF change of probability (i.e., the conditional First-Order Esscher Transform)[5] by means of the conditional

[5] The asset pricing literature has in general derived or specified $M_{t,t+1}(\underline{w}_{t+1})$ as an exponential-affine function of w_{t+1}. Indeed, this form naturally stands out in equilibrium models like CCAPM [see e.g. Cochrane (2005)], consumption-based asset pricing models with habit formation or with Epstein-Zin preferences [see, among others, Bansal and Yaron (2004), Campbell and Cochrane (1999), Bonomo, Garcia, Meddahi and Tedongap (2011), Garcia, Renault and Semenov (2006)]. Moreover, in general continuous-time security market models the discretized version of the SDF is exponential-affine [see Gourieroux and Monfort (2007)]. Finally, the exponential-affine specification is particularly well adapted to the Laplace Transform which is a central tool in discrete-time asset pricing theory [see e.g. Bertholon, Monfort and Pegoraro (2006), Darolles, Gourieroux and Jasiak (2006), Gourieroux, Jasiak and Sufana (2009), Gourieroux, Monfort and Polimenis (2006), Monfort and Pegoraro (2007)].

Second-Order Esscher Transform that is, by introducing the following exponential-quadratic SDF:

$$M_{t,t+1}^{(S)} = \exp\left[-r_{t+1}(\underline{w}_t) + \alpha_{1,t}'(\underline{w}_t)w_{t+1} + w_{t+1}'\alpha_{2,t}(\underline{w}_t)w_{t+1} - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w}_t)\right] , \quad (8)$$

with $\psi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w}_t) = \log \varphi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w}_t)$, $\varphi_{S,t}(\alpha_{1,t}, \alpha_{2,t}|\underline{w}_t) = E_t[\exp(\alpha_{1,t}' w_{t+1} + w_{t+1}' \alpha_{2,t} w_{t+1})]$ the conditional second-order Log-Laplace transform and where $\alpha_{2,t}$ is a (time-varying) $(n \times n)$ symmetric matrix ($\alpha_{2,t} \in \mathcal{S}_n(\mathbb{R})$).

The Risk-Neutral (R.N.) conditional distribution $\mathbb{Q}_{t+1}$ of w_{t+1}, given $\underline{w}_t$, has an exponential-quadratic (in w_{t+1}) p.d.f. with respect to $\mathbb{P}_{t+1}$ given by:

$$d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t) = \frac{M_{t,t+1}^{(S)}(\underline{w}_{t+1})}{E_t\left[M_{t,t+1}^{(S)}(\underline{w}_{t+1})\right]} = \exp\left[\alpha_{1,t}' w_{t+1} + w_{t+1}'\alpha_{2,t}w_{t+1} - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t})\right] , \quad (9)$$

and, therefore, the R.N. conditional p.d.f. (with respect to the same measure as the corresponding conditional historical probability) is $f_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t) = f_t(w_{t+1}|\underline{w}_t) d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t)$ and the R.N. conditional second-order Log-Laplace transform is:

$$\psi_{S,t}^{\mathbb{Q}}(u_1, u_2) = \psi_{S,t}(u_1 + \alpha_{1,t}, u_2 + \alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t}) , \quad u_1 \in \mathbb{R}^n , \quad u_2 \in \mathcal{S}_n(\mathbb{R}) . \quad (10)$$

Conversely, the p.d.f. of the conditional historical distribution with respect to the R.N. one is given by :

$$\begin{aligned} d_t^{\mathbb{P},S}(w_{t+1}|\underline{w}_t) = \frac{1}{d_t^{\mathbb{Q},S}(w_{t+1}|\underline{w}_t)} &= \exp\left[-\alpha_{1,t}' w_{t+1} - w_{t+1}'\alpha_{2,t}w_{t+1} + \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t})\right] \\ &= \exp\left[-\alpha_{1,t}' w_{t+1} - w_{t+1}'\alpha_{2,t}w_{t+1} - \psi_{S,t}^{\mathbb{Q}}(-\alpha_{1,t}, -\alpha_{2,t})\right] , \end{aligned} \quad (11)$$

since, using (10) with $u_1 = -\alpha_{1,t}$ and $u_2 = -\alpha_{2,t}$, we have $\psi_{S,t}^{\mathbb{Q}}(-\alpha_{1,t}, -\alpha_{2,t}) = -\psi_{S,t}(\alpha_{1,t}, \alpha_{2,t})$. We get the following :

Proposition 1 : *If we consider the exponential-quadratic stochastic discount factor* $M^{(S)}_{t,t+1}$*, the risk-neutral conditional distribution* $\mathbb{Q}_{t+1}$ *of* w_{t+1}*, conditionally to* $\underline{w_t}$*, is the conditional Second-Order Esscher Transform of* $\mathbb{P}_{t+1}$ *associated with* $(\alpha_{1,t}, \alpha_{2,t})$*, that is* $\mathbb{Q}_{t+1} = S_{(\alpha_{1,t},\alpha_{2,t})}(\mathbb{P}_{t+1})$*. Conversely, the historical conditional distribution* $\mathbb{P}_{t+1}$ *is the conditional Second-Order Esscher Transform of* $\mathbb{Q}_{t+1}$ *associated with* $(-\alpha_{1,t}, -\alpha_{2,t})$*, that is* $\mathbb{P}_{t+1} = S_{(-\alpha_{1,t},-\alpha_{2,t})}(\mathbb{Q}_{t+1})$.

We will see in the following sections that first, if the Back Modelling strategy is adopted, $\alpha_{1,t}$ and $\alpha_{2,t}$, namely the *stochastic* risk-correction coefficients, are allowed to be *any non-linear* function of the *present and past* values of the factor w_t, while keeping the historical factor dynamics computationally tractable (i.e. providing a likelihood function in closed form or by standard filtering techniques). This specification clearly provides a generalization of the changes of probability measure proposed by the continuous-time and discrete-time literature on Quadratic-Gaussian term structure models or on Wishart Autoregressive (WAR) asset pricing models where, for tractability reasons, the risk-correction coefficients are constant or deterministic functions of time or affine functions of the factor [see Singleton (2006, Chapter 12), Gourieroux and Sufana (2010, 2011)].

Second, we will also see that our exponential-quadratic change of probability measure involves a risk-neutral conditional variance-covariance matrix of the factor *different from* the historical one while keeping at the same time the probability measure $\mathbb{Q}$ *equivalent to* $\mathbb{P}$. This kind of result can not be obtained by a continuous-time (Girsanov-based) approach, since a risk-neutral diffusion term different from the historical one would imply $\mathbb{Q}$ and $\mathbb{P}$ mutually singular [see Cont and Tankov (2004)].

3.2.2. Structural justifications of $M^{(S)}_{t,t+1}$

It is interesting to observe that the asset pricing literature has also recently provided some structural justification for the exponential-quadratic SDF assumption characterizing our no-arbitrage approach. For instance, Bakshi and Madan (2007), working in a simple static and univariate setting and aggregating the marginal rate of substitution of power utility investors that are either long or short the market index, determine an exponential-quadratic (in the scalar index return) SDF when the risk-aversion parameter among agents ϕ (say) is normally distributed.

In a more realistic dynamic framework, characterized by a discrete-time recursive utility model *à la* Epstein and Zin (1989) in which the state of the economy follows a Gaussian VAR(1) process and the (geometric one-period) consumption growth is a linear function of that state vector, Hansen, Heaton and Li (2008) obtain an exponential-quadratic specification by linearizing the log-SDF around the log of the (explicit and exponential-affine) SDF computed when the inverse of the EIS (elasticity of intertemporal substitution) parameter (namely, ρ) is assumed to be equal to one. This quadratic term can, thus, be associated to an economy able to provide a (empirically suggested) time-varying wealth-consumption ratio, while the exponential-affine SDF case ($\rho = 1$) obliges this ratio to be unrealistically constant [see Hansen, Heaton, Li and Roussanov (2007) and the references therein for further details].

3.3. *Internal consistency conditions*

The no-arbitrage discrete-time asset pricing setting, based on an exponential-affine SDF $M_{t,t+1}$, conveniently provides explicit conditions, through the historical and R.N. Log-Laplace transforms ψ_t and $\psi_t^{\mathbb{Q}}$, to guarantee the internal consistency of the model [see BMP (2008) for details]. These Internal Consistency Conditions (ICC) are easily extended to the case of an exponential-quadratic SDF $M_{t,t+1}^{(S)}(\underline{w}_{t+1})$. Let us consider, for instance, the situation in which the factor w_{t+1} contains (at least) a geometric stock return and in which the short rate r_{t+1} is exogenous. If $w_{j,t+1} = e_j' w_{t+1}$ is a scalar geometric return (e_j being the j^{th} column of the identity matrix $I_{n\times n}$) we must have:

$$\begin{aligned}\exp(-r_{t+1})E_t^{\mathbb{Q}}[\exp(w_{j,t+1})] = 1 \qquad &\iff \quad r_{t+1} = \psi_{S,t}^{\mathbb{Q}}(e_j, 0) \\ &\iff \quad r_{t+1} = \psi_{S,t}(\alpha_{1,t} + e_j, \alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t})\,.\end{aligned} \tag{12}$$

4. Conditionally Gaussian economies

4.1. Pricing mean-based and variance-covariance-based sources of risk

Let us assume that the factor w_t is a n-dimensional vector of geometric stock returns of risky assets, that is $w_{i,t+1} = \log(S_{i,t+1}/S_{i,t})$ for each $i \in \{1, \ldots, n\}$, where $S_{i,t}$ is the price at t of asset i. If we follow the Direct Modelling strategy formalized by Bertholon, Monfort and Pegoraro (2008), we first have to specify the historical dynamics ($\mathbb{P}_{t+1}$) of w_{t+1}. Assuming conditional normality, that is:

$$w_{t+1}|\, \underline{w}_t \overset{\mathbb{P}}{\sim} N\left(\mu_t, \Sigma_t\right)\,, \tag{13}$$

we have to choose μ_t and Σ_t (for instance, VAR and VARMA models with GARCH-type noise). Second, we have to specify $\alpha_{1,t}$ and $\alpha_{2,t}$ appearing in the exponential-quadratic SDF (8) and to impose the ICC (12):

$$\begin{aligned} & r_{t+1} = \psi_{S,t}(e_i + \alpha_{1,t}, \alpha_{2,t}) - \psi_{S,t}(\alpha_{1,t}, \alpha_{2,t})\,, \text{ where} \\ & \psi_{S,t}(u_1, u_2) = -\frac{1}{2}\log\det\left(I - 2\Sigma_t u_2\right) - \frac{1}{2}\mu_t'\Sigma_t^{-1}\mu_t \\ & \qquad\qquad\qquad + \frac{1}{2}(\Sigma_t^{-1}\mu_t + u_1)'(\Sigma_t^{-1} - 2u_2)^{-1}(\Sigma_t^{-1}\mu_t + u_1)\,, \end{aligned} \tag{14}$$

which implies :

$$
\begin{aligned}
r_{t+1} &= \frac{1}{2}(\Sigma_t^{-1}\mu_t + e_i + \alpha_{1,t})'(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}(\Sigma_t^{-1}\mu_t + e_i + \alpha_{1,t}) \\
&\qquad -\frac{1}{2}(\Sigma_t^{-1}\mu_t + \alpha_{1,t})'(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}(\Sigma_t^{-1}\mu_t + \alpha_{1,t}) \\
&= \frac{1}{2}e_i'(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}e_i + e_i'(I - 2\Sigma_t\alpha_{2,t})^{-1}(\mu_t + \Sigma_t\alpha_{1,t}) \quad \forall\, i \in \{1, \ldots, n\}\,,
\end{aligned} \tag{15}
$$

that is :

$$
\frac{1}{2}\, vdiag\,[(\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}] + (I - 2\Sigma_t\alpha_{2,t})^{-1}(\mu_t + \Sigma_t\alpha_{1,t}) = r_{t+1}e\,, \tag{16}
$$

where e denotes the n-dimensional unitary vector.

Proposition 2 : *The specification of the historical dynamics (13) and of the exponential-quadratic SDF (8) implies the following R.N. dynamics ($\mathbb{Q}_{t+1}$):*

$$
w_{t+1}|\,\underline{w}_t \overset{\mathbb{Q}}{\sim} N\left[(I - 2\Sigma_t\alpha_{2,t})^{-1}(\mu_t + \Sigma_t\alpha_{1,t}), (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right]\,, \tag{17}
$$

that is, $\mathbb{Q}_{t+1} = S_{(\alpha_{1,t},\alpha_{2,t})}(\mathbb{P}_{t+1})$. *If we impose to (17) the ICC (16), we find that the R.N. dynamics compatible with no-arbitrage restrictions is:*

$$
N\left[r_{t+1}e - \frac{1}{2}vdiag\,((\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}), (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\right]\,. \tag{18}
$$

It is important to stress that this exponential-quadratic SDF change of probability measure induces three generalizations with respect to the exponential-affine one. First, it provides a different R.N. conditional mean and conditional variance-covariance matrix, namely:

$$
\begin{aligned}
\mu_t^{\mathbb{Q}} &= r_{t+1}e - \frac{1}{2}vdiag\,(\Sigma_t^{\mathbb{Q}}) \\
\Sigma_t^{\mathbb{Q}} &= (\Sigma_t^{-1} - 2\alpha_{2,t})^{-1}\,,
\end{aligned} \tag{19}
$$

because of the second-order risk-sensitivity function $\alpha_{2,t}$. On the contrary, in the continuous-time (Brownian motion-based) framework, the risk-neutral diffusion term has to be equal to the historical one ($\Sigma_t^{\mathbb{Q}} = \Sigma_t$, in our notation) in order to guarantee $\mathbb{Q}$ equivalent to $\mathbb{P}$, otherwise the two measures would be mutually singular.

Second, for a given historical dynamics (estimated using stock return observations) and under no-arbitrage restrictions, the Gaussian risk-neutral dynamics, contrary to the exponential-affine setting, still delivers free parameters (those specifying $\alpha_{2,t}$) adapted to match derivative prices.

Third, the time-varying risk-sensitivity functions characterizing the SDF are given by :

$$\begin{aligned} \alpha_{2,t} &= \frac{1}{2}\Sigma_t^{-1}\left[\Sigma_t^{\mathbb{Q}} - \Sigma_t\right](\Sigma_t^{\mathbb{Q}})^{-1} = \frac{\Sigma_t^{-1} - (\Sigma_t^{\mathbb{Q}})^{-1}}{2}\,, \text{ and} \\ \alpha_{1,t} &= (\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} - \Sigma_t^{-1}\mu_t\,, \end{aligned} \tag{20}$$

and, therefore, they can be seen respectively as a normalized measure of the historical and risk-neutral variance-covariance spread, and as a (variance-weighted) measure of the historical and risk-neutral mean spread. Observe that the (explicit) ICC (16) makes $\alpha_{1,t}$ a function of $\alpha_{2,t}$ and the latter can be *any function* of the date t information (i.e., a stochastic risk-correction coefficient) such that $\Sigma_t^{\mathbb{Q}} \in \mathcal{S}_n^+(\mathbb{R})$.

4.2. Generalized market risk premium (GMRP) and second-order Black and Scholes pricing formula

In order to provide a more precise interpretation of the risk-sensitivity functions $\alpha_{1,t}$ and $\alpha_{2,t}$, let us define the scalar market risk premium between t and $t+1$, associated to any stock return $w_{i,t+1}$ for $i \in \{1, \ldots, n\}$, in the following way:

$$\lambda_{t,t+1}^{(i)} = \log E_t[\exp(w_{i,t+1})] - r_{t+1}\,,$$

and let us collect them in the vector $\lambda_{t,t+1} = \left[\lambda_{t,t+1}^{(1)}, \ldots, \lambda_{t,t+1}^{(n)}\right]'$. Then, using (16), we can write the following n-dimensional generalized market risk premium ($GMRP$):

$$\begin{aligned}
\lambda_{t,t+1} &= \mu_t + \frac{1}{2} vdiag\,(\Sigma_t) - r_{t+1}\, e \\
&= \mu_t - \mu_t^{\mathbb{Q}}(\alpha_{2,t}) - \frac{1}{2} vdiag\,(\Sigma_t^{\mathbb{Q}}(\alpha_{2,t}) - \Sigma_t) \\
&= \lambda_{t,t+1}^F + \left[\mu_t^{\mathbb{Q}}(0) - \mu_t^{\mathbb{Q}}(\alpha_{2,t})\right] - \frac{1}{2} vdiag\,(\Sigma_t^{\mathbb{Q}}(\alpha_{2,t}) - \Sigma_t)\,,
\end{aligned} \tag{21}$$

with $\lambda_{t,t+1}^F := (\mu_t - \mu_t^{\mathbb{Q}}(0)) = -\,\Sigma_t\,\alpha_{1,t}$ denoting the n-dimensional (first-order) risk premium associated to an exponential-affine SDF (and e denoting the n-dimensional unitary vector). Relation (21) shows the role played by $\alpha_{2,t}$, that is, the consequences on the risk premium of the introduction of a quadratic term in the SDF:

i) if we assume $\alpha_{2,t} = 0$ (an exponential-affine SDF) we find $\lambda_{t,t+1} = \lambda_{t,t+1}^F$, that is, the risk premium is (classically) determined comparing only historical and risk-neutral factor conditional means and $-\alpha_{1,t}$ can be interpreted as a first moment-based risk premium per unit of conditional variance-covariance;

ii) if $\alpha_{2,t} \neq 0$, the size of $\lambda_{t,t+1}$ depends positively on the mean spread $(\mu_t - \mu_t^{\mathbb{Q}}(\alpha_{2,t}))$ and negatively on the variance spreads $vdiag\,(\Sigma_t^{\mathbb{Q}}(\alpha_{2,t}) - \Sigma_t)$, that empirical evidence finds to be (both) positive [see, among the others, Bakshi and Madan (2006)]. Alternatively, $\lambda_{t,t+1}$ differs from $\lambda_{t,t+1}^F$ because $\alpha_{2,t}$ introduces in the $GMRP$ not only a second moment-based source of risk information $(\Sigma_t \neq \Sigma_t^{\mathbb{Q}}(\alpha_{2,t}))$, but also because it also modifies the role played by the first moment-based source of risk information $(\mu_t^{\mathbb{Q}}(\alpha_{2,t}) \neq \mu_t^{\mathbb{Q}}(0))$.

It is also relevant to observe that, when considering the particular scalar ($n = 1$) static case ($r_{t+1} = r, \sigma_t = \sigma, \alpha_{2,t} = \alpha_2$), we immediately find a (discrete-time) generalization of the Black and Scholes (1973) setting and an associated European Call option pricing formula $C_{BS}(t, h; K, S_t, r, \sigma^2)$ (say), where K is the strike price and h denotes the residual maturity. Indeed, the Gaussian stock return risk-neutral dynamics, namely $IIN\left[r - (\sigma^{\mathbb{Q}})^2(\alpha_2)/2,\right.$

$(\sigma^{\mathbb{Q}})^2(\alpha_2)\big]$, immediately delivers the following explicit *Second-Order Black and Scholes pricing formula* (for European Call options):

$$C_{BS}^{(S)}(t,h;K,S_t,r,\sigma^2,\alpha_2) = C_{BS}(t,h;K,S_t,r,(\sigma^{\mathbb{Q}})^2(\alpha_2))\,, \tag{22}$$

in which α_2 is an additional degree of freedom with respect to the classical Black and Scholes one ($\alpha_2 = 0$ implies $C_{BS}^{(S)}(t,h;K,S_t,r,\sigma^2,0) = C_{BS}(t,h;K,S_t,r,\sigma^2)$). This source of flexibility can be further exploited by specifying $\alpha_{2,t}$ as a deterministic function of time, still leading to an explicit pricing formula. Moreover, we can easily propose, in a dynamic setting, richer Call option pricing formulas if we assume σ_t^2 and $\alpha_{2,t}$ functions of the date t information. In that case, the pricing formula has no longer a closed form but it can be easily determined by simulation for any residual maturity h.

4.3. *The back modelling approach to conditionally Gaussian economies*

Let us maintain the conditionally Gaussian setting of the previous section, but let us now adopt the Back Modelling strategy of Bertholon, Monfort and Pegoraro (2008) opening the way for a *tractable and flexible* specification of the asset pricing model of interest. More precisely, let us assume that the R.N. dynamics ($\mathbb{Q}_{t+1}$) of w_{t+1} is given by:

$$w_{t+1}|\,\underline{w}_t \overset{\mathbb{Q}}{\sim} N\left(\mu_t^{\mathbb{Q}}, \Sigma_t^{\mathbb{Q}}\right)\,, \tag{23}$$

with the associated conditional second-order Log-Laplace transform

$$\begin{aligned} \psi_{S,t}^{\mathbb{Q}}(u_1,u_2) \;=\; & -\frac{1}{2}\log\det\left(I - 2\Sigma_t^{\mathbb{Q}}u_2\right) - \frac{1}{2}\mu_t^{\mathbb{Q}\prime}(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} \\ & + \frac{1}{2}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + u_1]'[(\Sigma_t^{\mathbb{Q}})^{-1} - 2u_2]^{-1}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}} + u_1]\,, \end{aligned} \tag{24}$$

and we impose the ICC $\psi^{\mathbb{Q}}_{S,t}(e_i,0)=r_{t+1}$ for all $i\in\{1,\ldots,n\}$, that is:

$$\begin{aligned} r_{t+1} &= -\frac{1}{2}\mu_t^{\mathbb{Q}\prime}(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}}+\frac{1}{2}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}}+e_i]'\Sigma_t^{\mathbb{Q}}[(\Sigma_t^{\mathbb{Q}})^{-1}\mu_t^{\mathbb{Q}}+e_i] \\ &= \frac{1}{2}e_i'\Sigma_t^{\mathbb{Q}}e_i+e_i'\mu_t^{\mathbb{Q}} \quad \forall\, i\in\{1,\ldots,n\}\,. \end{aligned} \tag{25}$$

From (25) we have $\mu_t^{\mathbb{Q}} = r_{t+1}e-\frac{1}{2}vdiag(\Sigma_t^{\mathbb{Q}})$ and, therefore, we find the (no-arbitrage) risk-neutral dynamics:

$$N\left[r_{t+1}e-\frac{1}{2}vdiag\,(\Sigma_t^{\mathbb{Q}}),\Sigma_t^{\mathbb{Q}}\right]\,, \tag{26}$$

which is easily made Compound Autoregressive (Car or discrete-time affine)[6] simply by assuming, for instance, $\Sigma_t^{\mathbb{Q}}=\Sigma^{\mathbb{Q}}$ and without making any assumption or imposing any restriction on the risk-correction coefficients.

Proposition 3 : *The historical dynamics $\mathbb{P}_{t+1}$ is given, for any non-linear stochastic risk-correction coefficients $\alpha_{1,t}$ and $\alpha_{2,t}$, by $\mathbb{P}_{t+1}=S_{(-\alpha_{1,t},-\alpha_{2,t})}(\mathbb{Q}_{t+1})$ and we have:*

$$\begin{aligned} & w_{t+1}|\,\underline{w}_t \overset{\mathbb{P}}{\sim} N\,(\mu_t,\Sigma_t)\ , \\ & \mu_t=(I+2\Sigma_t^{\mathbb{Q}}\alpha_{2,t})^{-1}(r_{t+1}e-\frac{1}{2}vdiag(\Sigma_t^{\mathbb{Q}})-\Sigma_t^{\mathbb{Q}}\alpha_{1,t})\,, \\ & \Sigma_t=((\Sigma_t^{\mathbb{Q}})^{-1}+2\alpha_{2,t})^{-1}\,. \end{aligned} \tag{27}$$

So, for any given R.N. conditionally Gaussian dynamics (possibly affine) and for any risk-correction coefficients specifications i) the historical dynamics is still conditionally Gaussian, thus providing to the asset pricing model a relevant computational tractability (i.e. a likelihood function known exactly in closed form or by standard filtering techniques) and ii) any conditional mean and any conditional variance-covariance matrix (both, non-linear functions of the date t information) can be reached thanks to the non-linearities of the risk-correction coefficients.

This asset pricing framework clearly provides an important generalization of the changes

[6]See Darolles, Gourieroux and Jasiak (2006) for details.

of probability measures proposed by the continuous-time and discrete-time literature on Quadratic-Gaussian (QG) or on Wishart Autoregressive (WAR) asset pricing models where, for tractability reasons, the risk-correction coefficients are given by constant parameters or are specified as deterministic function of time or as affine functions of the factor [see Buraschi, Porchia and Trojani (2010), Cheng and Scaillet (2007), Dumas, Kurshev and Uppal (2009), Gourieroux, Jasiak and Sufana (2009), Gourieroux and Sufana (2011) and Leippold and Wu (2002)].

4.4. Calibrating the second-order GARCH option pricing model

4.4.1. The model

The purpose of this section is to consider a classical GARCH option pricing model with leverage and exponential-affine SDF [see, among the others, Amin and Ng (1993), Engle and Mustafa (1992), Duan (1995), Heston and Nandi (2000), and Christoffersen and Jacobs (2004)] and to show, thanks to a calibration exercise, that the generalization obtained by adopting an exponential-quadratic SDF may provide implied volatility smiles and skews much closer to the observed ones [see, for instance, Pan (2002)].

In this first subsection we specify and present, following a Direct Modelling strategy, the Second-Order GARCH Option Pricing Model and then, in the second subsection, we focus on the calibration of the model.

Let us assume that the one-period geometric stock return $w_{t+1} = y_{t+1} = \log(S_{t+1}/S_t)$, where S_t is the price at date t of the risky asset, is characterized, conditionally to $\underline{y}_t$, by the following historical distribution ($\mathbb{P}_{t+1}$, say):

$$
\begin{aligned}
& y_{t+1} = \mu_0 + \mu_1 \, y_t + \sigma_{t+1}\varepsilon_{t+1} \, , \\
& \sigma^2_{t+1} = \omega_0 + \omega_1 \, (\sigma_t \, \varepsilon_t - \omega_2)^2 + \omega_3 \sigma^2_t \, , \\
& \varepsilon_{t+1} | \underline{\varepsilon_t} \overset{\mathbb{P}}{\sim} N(0,1) \, .
\end{aligned}
\tag{28}
$$

that is, y_{t+1} follows a AR(1)-GARCH(1,1) process with leverage effect. From Proposition 2, we easily obtain that the no-arbitrage risk-neutral conditional distribution $\mathbb{Q}_{t+1} = S_{(\alpha_{1,t},\alpha_{2,t})}(\mathbb{P}_{t+1})$ of y_{t+1}, conditionally to $\underline{y}_t$, is given by:

$$\begin{aligned} & y_{t+1} = r_{t+1} - \frac{1}{2}\,(\sigma^{\mathbb{Q}}_{t+1})^2 + \sigma^{\mathbb{Q}}_{t+1}\,\xi_{t+1}\,, \\ & (\sigma^{\mathbb{Q}}_{t+1})^2 = \frac{\sigma^2_{t+1}}{(1 - 2\sigma^2_{t+1}\alpha_{2,t})}\,, \\ & \xi_{t+1} \,|\, \underline{\xi_t} \overset{\mathbb{Q}}{\sim} N(0,1)\,. \end{aligned} \tag{29}$$

The price at date t of a European Call option with underlying stock price S_t, residual maturity τ and moneyness strike $\kappa_t = \frac{K}{S_t}$ is given, assuming an exogenous short rate, by :

$$\begin{aligned} C(t,\tau;\kappa_t) &= \exp\left(-\textstyle\sum_{i=1}^{\tau} r_{t+i}\right)\, E^{\mathbb{Q}}_t[S_{t+\tau} - K]^+\,, \\ &= \exp\left(-\textstyle\sum_{i=1}^{\tau} r_{t+i}\right)\, S_t\, E^{\mathbb{Q}}_t[\exp(y_{t+1} + \ldots + y_{t+\tau}) - \kappa_t]^+\,, \\ &\approx \exp\left(-\textstyle\sum_{i=1}^{\tau} r_{t+i}\right)\, S_t\, \frac{1}{\mathcal{S}} \sum_{s=1}^{\mathcal{S}} [\exp(y^{(s)}_{t+1} + \ldots + y^{(s)}_{t+\tau}) - \kappa_t]^+\,, \end{aligned} \tag{30}$$

where, for each $s \in \{1, \ldots, \mathcal{S}\}$, the values $(y^{(s)}_{t+1}, \ldots, y^{(s)}_{t+\tau})$ entering in the pricing formula are simulated from the no-arbitrage risk-neutral conditional distribution (29).

4.4.2. A calibration exercise

The implementation of the Second-Order GARCH Option Pricing Model is based on the following steps. First, we re-parameterize α_{2t} is a such a way that the positiveness of the risk-neutral conditional variance is satisfied. More precisely, we assume that:

$$\begin{aligned} & \alpha_{2t} = \frac{1}{2\,\sigma^2_{t+1}}\,[1 - \exp(\varphi_t)] \\ & \text{with } \ \varphi_t = \beta_0 + \beta_1\, y_t + \beta_2\, \varphi_{t-1}\,, \ \ \beta_0,\, \beta_1,\, \beta_2 \ \in \ \mathbb{R}\,, \end{aligned} \tag{31}$$

where the market price of variance risk α_{2t} is made dependent not only on an affine function of the factor y_t, like classically adopted in the continuous-time and discrete-time asset pricing literature [see Singleton (2006) and the references therein], but also on an extra term introducing a *dependence on its lagged value* ($\beta_2 \varphi_{t-1}$). This lag dependence, specific to discrete-time models, turns out to provide a useful source of flexibility in order to replicate implied volatility smiles and skews.

Second, on the basis of S&P 500 daily stock returns observed from July 1, 1962 to December 31, 2001, we estimate the historical parameters $\theta_{\mathbb{P}} = (\mu_0, \mu_1, \omega_0, \omega_1, \omega_2, \omega_3)'$ (say) by the maximum likelihood method [see Table 1] and third, given this estimator $\widehat{\theta}_{\mathbb{P}}$, we calibrate the risk-neutral parameters $\theta_{\mathbb{Q}} = (\beta_0, \beta_1, \beta_2)'$ characterizing φ_t (i.e. α_{2t}) in order to study the larger set of implied volatility shapes that our model is able to provide, compared to the one provided by the classical GARCH(1,1) case with $\alpha_{2t} = 0$ (our benchmark). Indeed, in the latter case, for any given set of historical parameter estimates, the option pricing formula does not show any free parameter that can be used to match a given cross-section of options prices and, thus, the implied volatility curve (smile or smirk) turns out to be a function of $\widehat{\theta}_{\mathbb{P}}$ only. Geometric stock returns are simulated from the risk-neutral dynamics (29) adopting the empirical martingale simulation method of Duan and Simonato (1998), we have assumed a constant 5% annual short rate, leading to a daily rate of $\frac{0.05}{365} = 0.000137$, and $\mathcal{S} = 10000$ simulations.

$\theta_{\mathbb{P}}$	μ_0	μ_1	ω_0	ω_1	ω_2	ω_3
Estimate	0.00022	0.13899	-4.54E-07	0.07105	0.00405	0.92353
Standard error	0.00008	0.01051	1.43E-07	0.00198	0.00031	0.00226
Ln Likelihood	34182.84					

Table 1: Maximum likelihood estimates of GARCH(1, 1) model (28) using S&P 500 daily stock returns observed from July 1, 1962 to December 31, 2001 (9942 observations). Standard errors are calculated from the outer product of the gradient at the optimum parameter values.

The Black and Scholes (1973) implied volatilities $\sigma_{IV,t} = \sigma_{IV,t}(\kappa_t, \tau; \alpha_{2t})$ (say) provided by our calibration exercise for an European Call option with $S_t = 1$, $\kappa_t \in (0.8, 1.2)$ and $\tau = 30$ days, are shown in Figures 1 to 3. Figure 1 shows $\sigma_{IV,t}$ obtained by assuming in (31)

$\varphi_t = \beta_0$, with $\beta_0 \in \{0, -0.25, -0.50, -0.75\}$ (left panel) and $\varphi_t = \beta_0 + \beta_1\, y_t$, with $\beta_0 = -0.25$ and $\beta_1 \in \{-25, 0, 25\}$ (right panel). The former case, characterized by a constant market price of variance risk, is able to move the implied volatility smile of the classical GARCH option pricing formula (the case $\alpha_{2t} = 0$, that will be always denoted by a solid blue line) at different levels, with at-the-money annualized values ranging from 0.22 to 0.36. The latter case, featuring a stochastic second-order risk-correction coefficient, given the presence in φ_t of the geometric return, highlights that our option pricing formula is potentially able to match implied volatility smiles and smirks with different degrees of curvature and steepness at larger levels than the benchmark case $\alpha_{2t} = 0$.

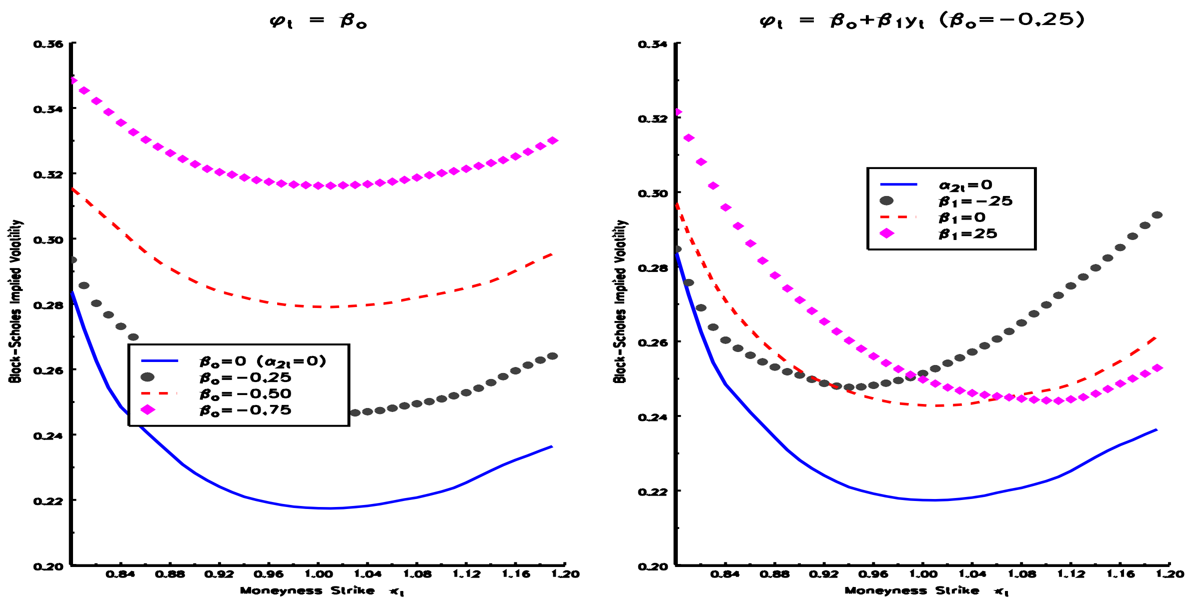

Figure 1: Implied volatilities $\sigma_{IV,t} = \sigma_{IV,t}(\kappa_t, \tau; \alpha_{2t})$ when $S_t = 1$, $\kappa_t \in (0.8, 1.2)$, $\tau = 30$ days and $\alpha_{2t} = [1 - \exp(\varphi_t)]/2\, \sigma^2_{t+1}$ with $\varphi_t = \beta_0$ (left panel) and $\varphi_t = \beta_0 + \beta_1\, y_t$ (right panel).

In Figure 2, exploiting the flexibility of the discrete-time setting, we generalize the specification of φ_t by introducing the lagged term $\beta_2\, \varphi_{t-1}$. We consider $\varphi_t = \beta_0 + \beta_1\, y_t + \beta_2\, \varphi_{t-1}$, with $\beta_0 = 0$, $\beta_2 \in \{-0.75, 0, 0.50\}$ and $\beta_1 = 20$ in the left panel, while $\beta_1 = -20$ in the right panel. We assume $\beta_0 = 0$ in order to provide a direct comparison with the benchmark (the case $\beta_0 \neq 0$ is shown in Figure 3). This example shows the usefulness that the discrete-time setting may provide in the specification of an asset pricing model: it opens the way to a more general change of probability measure *and* a more flexible specification of the associated stochastic market price (of variance) risk. Both features are shown to be relevant. Indeed, in the left panel we see that, as far as β_2 moves from -0.75 to 0.50, the degree of steepness of $\sigma_{IV,t}$ for out-of-the-money Call options reduces, thus modifying the volatility smile into a volatility skew. The right panel shows that, when β_1 moves from 20 to -20, the same different values of β_2 modify the entire curvature degree of the implied volatility smile. Figure 3 shows that smiles and skews with different degrees of steepness and curvatures may also be provided when levels of volatility, larger than the benchmark ones, are required.

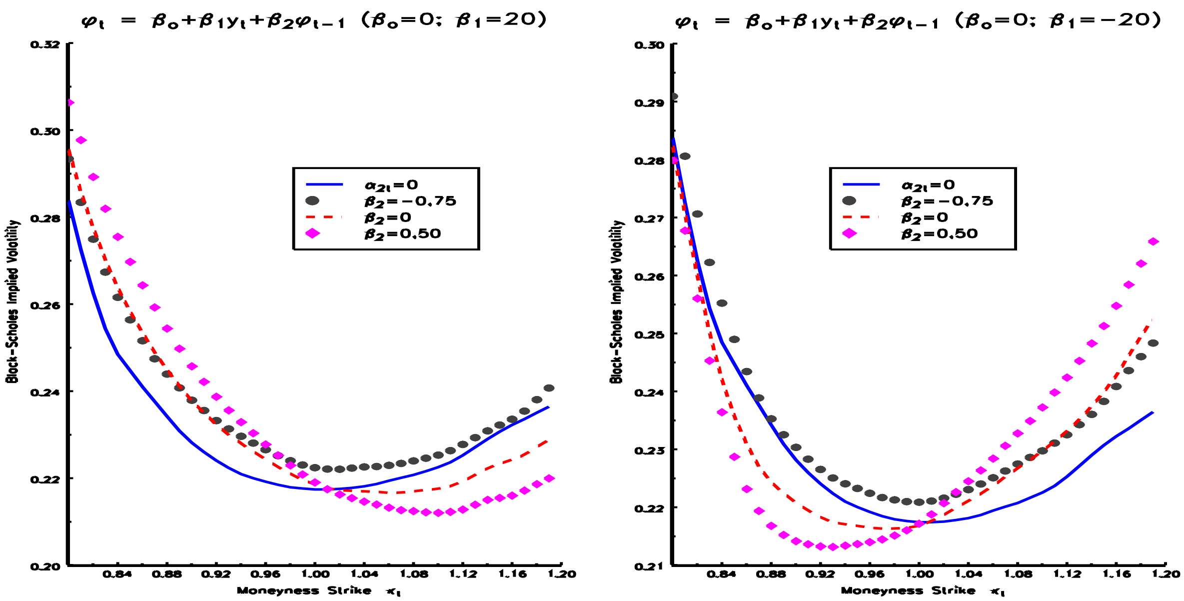

Figure 2: Implied volatilities $\sigma_{IV,t} = \sigma_{IV,t}(\kappa_t, \tau; \alpha_{2t})$ when $S_t = 1$, $\kappa_t \in (0.8, 1.2)$, $\tau = 30$ days and $\alpha_{2t} = [1 - \exp(\varphi_t)]/2\,\sigma^2_{t+1}$ with $\varphi_t = \beta_0 + \beta_1\, y_t + \beta_2\, \varphi_{t-1}$ and $\beta_0 = 0$. $\beta_1 = 20$ (left panel) and $\beta_1 = -20$ (right panel).

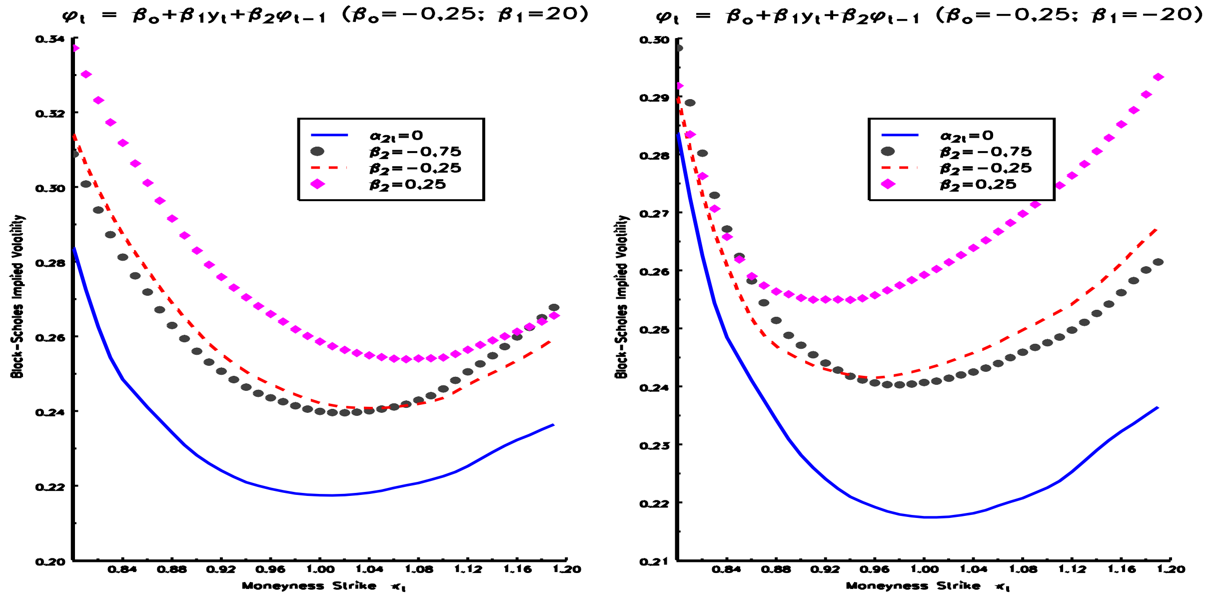

Figure 3: Implied volatilities $\sigma_{IV,t} = \sigma_{IV,t}(\kappa_t, \tau; \alpha_{2t})$ when $S_t = 1$, $\kappa_t \in (0.8, 1.2)$, $\tau = 30$ days and $\alpha_{2t} = [1 - \exp(\varphi_t)]/2\,\sigma^2_{t+1}$ with $\varphi_t = \beta_0 + \beta_1\, y_t + \beta_2\, \varphi_{t-1}$ and $\beta_0 = -0.25$. $\beta_1 = 20$ (left panel) and $\beta_1 = -20$ (right panel).

4.5. Extension to conditionally Gaussian switching regime economies

It is important to highlight that the results presented in the previous sections can be generalized to the case of a Conditionally Gaussian Switching Regime ($CGSR$, say) Economy, namely a security market model in which the dynamics of the quantitative factor is described by a conditionally Gaussian regime-switching model with a conditional mean and conditional variance featuring a general dependence on contemporaneous and past factor values as well as on the regime-indicator function. More precisely, it is possible to determine, first, the conditional Second-Order Esscher Transform of a $CGSR$ model and then to consider the no-arbitrage pricing within $CGSR$ economies with an exponential-quadratic SDF, which prices not only mean-based and variance-based sources of risk, but also regime-shift risk. We can also focus on a $CGSR$ economy in which the risk-neutral conditional mean and conditional variance of the factor are specified as an *additive* function of the contemporaneous and past regimes, in order to guarantee a Car $\mathbb{Q}$-dynamics able to provide tractable pricing formulas [see Monfort and Pegoraro (2010) for details].

5. Conclusions and further developments

In this paper we propose, working with discrete-time no-arbitrage asset pricing models, to widen the bridge between the historical and the risk-neutral factor distribution, while keeping, respectively, flexible and tractable the modelling of both dynamics. The key tools behind this more general change of probability measure are the Second-Order Esscher Transform or, equivalently, the Exponential-Quadratic Stochastic Discount Factor, specified by first-order and second-order stochastic risk-sensitivity coefficients.

We show the large flexibility of this new approach in the case of multivariate conditionally Gaussian dynamics and, in order to testify the usefulness of the methodology, we define and calibrate the Second-Order GARCH Option Pricing Model and we show its ability to provide implied volatility curves characterized by several degrees of steepness (smirks) and curvature (smiles). In other words, we highlight how the exponential-quadratic specification of the SDF is important in generating free risk-neutral parameters (associated to the market price of variance risk) able to match the observed variability of the cross-section of option prices.

Our approach can be coupled with a Back Modelling strategy assuming a Car risk-neutral factor dynamics and then obtaining an historical dynamics by means of a Second-Order Esscher Transform with risk-sensitivity coefficients specified as any functions of the state vector. In this case we have at the same time explicit or quasi explicit pricing formulas for several derivative assets and a very large set of possible historical dynamics that remain computationally tractable.

Although we illustrate our approach using security market models, our results are much more general than the RNVR or LRNVR ones, since they could be applied in many other asset pricing contexts like finance and macro-finance yield curve models, credit risk models, longevity risk and exchange rate models. We leave these developments to future research.

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Appendix: Computation of second-order Esscher transforms

Computation of the second-order Esscher transform of a Gaussian distribution

The proofs of the examples presented in Section 2 are based on the following result. If we consider the p.d.f. of a n-dimensional Gaussian random variable $N(\mu, \Sigma)$:

$$\begin{aligned} f(y) &= \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} \exp\left[-\frac{1}{2}(y-\mu)'\Sigma^{-1}(y-\mu)\right] , \\ &\propto \exp\left[-\frac{1}{2}y'\Sigma^{-1}y + \mu'\Sigma^{-1}y\right] \end{aligned} \tag{A.1}$$

then, from Definition 2 we have :

$$\begin{aligned} g(y;\theta_1,\theta_2) &\propto \exp\left[-\frac{1}{2}y'\Sigma^{-1}y + \mu'\Sigma^{-1}y + \theta_1 y + y'\theta_2 y\right] , \\ &\propto \exp\left[-\frac{1}{2}y'\left(\Sigma^{-1} - 2\theta_2\right)y + \left(\Sigma^{-1}\mu + \theta_1\right)' y\right] , \\ &\propto \exp\left[-\frac{1}{2}y'\left(\Sigma^{-1} - 2\theta_2\right)y + \left(\Sigma^{-1}\mu + \theta_1\right)'\left(\Sigma^{-1} - 2\theta_2\right)^{-1}\left(\Sigma^{-1} - 2\theta_2\right)y\right] , \end{aligned} \tag{A.2}$$

and, therefore, $g(y;\theta_1,\theta_2)$ is the p.d.f. of the n-dimensional Gaussian random variable

$$N\left[\left(\Sigma^{-1} - 2\theta_2\right)^{-1}\left(\Sigma^{-1}\mu + \theta_1\right), \left(\Sigma^{-1} - 2\theta_2\right)^{-1}\right] \tag{A.3}$$

proving relation (5) in Section 2.3, and relation (4) in Section 2.2 when $n = 1$.

Computation of the second-order Laplace transform of a Gaussian distribution

From relations (2) and (5) we see that the Second-Order Laplace Transform of the Gaussian random vector $y \sim N(\mu, \Sigma)$ is given by:

$$\begin{aligned}\varphi_S(\theta_1, \theta_2) &= \int_{\mathbb{R}^n} f(y) \exp(\theta_1' y + y'\theta_2 y) dy = \frac{f(y) \exp(\theta_1' y + y'\theta_2 y)}{g(y; \theta_1, \theta_2)} \\ &= \det (I - 2\Sigma\theta_2)^{-\frac{1}{2}} \exp \left[-\frac{1}{2}\mu'\Sigma^{-1}\mu + \frac{1}{2}(\Sigma^{-1}\mu + \theta_1)'(\Sigma^{-1} - 2\theta_2)^{-1}(\Sigma^{-1}\mu + \theta_1) \right] .\end{aligned} \tag{A.4}$$

If we consider the case of a scalar ($n = 1$) Gaussian random variable $N(\mu, \sigma^2)$, the Second-Order Gaussian Laplace Transform (A.4) takes the following particular form:

$$\begin{aligned}\varphi_S(\theta_1, \theta_2) &= \int_{\mathbb{R}} f(y) \exp(\theta_1 y + \theta_2 y^2) dy \\ &= (1 - 2\sigma^2\theta_2)^{-\frac{1}{2}} \exp \left[-\frac{1}{2}\frac{\mu^2}{\sigma^2} + \frac{1}{2} \left(\frac{\sigma^2}{1 - 2\sigma^2\theta_2} \right) \left(\frac{\mu}{\sigma^2} + \theta_1 \right)^2 \right] .\end{aligned} \tag{A.5}$$

Computation of the second-order Esscher transform of a mixture of Gaussian distributions

Denoting by $n(y; \mu_j, \Sigma_j)$ the p.d.f. of the Gaussian random vector $y \sim N(\mu_j, \Sigma_j)$, we want to find the Second-Order Esscher Transform of the density:

$$\sum_{j=1}^{J} \lambda_j \, n(y; \mu_j, \Sigma_j) \, , \tag{A.6}$$

which is given, following Definition 2, by the family of probability distributions with p.d.f.:

$$g(y;\theta_1,\theta_2)=\frac{\displaystyle\sum_{j=1}^{J}\lambda_j\,\exp(\theta_1' y+y'\,\theta_2\,y)\,n(y;\mu_j,\Sigma_j)}{\displaystyle\sum_{j=1}^{J}\lambda_j\,\varphi_{S,j}(\theta_1,\theta_2)}\,, \tag{A.7}$$

where $\varphi_{S,j}(\theta_1,\theta_2)$ is the Second-Order Laplace Transform of $y \sim N(\mu_j,\Sigma_j)$ given by (A.4) with $\mu=\mu_j$ and $\Sigma=\Sigma_j$. From the results proved above we obtain:

$$\begin{aligned} g(y;\theta_1,\theta_2) &= \sum_{j=1}^{J}\lambda_j^*\,n\left[y;\left(\Sigma_j^{-1}-2\theta_2\right)^{-1}\left(\Sigma_j^{-1}\mu_j+\theta_1\right),\left(\Sigma_j^{-1}-2\theta_2\right)^{-1}\right]\,, \\ \text{with }\ \lambda_j^* &= \frac{\lambda_j\,\varphi_{S,j}(\theta_1,\theta_2)}{\displaystyle\sum_{j=1}^{J}\lambda_j\,\varphi_{S,j}(\theta_1,\theta_2)} \end{aligned} \tag{A.8}$$

proving relation (6) of Section 2.4.

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