Abstract: In this paper, we present a general discrete-time affine framework aimed at jointly modeling yield curves associated with different debtors. The underlying fixed-income securities may differ in terms of credit quality and/or in terms of liquidity. The risk factors follow conditionally Gaussian processes, with drifts and variance-covariance matrices that are subject to regime shifts described by a Markov chain with (historical) non-homogenous transition probabilities. Importantly, bond prices are given by quasi-explicit formulas, ensuring the tractability of the framework. This tractability is illustrated by the estimation of a term-structure model of the spreads between U.S. BBB-rated corporate bonds and Treasuries. Alternative applications are proposed, including a sector-contagion model as well as the explicit modeling of credit-rating transitions.

JEL codes: E43, E44, E47, G12, G24.

Keywords: credit risk, liquidity risk, term structure, affine model, regime switching, Car process.
Introduction

There is strong evidence of regime switching in the dynamics of interest rates (see, e.g., Hamilton, 1988 [32] or Cai, 1994 [10]). Regime shifts have been successfully introduced in term-structure models of risk-free interest rates by, amongst others, Bansal and Zhou (2002) [5], Monfort and Pegoraro (2007) [46], Dai, Singleton and Yang (2007) [15] or Ang Bekaert and Wei (2008) [3]. Whereas these contributions put forward the importance of modeling regime switching in yield-curve models, a few has been done to integrate such a feature in term-structure models of defaultable bonds. However, empirical studies point to the existence of different regimes in the default risk valuation (see, e.g., Davies, 2004 [18] or Alexander and Kaeck, 2008 [1]). From a theoretical point of view, Hackbarth, Miao and Morellec (2006) [31] provide a theoretical model to explain the dependence of credit spreads on business-cycle regimes. In the same vein, Bhamra, Kuehn and Strebulaev (2007) [8] and David (2008) [17] adopt structural models including regime switching to assess the influence of different states of the economic cycles on the credit-risk premia.

The main aim of the present paper is to propose a general discrete-time dynamic modeling framework including switching regimes, both in the historical and the risk-neutral worlds. Particular attention is paid to the tractability of the model and its estimation. Tractability is notably obtained through an extensive use of Car’s –Compound autoregressive processes – properties (see, e.g. Darolles, Gourieroux and Jasiak, 2006 [16]), which leads to quasi-explicit formulas for riskless and defaultable bond prices. Both historical and risk-neutral dynamics are explicitly modeled, which is helpful for choosing appropriate specifications under the historical measure, for dealing simultaneously with pricing and forecasting, for Value-at-Risk calculations or for Sharpe-ratio computations.\(^1\)

Our modeling of defaults is based on the so-called “doubly-stochastic” assumption: correlations between default events arise solely through dependence on some common underlying stochastic factors –also termed with “risk factors”– which influence the default probabilities of every single loans. Some of the factors may be unobserved. In this sense, our model accommodates frailty. This feature is advocated by recent papers suggesting that including only observable covariates in default-intensity specifications results in poorly-estimated conditional probabilities of default (see e.g. Lando and Nielsen, 2008 [39] or Duffie et al., 2009 [21]).

In our approach, regime shifts may affect pricing through several channels: (i) regimes
affect the historical and risk-neutral dynamics of the risk factors, (ii) regimes appear in
the stochastic discount factor (s.d.f.) –which implies that regime-transition risk is priced–
and (iii) regimes appear in the default-intensity functions. This results in a large degree of
flexibility in the model specifications, which is illustrated by several numerical examples in
the paper. In particular, since default intensities can be affected by the regime variable,
our model is appropriate to capture default clustering.

In order to show some of the framework advantages and to illustrate its tractability,
we estimate a simple model of the term structure of the spreads between U.S. BBB-rated
corporate bonds and Treasuries. In particular, a comparison of this model with purely
Gaussian model highlights the potential of regime switching to capture salient features of
the spread distributions.

Beyond the enrichment of the specifications of the risk factors and those of the default
intensities by introducing nonlinearities, the regime-switching feature can be further ex-
ploited to handle specific forms of contagions. Contagion effects, whose consequences are
cascades of subsequent spread changes, is explained by the existence of close ties between
firms (see, e.g., Jarrow and Yu, 2001 [36], Davies and Lo, 2001 [19] or Giesecke, 2004 [28]).
Contagion takes place when the default probability of any debtor can be affected by the
default event of another one. Given that our baseline model relies on the doubly-stochastic
or conditional-independence assumption –which states that, conditional to the underlying
factors and regimes, the default events of the firms in a portfolio are independent– direct
contagion effects is not be captured. Nevertheless, we can model specific contagion effects in
two distinct ways. First, our framework can accomodate the specific contagion case where
one entity –or, for the sake of tractability, a small number of them– affects the default prob-
ability of the others: it suffices to make one of the regimes corresponds to the default state
of this entity. Second, the regime-switching feature can be exploited in order to capture
“sector-contagion” phenomena. The sectors can represent different industries or different
geographical areas. Each sector can be “infected” or not, and when a sector gets infected,
the default intensities of its constituents (the debtors) shift upwards. In this context, sector
contagion stems from the parameterizations of the matrix of regime-transition probabil-
ities. For instance, you can easily model infection probabilities that depend positively on the
number of sectors already infected.

Our baseline model considers only one credit event: the default of the debtor. However,
credit events include more generally the changes in credit ratings like these attributed by agencies like Moody’s, Standard & Poor’s or Fitch. There are several reasons why it may be desirable to model not only default events but also rating transitions (see Cantor, 2004 [11] or Gagliardini and Gourieroux, 2001 [27]). It turns out that our framework can be adapted to accommodate time-varying credit-rating migrations’ probabilities along the lines of Lando (1998) [38] while keeping quasi-explicit bond-pricing formulas.

The remainder of the paper is organized as follows. Sections 1 and 2 respectively present the historical and risk-neutral dynamics of the variables. Section 3 gives the bond-pricing formulas with zero or non-zero recovery rates. Section 4 deals with internal-consistency restrictions that arise when yields or asset returns are included amongst the risk factors. In Section 5, we discuss the estimation of such models, which is illustrated by an estimation of a term-structure model of spreads between U.S. BBB-rated corporate bonds and Treasuries. Section 6 shows how the model accommodates the pricing of liquidity. Section 7 investigates possible extensions of the framework: Subsection 7.1 deals with multi-lag dynamics of the risk factors; Subsection 7.2 deals with the specific case where one of the Markov chains coincides with the default state of a given entity; Subsection 7.3 presents a sector-contagion model and Subsection 7.4 shows how to introduce rating-migration modeling in the framework. Section 8 concludes.

1. Information and historical dynamics

1.1. Information

The new information of the investors at date \( t \) is \( w' = (z'_t, y'_t, x'_{1,t}, \ldots, x'_{N,t}) \) where \( z_t \) is a regime variable that can take a finite number \( J \) of values, \( y_t \) is a multivariate macroeconomic factor, \( x'_t = (x'_{1,t}, \ldots, x'_{N,t}) \) is a set of specific multivariate factors \( x_{n,t} \) associated with debtor \( n \), and \( d'_t = (d'_{1,t}, \ldots, d'_{N,t}) \) is a set of binary variables indicating the default \( (d_{n,t} = 1) \) or the non-default \( (d_{n,t} = 0) \) state of entity \( n \). The whole information set of the investors at date \( t \) is \( w'_t = (w'_1, \ldots, w'_t) \). At this stage, we do not make any assumption about the observability of these variables by the econometrician (this is done below in Section 5). These regimes influence bond pricing through different channels (they will appear in the dynamics of the risk factors \( y_t \) and \( x_{n,t} \)'s, in the stochastic discount factor and in the default-intensity functions). In the baseline framework, the regimes are viewed as transitory: none of these
regimes is absorbing but this restriction is relaxed in a specific case presented in Subsection 7.2.

1.2. Historical dynamics

It is convenient to make the regime variable $z_t$ valued in \( \{e_1, \ldots, e_J\} \), the set of column vectors of the identity matrix $I_J$. The conditional distribution of $z_t$ given $w_{t-1}$ is characterized by the probabilities:

$$ p\left(z_t \mid w_{t-1}\right) = \pi\left(z_t \mid z_{t-1}, y_{t-1}\right). $$

(1)

The probability $\pi(e_j \mid e_i, y_{t-1})$ that $z_t$ shifts from regime $i$ to regime $j$ between period $t-1$ and $t$, conditional on $y_{t-1}$, is also denoted by $\pi_{ij,t-1}$. These specifications allow for state-dependent transition probabilities, as in Ang and Bekaert (2002) [2] or Dai, Singleton and Yang (2007) [15].

The conditional distribution of $y_t$ given $z_t$ and $w_{t-1}$ is Gaussian and given by:

$$ y_t = \mu(z_t, z_{t-1}) + \Phi y_{t-1} + \Omega(z_t, z_{t-1}) \epsilon_t $$

(2)

where the $\epsilon_t$ are independently and identically $N(0, I)$ distributed. Specifications (1) and (2) imply that, in the universe $(z_t, y_t)$, $z_t$ Granger-causes $y_t$, $y_t$ causes $z_t$ and there is instantaneous causality between $z_t$ and $y_t$. Moreover, in the universe $w_t = (z_t, y_t, x_t, d_t)$, $(x_t, d_t)$ does not cause $(z_t, y_t)$. As noted by Ang, Bekaert and Wei (2008) [3], instantaneous causality between $z_t$ and $y_t$ implies that the variances of the factors $y_t$, conditional on $w_{t-1}$, embed a jump term reflecting the difference in drifts $\mu$ across regimes. Such a feature, that allows for conditional heteroskedasticity, is absent from the Dai, Singleton and Yang (2007) [15] setting. However, it should be noted that our framework nests the case where there is no instantaneous causality between $z_t$ and $y_t$ in the historical dynamics.\(^6\) Contrary to Bansal and Zhou (2002) [5], matrix $\Phi$ is not regime-dependent: this is for the sake of tractability when it comes to bond pricing.\(^7\)

The $x_{n,t}$’s, $n = 1, \ldots, N$ are assumed to be independent conditionally to $(z_t, y_t, w_{t-1})$. The conditional distribution of $x_{n,t}$ is Gaussian and defined by:

$$ x_{n,t} = q_{1n}(z_t, z_{t-1}) + Q_{2n}y_t + Q_{3n}y_{t-1} + Q_{4n}x_{n,t-1} + Q_{5n}(z_t, z_{t-1}) \eta_{n,t} $$

(3)

where the shocks $\eta_{n,t}$ are $IIN(0, I)$. Specifications (1), (2) and (3) imply that, in the uni-
verse \((z_t, y_t, x_{n,t})\), \((z_t, y_t)\) causes \(x_{n,t}\), \(x_{n,t}\) does not cause \((z_t, y_t)\) and there is instantaneous causality between \((z_t, y_t)\) and \(x_{n,t}\). Moreover, denoting by \(\mathbf{x}_{n,t}\) the vector \(x_t\) excluding \(x_{n,t}\), \((\mathbf{x}_{n,t}, d_t)\) does not cause \((z_t, y_t, x_{n,t})\) in the whole universe \(w_t\).

Finally, the \(d_{n,t}\)'s, \(n = 1, \ldots, N\), are independent conditionally to \((z_t, y_t, x_t, w_{t-1})\) and the conditional distribution of \(d_{n,t}\) is such that:

\[
p(d_{n,t} = 1 \mid z_t, y_t, x_t, w_{t-1}) = \begin{cases} 
1 & \text{if } d_{n,t-1} = 1, \\
1 - \exp(-\lambda_{n,t}) & \text{otherwise},
\end{cases}
\]

with \(\lambda_{n,t} = \alpha_n z_t + \beta_n y_t + \gamma_n x_{n,t}\).

In other words, state 1 of \(d_{n,t}\) is an absorbing state and \(\exp(-\lambda_{n,t})\) is the survival probability. Since the default probability \(1 - \exp(-\lambda_{n,t})\) is close to \(\lambda_{n,t}\) if \(\lambda_{n,t}\) is small, \(\lambda_{n,t}\) is called the default intensity. The default intensity is expected to be positive, which is not necessarily the case since the \(\varepsilon_t\)'s are Gaussian. However, the parameterization of the model may make this extremely unfrequent.

So, in the universe \((z_t, y_t, x_{n,t}, d_{n,t})\), \((z_t, y_t, x_{n,t})\) causes \(d_{n,t}\) whereas \(d_{n,t}\) does not causes \((z_t, y_t, x_{n,t})\) and there is instantaneous causality. In the whole universe \(w_t\), \((\mathbf{x}_{n,t}, \mathbf{d}_{n,t})\) does not cause \((z_t, y_t, x_{n,t}, d_{n,t})\). The causality scheme is summarized in Figure 1.

[Insert Figure 1 about here]

In the proposition below, we consider the historical conditional Laplace transform of the distribution of \((z_t, y_t)\) given \(w_{t-1}\), that is \(\phi_{t-1}(u, v) = E_{t-1} [\exp (u' z_t + v' y_t)]\).

**Proposition 1.** The historical conditional Laplace transform of \((z_t, y_t)\) given \(w_{t-1}\) is:

\[
\phi_{t-1}(u, v) = \exp \left( v' \Phi y_{t-1} + [l_1, \ldots, l_J] z_{t-1} \right),
\]

where \(l_i = \log \sum_{j=1}^J \pi_{ij,t-1} \exp \{ u_j + v' \mu(e_j, e_i) + \frac{1}{2} v' \Omega (e_j, e_i) \Omega' (e_j, e_i) v \}\).

**Proof.** See Appendix A.1.

This Laplace transform is not, in general, exponential affine in \((z_{t-1}, y_{t-1})\), since \(y_{t-1}\) appears in the \(\pi_{ij,t-1}\)'s. However, this is the case if the \(\pi_{ij,t-1}\)'s do not depend on \(y_{t-1}\) and then, by definition, \((z_t, y_t)\) is Car(1). As detailed in Darolles, Gourieroux and Jasiak (2006)[16] or Bertholon, Monfort and Pegoraro (2008) [7], Car processes are omnipresent
in asset-pricing literature due to the fact that there exist recursive algorithms to compute multi-horizon Laplace transforms, which is key for term-structure modeling. As will be illustrated below, we will rely heavily on the properties of the Car processes to ensure tractability in bond pricing (Section 3).

2. Stochastic discount factor and risk-neutral dynamics

2.1. Stochastic discount factor

We assume that the riskless short-term rate between \( t - 1 \) and \( t \) is:

\[
    r_t = a'_t z_{t-1} + b'_t y_{t-1}.
\]

Then, we define the stochastic discount factor \( M_{t-1,t} \) between \( t - 1 \) and \( t \) by:

\[
    M_{t-1,t} = \exp \left[ -a'_t z_{t-1} - b'_t y_{t-1} - \frac{1}{2} \nu' (z_t, z_{t-1}, y_{t-1}) \nu (z_t, z_{t-1}, y_{t-1}) + \nu' (z_t, z_{t-1}, y_{t-1}) \epsilon_t + \delta' (z_{t-1}, y_{t-1}) z_t \right],
\]

The \( \nu \) and \( \delta \) vectors can be seen respectively as the prices of risks associated with the (standardized) innovations of the process \( y_t \) and the regimes \( z_t \). Regarding the latter, the fact that we must have \( E_{t-1} (M_{t-1,t}) = \exp(-a'_t z_{t-1} - b'_t y_{t-1}) \) implies that \( E_{t-1} \exp \left( \delta' (z_{t-1}, y_{t-1}) z_t \right) = 1 \), which is equivalent to:

\[
    \sum_{j=1}^J \pi_{ij,t-1} \exp [\delta_j (e_i, y_{t-1})] = 1, \ \forall i, y_{t-1},
\]

where \( \delta_j \) is the \( j^{th} \) component of \( \delta \).

In our framework, the variables \( (x_{n,t}, d_{n,t}) \), specific to entity \( n \), do not appear in the stochastic discount factor, which reflects the fact that these entities have no impact at the macroeconomic level (in Subsection 7.2, we discuss the case where one entity has a “systemic” status).
2.2. Risk-neutral dynamics

2.2.1. The conditional risk-neutral distribution of \((z_t, y_t)\) given \(w_{t-1}\)

Let us now consider the conditional risk-neutral Laplace transform of \((z_t, y_t)\) given \(w_{t-1}\),
\[\varphi^Q_{t-1}(u, v) := E^Q_{t-1}(\exp[u'z_t + v'y_t]),\]
and let us introduce the simplified notations:
\[
\begin{align*}
\mu_t &= \mu(z_t, z_{t-1}) \\
\Omega_t &= \Omega(z_t, z_{t-1}), \Sigma(z_t, z_{t-1}) = \Omega_t \Omega'_t = \Sigma_t \\
\nu_t &= \nu(z_t, z_{t-1}, y_{t-1}) \\
\delta_{t-1} &= \delta(z_{t-1}, y_{t-1}).
\end{align*}
\]

**Proposition 2.** The conditional risk-neutral Laplace transform of \((z_t, y_t)\) given \(w_{t-1}\) is:
\[
\varphi^Q_{t-1}(u, v) = \exp\left[v'\Phi_{y_{t-1}} + \left(A_{1,t-1}(u, v) \ldots A_{J,t-1}(u, v) \right)z_{t-1}\right],
\]
where
\[
A_{i,t-1}(u, v) = \log(\sum_{j=1}^{J} \pi_{ij,t-1} \exp \left\{ v'\Omega(e_j, e_i) \nu(e_j, e_i, y_{t-1}) + \frac{1}{2} v'\Sigma(e_j, e_i) v + v'\mu(e_j, e_i) + u_j + \delta_j(e_i, y_{t-1}) \right\}).
\]

**Proof.** See Appendix A.2.

As mentioned above, Car processes are particularly convenient for the sake of asset pricing (and in particular for term-structure modeling) because the computation of their multi-horizon Laplace transforms is straightforward, as will be shown below. This motivates the next Corollary.

**Corollary 1.** The risk-neutral dynamics of \((z_t, y_t)\) is Car(1) if the risk sensitivities \(\delta\) and \(\nu\), appearing in the s.d.f., satisfies the constraints (for any \(i, j\) and \(t\)):
\[
\begin{align*}
\delta_j(e_i, y_{t-1}) &= \log \left[ \frac{\pi^*_j}{\pi(e_j | e_i, y_{t-1})} \right] \\
\nu(e_j, e_i, y_{t-1}) &= \Omega(e_j, e_i)^{-1} [\Phi^*y_{t-1} + \mu^*(e_j, e_i)],
\end{align*}
\]


for any transition matrix $\pi_{ij}^* = \pi^*(e_j \mid e_i)$, any matrix $\Phi^*$ and any function $\mu^*$.

It is important to note that these constraints still allow for a large number of degrees of freedom in the specification of the s.d.f., since the transition matrix $\{\pi_{ij}^*\}$, the matrix $\Phi^*$ and the vectors $\mu^*(e_j, e_i)$ are arbitrary. If the constraints (10) are satisfied, the risk-neutral conditional Laplace transform becomes:

$$
\varphi^Q_{t-1}(u, v) = \exp \left[ v' (\Phi + \Phi^*) y_{t-1} + \left( A^*_1(u, v) \ldots A^*_J(u, v) \right) z_{t-1} \right],
$$

where, for any $i$, $A^*_i(u, v) = \log \left( \sum_{j=1}^J \pi_{ij}^* \exp \left\{ u_j + v' [\mu(e_j, e_i) + \mu^*(e_j, e_i)] + \frac{1}{2} v' \Sigma(e_j, e_i) v \right\} \right)$.

Comparing with equation (5), we deduce that the risk-neutral dynamics of $(z_t, y_t)$ is then defined by:

$$
y_t = \mu(z_t, z_{t-1}) + \mu^*(z_t, z_{t-1}) + (\Phi + \Phi^*) y_{t-1} + \Omega(z_t, z_{t-1}) \epsilon_t^r,
$$

where, under $Q$, $z_t$ is an homogenous Markov chain defined by the transition matrix $\{\pi_{ij}^*\}$, and $\epsilon_t^r$—defined by $\epsilon_t^r = \epsilon_t - \Omega^{-1}(z_t, z_{t-1}) [\mu^*(z_t, z_{t-1}) + \Phi^* y_{t-1}]$—is $\mathcal{N}^Q(0, I)$. Note that $\bar{\mu} = \mu + \mu^*$ and $\tilde{\Phi} = \Phi + \Phi^*$ are arbitrary and that the $\Omega$ function is the same in the historical and risk-neutral worlds.

### 2.2.2. The risk-neutral distribution of $(x_t, d_t)$ given $(z_t, y_t, w_{t-1})$

**Lemma 1.** Let us consider a partition of $w_t = (w_{1,t}, w_{2,t})'$. If $M_{t-1,t}$ is a function of $(w_{1,t}, w_{t-1})$, the risk-neutral probability density function, or p.d.f., of $w_{1,t}$ given $w_{t-1}$ is:

$$
f^Q(w_{1,t} \mid w_{t-1}) = f(w_{1,t} \mid w_{t-1}) M_{t-1,t} \exp (-r_t)
$$

(where $f$ is the historical conditional p.d.f. of $w_{1,t}$ given $w_{t-1}$) and the conditional risk-neutral distribution of $w_{2,t}$ given $(w_{1,t}, w_{t-1})$ is the same as the corresponding historical distribution.

**Proof.** See Appendix A.3. $\square$

Since $M_{t-1,t}$ is a function of $(z_t, y_t)$ but not of $(x_t, d_t)$, the previous lemma shows that the risk-neutral distribution of $(x_t, d_t)$ given $(z_t, y_t, w_{t-1})$ is the same as the historical one.
and it is given by equations (3) and (4). In particular, the functional forms of the default intensities $\lambda_{n,t}$ are the same as in the historical world. Of course, since the dynamics of $(z_t, y_t)$ are different in the two worlds, the same is true for the $x_{n,t}$'s and the $\lambda_{n,t}$'s.

In addition, it can be shown that $(z_t, y_t, x_{n,t})$ is Car(1) under the risk-neutral measure (see Appendix A.4). However, it is not the case for $(z_t, y_t, x_{n,t}, d_{n,t})$.

It is also clear that the causality structure of the risk-neutral dynamics is similar to the historical one, the only difference being the non-causality from $y_t$ to $z_t$ implied by the homogeneity of the matrix $\{ \tau_{ij}^* \}$.

3. Pricing

3.1. Defaultable (zero-recovery-rate) bond pricing with non-zero recovery rate

The price at $t$ of a riskless zero-coupon bond with residual maturity $h$ is given by:

$$ B(t, h) = E_t^Q \left[ \exp \left( -r_{t+1} - \ldots - r_{t+h} \right) \right], $$

where $r_{t+i} = a^*_1 z_{t+i-1} + b^*_1 y_{t+i-1}$, $i = 1, \ldots, h$. The following proposition shows that thanks to the risk-neutral causality structure of our model, there exists an analogous formula to price defaultable bonds with zero recovery rates. Naturally, the case of risk-free bond pricing is nested within the more general defaultable-bond pricing case (with a zero default intensity).

**Proposition 3.** The price of a zero-recovery-rate zero-coupon defaultable bond issued by debtor $n$ is given by:

$$ B^D_n(t, h) = E_t^Q \left[ \exp \left( -\left( r_{t+1} + \lambda_{n,t+1} \right) - \ldots - \left( r_{t+h} + \lambda_{n,t+h} \right) \right) \right], $$

which is exponential linear in $(z_t, y_t, x_{n,t})$:

$$ B^D_n(t, h) = \exp \left( -c^*_{n,h} z_t - f^*_{n,h} y_t - g^*_{n,h} x_{n,t} \right) $$

$$ R^D_n(t, h) = \frac{1}{h} \left( c^*_{n,h} z_t + f^*_{n,h} y_t + g^*_{n,h} x_{n,t} \right), $$

where $c^*_{n,h}$, $f^*_{n,h}$, and $g^*_{n,h}$ are determined by the model parameters.
In Equations (15) and (16), \((c'_{n,h}, f'_{n,h}, g'_{n,h})\) is computed recursively by:

\[
(c'_{n,h}, f'_{n,h}, g'_{n,h}) = (a'_1, b'_1, 0) - \pi \left( \omega_{H-h+1} - (c'_{n,h-1} - a'_1, f'_{n,h-1} - b'_1, -g'_{n,h-1}) \right)
\]

where

- the sequence \(\omega_h, h = 1, \ldots, H\) is defined by \(\omega_H = (\alpha'_n, -\beta'_n, -\gamma'_n)\) and \(\omega_h = (-\alpha'_n - a'_1, -\beta'_n - b'_1, -\gamma'_n)\) for \(h = 1, \ldots, H - 1\), with \(c_{n,0} = a_1, f_{n,0} = b_1, g_{n,0} = 0\),

- The function \(\tilde{a}\) is defined by \(\tilde{a}(u,v,w) = [\tilde{A}_1, \ldots, \tilde{A}_j], (v' + w'Q2n)(\Phi + \Phi^*) + w'Q3n, w'Q4n\], where

\[
\tilde{A}_i(u,v,w) = \log(\sum_{j=1}^{J} \pi_{ij}^* \exp\{u_j + (v' + w'Q2n)\mu(e_j,e_i) + \mu^*(e_j,e_i)\} + w'q_{1n}(e_j,e_i) + \frac{1}{2}(v' + w'Q2n)\Sigma(e_j,e_i)(v + Q2n w) + \frac{1}{2} w'Q5n(e_j,e_i) Q5n(e_j,e_i) w).
\]

**Proof.** The price of a zero-coupon bond providing one money unit at \(t+h\) if entity \(n\) is still alive at \(t+h\) and zero otherwise is:

\[
B^D_n(t,h) = E_t^Q \left[ \exp(-r_{t+1} - \cdots - r_{t+h}) \mathbb{I}_{\{d_{n,t+h}=0\}} \right]
\]

\[
= E_t^Q \left[ E_t^Q \left( \exp(-r_{t+1} - \cdots - r_{t+h}) \mathbb{I}_{\{d_{n,t+h}=0\}} \mid \tilde{z}_{t+h}, \tilde{y}_{t+h}, \tilde{x}_{n,t+h}, d_{n,t} = 0 \right) \right]
\]

\[
= E_t^Q \left[ \exp(-r_{t+1} - \cdots - r_{t+h}) Q(d_{n,t+h} = 0 \mid \tilde{z}_{t+h}, \tilde{y}_{t+h}, \tilde{x}_{n,t+h}, d_{n,t} = 0) \right].
\]

Moreover,

\[
Q(d_{n,t+h} = 0 \mid \tilde{z}_{t+h}, \tilde{y}_{t+h}, \tilde{x}_{n,t+h}, d_{n,t} = 0) = \prod_{i=1}^{h} Q(d_{n,t+i} = 0 \mid \tilde{z}_{t+h}, \tilde{y}_{t+h}, \tilde{x}_{n,t+h}, d_{n,t+i-1} = 0)
\]

and, since \(d_{n,t}\) does not \(Q\)-cause \((z_t, y_t, x_{n,t})\) in the Granger’s or Sims’ sense, we have:9

\[
Q(d_{n,t+i} = 0 \mid \tilde{z}_{t+h}, \tilde{y}_{t+h}, \tilde{x}_{n,t+h}, d_{n,t+i-1} = 0) = Q(d_{n,t+i} = 0 \mid \tilde{z}_{t+i}, \tilde{y}_{t+i}, \tilde{x}_{n,t+i}, d_{n,t+i-1} = 0) = \exp(-\lambda_{n,t+i}).
\]

where the last equality comes from the fact that the conditional historical and risk-neutral
distributions of $d_{n,t}$ are the same (see Subsection 2.2.2). Hence, Equation (14) holds. The latter suggests that since the $r_{t+i}$’s and the $\lambda_{n,t+i}$’s are linear in the $(z_{t+i}, y_{t+i}, x_{n,t+i})$’s, the bond prices are multi-horizon Laplace transforms.

Besides, it can be shown (see Appendix A.4) that $(z_t, y_t, x_{n,t})$ is Car(1) under $\mathbb{Q}$, with a conditional Laplace transform of the type $\exp[\tilde{a}'(u, v, w)(z_t, y_t, x'_{n,t})]$. The recursive formulas presented in Proposition 3 directly stem from Appendix A.5, where it is explained how to exploit the Car(1) property of $(z_t, y_t, x_{n,t})$ to compute its multi-horizon Laplace transforms.

3.2. Defaultable (zero-recovery-rate) bond pricing with non-zero recovery rate

In the next Proposition, we present conditions under which quasi-explicit formulas are still available in the case of non-zero recovery rates.

Proposition 4. If, for any bond issued by debtor $n$ before $t$, the recovery payoff –that is assumed to be paid at time $t$ in case of default between $t-1$ and $t$ of debtor $n$– is equal to the product of a function $\zeta_{n,t}$ (with $0 \leq \zeta_{n,t} \leq 1$) of the information available at time $t$ by the survival-contingent market value of the bond at $t$, the price at $t$ of a bond with residual maturity $h$ is:

$$B^D_n(t, h) = E^Q_t \left[ \exp(-r_{t+h} - \ldots - r_{t+h} - \tilde{\lambda}_{n,t+1} - \ldots - \tilde{\lambda}_{n,t+h}) \right],$$

(17)

where $\tilde{\lambda}_{n,s}$ is defined by (for any $s$):

$$\exp(-\tilde{\lambda}_{n,s}) = \exp(-\lambda_{n,s}) + (1 - \exp(-\lambda_{n,s})) \zeta_{n,s}.$$  

Proof. See Appendix A.6. 

The assumption of Proposition 4 is similar to the “Recovery of Market Value” assumption made by Duffie and Singleton (1999) [22] except that, in their discrete-time approach, they assume that $\zeta_t$ is known at time $t-1$, and that conditionally to the information at $t-1$, $d_{n,t}$ is independent of the recovery payoff at $t$. 


4. Internal consistency (IC) conditions

4.1. IC conditions based on riskless yields

If the short rate $r_{t+1}$ is a component of $y_t$, for instance the first one, we have to impose an internal consistency condition implying that $r_{t+1} = a'_1 z_t + b'_1 y_t$ is equal to the first component of $y_t$, that is:

$$a_1 = 0, \quad b_1 = \tilde{e}_1,$$

where $\tilde{e}_i$ is the vector selecting the $i^{th}$ component of $y_t$.

Moreover, if another component of $y_t$, for instance the second one, is equal to a riskless yield of maturity $h_0$ – i.e. $R(t, h_0)$ – we have to impose that $(1/h_0) \left( a'_{h_0} z_t + b'_{h_0} y_t \right)$ is equal to the second component of $y_t$, that is

$$\begin{cases}
    a_{h_0} = 0 \\
    b_{h_0} = h_0 \hat{e}_2.
\end{cases}$$

4.2. IC conditions based on defaultable yields

Similarly, if the first component of $x_{n,t}$ is a defaultable yield with residual maturity $h_0$, equation (15) implies that we have to impose:

$$\begin{cases}
    c_{n,h_0} = 0 \\
    f_{n,h_0} = 0 \\
    g_{n,h_0} = h_0 \hat{e}_1.
\end{cases}$$

where $\hat{e}_i$ denotes the vector selecting the $i^{th}$ component of $x_{n,t}$.

4.3. IC conditions based on asset returns

If the first component of $y_t$ is the geometric return of a market index, we have to impose

$$\exp (-r_{t+1}) E_t^Q \left( \exp (y_{1,t+1}) \right) = 1.$$
Using equation (11), this gives

\[
\begin{pmatrix}
    A_{1,0}^* & \ldots & A_{J,0}^*
\end{pmatrix}
z_t + (\Phi_1 + \Phi_1^*) y_t = a_1' z_t + b_1' y_t,
\]

with \(A_{i,0}^* = \log \left\{ \sum_{j=1}^J \pi_{ij}^* \exp \left[ \mu_1 (e_j, e_i) + \mu_1^* (e_j, e_i) + \frac{1}{2} \sigma_1^2 (e_j, e_i) \right] \right\} \), \(\mu_1\) and \(\mu_1^*\) being the first components of \(\mu\) and \(\mu^*\) respectively, \(\sigma_1^2\) being the \((1,1)\) entry of \(\Sigma\) and \(\Phi_1\) and \(\Phi_1^*\) the first rows of \(\Phi\) and \(\Phi^*\) respectively. Then we get

\[
\begin{cases}
    a_1' = \left( A_{1,0}^* \ldots A_{J,0}^* \right)' \\
    b_1' = (\Phi_1 + \Phi_1^*)'
\end{cases}
\]

Similarly, if the first component of \(x_{n,t}\) is the return of a stock attached to entity \(n\), we must have:

\[
\exp (-r_{t+1}) E_t^Q \left( \exp (x_{1,n,t+1}) \right) = 1
\]

or

\[
r_{t+1} = \log \left[ E_t^Q \left( \exp (x_{1,n,t+1}) \right) \right].
\]

Using the fact that \((z_t, y_t, x_{n,t})\) is Car(1) under \(Q\) (see Appendix A.4), it is readily seen that \(\log \left[ E_t^Q \left( \exp (x_{1,n,t+1}) \right) \right]\) is linear in \(z_t, y_t, x_{n,t}\) and the IC constraint follows.

5. Inference

5.1. Observability

We assume that \(z_t, y_t\) and the \(x_{n,t}\)'s are partitioned into \(z_t = (z_{1,t}, z_{2,t})', y_t = (y_{1,t}, y_{2,t})'\) and \(x_{n,t} = (x_{1,n,t}', x_{2,n,t}')\), that \(z_{1,t}, y_{1,t}, x_{1,n,t}\) are observed by the econometrician and \(z_{2,t}, y_{2,t}\) and \(x_{2,n,t}\) are not. Typically, \(z_{1,t}\) and \(z_{2,t}\) will be two regime processes valued respectively in \(E_1 = \{e_1, \ldots, e_{J_1}\}\) and \(E_2 = \{e_1, \ldots, e_{J_2}\}\) so \(z_t\) will be equal to \(z_{1,t} \otimes z_{2,t}\), where \(\otimes\) denotes the Kronecker product operator. Moreover, we observe at each date \(t\) a vector of risk-free yields denoted by \(R_t\) and, for each obligor \(n\), a vector of defaultable yields denoted by \(R_{n,t}^D\). Note that if some yields are included in the vectors \(y_t\) or \(x_{n,t}\), they do not enter the vectors \(R_t\) and \(R_{n,t}^D\).
5.2. Estimation methods

Regarding the estimation of models based on that framework, it is convenient to distinguish two main kinds of equations. While the first kind of equations defines the dynamics of the factors (i.e., equations 2 and 3), the second kind is concerned with the fit of observed yields. If the number of unobserved factors is lower than the number of yields to fit, some pricing errors arise. Obviously, if one wants to compute the log-likelihood of the model, one has to specify a distribution type for these pricing errors. Usually, these are supposed to be (i.i.d.) normally distributed in the affine term-structure literature.

In the absence of latent factors or regimes, the computation of the likelihood of the model is straightforward. On the contrary, specific techniques are required as soon as some factors and/or some Markov chains are unobserved. Table 1 proposes techniques that can be implemented in the different possible cases. For instance, when the model includes latent variables but no unobserved Markov chains, the log-likelihood can be computed by means of the Kalman filter or the so-called inversion techniques (see Chen and Scott, 1993 [13]) may be resorted to. Absent unobserved factors $y_{2,t}$, the Kitagawa-Hamilton filter can be used if some regimes are unobserved. Finally, if there are both unobserved regimes and factors, two techniques can be implemented. First, one can use Kim’s (1994) [37] filter that allows to approximate the log-likelihood in the presence of both kinds of unobserved processes. Second, inversion techniques à la Chen and Scott (1993) [13] may still be used; the implied adjustments to deal with unobserved regimes being detailed in Appendix B.

[Insert Table 1 about here]

5.3. Estimation example: a simple model of the BBB-Treasury spreads

In this subsection, we illustrate the flexibility and the tractability of the framework by estimating a model using real data. Note that this example reflects only one, out of many, possible use of the framework. (the multiplicity of its applications is addressed in Section 7.)

We consider one defaultable entity whose funding costs are representative of those of BBB-rated corporates. As commonly assumed, bonds issued by the U.S. Treasury are supposed to be riskfree (i.e., the U.S.-Treasury default intensity is zero). Dropping the debtor index in that subsection (since we consider only one risk entity), the BBB-rated-
firms’ default intensity is defined as:

$$\lambda_t = y_{1,t} + y_{2,t}$$

where the $y_{i,t}$’s are some risk factors following:

$$
\begin{bmatrix}
y_{1,t} \\
y_{2,t}
\end{bmatrix} = 
\begin{bmatrix}
\mu_1 & \mu_1 & \mu_1 \\
0 & 0 & \mu_2
\end{bmatrix}
\begin{bmatrix}
z_t \\
y_{1,t-1} \\
y_{2,t-1}
\end{bmatrix} + 
\begin{bmatrix}
\varphi_1 & 0 \\
0 & \varphi_2 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
y_{1,t-1} \\
y_{2,t-1} \\
\omega \varepsilon_t
\end{bmatrix}
$$

with $\varepsilon_t \sim \text{i.i.d. } \mathcal{N}(0,1)$ and where $z_t$ is a three-state Markov chain which is independent from $\varepsilon_t$ and has a matrix of transition probabilities of the form:

$$P = 
\begin{bmatrix}
p_{11} & (1 - p_{11}) & 0 \\
(1 - p_{22} - p_{23}) & p_{22} & p_{23} \\
0 & (1 - p_{33}) & p_{33}
\end{bmatrix}
$$

While the first regime is conceived as being a “tranquil” regime, the third is supposed to correspond to a “crisis” regime. The second acts as an intermediary regime: under this regime, the risk factors $y_{1,t}$ and $y_{2,t}$ have the same dynamics as under the tranquil regime, but with the threat of switching to the third regime (such a threat does not exist under the first regime since the probability of switching from the first to the third regime is null). The “crisis” nature of the third regime stems from the fact that the factor $y_{2,t}$ increases only under this last regime whereas it stagnates or decreases under the other two regimes (assuming that $\mu_2 > 0$ and $0 < \varphi_2 < 1$).

It is important to note that conditionally on the information available at time $t$, the means and variances of future hazard rates $\lambda_{t+k}$ depend on the current regimes. For instance, whereas the one-period-ahead variance of the intensity is $\omega^2$ under the first regime, it is $\mu_2^2p_{23}(1 - p_{23}) + \omega^2$ ($> \omega^2$) under the second regime. This illustrates in particular that the model is able to generate some forms of stochastic volatility.

The risk-neutral dynamics of $y_{1,t}$ and $y_{2,t}$ is assumed to be similar to its historical counterparts (equations 18 and 19), except that parameters $\varphi_1$, $\varphi_2$, $\mu_1$, $\mu_2$ and the $p_{ij}$’s are respectively replaced by $\varphi_1^*, \varphi_2^*, \mu_1^*, \mu_2^*$ and by some $p_{ij}^*$’s. Besides, the $\varepsilon_t$’s in equation (18) are replaced by some $\varepsilon_t^*$’s that are normal in the risk-neutral world.

For the sake of simplicity, we assume that $y_{1,t}$ and $y_{2,t}$ are independent from the factors.
driving the short-term risk-free rate under both historical and the risk-neutral measure. This implies that we can estimate the dynamics of \( y_{1,t} \) and \( y_{2,t} \) without defining a process for the short rate and that the estimation requires only spreads data.\(^\text{12}\)

The data are weekly and cover the period from 17 March 1995 to 1 July 2011. The spreads are computed by subtracting from the corporate yields the Treasury zero-coupon rates of the same maturities.\(^\text{13}\) We consider four maturities: 1, 2, 3 and 5 years. The spreads are assumed to be observed with i.i.d. measurement errors. The model can be seen as a state-space model with (a) four measurement equations (relating the observed spreads to the modeled ones, the discrepancy being the measurement –or pricing– errors) and (b) transition equations defined by (18) and (19). The parameters are estimated by maximizing the log-likelihood, using the approximation proposed by Kim’s (1994) [37]. Additional details regarding the estimation –including the developed state-space version of the model and the parameter estimates– are presented in Appendix C.

The upper panel of Figure 2 displays the estimated components of the default intensity (i.e. factors \( y_{1,t} \) and \( y_{2,t} \)). The second panel shows the (smoothed) probabilities of being in each of the three regimes. As expected, the failure of LTCM (Fall 1998), the bursting out of the internet bubble (2001) or the recent financial crisis (2007-2009) are associated with the crisis-regime periods (see the black areas of this second panel).\(^\text{14}\) The lowest two panels of Figure 2 display respectively the 2-year and 5-year observed spreads together with their model-implied counterparts, showing that the model captures most of the spread fluctuations (close to 99% of the spread fluctuations are accounted for by the model).

An important feature of the model is that it is not only capable of fitting the data, but it is also relevant to simulate realistic ones. Obviously, this is key if one wants to use the model to compute Value-at-Risks, for instance. In order to illustrate this, we have compared our estimated regime-switching model (RS model hereinafter) with two alternative (purely) Gaussian models. In the first alternative model, the default intensity is a simple AR(1); in the second model, the default intensity is a sum of two independent Gaussian AR(1) processes. The standard deviations of the spread pricing errors obtained with the RS model, the 1-factor Gaussian model and the 2-factor Gaussian model are respectively of 8, 10 and 7 basis points. However, the quality of the data resulting from Monte-Carlo simulations of the Gaussian models is poor in comparison with the RS model. This is illustrated in Figure 3. The first row of charts shows, for the 2-year and the 5-year maturities, the unconditional
distributions of spreads simulated by the three models mentioned above. These distributions are compared with the sample-data ones. The charts show that the Gaussian models are inappropriate to capture the tails’ shapes: in particular, while the one-factor Gaussian model often generates negative spreads, the two-factor Gaussian model fails to generate high spreads. By contrast, the RS model is able to keep the number of simulated negative spreads at a minimum while allowing for frequent high (crisis) spreads. The skewness and kurtosis of the sample data are impressively well reproduced by the RS model: considering the 5-year maturity, the skewness of the simulated spread is of 1.78 vs. 1.83 for the sample data and the kurtosis are respectively of 7.45 vs. 7.46. The superiority of the RS model in terms of simulation of plausible spreads is further highlighted by the lower plots in Figure 3: according to these charts, the RS model performs well in terms of fitting the 5th, 50th and 95th quantiles of the spreads (as well as their mean).

6. Liquidity risk

There is compelling evidence that yields and spreads contain components that are closely linked to liquidity. The estimation of the liquidity premium is of concern for several reasons. First, gauging the liquidity-risk premium provides policy makers –central bankers in particular– with insights on the valuation of liquidity by the markets (see Taylor and Williams, 2008 [51] or Michaud and Upper, 2008 [44]). Second, if one wants to extract default probabilities from market data, one has to distinguish between what is related to default and what is caused by the liquidity of the considered bonds.

However, the identification of the liquidity premium, that is, distinguishing between the default-related and the liquidity-related components of yield spreads, remains a challenging task. Empirical evidence points to the existence of commonality amongst the liquidity components of prices of different bonds (see e.g. Fontaine and Garcia, 2009 [26]). Therefore, the identification of the liquidity component relies on the ability to exhibit risk factors that reflects liquidity valuation. Liu, Longstaff and Mandell (2006) [40] and Feldhütter and Lando (2008) [24] develop affine term-structure models where a liquidity factor is latent and the identification is based on assumptions regarding the relative liquidity of different interest-rate instruments. Using the present framework, Monfort and Renne (2011) [47]
identify a liquidity latent factor by exploiting the term-structure of the KfW-Bund spreads. KfW is a German public agency whose issuances are fully and explicitly guaranteed by the Federal Republic of Germany. Accordingly, the spreads between the yields of bonds issued by KfW and those issued by the German government (called “Bunds”) mainly reflects liquidity-pricing effects. Further, Monfort and Renne (2011) show that this liquidity-pricing factor significantly contributes to the fluctuations of various euro-area sovereign spreads. Alternatively, the liquidity factor could be proxied by some observable factors. One may resort to intermediate –or mixed– approach, where part of the liquidity-factor dynamics is observable (through observed proxies) and part of it is latent.

Let us come back to our modeling framework. We have seen in section 3 that incorporating default risk in the pricing methodology implies to replace the short rate \( r_{t+1} \) by a “default-adjusted” short-rate \( r_{t+1} + \lambda_n, t+1 \). Besides, in order to take into account recovery-rate effects, \( \lambda_n, t+1 \) can be seen as a “recovery adjusted” default intensity between \( t \) and \( t+1 \) (see Appendix A.6). So the price at \( t \) of a defaultable asset providing the payoff \( g(w_{t+h}) \) at \( t+h \) in case of absence of default, is:

\[
E_t^Q \left[ \exp \left( -r_{t+1} - \lambda_n, t+1 - \ldots - r_{t+h} - \lambda_n, t+h \right) g \left( w_{t+h} \right) \right].
\]

As suggested by Duffie and Singleton (1999) [22], intensity-based model can also account for liquidity effects by introducing a stochastic process that is interpreted as the carrying cost of non-liquid defaultable securities. This process then appears alongside the default intensity in the spread between the “pure” –i.e. default and liquidity-adjusted– short rate and the short rate associated with a defaultable bond. The affine term-structure literature is relatively silent on the interpretation or the microfoundations of the illiquidity intensity. In a theoretical paper analyzing interactions between credit and liquidity risks, He and Xiong (2011) [33] show that such an illiquidity intensity may reflect the probability of occurrence of a liquidity shock; upon the arrival of this shock, the bond investor has to exit by selling his bond at a fractional cost (i.e. the selling price is equal to a fraction of the price that would have prevailed in the absence of the liquidity shock); the fractional cost is the analogous to the fractional loss \( 1 - \zeta \) in the default case (see also Ericsson and Renault, 2006 [23] for a similar interpretation). Accordingly, let us introduce an “illiquidity intensity” between \( t \)
and \( t + 1 \), denoted with \( \lambda_{n,t+1}^L \). If \( \lambda_{n,t+1} \) and \( \lambda_{n,t+1}^L \) are specified in an affine way,

\[
\begin{align*}
\lambda_{n,t+1} &= \alpha_n z_{t+1} + \beta_n y_{t+1} + \gamma_n x_{n,t+1} \\
\lambda_{n,t+1}^L &= \alpha_n^L z_{t+1} + \beta_n^L y_{t+1} + \gamma_n^L x_{n,t+1},
\end{align*}
\]

we could price not only riskless bonds \( B_n(t,h) \) and defaultable bonds \( B_n^D(t,h) \) as above, but also bonds facing liquidity risk \( B_n^L(t,h) \) and bonds facing both default and liquidity risk \( B_n^{DL}(t,h) \). We would have:

\[
\begin{align*}
B(t,h) &= E_t^Q \left[ \exp \left( -r_{t+1} - \ldots - r_{t+h} \right) \right] \\
B_n^D(t,h) &= E_t^Q \left[ \exp \left( -r_{t+1} - \lambda_{n,t+1} - \ldots - r_{t+h} - \lambda_{n,t+h} \right) \right] \\
B_n^L(t,h) &= E_t^Q \left[ \exp \left( -r_{t+1} - \lambda_n^L - \ldots - r_{t+h} - \lambda_n^L_{n,t+h} \right) \right] \\
B_n^{DL}(t,h) &= E_t^Q \left[ \exp \left( -r_{t+1} - \lambda_{n,t+1}^L - \ldots - r_{t+h} - \lambda_{n,t+h}^L - \lambda_n^L_{n,t+h} \right) \right].
\end{align*}
\]

In the context of a Car(1) risk-neutral dynamics of \((z_t, y_t, x_{n,t})\), these prices are exponential linear in \((z_t, y_t, x_{n,t})\) and the corresponding yields are linear in \((z_t, y_t, x_{n,t})\).

7. Model extensions

7.1. Multi-lag dynamics for \( y_t \) and \( x_{n,t} \) processes

The model can easily be extended to allow for \( y_t \) and \( x_{n,t} \) dynamics that include several lags.

In particular, when observed data are used in the estimation process—the \( y_{1,t} \) and \( x_{1,n,t} \) defined in Section 5—, preliminary analysis of the data could point to the need of taking different lags into account to model the historical dynamics of these variables. The flexibility in the choice of the lag structure constitutes an advantage of working in discrete-time over most continuous-time models (see, e.g., Monfort and Pecoraro, 2007 [45] or Gourieroux, Monfort and Polimenis, 2006 [30]).

Equations (2) and (3) imply that the multivariate factors \( y_t \) and \( x_t \) follow auto-regressive process of order one. However, to the extent that a VAR\((p)\) amounts to a VAR\((1)\) once the last \( p \) lags of the endogenous variable are stacked in the same vector, the pricing techniques of the bonds—namely equation (16)—are not affected if \( y_t \) and \( x_t \) follow VAR\((p)\). However, in order to make the estimation strategy presented in Section 5 still effective—in particular
regarding inversion techniques—, the unobserved vector variables \( y_{2,t} \) and \( x_{2,n,t} \) should not enter equations (2) and (3) with lags larger than one. To the extent that this restriction only applies to the unobserved factors—for which insights on the appropriate distributions are a priori not readily available—such a constraint is not really restrictive.

7.2. Interpretation of a regime as the default state of an entity

In this subsection, we consider the specific case where one the Markov chain included in \( z_t \) corresponds to the default state of a given entity.\(^{18}\) The specificity of that situation lies in the fact that the default of this entity then enters the s.d.f.. Therefore, we leave the framework described in Subsection 2.1 where all defaultable entities were small enough to have no impact at the macroeconomic level. As a consequence, the “zero” entity may represent a whole industry or a very big institution. This could be extended to a few major entities but one has to bear in mind that increasing their number results in an exponential growth in the dimension of \( z_t \).

The fact that this default enters the s.d.f. results in new components in bond prices. As pointed out by Yu (2002) [53] and Jarrow, Lando and Yu (2005) [35], such components arise only when the default-event risk is not diversifiable.

As mentioned in the introduction, this interpretation is also linked with previous studies attempting to introduce contagion effects in affine term-structure models. Indeed, the default of entity zero may lead to a simultaneous increase in the default intensities of any other debtor (through the regime variable \( z_t \) that may enter all default intensities).

For sake of simplicity, let us assume that such a crisis variable is the only regime captured by \( z_t \), which can be observable or not. In this case, assuming that the state \( e_2 = (0, 1)' \) is the absorbing crisis state, we have:

\[
\pi (e_2 \mid e_2, y_{t-1}) = 1 \\
\pi (e_1 \mid e_2, y_{t-1}) = 0.
\]

Moreover, we could specify:

\[
\pi (e_1 \mid e_1, y_{t-1}) = \exp (-\lambda_{0,t-1}),
\]

with \( \lambda_{0,t-1} = \alpha_0 + \beta_0 y_{t-1} \). In this case, \( \lambda_{0,t-1} \) can be interpreted as a systemic-risk intensity.
Conditions (10) \( \{ \pi_j(e_i,t-1) \exp[\delta_j(e_i,t-1)] = \pi_{ij}^* \} \) imply the followings:

- \( \pi_{21}^* = 0, \pi_{22}^* = 1, \delta_1(e_2,t-1) \) is undefined, \( \delta_2(e_2,t-1) = 0 \) and, therefore, \( \delta'(e_2,t-1)z_t = 0 \).

- \( \exp[\delta_1(e_1,t-1)] = \pi_{11}^* \exp(\lambda_{0,t-1}) \) or \( \delta_1(e_1,t-1) = \log \pi_{11}^* + \alpha_0 + \beta'_{0,yt-1} \).

- \( \exp[\delta_2(e_1,t-1)] = (1 - \pi_{11}^*) [1 - \exp(-\lambda_{0,t-1})]^{-1} \), or \( \delta_2(e_1,t-1) = \log(1 - \pi_{11}^*) - \log[1 - \exp(-\alpha_0 - \beta'_{0,yt-1})] \).

Denoting \( \pi_{11}^* = \exp(-\lambda_{0}^*) \), \( \lambda_{0}^* \) being the systemic-risk intensity in the risk-neutral world, we get:

\[
\begin{align*}
\delta_1(e_1,t-1) &= \lambda_{0,t-1} - \lambda_{0}^* \\
\delta_2(e_1,t-1) &= \log[1 - \exp(-\lambda_{0}^*)] - \log[1 - \exp(-\lambda_{0,t-1})] \\
&\approx \log(\lambda_{0}^*) - \log(\lambda_{0,t-1}) \text{ if } \lambda_{0}^*, \lambda_{0,t-1} \text{ are small}.
\end{align*}
\]

In particular, the risk-neutral intensity \( \lambda_{0}^* \) and the historical intensity \( \lambda_{0,t-1} \) are different functions, contrary to what happened in the previous sections. Both the riskless yields:

\[
R(t,h) = \frac{1}{h} (a'_h z_t + b'_h y_t)
\]

and the defaultable yields:

\[
R^D_n(t,h) = \frac{1}{h} (c'_n,z_t + f'_n,y_t + g'_{n,t}x_{n,t})
\]

will be different functions of \( y_t \) (and of \( x_{n,t} \) for \( R^D_n(t,h) \)) before and after the systemic crisis. The term structure of the impact of the systemic crisis will be:

\[
\begin{align*}
\begin{cases}
\quad a_{2,h} - a_{1,h} & \text{for the riskless yield of residual maturity } h, \\
\quad c_{2,n,h} - c_{1,n,h} & \text{for the defaultable yield of residual maturity } h, \text{ for the } n^{th} \text{entity.}
\end{cases}
\end{align*}
\]

### 7.3. A sector-contagion model

#### 7.3.1. General approach

In this subsection, we propose another specific use of the regimes that makes it possible to model sector-contagion phenomena. As explained in the introduction, our assumptions
prevent us from making the default intensity of any entity depend on the default event of other entities. In other words, the baseline framework does not allow us to account for contagion at the debtor level (except in the specific case presented in 7.2). Nevertheless, as shown here, this can be done at a sector level, the sectors representing for instance different industries or different geographical areas.

Specifically, in this model, each debtor belongs to one of the sectors. At each period, a sector is either "infected" or not infected. When a sector is infected, the default intensities of its constituent entities tend to be higher. Let us denote by $S_{i,t}$ the state the $i^{th}$ sector at time $t$: $S_{i,t}$ is equal to $[1,0]'$ if the $i^{th}$ sector is infected at time $t$, and is equal to $[0,1]'$ otherwise. If we have $N_S$ sectors, then we have to consider $2^{N_S}$ regimes, the regime variable $z_t$ being given by:

$$z_t = S_{1,t} \otimes S_{2,t} \otimes \ldots \otimes S_{N_S,t}$$

where $\otimes$ denotes the Kronecker product. In such a model, one can make the default intensity of any firm depend on the state of the sectors (and, in particular, on the state of its own sector). Further, the sector-contagion phenomena can be obtained through the specifications of the regime-transition matrix. Indeed, this matrix contains the probabilities that any sector gets infected (or cured) given the states of the other sectors.

### 7.3.2. Numerical example

In this example, we make use of processes $y_{r,t}$ and $z_t$ whose dynamics are defined in Table 2. We consider three homogeneous sectors. The probability that a sector gets cured/infected at time $t$ depends on the number of infected sectors at the previous period. In that case, the regime-transition matrix is defined by a set of probabilities like the one reported in Table 2 (Panel B). In our example, the probability of getting infected is far higher when at least one sector is already infected than when none of them is infected.

[Insert Table 2 about here]

The default intensities of sector-\(i\) firms are given by:

$$\lambda_{i,t} = 0.01 + 0.02 \times I_{\{S_{1,t}=1\}} + 0.02 \times I_{\{S_{2,t}=1\}} \times I_{\{S_{3,t}=1\}} + 0.002 y_{r,t},$$

which implies that the default intensity of a Sector-\(i\) entity increases by two percentage points when Sector \(i\) gets infected and increases by an additional two percentage points if
all sectors become infected simultaneously.

[Insert Figure 5 about here]

Let us now consider a portfolio of 600 debtors, with 200 debtors in each sector. Figure 2 shows a simulation of the timing of defaults for this portfolio. Each panel corresponds to one of the three sectors. At one point, Sector 1 gets infected (see the grey area in the first panel of Figure 2). While the default intensities of Sector-2 and Sector-3 firms are not contemporaneously impacted by the infection of the first sector, 5-year default probabilities of Sector-2 and Sector-3 firms shift upwards. This is a consequence of the fact that once Sector 1 is infected, the probability that Sector 2 and Sector 3 get infected over the next periods is higher. A few periods later, Sector 3 and then Sector 2 get infected.

7.4. Modeling credit-rating transitions

In their seminal study of credit spread, Jarrow, Lando and Turnbull (1997) [34] model rating transitions as a time-homogenous Markov chain. That is, in their model, whether a firm’s rating will change in the next period depends on its current rating only and the probability of changing from one rating to the other remains the same over time. Different studies suggest however that –per-period– transition probabilities are time-varying and that simple Markov processes are not appropriate to model credit migrations (see e.g. Lucas and Lonski, 1992 [43] or Feng, Gourieroux and Jasiak, 2008 [25] or Bangia et al., 2002 [4]).

In the present subsection, we show how our framework can be adapted in order to account explicitly for rating migration. Building on Lando’s (1998) [38] approach (see also Feldhütter and Lando, 2008 [24]), the structure accommodates a time-varying rating-migration matrix while allowing different ratings to respond in a correlated yet different fashion to the same change in the general economic conditions. The time variability of the rating-migration probabilities results from Gaussian shocks as well as from regime shifts. Note that the model that we propose here is very general and may be suited to address various features of empirical evidences regarding credit-rating transitions. In particular, this framework is such that the marginal dynamics of the credit ratings (once the regime variable and the factors have been integrated out) depends on the whole history of the past ratings and therefore is not Markovian.
7.4.1. Adaptation of the framework

While most of the previous framework is still valid, some changes regard the modeling of the default intensity. Specifically, the historical dynamics of \((z_t, y_t, x_{n,t})\), as well as the s.d.f. specifications are still given by equations (1), (2), (3) and (7). However, in this adapted framework, each firm \(n\) is also characterized by a credit-rating process, denoted by \(\tau_{n,t}\). For any firm \(n\) and period \(t\), \(\tau_{n,t}\) can take one of \(K\) values: the first \(K - 1\) values correspond to credit ratings and the \(K^{th}\) corresponds to the default state. For instance, rating 1 can be the highest (Aaa in Moody’s rankings) and \(K-1\) can be the lowest (C in Moody’s rankings). In addition, we have, \(d_{n,t} = \mathbb{I}(\tau_{n,t} = K)\). Like the \(d_{n,t}\)’s, the \(\tau_{n,t}\)’s, \(n = 1, \ldots, N\), are independent conditionally to \((z_t, y_t, x_t, w_{t-1})\). In addition, we assume that the rating transition probabilities, for firm \(n\) and from period \(t - 1\) to period \(t\), is a function of \((z_t, y_t, x_{n,t})\). Accordingly, this transition matrix is denoted with \(\Pi(z_t, y_t, x_{n,t})\) and we have:

\[
P(\tau_{n,t} = j \mid \tau_{n,t-1} = i, z_t, y_t, x_{n,t}) = \Pi_{i,j}(z_t, y_t, x_{n,t}),
\]

where \(\Pi_{i,j}(z_t, y_t, x_{n,t})\), the \((i, j)\) entry of the transition matrix \(\Pi(z_t, y_t, x_{n,t})\), represents the actual probability of going from state \(i\) to state \(j\) in one time step. Each of these entries must be in \([0, 1]\) and for each line, the sum of the entries must sum to one. In other words, \(\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T\) is an eigenvector of \(\Pi(z_t, y_t, x_{n,t})\) associated with the eigenvalue 1. In addition, the default state being absorbing, the bottom row of \(\Pi(z_t, y_t, x_{n,t})\) is equal to \(\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}\). Importantly, the entries of \(\Pi\) are the same function of \((z_t, y_t, x_{n,t})\) under both measures (as the hazard rates in the baseline model).

In this context, a defaultable zero-coupon bond providing one money unit at \(t + h\) if entity \(n\) is still alive in \(t + h\) and zero otherwise has a price, in period \(t\), that is given by (assuming that entity \(n\) has not defaulted before \(t\)):

\[
B_n^D(t, h) = E_t^Q \left[ \exp \left( -r_{t+1} - \cdots - r_{t+h} \right) \mathbb{I}_{\{\tau_{n,t+h} < K\}} \right]. \tag{20}
\]

In order to keep a quasi-explicit formula for defaultable zero-coupon bonds, we assume that \(\Pi(z_t, y_t, x_{n,t})\) admits the diagonal representation:

\[
\Pi(z_t, y_t, x_{n,t}) = V \Psi(z_t, y_t, x_{n,t}) V^{-1},
\]
where the columns of \( V \) are the eigenvectors of \( \Pi(z_t, y_t, x_{n,t}) \) and constitute a basis in \( \mathbb{R}^K \) and \( \Psi(z_t, y_t, x_{n,t}) \) is a diagonal matrix of real eigenvalues that are positive and smaller than one. Given that 1 is an eigenvalue of \( \Pi(z_t, y_t, x_{n,t}) \), \( \Psi(z_t, y_t, x_{n,t}) \) can be written in the following manner:

\[
\Psi(z_t, y_t, x_{n,t}) = 
\begin{bmatrix}
\exp[-\psi_1(w_t)] & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \exp[-\psi_{K-1}(w_t)] & 0 \\
0 & \cdots & 0 & 1
\end{bmatrix},
\]

with, for any \( i < K \), \( \psi_i(w_t) \geq 0 \). Then, it is easily seen that, conditionally on \( (z_{t+h}, y_{t+h}, x_{n,t+h}, \tau_{n,t} = i) \) the probability of defaulting before \( t + h \) corresponds to the entry \((i, K)\) of the matrix that is given by:

\[
V.\Psi(z_{t+1}, y_{t+1}, x_{n,t+1}) \ldots \Psi(z_{t+h}, y_{t+h}, x_{n,t+h}).V^{-1}.
\]

This probability is therefore:

\[
P(\tau_{n,t+h} = K | z_{t+h}, y_{t+h}, x_{n,t+h}, \tau_{n,t} = i) = \sum_{j=1}^{K} V_{i,j} V_{j,K}^{-1} \exp \left[ -\sum_{p=1}^{h} \psi_{j}(w_{t+p}) \right],
\]

where \( V_{i,j} \) and \( V_{i,j}^{-1} \) are the entries \((i, j)\) of \( V \) and \( V^{-1} \). Since \( V_{i,K} V_{K,K}^{-1} = 1 \) (see Appendix D) using \( \psi_{K} \equiv 0 \), we get:

\[
P(\tau_{n,t+h} < K | z_{t+h}, y_{t+h}, x_{n,t+h}, \tau_{n,t} = i) = -\sum_{j=1}^{K-1} V_{i,j} V_{j,K}^{-1} \exp \left[ -\sum_{p=1}^{h} \psi_{j}(w_{t+p}) \right]. \quad (21)
\]

If the \( \psi_{j}'s \) are some linear combinations of \((z_t, y_t, x_{n,t})\), equations (20) and (21) imply that the price of a bond is a sum of \( K - 1 \) multi-horizon Laplace tranforms. As a consequence, the bond prices can be obtained using the algorithm presented in Lemma 2. However, it should be noted that in this context, the yields are no longer affine in the factors, which implies in particular that the Kalman filter has to be adapted so as to accomodate the nonlinearity of the state-space measurement equations. In such a context, Feldhütter and Lando (2008) \[24\] use the extended Kalman filter. As an alternative, the unscented Kalman filter can be implemented.
7.4.2. Numerical example

Let us consider again the processes $r_t$ and $z_t$ whose dynamics are specified in Table 3. In the present model, the credit-migration matrices are of the form:

$$
\Pi(z_t, y_t, x_{n,t}) = V. 
\begin{bmatrix}
\exp[-\alpha_1 z_t - \beta_1 y_{r,t}] & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \exp[-\alpha_{K-1} z_t - \beta_{K-1} y_{r,t}] & 0 \\
0 & \cdots & 0 & 1
\end{bmatrix}.V^{-1}
$$

In order to get plausible matrices, the first-regime calibration—that involves the $\alpha_{i,1}$’s—is based on the one-year-average rating-migration matrix for European corporates provided by Moody’s (Moody’s, 2010 [49]). This matrix is reported in Table 4. The spectral decomposition of this matrix provides us with the matrix of eigenvectors $V$. The eigenvalues are real and comprised between 0 and 1. Accordingly, they are of the form $\exp(-\alpha_{i,1})$. The $\alpha_{i,1}$ are reported in Table 5. The definition of the second regime requires a second set of $\alpha_i$’s, denoted by $\{\alpha_{i,2}\}_{i=1 \ldots K-1}$. We calibrate the latter in order to have 5-year default probabilities that are higher than those obtained with the first-regime transition matrix (see Table 5). Finally, the $\beta_i$’s are given by $(\alpha_{i,1} - \alpha_{i,2})/5$.

[Insert Table 3 about here]

[Insert Table 4 about here]

[Insert Table 5 about here]

Figure 5 displays yield curves for selected ratings under both regimes (for $y_{r,t} = 0$). Figure 6 presents some simulation results. The upper panel shows the time fluctuations of downgrade probabilities for two different ratings. The lower panel displays yield spreads between 10-year zero-coupon bonds issued by A-rated or Baa-rated firms and 10-year zero-coupon bonds issued by Aaa-rated firms.

[Insert Figure 5 about here]

[Insert Figure 6 about here]
8. Conclusion

In this paper, we have proposed an econometric framework aimed at jointly modeling yield curves associated with different defaultable issuers. Default intensities and yields are affine functions of a multivariate process which is Compound autoregressive (Car) in the risk-neutral world and thus provides us with quasi-explicit (recursive) formulas for both risk-free and defaultable bond prices.

The risk factors follow discrete-time conditionally Gaussian processes, with drifts and variance-covariance matrices that are subject to regime shifts described by a Markov chain with (historical) non-homogenous transition probabilities. The regime-switching feature is relevant for credit models in several respects. First, it makes it possible to capture non-linear behaviors of yields and spreads, which is consistent with empirical evidence. Second, it is appropriate to capture default clusters. Third, it offers some ways of dealing with specific forms of contagion. In this respect, we show how the framework can be used to capture sector-contagion phenomena. An other extension accommodates credit-rating migrations. While flexible, the model remains tractable and amenable to empirical estimation. To that end, a sequential estimation strategy is proposed in the paper.

References


**Notes**

1 Regarding the latter point, see Duffee (2010) [20]. The fact that our framework is defined in discrete time makes it easier (compared with continuous-time models) to properly specify the dynamics of the observable risk factors under the historical probability measure (see e.g. Duffie and Singleton, 1999 [22]) or Gourieroux, Monfort and Polimenis, 2006 [30]).

2 Several of the main credit models currently being used in the industry draw on the credit-migration approach. For presentation, comparison and evaluation of these models, see e.g. Gordy (2000) [29].

3 Other examples of term-structure models allowing for time-varying rating-migration probabilities include Bielecki and Rutkowski (2000) [9] and Wei (2003) [52].
Indeed, this implies that any function of the regimes taking the value \( f_j \) in the \( j^{th} \) regime, say, is the linear function of \( z_i \): \( f' z_i \) with \( f' = (f_1 \ldots f_J) \).

5. These specifications allow for various and rich dynamics of the risk factors \( y_t \) such as, notably, threshold auto-regressive dynamics (TAR) or self-exciting TAR (SETAR).

6. Formally, this corresponds to \( \mu(z_{t-1}, z_{t-1}) = \mu(z_{t-1}) \) and \( \Omega(z_{t-1}, z_{t-1}) = \Omega(z_{t-1}) \).

7. Indeed, the model of Bansal and Zhou (2002) [5] does not admit a closed-form exponential affine solution (they proceed by linearizing the discrete-time Euler equations and by solving the resulting linear relations for prices).

8. Recall that a random process \( \Lambda_t \) is Car(1) if its conditional Laplace transform (given information available up to date \( t - 1 \)) is exponential affine in \( \Lambda_{t-1} \).

9. A process \( X_t \) does not cause \( Y_t \) in Granger’s sense if and only if, for any \( t \), \( Y_t \) is independent of \( (X_{t-1}, \ldots, X_1) \) conditionally on \( (Y_{t-1}, \ldots, Y_1) \). This is equivalent to the non-causality in Sims’ sense \( (X_t \) does not cause the stochastic process \( Y_t \) in Sims’ sense if \( X_t \) is independent from \( (Y_{t+1}, Y_{t+2}, \ldots, Y_T) \) conditionally on \( (Y_1, X_{t-1}, Y_{t-1}, \ldots, X_1, Y_1) \)).

10. In the working paper version of the present paper (Monfort and Renne, 2011[48]), we explain how the basic Kitagawa-Hamilton filter can be adapted in order to deal with partially-hidden Markov chains (when there are both \( z_{1,t} \) and \( z_{2,t} \)).

11. Among others, Feldhütter and Lando (2008) [24] also consider firms that are representative of some credit-rating classes.

12. Such an assumption is for instance made by Pan and Singleton (2008) [50] or Longstaff et al. (2011) [41].

13. Zero-coupon yield curve yields have been obtained by applying bootstrap techniques on the BBB (coupon) yield curve provided by Bloomberg (tickers C009). The risk-free yields are US STRIPS yields extracted from Bloomberg (tickers C079).

14. Looking both at the first and second panel in Figure 2, one can check that the second factor \( y_{2,t} \) is pushed upwards during crisis periods.

15. The influence of liquidity effects on bond pricing has been investigated, amongst others, by Longstaff (2004) [42], Chen, Lesmond and Wei (2007) [12], Covitz and Downing (2007) [14].

16. In both studies, the liquidity factor that is estimated corresponds to the so-called “convenience yield”, that can be seen as a premium that one is willing to pay when holding Treasuries. This premium stems from various features of Treasury securities, such as repo specialness (see Feldhütter and Lando, 2008).

17. Among which: bid-ask spreads, market-depth measures, bond supply, spread between bonds of the same maturity but with different ages or spread between off-the-run and on-the -run Treasuries (see, e.g., Longstaff, 2004[42] or Beber, Brandt and Kavajecz, 2009 [6]).

18. We can deal with several Markov chains by writing vector \( z_t \) as a Kronecker product of several chains.

19. Note however that this does not imply that the distributions of these entries are the same under both measures (since the distributions of \( (z_t, y_t, x_n,t) \) differ under \( \mathbb{Q} \) and \( \mathbb{P} \)).

A. Proofs of Sections 2 and 3

A.1. Proof of Proposition 1

We have

\[
\varphi_{t-1}(u, v) = \begin{align*}
E_{t-1} & \left( \exp \left[ u' z_t + v' y_t \right] \right) \\
& = E_{t-1} \left( \exp \left[ u' z_t + v' \mu(z_t, z_{t-1}) + v' \Phi y_{t-1} + v' \Omega(z_t, z_{t-1}) \varepsilon_t \right] \right) \\
& = E \left( E \left( \exp \left[ u' z_t + v' \mu(z_t, z_{t-1}) + v' \Phi y_{t-1} + v' \Omega(z_t, z_{t-1}) \varepsilon_t \right] \mid w_{t-1}, z_t \right) \mid w_{t-1} \right) \\
& = \exp(v' \Phi y_{t-1}) E \left( \exp \left[ u' z_t + v' \mu(z_t, z_{t-1}) \right] \times \exp \left[ v' \Omega(z_t, z_{t-1}) \varepsilon_t \mid w_{t-1}, z_t \right] \mid w_{t-1} \right) \\
& = \frac{1}{2} v' \Omega(z_t, z_{t-1}) \Omega'(z_t, z_{t-1}) v \mid w_{t-1} \\
& = \exp(v' \Phi y_{t-1} + [l_1, \ldots, l_J] z_{t-1}).
\]

Using the expression given for the \( l_i \)'s leads to the result.
A.2. Proof of Proposition 2

\[ \varphi_{t-1}^Q(u, v) = E_{t-1}^Q \left( \exp \left[ u' z_t + v' y_t \right] \right) \]

\[ = E_{t-1} \left( \exp \left[ \frac{1}{2} \nu'_t \nu_t + \nu'_t \varepsilon_t + \delta_{t-1} z_t + u' z_t + v' y_t \right] \right) \]

\[ = \exp \left( v' \Phi_{yt-1} \right) \times \]

\[ E_{t-1} \left( \exp \left[ \frac{1}{2} \nu'_t \nu_t + \nu'_t \varepsilon_t + \delta_{t-1} z_t + u' z_t + v' \mu_t + v' Q_t \varepsilon_t \right] \right) \]

\[ = \exp \left( v' \Phi_{yt-1} \right) \times \]

\[ E_{t-1} \left( \exp \left[ \frac{1}{2} \nu'_t \nu_t + \frac{1}{2} \left( \nu'_t + v' Q_t \right) \left( \nu'_t + v' Q_t \right) + v' \mu_t + u' z_t + \delta_{t-1} z_t \right] \right) \]

Using the expression given for \( A_{i,t-1}(u, v) \) in 2.2.1 leads to the result.

A.3. P.d.f. under the risk-neutral world (Proof of Lemma 1)

Let us consider a pair \( (X, Y) \) of multivariate random vectors. Let denote with \( f^\mathbb{H}(X, Y) \) and \( f^Q(X, Y) \) their respective joint p.d.f. under the probability measure \( \mathbb{H} \) and \( Q \) and assume that the Radon-Nikodym derivative that relates \( \mathbb{H} \) and \( Q \) depends on \( X \) only and is proportional to \( M(X) \). We have (assuming, for the sake of notational simplicity, that the dominating measures are Lebesgue):

\[ f^Q(X, Y) = \frac{f^\mathbb{H}(X, Y) M(X)}{\int f^\mathbb{H}(X, Y) M(X) dX dY} \]

\[ = \frac{f^\mathbb{H}(X) f^\mathbb{H}(Y | X) M(X)}{\int f^\mathbb{H}(X) f^\mathbb{H}(Y | X) M(X) dX dY} \]

\[ = \frac{f^\mathbb{H}(X) f^\mathbb{H}(Y | X) M(X)}{\int f^\mathbb{H}(X) M(X) \int f^\mathbb{H}(Y | X) dY dX} \]

\[ = \frac{f^\mathbb{H}(X) M(X)}{\int f^\mathbb{H}(X) M(X) dX} \]

\[ = f^Q(X) f^\mathbb{H}(Y | X). \]

A.4. The risk-neutral Laplace transform of \( (z_t, y_t, x_{n,t}) \)

In this appendix, we compute \( E_{t-1}^Q \left( \exp \left[ u' z_t + v' y_t + w' x_{n,t} \right] \right) \) and show that it is exponential affine in \( (z_{t-1}, y_{t-1}, x_{n,t-1}) \), that is, we show that \( (z_t, y_t, x_{n,t}) \) is Car(1) (see Darolles, Gouriou and Jasiak, 2006 [16]).

\[ E_{t-1}^Q \left( \exp \left[ u' z_t + v' y_t + w' x_{n,t} \right] \right) = E_{t-1}^Q \left( \exp \left[ u' z_t + v' y_t + w' q_{1n}(z_t, z_{t-1}) + Q_{2n} y_t + Q_{3n} y_{t-1} + Q_{4n} x_{n,t-1} + Q_{5n}(z_t, z_{t-1}) \eta_{n,t} \right] \right) \]

\[ = \exp \left( w' q_{3n} y_{t-1} + w' q_{4n} x_{n,t-1} \right) \times \]

\[ E_{t-1}^Q \left( \exp \left[ u' z_t + (v' + w' Q_{2n}) y_t + w' q_{1n}(z_t, z_{t-1}) + w' Q_{5n}(z_t, z_{t-1}) \eta_{n,t} \right] \right) \]

\[ = \exp \left( w' Q_{3n} y_t + w' Q_{4n} x_{n,t-1} \right) \times \]

\[ E_{t-1}^Q \left( \exp \left[ u' z_t + w' q_{1n}(z_t, z_{t-1}) + w' Q_{5n}(z_t, z_{t-1}) \eta_{n,t} + (v' + w' Q_{2n}) \left( \mu_t + \mu_{t-1} + (\Phi + \Phi^*) y_t + \Omega \varepsilon_t^* \right) \right] \right) \]

\[ = \exp \left[ \left( v' + w' Q_{2n} \right) \left( \Phi + \Phi^* \right) + w' Q_{3n} y_t + w' Q_{4n} x_{n,t-1} + \left( \hat{A}_1(u, v, w) \ldots \hat{A}_j(u, v, w) \right) z_t \right] \]
Using the expression given for $\tilde{A}((u, v, w)$ in Proposition 3 leads to the result.

A.5. Multi-horizon Laplace transform of a Car(1) process

Let us consider a multivariate Car(1) process $Z_t$ and its conditional Laplace transform given by $\exp[a'(s)Z_t + b(s)]$. Let us further denote by $L_{t,h}(\omega)$ its multi-horizon Laplace transform given by:

$$L_{t,h}(\omega) = E_t \left[ \exp \left( \omega_{h - h+1} Z_{t+1} + \ldots + \omega_H Z_{t+h} \right) \right], \quad t = 1, \ldots, T, \ h = 1, \ldots, H,$$

where $\omega = (\omega_1, \ldots, \omega_H)$ is a given sequence of vectors. We have, for any $t$,

$$L_{t,h}(\omega) = \exp(A'_h Z_t + B_h), \ h = 1, \ldots, H,$$

where the sequences $A_h, B_h, h = 1, \ldots, H$ are obtained recursively by:

$$A_h = a(\omega_{h-1} + A_{h-1}),$$

$$B_h = b(\omega_{h-1} + A_{h-1}) + B_{h-1},$$

with the initial conditions $A_0 = 0$ and $B_0 = 0$.

**Proof.** The formula is true for $h = 1$ since:

$$L_{t,1}(\omega) = E_t (a'_1 Z_{t+1}) = \exp[a'(\omega_H)Z_t + b(\omega_H)]$$

and therefore $A_1 = a(\omega_H)$ and $B_1 = b(\omega_H)$.

If it is true for $h - 1$, we get:

$$L_{t,h}(\omega) = E_t \left[ \exp \left( a'(\omega_H Z_{t+1}) E_{t+1} \left( \exp \left( \omega_{h-1} Z_{t+2} + \ldots + \omega_H Z_{t+h} \right) \right) \right) \right]$$

$$= E_t \left[ \exp \left( a'(\omega_{h-1} Z_{t+1}) L_{t+1,h-1}(\omega) \right) \right]$$

$$= \exp \left[ a(\omega_{h-1} + A_{h-1}) Z_t + b(\omega_{h-1} + A_{h-1}) + B_{h-1} \right]$$

and the result follows.

A.6. Proof of Proposition 6 (Pricing of defaultable bonds with nonzero recovery rates)

Section 3 gives quasi-explicit formulas for the pricing of bonds with zero recovery rates. In the current appendix, we present conditions under which one can derive formulas for nonzero-recovery-rate bond pricing. If a debtor defaults between $t - 1$ and $t$ (with $t < T$, where $T$ denotes the contractual maturity of a bond issued by this debtor), recovery is assumed to take place at time $t$. In addition, we assume that the recovery payoff – i.e. one minus the loss-given-default – depends on $(z_t, y_t, x_t)$. This recovery payoff is denoted by $R^{DR}_{n-1} = R(z_t, y_t, x_t, T - t)$.

Let us consider the price $B^{DR}_n(T - 1, 1)$, in period $T - 1$, of a one-period nonzero-recovery-rate bond issued by a given debtor (before $T - 1$). We distinguish three cases:

1. The debtor had defaulted before $T - 2$, then: $B^{DR}_n(T - 1, 1) = 0$.
2. The debtor defaulted between $T - 2$ and $T - 1$, then: $B^{DR}_n(T - 1, 1) = R^1_{n,T-1}$.
3. The debtor has not defaulted before $T - 1$, then:

$$B^{DR}_n(T - 1, 1) = \exp(-r_T)E^Q \left[ \mathbb{1}_{\{d_{n,T-1} = 0\}} R^0_{n,T-1} | \mathbb{1}_{\{d_{n,T-1} = 1\}} R^0_{n,T} | \mathbb{1}_{\{d_{n,T-1} = 0\}} R^0_{n,T} | \mathbb{1}_{\{d_{n,T-1} = 1\}} R^0_{n,T} | \mathbb{1}_{\{d_{n,T-1} = 0\}} R^0_{n,T} | \mathbb{1}_{\{d_{n,T-1} = 1\}} R^0_{n,T} \right]$$

$$= \exp(-r_T)E^Q \left[ \mathbb{1}_{\{d_{n,T-1} = 0\}} R^0_{n,T-1} + \mathbb{1}_{\{d_{n,T-1} = 1\}} R^0_{n,T - 1} \mathbb{1}_{\{d_{n,T-1} = 0\}} R^0_{n,T} \mathbb{1}_{\{d_{n,T-1} = 1\}} R^0_{n,T} \mathbb{1}_{\{d_{n,T-1} = 0\}} R^0_{n,T} \mathbb{1}_{\{d_{n,T-1} = 1\}} R^0_{n,T} \right]$$

$$= \exp(-r_T)E^Q \left[ \exp(-\lambda_{n,T}) \mathbb{1}_{\{d_{n,T-1} = 0\}} \left( 1 - \exp(-\lambda_{n,T}) \right) R^0_{n,T-1} \mathbb{1}_{\{d_{n,T-1} = 1\}} R^0_{n,T} \mathbb{1}_{\{d_{n,T-1} = 0\}} R^0_{n,T} \mathbb{1}_{\{d_{n,T-1} = 1\}} R^0_{n,T} \right]$$
and, defining the random variable \( \hat{\lambda}_{n,T}^0 \) by \( \exp(-\hat{\lambda}_{n,T}^0) = \exp(-\lambda_{n,T}) + (1 - \exp(-\lambda_{n,T})) R_{n,T}^0 \), we have (still in case 3):

\[
B_{n}^{DR}(T-1,1) = E^Q \left[ \exp(-r_T - \hat{\lambda}_{n,T}^0) \mid \tilde{z}_{T-1}, \tilde{y}_{T-1}, \tilde{z}_{n,T-1} \right].
\]

Further, let us consider the price of the same bond in period \( T - 2 \). Assuming that there was no default before \( T - 2 \):

\[
B_{n}^{DR}(T-2,2) = \exp(-r_{T-1}) \times E^Q \left[ \mathbb{I}_{(d_{n,T-1}=0)} \left( E^Q \left[ \exp(-r_T - \hat{\lambda}_{n,T}^0) \mid \tilde{z}_{T-2}, \tilde{y}_{T-2}, \tilde{z}_{n,T-2} \right] \right) + \mathbb{I}_{(d_{n,T-1}=1)} R_{n,T-1}^1 \tilde{z}_{T-2}, \tilde{y}_{T-2}, \tilde{z}_{n,T-2}, d_{n,T-2} = 0 \right].
\]

(22)

Let us introduce a random variable \( \zeta_{n,T-1}^1 \) that is defined through:

\[
R_{n,T-1}^1 = \zeta_{n,T-1}^1 E^Q \left[ \exp(-r_T - \hat{\lambda}_{n,T}^0) \mid \tilde{z}_{T-1}, \tilde{y}_{T-1}, \tilde{z}_{n,T-1} \right].
\]

With this notation, Equation (22) reads:

\[
B_{n}^{DR}(T-2,2) = E^Q \left[ \exp(-r_{T-1} - r_T - \hat{\lambda}_{n,T}^0) \left( \mathbb{I}_{(d_{n,T-1}=0)} + \zeta_{n,T-1}^1 \mathbb{I}_{(d_{n,T-1}=1)} \right) \right] = E^Q \left[ \exp(-r_{T-1} - r_T - \hat{\lambda}_{n,T}^0) \left( \exp(-\lambda_{n,T-1}) + \zeta_{n,T-1}^1 (1 - \exp(-\lambda_{n,T-1})) \right) \tilde{z}_{T-2}, \tilde{y}_{T-2}, \tilde{z}_{n,T-2} \right].
\]

Then, defining the random variable \( \hat{\lambda}_{n,T-1}^1 \) by:

\[
\exp(-\hat{\lambda}_{n,T-1}^1) = \exp(-\lambda_{n,T-1}) + (1 - \exp(-\lambda_{n,T-1})) \zeta_{n,T-1}^1,
\]

we get (conditionally on \( d_{n,T-2} = 0 \)):

\[
B_{n}^{DR}(T-2,2) = E^Q \left[ \exp(-r_{T-1} - r_T - \hat{\lambda}_{n,T}^0 - \hat{\lambda}_{n,T-1}^1) \tilde{z}_{T-2}, \tilde{y}_{T-2}, \tilde{z}_{n,T-2} \right].
\]

Applying this methodology recursively, it is easily seen that the price of a non-zero-recovery-rate defaultable bond of maturity \( h \) is given by (assuming no default before \( t \), i.e. conditionally on \( d_{n,t} = 0 \)):

\[
B_{n}^{DR}(t,h) = E^Q \left[ \exp(-r_{t+h} - \ldots - r_{t+1} - \hat{\lambda}_{n,t+h}^0 - \hat{\lambda}_{n,t+h} - \ldots - \hat{\lambda}_{n,t+h}^1) \tilde{z}_{t+h}, \tilde{y}_{t+h}, \tilde{z}_{n,t+h} \right]
\]

(23)

where the \( \hat{\lambda}_{n,t+i}^h \)'s are defined recursively in \( i \) by the backward equation:

\[
\exp(-\hat{\lambda}_{n,t+i}^h) = \exp(-\lambda_{n,t+i}) + (1 - \exp(-\lambda_{n,t+i})) \zeta_{n,t+i}^h
\]

where

\[
\zeta_{n,t+i}^h = \begin{cases} 
E^Q \left[ \exp(-r_{t+h} - \ldots - r_{t+i+1} - \hat{\lambda}_{n,t+h}^0 - \hat{\lambda}_{n,t+h} - \ldots - \hat{\lambda}_{n,t+h}^1) \tilde{z}_{t+i}, \tilde{y}_{t+i}, \tilde{z}_{n,t+i} \right] & \text{if } i < h \\
R_{t+h,0}^h \hat{\lambda}_{n,t+i}^h & \text{if } i = h.
\end{cases}
\]

Looking at Equation (23), it is tempting to interpret the \( \hat{\lambda}_{n,t+i}^h \)'s as “recovery-adjusted” hazard rates for debtor \( n \). However, the dependency of these intensities on the maturity \( h \) of the considered bond is problematic. Indeed, by analogy with the standard default intensities \( \lambda_{n,t} \), one would like to have, at each period, only one adjusted intensity by debtor (and not a collection of adjusted intensities associated with the different bonds that have been issued by the considered debtor). To that end, Duffie and Singleton (1999) [22] propose a “recovery of market value” assumption. Under this assumption, the variable \( R_{n,s}^m \)—that is, the recovery at time \( s \) of a bond with residual maturity
m, in the event of default between s – 1 and s – is equal to the product of a factor common to all maturities with the survival-contingent market value at time s. In the same spirit, let us assume that the \( \zeta_{m,n,s} \)'s do no longer depend on m. Then, the \( \lambda_{n,s}^m \) do not depend on the maturity any longer and are simply given by:

\[
\exp(-\lambda_{n,s}^m) = \exp(-\lambda_{n,s}) + (1 - \exp(-\lambda_{n,s})) \zeta_{n,s}.
\]

Actually, this formulation is more general than the one considered by Duffie and Singleton (1999) when they expose a discrete-time motivation. Indeed, in the latter case, they assume that \( \zeta_{n,s} \) is known at time s – 1, which is not necessarily the case in the framework described above.

**B. Inversion techniques in the presence of unobserved regimes**

In this appendix, we detail an approach using jointly the Kitagawa-Hamilton filter and the so-called inversion techniques à la Chen and Scott (1993) [13]. Such an approach is aimed at estimating models in which there are both latent factors \( (y_{2,t}) \) and latent regimes \( (z_{2,t}) \) (see Section 5 for notations). Note that the implementation of the following estimation strategy requires that the transition probabilities do not depend on the unobserved vectors \( y_{2,t-1} \). The period of observation is \( \{1, \ldots, T\} \).

**B.1. Decomposition of the joint p.d.f. and estimation strategy**

Let us denote by \( \theta^{zy} \) the vector of parameters defining the historical dynamics of \( (z_t, y_t) \), by \( \theta^x_{n,t} \) the vector of parameters defining the conditional p.d.f. of \( x_{n,t} \) given \( z_t, y_t, z_{n,t-1} \) and by \( \theta^d_{n,t} \) the vector of parameters defining the conditional p.d.f. of \( d_{n,t} \) given \( z_t, y_t, z_{n,t}, y_{n,t} \).

The joint p.d.f. of \( w_T \) is:

\[
f(w_T, \theta) = \prod_{t=1}^{T} f(z_t, y_t | z_{t-1}, y_{t-1}; \theta^{zy}) \\
\times \prod_{n=1}^{N} \prod_{t=1}^{T} f(x_{n,t} | z_t, y_t, z_{n,t-1}; \theta^x_{n,t}) \\
\times \prod_{n=1}^{N} \prod_{t=1}^{T} f(d_{n,t} | z_t, y_t, z_{n,t}, y_{n,t}; \theta^d_{n,t}).
\]

The parameters appearing in \( M_{t-1,t} \) are denoted by \( \theta^* \). The theoretical values of \( R_t \) and \( R_{nt}^D \) given by the model are denoted by \( R_t(\theta^{zy}, \theta^*) \) and \( R_{nt}^D(\theta^{zy}, \theta^*, \theta^x_{n,t}, \theta^d_{n,t}) \) respectively. A sequential strategy of estimation is the following:

1. Estimate \( \theta^{zy} \) and \( \theta^* \) from the observations of \( y_{1,t}, z_{1,t}, R_t, t = 1, \ldots, T \).
2. Estimate the \( \theta^x_{n,t} \)'s and the \( \theta^d_{n,t} \)'s from the observations of \( x_{1n,t} \) and \( R_{nt}^D, t = 1, \ldots, T \), taking as given the values of \( \theta^{zy} \) and \( \theta^* \), and the values of \( y_{2,t} \) and \( z_{2,t} \) being fixed at the approximated values obtained from step 1.

The remaining of the current section details these two steps. The methodology that is proposed builds on the so-called inversion technique developed by Chen and Scott (1993) [13]. This technique is adapted in order to accomodate regime switching.

**B.2. Estimation of the parameters \( (\theta^{zy}, \theta^*) \)**

Using equation (16), we have, with obvious notations:

\[
R_t(\theta^{zy}, \theta^*) = A_{2t} + B_1 y_{1,t} + B_2 y_{2,t}.
\]

If \( m \) is the dimension of \( y_{2,t} \), let us partition \( R_t \) in \( \left( R_{1,t}^{1,t}, R_{2,t}^{2,t} \right) \) where \( R_{2,t} \) is of dimension \( m \). With obvious notations, we get:

\[
R_{2,t}(\theta^{zy}, \theta^*) = A_{22t} + B_{21} y_{1,t} + B_{22} y_{2,t}.
\]
and denoting \((y_{1,t}', R_{2,t}')\) by \(\tilde{y}_t\) we get:

\[
\tilde{y}_t = \begin{pmatrix} I & 0 \\ B_{21} & B_{22} \end{pmatrix} y_t + \begin{pmatrix} 0 \\ A_2 \end{pmatrix} z_t
\]

or

\[
\tilde{y}_t = \tilde{B} y_t + \tilde{A} z_t
\]

and

\[
y_t = \tilde{B}^{-1} \left( \tilde{y}_t - \tilde{A} z_t \right)
\]

and from equation (2) we get:

\[
\tilde{B}^{-1} \left( \tilde{y}_t - \tilde{A} z_t \right) = \mu (z_t, z_{t-1}) + \Phi \left[ \tilde{B}^{-1} \left( \tilde{y}_{t-1} - \tilde{A} z_{t-1} \right) \right] + \Omega (z_t, z_{t-1}) \varepsilon_t
\]

or

\[
\tilde{y}_t = \tilde{A} z_t + \tilde{B} \mu (z_t, z_{t-1}) + \tilde{B} \Phi \left[ \tilde{B}^{-1} \left( \tilde{y}_{t-1} - \tilde{A} z_{t-1} \right) \right] + \tilde{B} \Omega (z_t, z_{t-1}) \varepsilon_t
\]

or

\[
\tilde{y}_t = \tilde{\mu} (z_t, z_{t-1}) + \tilde{\Phi} \tilde{y}_{t-1} + \tilde{\Omega} (z_t, z_{t-1}) \varepsilon_t,
\]

with

\[
\begin{align*}
\tilde{\mu} (z_t, z_{t-1}) &= \tilde{A} z_t + \tilde{B} \mu (z_t, z_{t-1}) - \tilde{B} \Phi \tilde{B}^{-1} \tilde{A} z_{t-1} \\
\tilde{\Phi} &= \tilde{B} \Phi \tilde{B}^{-1} \\
\tilde{\Omega} (z_t, z_{t-1}) &= \tilde{B} \Omega (z_t, z_{t-1}).
\end{align*}
\]

The conditional distribution of \(\tilde{y}_t\) given \(z_t, z_{t-1}\) is similar to that of \(y_t\) given \(z_t, \tilde{y}_{t-1}\), and in particular is Gaussian, the difference being that \(\tilde{y}_t\) is fully observable. Assuming moreover that the \(R_{1,t}\) are observed with Gaussian errors we get, with obvious notations:

\[
R_{1,t} = A_1 z_t + B_{11} y_{1,t} + B_{12} y_{2,t} + \xi_t
\]

\[
= A_1 z_t + B_{11} y_{1,t} + B_{12} B_{22}^{-1} (R_{2t} - A_2 z_t - B_{21} y_{1,t}) + \xi_t,
\]

with \(\xi_t \sim IN \left(0, \sigma^2 I\right)\).

Putting equations (24), (25) and (1) together, we have a dynamic model in which the only latent variables are \(z_{2,t}\) and which can be estimated by the maximum likelihood methods using Hamilton’s approach. At this stage, IC constraints on \((\theta^{\nu}, \theta^*)\) must be taken into account.

**B.3. Estimation of \((\theta^{\nu}_n, \theta^d_n)\)**

From the inversion method of B.2, we can get approximations of the \(y_{2,t}\)’s and smoothing algorithms provide approximations of the \(z_{2,t}\)’s (see Kim, 1994 [37]). The \(z_{2t}\) are replaced by those states presenting the highest smoothed probabilities. Then using equation (16), we get:

\[
R_{1,n}^D = C^n_{11} z_{1,t} + C^n_{21} z_{2,t} + D^n_1 y_{1,t} + D^n_2 y_{2,t} + F^n_{1} x_{1,n,t} + F^n_{2} x_{2,n,t}.
\]

and using equations (2), (3) and (26) and replacing \(y_{2,t}\) and \(z_{2,t}\) by their approximations, we get a system in which the only latent variables are the \(x_{2,n,t}\). Taking \(\theta^{\nu}_n\) and \(\theta^d_n\) as given, the parameters \(\theta_n^\nu\) and \(\theta_n^d\) can be estimated either by an inversion technique or by Kalman filtering, taking into account IC conditions.

Note that in this strategy, the observable variables \(d_{n,t}\)’s have not been used. If the recovery rate was effectively zero, \(\lambda_{n,t}\) would be the default intensity and the conditional p.d.f. of \(d_{n,t}\) given \(\tilde{z}_t, \tilde{y}_t, z_{n,t}, d_{n,t-1}\) would be:

\[
d_{n,t} | d_{n,t-1} + (1 - d_{n,t-1}) \exp \left[-(1 - d_{n,t}) \lambda_{n,t}\right] \times [1 - \exp \left(-\lambda_{n,t}\right)]^{d_{n,t}}.
\]

This p.d.f. could be incorporated in the likelihood function. However, in the more realistic case of non-zero recovery rate, we have seen that (see Subsection ??) the \(\lambda_{n,t}\)’s must be interpreted as risk-neutral “recovery adjusted” default intensities and, therefore, they cannot be used for describing the historical dynamics of the \(d_{n,t}\)’s.
C. Estimation example: U.S. BBB-AAA corporate spreads

C.1. State-space model

The model introduced in 5.3 can be written as a state-space system for the purpose of estimation. Let denote by $s_t$ the $4 \times 1$ vector containing the BBB-Treasury spreads with respective maturities of 1, 2, 3 and 5 years. Using a matrix representation, the measurement equations of the state-space model are given by:

$$s_t = cz_t + fy_t + \varepsilon_{err,t},$$

where the $\varepsilon_{err,t}$ are some i.i.d. pricing (measurement) errors and where the matrices $c$ and $f$, that are respectively of dimension $4 \times 3$ (because there are three regimes) and $4 \times 2$ (because there are two factors $y_{1,t}$), are computed by applying the recursive pricing formulas introduced in Proposition 3. Because both $y_{1,t}$ and $y_{2,t}$ are unobserved, the transition equations read:

$$y_t = \mu z_t + \Phi y_{t-1} + \Omega \varepsilon_t,$$

where $\mu$, $\Phi$ and $\Omega$ are constrained along the lines presented in 5.3. The state-space model is completed by the specification of the matrix of regime-switching probabilities $\{\pi_{i,j}\}$.

C.2. Estimation results

The estimation is conducted by maximizing the log-likelihood (approximated by the filter proposed by Kim, 1994 [37]). Some of the parameters are calibrated. First, the unconditional variance of the first (purely Gaussian) factor $y_{1,t}$ is constrained to be relatively small with comparison to the overall standard deviation of the spreads, so as to make sure that most of the spread fluctuations are to be explained by the second factor $y_{2,t}$. Specifically, the standard deviation of $y_{1,t}$ is set to 10 bp. Alternative estimations have shown that the results are fairly robust to this first choice. Second, the matrix of probabilities of regime shifts is parameterized so as to be consistent with the regimes’ interpretation. The “tranquil” regime is supposed to be persistent and to prevail 50% of the time. By contrast, the crisis regime is supposed to be relatively short-lived (with an average length of 4 weeks) and to prevail only 5% of the time. Formally, these constraints mean that (a) the ergodic distribution of the Markov chain is $[50\%, 45\%, 5\%]$ and that (b) the third diagonal entry of the matrix of transition probabilities (i.e. $p_{33}$) is such that $4 = 1/(1-p_{33})$ (4 weeks = average length of the third regime). The resulting matrix of transition probabilities (under the historical measure) is:

$$
\begin{bmatrix}
0.976 & 0.024 & 0 \\
0.027 & 0.945 & 0.028 \\
0 & 0.250 & 0.750
\end{bmatrix}.
$$

The estimated dynamics of $y_t$ under the historical and the risk-neutral measures are, respectively (standard errors, based on the outer-product estimate of the Fisher information matrix, are reported in parentheses below the point estimates):

$$
y_t = 
\begin{bmatrix}
0.023_{(2111)} & 0.023_{(2111)} & 0.023_{(2111)} \\
0 & 0 & 0.219_{(0.007)}
\end{bmatrix}
z_t + 
\begin{bmatrix}
0.841_{(0.010)} & 0 \\
0 & 0.981_{(0.001)}
\end{bmatrix}y_{t-1} + 
\begin{bmatrix}
0.053_{(0.001)} & 0 \\
0 & 0
\end{bmatrix}\varepsilon_t
$$

$$
y_t^* =
\begin{bmatrix}
-0.0029_{(0.0001)} & -0.0029_{(0.0001)} & -0.0029_{(0.0001)} \\
0 & 0 & 0.0063_{(0.0001)}
\end{bmatrix}z_t + 
\begin{bmatrix}
1_{(1.000)} & 0 \\
0 & 1_{(1.000)}
\end{bmatrix}y_{t-1} + 
\begin{bmatrix}
0.053_{(0.001)} & 0 \\
0 & 0
\end{bmatrix}\varepsilon_t^*.
$$

where $\varepsilon_t$ and $\varepsilon_t^*$ are i.i.d. normally distributed shocks under the historical and the risk-neutral measures, respectively. Besides, the hazard rate is

$$\lambda_t = 0.622_{(0.024)} + y_{1,t} + y_{2,t}.$$

The risk-neutral probabilities of transition (the $\pi_{i,j}^*$‘s) are estimated via MLE (together with the parameterization of the dynamics of $y_t$):
Finally, the pricing-error standard-deviation estimate (i.e. the standard deviation of the $\varepsilon_{err,t}$’s defined in C.1) is 0.08%, or 8 basis points (the standard deviation of the parameter estimate is 0.001%, or 0.1 bp).

**D. About the eigenvectors of the rating-migration matrix $\Pi$**

In this appendix, using the notations presented in Subsection 7.4, we outline some properties of matrices $\Pi$ and $V$. For notational simplicity, we drop arguments and time subscripts associated with these matrices.

- As the sum of the entries of each line of $\Pi$ is equal to 1, the vector $[1 \cdots 1]'$ is an eigenvector of $\Pi$ associated with the eigenvalue 1. Consequently, since this eigenvalue is supposed to be the last one appearing in $\Psi$, the last column of $V$ – that collects the eigenvectors of $\Pi$ – is proportional to $[1 \cdots 1]'$.

- The fact that default is an absorbing state implies that the last row of $\Pi$ is $[0 \cdots 0 1]$. Since we have $\Pi V = V \Psi$, it comes (considering the last line of this equation):

  $\forall j \quad V_{K,j} = V_{K,j} \exp(-\psi_j),$

  which implies: $\forall j < K, V_{K,j} = 0$.

- The two previous points imply that the matrix $V$ admits the following form:

  $$V = \begin{bmatrix}
  V_{1,1} & \cdots & V_{1,K-1} & \gamma \\
  \vdots & \ddots & \vdots & \vdots \\
  V_{K-1,1} & \cdots & V_{K-1,K-1} & \gamma \\
  0 & \cdots & 0 & \gamma
  \end{bmatrix}$$

  Since $VV^{-1} = I$, we have (considering the last line and using the notation $V^{-1}_{i,j}$ for the entry $(i,j)$ of $V^{-1}$)

  $$\begin{bmatrix}
  V_{K,1}^{-1} & \cdots & V_{K,K-1}^{-1} & V_{K,K}^{-1} \\
  \end{bmatrix} = \begin{bmatrix}
  0 & \cdots & 0 & \frac{1}{\gamma}
  \end{bmatrix}$$

  and, therefore, for $i = 1, \ldots, K$, we have $V_{i,K}^{-1} = 1.$
Figure 1: Causality scheme

\[ z_t \rightarrow y_t \rightarrow x_{1t} \rightarrow d_{1t} \]
\[ x_{2t} \rightarrow d_{2t} \]
\[ x_{nt} \rightarrow d_{nt} \]

- \( z_t \): Markov-Switching (MS) regime
- \( y_t \): Macroeconomic risk factors
- \( x_{nt} \): Obligor-specific risk factors

\[ \rightarrow \quad \text{Granger Causality} \]
\[ \quad \rightarrow \quad \text{Instantaneous Causality} \]

Figure 2: BBB vs. Treasury Spreads, Estimation results

Notes: The upper panel presents the smoothed (using Kim’s (1994) filter) estimates of the two factors \( y_{1t} \) and \( y_{2t} \) that are such that the default intensities \( \lambda_t \) of BBB-rated corporates is given by \( \lambda_t = y_{1t} + y_{2t} \). Grey-shaded areas correspond to 95% confidence intervals. The second panel reports the (smoothed) probabilities of being in the “tranquil-times” regime 1 (white), the “intermediary” regime 2 (in grey) or the “crisis” regime 3 (in black). The lowest two panels display model-implied spreads together with observed ones for two respective maturities: 2 years and 5 years.
Figure 3: BBB vs. Treasury Spreads, Simulations

Notes: This Figure compares the distributions of spreads simulated by different models (with the sample distributions of spreads, the sample covering the period from March 1995 to July 2011). Three alternative models are used: the regime-switching one (presented in C) and two “purely Gaussian” models (involving respectively one and two AR(1) factors). Simulations are based on 50,000 replications of each models. The lower row of panels present the term-structures of the spreads (observed for the left plot and implied by the models for the other plots); for each panel, the grey shaded area is delimited by the 5th and the 95th percentiles of the spreads at each considered maturity. In addition, the lower-row plots present the term structures of medians and means of the spreads.
Figure 4: Simulated sample of the sector-contagion model

Notes: Each panel corresponds to one sector. There are 600 debtors in the portfolio (200 per sector). The vertical bars represent the number of firms that have defaulted during the considered period. At the end of each period, defaulted firms are replaced by new ones (of the same sector). Grey-shaded areas indicate periods during which the considered sector is in distress. Darker areas indicate periods when all three sectors are in distress.

Figure 5: Yield curves for selected ratings (with impact of regimes)

Notes: The left plot shows yield curves for selected ratings, with $y_{t,1} = 0$ and $z_t = [1, 0]'$ (solid lines) or $z_t = [0, 1]'$ (dashed lines). The right plot shows the term structure of spreads vs. Aaa-rated bonds.
Notes: The upper plot shows simulated downgrade probabilities for two ratings (the downgrade can be of one or more notches). Formally, for rating $j$, the upper panel plots $P(\tau_{n,t} > \tau_{n,t-1} | \tilde{z}_t, y_t, \tilde{x}_{n,t}, \tau_{n,t-1} = j)$. The grey-shaded areas indicate "crisis" periods. The lower plot shows the yield spreads between 10-year zero-coupon bonds issued by A-rated or Baa-rated debtors and zero-coupon bonds issued by Aaa-rated issuers.
Table 1 – Estimation methods
Notes: This Table sums up the different estimation procedures that can be implemented depending on the observability of the regimes ($z_t$) and of the factors ($y_t$). The unobserved regimes and factors (if any) are respectively denoted by $z_{2,t}$ and $y_{2,t}$. In the Table, the notation $y_{2,t} = \varnothing$ (respectively $z_{2,t} = \varnothing$) corresponds to those models in which there are no latent factors (respectively no latent regimes).

<table>
<thead>
<tr>
<th>$z_{2,t} = \varnothing$</th>
<th>$y_{2,t} = \varnothing$</th>
<th>$y_{2,t} \neq \varnothing$</th>
</tr>
</thead>
<tbody>
<tr>
<td>“Standard techniques”</td>
<td>Kalman filter /</td>
<td>Kalman filter /</td>
</tr>
<tr>
<td></td>
<td>Inversion techniques</td>
<td>Kim filter /</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Kitagawa-Hamilton filter + Inv. techniques</td>
</tr>
</tbody>
</table>

Table 2 – Calibration of the sector-contagion model
Notes: The second (respectively the third) lines reports the probabilities, for any sector, of getting infected (resp. cured), depending on the number of infected sectors during the previous period.

<table>
<thead>
<tr>
<th>Numb. of infected sect. ($\sum_{i \in [0,1]} S_{i,t}$)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proba. of getting infected (in $t+1$)</td>
<td>0.25%</td>
<td>10%</td>
<td>10%</td>
<td>–</td>
</tr>
<tr>
<td>Proba. of getting cured (in $t+1$)</td>
<td>–</td>
<td>10%</td>
<td>10%</td>
<td>10%</td>
</tr>
</tbody>
</table>

Table 3 – Dynamics of risk factors under both measures
Notes: The shocks $\varepsilon_t$ and $\varepsilon^*_t$ are i.i.d., $\varepsilon_t \sim N^P(0,1)$ and $\varepsilon^*_t \sim N^Q(0,1)$. The risk-free short-term rate is $r_{t+1} = 4\% + y_{r,t}/100$.

<table>
<thead>
<tr>
<th>Dynamics of</th>
<th>Under $P$</th>
<th>Under $Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_t = 0.6y_{t-1} + \varepsilon_t$</td>
<td>$y_t = 0.3 + 0.8y_{t-1} + \varepsilon^*_t$</td>
<td></td>
</tr>
<tr>
<td>Transition proba.</td>
<td>${\pi_{i,j}} = \begin{bmatrix} 0.98 &amp; 0.02 \ 0.25 &amp; 0.75 \end{bmatrix}$</td>
<td>${\pi^*_{i,j}} = \begin{bmatrix} 0.98 &amp; 0.02 \ 0.01 &amp; 0.99 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Table 4 – Baseline matrix of rating-migration probabilities
Notes: This matrix is based on Moody’s (2010) [49] (Exhibit 12: One-year average ratings-transition for European corporates 1985-2009). According to the industry standard, the probability of transitions to the “not rated” state is distributed among all states in proportion to their values (see Bangia et al., 2002 [4]).

<table>
<thead>
<tr>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa-C</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.911</td>
<td>0.084</td>
<td>0.004</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Aa</td>
<td>0.009</td>
<td>0.902</td>
<td>0.083</td>
<td>0.005</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>A</td>
<td>0.000</td>
<td>0.042</td>
<td>0.898</td>
<td>0.055</td>
<td>0.003</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Baa</td>
<td>0.000</td>
<td>0.004</td>
<td>0.072</td>
<td>0.868</td>
<td>0.041</td>
<td>0.009</td>
<td>0.003</td>
</tr>
<tr>
<td>Ba</td>
<td>0.000</td>
<td>0.000</td>
<td>0.007</td>
<td>0.074</td>
<td>0.788</td>
<td>0.107</td>
<td>0.012</td>
</tr>
<tr>
<td>B</td>
<td>0.000</td>
<td>0.000</td>
<td>0.004</td>
<td>0.004</td>
<td>0.073</td>
<td>0.794</td>
<td>0.092</td>
</tr>
<tr>
<td>Caa-C</td>
<td>0.000</td>
<td>0.003</td>
<td>0.001</td>
<td>0.000</td>
<td>0.007</td>
<td>0.106</td>
<td>0.706</td>
</tr>
</tbody>
</table>

Table 5 – Eigenvalues of the transition matrix under both regimes
Notes: “Regime 1” is consistent with the transition matrix reported in Table 3. Regime 2 is intended to depict a “crisis” regime. The $\alpha_{i,j}$’s ($i = 1, \ldots, 7$, $j = 1, 2$) are such that the $\exp(-\alpha_{i,j})$’s are the eigenvalues –those different from 1– of the rating-transition matrix obtained under regime $j$ (when $y_{r,t} = 0$). The 5-year default probabilities are computed conditionally on the absence of regime switching (i.e. as if the current regime is to last 5 years).

<table>
<thead>
<tr>
<th>5-yr default prob.</th>
<th>Aaa</th>
<th>Aa</th>
<th>A</th>
<th>Baa</th>
<th>Ba</th>
<th>B</th>
<th>Caa-C</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regime 1</td>
<td>0.057%</td>
<td>0.24%</td>
<td>0.80%</td>
<td>1.91%</td>
<td>8.72%</td>
<td>21.8%</td>
<td>52.0%</td>
<td></td>
</tr>
<tr>
<td>Regime 2</td>
<td>0.077%</td>
<td>1.79%</td>
<td>3.01%</td>
<td>6.40%</td>
<td>16.74%</td>
<td>32.6%</td>
<td>63.2%</td>
<td></td>
</tr>
<tr>
<td>$-\log($eigenvalues)</td>
<td>$1^{st}$</td>
<td>$2^{nd}$</td>
<td>$3^{rd}$</td>
<td>$4^{th}$</td>
<td>$5^{th}$</td>
<td>$6^{th}$</td>
<td>$7^{th}$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{1,1}$ ($i = 1, \ldots, K - 1$)</td>
<td>0.009</td>
<td>0.069</td>
<td>0.097</td>
<td>0.143</td>
<td>0.213</td>
<td>0.311</td>
<td>0.464</td>
<td></td>
</tr>
<tr>
<td>$\alpha_{1,2}$ ($i = 1, \ldots, K - 1$)</td>
<td>0.017</td>
<td>0.110</td>
<td>0.146</td>
<td>0.205</td>
<td>0.294</td>
<td>0.463</td>
<td>0.807</td>
<td></td>
</tr>
</tbody>
</table>