OPTIMAL PORTFOLIO ALLOCATION UNDER ASSET AND SURPLUS VaR CONSTRAINTS

A. MONFORT[1]

[1] Alain MONFORT, professor at CNAM and CREST, Paris, France, mailing address - CREST, 15 Boulevard Gabriel Péri, 92245 Malakoff Cedex France, phone : +33 1 41 17 77 28, email : monfort@ensae.fr

ABSTRACT

In this paper we propose an approach to Asset Liability Management of various institutions, in particular in insurance companies, based on a dual VaR constraint for the asset and the surplus. A key ingredient of this approach is a flexible modelling of the term structure of interest rates leading to an explicit formula for the returns of bonds. VaR constraints on the asset and on the surplus also take tractable forms, and graphical illustrations of the impact and of the sensitivity of these constraints are easily explicited in terms of various parameters : share of stocks, duration and convexity of the bonds on the asset and liability sides, expected return and volatility of the asset...

1 INTRODUCTION

This paper proposes a flexible and tractable modelling of four important aspects of Asset Liability Management (ALM); i) the trade-off between asset performance and liability hedging ii) the measures of risks, iii) the shape and the dynamics of the interest rate curve, iv) the modelling of the coupon structure of the bonds (on the asset and liability side).

The first point is a key issue in management of many institutions, in particular insurance companies, and it is becoming even more important because of the new accounting standards and regulation rules (Basle committees, Solvency I and II) [see Amenc-Martellini-Foulquier-Sender (2006) for a discussion of the impact of Solvency II ou ALM]. Building on a series of papers by Leibowitz, Bader and Kogelman [see Leibowitz-Bader-Kogelman (1996) and the references therein] , our approach is based on a joint modelling of the asset return and of the surplus return.

The second point is also central because it is now well documented that symmetric measures of risk like variance or standard error (or volatility) may be misleading when used at the decision stage. So we use the more appropriate VaR approach based on the measure of extreme risks and which is now recommended by regulatory authorities.

The third point is obviously crucial since the shape and the dynamics of the interest rate have decisive impacts both on the asset and liability side. Roughly speaking the modern literature on interest rates is divided in two streams. The first one uses factor models and the pioneering works in these are those of Vasicek (1977), Cox-Intersoll-Ross (1985) and Duffie-Kan (1996) [see also Gourieroux-Monfort-Polimenis (2003), Gourieroux-Monfort (2007) and Monfort-Pegoraro (2007) for more flexible models]. In the second stream, the yied curve is assumed to be as a linear combination of basic functions of the maturity, the coefficients of the combination being specified as a stochastic processes [see e.g. Diebold-Li (2006)]. In this paper we adopt this second approach and, in order to have explicit formulas for the bond returns, we propose to use polynomials as basic functions of maturity. More precisely the shape of the curve is determined at each date by a level, a slope and a convexity parameter, the level (and possibly the slope in the extended version of the model) being a stochastic process.

Finally the structure of the coupons of the bonds appearing on the asset and liability sides, is summarized by two parameters (or four parameters in the extended version) interpreted as the two (or four) first empirical moments of the maturities (weighted by the actualized coupons). In particular, we introduce the extent parameter, which is the standard error of these maturities (and could be seen as one of the many definitions of the convexity of a coupon bond). It turns out that, under this approach, and using a very accurate expansion technique, we obtain an explicit formula for the annual return of coupon bonds as a function of the parameters of the yield curve and of the parameters of the coupon structure. The accuracy of this approximation is assessed by simulation and kernel non-parametric techniques. Moreover, introducing assumptions on the stock returns and on the correlation of this returns with bond returns, we obtain explicit formulas for asset return, liability return, surplus return, and in particular for the means, the volatilities, the quantiles and the VaR of these random variables. This allows for a simple and illuminating graphical presentation (in 2 and 3 dimensions) of these quantities. Then the problem of optimal allocation of the asset under constraints on the asset VaR and the surplus VaR can be tackled, and, playing with the parameters, a sensitivity analysis of the admissible and optimal allocations can be easily performed.

The paper is organized as follows. In section 2 we consider the modelling of the yield curve and of the bond return; in particular we develop a careful study of the evaluation of the approximation based on simulations and kernel non-parametric techniques. Section 3 is devoted to asset modelling. Section 4 deals with liability and surplus modelling. Section 5 considers extensions. Section 6 concludes and two appendices gather the technical proofs.

2 Interest rates and return of coupon bonds

2.1 Shape and dynamics of the interest rate curve.

In this paper we assume that the interest rate curve is quadratic. More precisely the (arithmetic) interest rate at time t of maturity h years is given by :

$$R(t,h) = R(t,1) + \alpha \times (h-1) + \beta \times (h-1)^2 \quad (1)$$

The parameters α and β are interpreted repectively as a slope and curvature parameter. This modelling is obviously not valid when h goes to infinity, but it is flexible enough to provide good approximations for the usual values of h and, moreover, we shall see that it leads to tractable formulas.

The dynamics of the short rate $R(t,1)$ is first assumed to be that of a random walk :

$$\begin{aligned} R(t+1,1) &= R(t,1)+\sigma\varepsilon_{t+1} \\ \text{and, therefore, } R(t+1,h) &= R(t,1)+\alpha(h-1)+\beta(h-1)^2+\sigma\varepsilon_{t+1} \quad (2) \end{aligned}$$

where ε_{t+1} is $N(0,1)$ distributed.

In other words, at horizon one (which is the horizon we are interested in) the random change of the curve is a translation (see section 5 for an extension).

2.2 Return of coupon bonds, duration and extent.

Let us denote by $C_{t+h}, h=1,\ldots,H$ the coupons of the bond at dates $t+h, h=1,\ldots,H$. The (arithmetic) return of this bond is :

$$\begin{aligned} r_b &= \frac{p_{t+1}}{p_t}-1 \\ \text{with } p_{t+1} &= \sum_{h=1}^{H}\frac{C_{t+h}}{[1+R(t,1)+\alpha(h-2)+\beta(h-2)^2+\varepsilon_{t+1}]^{h-1}} \\ p_t &= \sum_{h=1}^{H}\frac{C_{t+h}}{[1+R(t,1)+\alpha(h-1)+\beta(h-1)^2]^{h}} \end{aligned}$$

Using a Taylor expansion it can be shown (see appendix 1) that r_b can be approximated by :

$$r_b^{D,E}=R(t,1)+2\alpha D+\beta(3D^2-D+3E^2)+\psi\frac{\sigma^2}{2}(D^2+D+E^2)-D\sigma\varepsilon_{t+1} \quad (3)$$

$$\text{where } D = \frac{\sum_{h=1}^{H}C^*_{t+h}(h-1)}{\sum_{h=1}^{H}C^*_{t+h}} \quad (4)$$

$$
\begin{aligned}
D_2 &= \frac{\sum_{h=1}^{H} C^*_{t+h}(h-1)^2}{\sum_{h=1}^{H} C^*_{t+h}} && (5)\\
E^2 &= D_2 - D^2 && (6)\\
C^*_{t+h} &= \frac{C_{t+h}}{[1+R(t,1)+\alpha(h-1)+\beta(h-1)^2]^h} && (7)
\end{aligned}
$$

ψ is a correction parameter (we shall take $\psi = 0.95$, see appendix 1).

D is the duration at $t+1$ of the bond and E, called the extent, is a standard error of the maturity. In particular a zero-coupon bond has an extend equal to zero. [2]

In particular, the mean of $r_b^{D,E}$ is :

$$m_b = R(t,1) + 2\alpha D + \beta(3D^2 - D + 3E^2) + \psi\frac{\sigma^2}{2}(D^2 + D + E^2) \quad (8)$$

and its standard error, or volatility :

$$\sigma_b = D\sigma \quad (9)$$

In the volatility-mean plan, the pairs (σ_b, m_b), are, for different E, on parallel parabolas : (for $D > 1$, or $\sigma_b > \sigma$, since for $D = 1$, we have $E = 0$) :

$$m_b = R(t,1) + \left(3\beta + \psi\frac{\sigma^2}{2}\right)E^2 + \left(2\alpha - \beta + \psi\frac{\sigma^2}{2}\right)\frac{\sigma_b}{\sigma} + \left(3\beta + \psi\frac{\sigma^2}{2}\right)\frac{\sigma_b^2}{\sigma^2} \quad (10)$$

In particular, for a zero-coupon bond, the pairs (σ_b, m_b) are on the parabola :

$$m_b = R(t,1) + \left(2\alpha - \beta + \psi\frac{\sigma^2}{2}\right)\frac{\sigma_b}{\sigma} + \left(3\beta + \psi\frac{\sigma^2}{2}\right)\frac{\sigma_b^2}{\sigma^2} \quad (11)$$

The convexity of the parabolas depend on the convexity of the rate curve and on the variance of the noise.

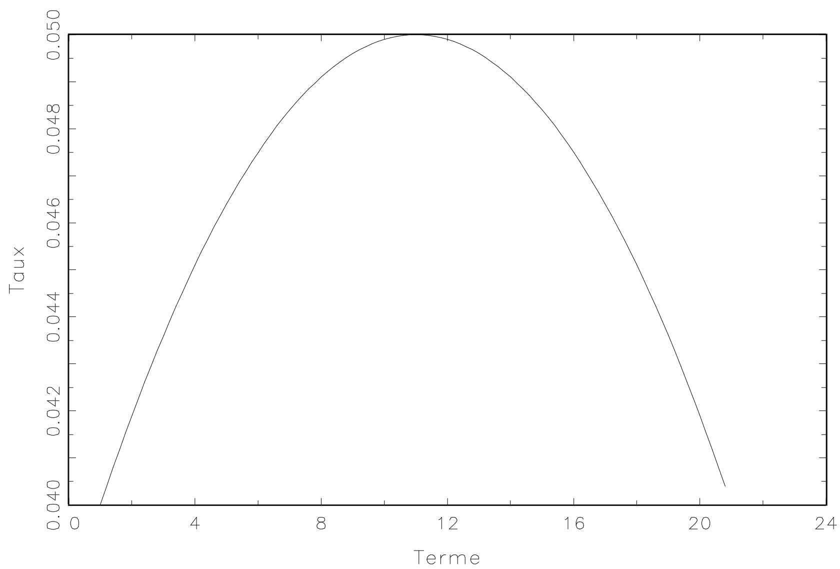
[2] E is sometimes called convexity in the literature; however since there are many different definitions of the convexity we prefer to introduce this new term.

2.3 Precision of the approximation

Formula (3.8) is based on an expansion assuming α, β, σ small. So it is natural to evaluate this approximation for realistic values of α, β, σ.

We consider a humped rate curve similar to those observed recently, corresponding to $R(t, 1) = 4.10^{-2}, \alpha = 2.10^{-3}$ et $\beta = -10^{-4}$ (see figure 1) and we take $\sigma = 10^{-2}$.

FIGURE 1:Yield Curve(Quadratic)



For a given bond we can compute the exact probability density function (pdf) of the return using simulations and nonparametric kernel estimation methods (the bandwidth is chosen according to Silverman's rule). We can, in particular, compute the mean and the standard error (volatility) of this exact distribution and compare them with the values given by formula (8) and (9). We can also compute the skewness and the kurtosis and compare them with the values corresponding to a normal distribution, namely 0 and 3.

In the following table we give the result of this study for various zero-coupon bonds with maturity between 2 and 21, i.e. with durations D (at $t+1$) between 1 and 20.

D	exact m	m_b	exact vol	σ_b	skewness	kurtosis
1	0,0440	0,0439	0,0100	0,0100	0,04	2,96
2	0,0476	0,0472	0,0201	0,0200	0,10	3,02
3	0,0503	0,0502	0,0304	0,0300	0,11	3,02
4	0,0524	0,0525	0,0402	0,0400	0,14	3,02
5	0,0546	0,0544	0,0503	0,0500	0,17	3,04
8	0,0579	0,0570	0,0810	0,0800	0,24	3,04
10	0,0564	0,0562	0,1016	0,1000	0,35	3,18
12	0,0532	0,0534	0,1202	0,1200	0,36	3,29
15	0,0459	0,0454	0,1511	0,1500	0,46	3,38
18	0,0339	0,0329	0,1809	0,1800	0,53	3,45
20	0,0221	0,0219	0,1988	0,2000	0,59	3,62

We see that the approximations of the means and of the volatilities are very good. The skewness and the kurtosis slightly increase with the maturity; in other words, when the maturity increases, we observe a slightly increasing positive asymmetry and also a slightly increasing thickness of the tails.

Figures 2 and 3 show the exact and approximated pdf for bonds with short and long duration. The short term bond has identical coupons at $t+2$ and $t+3(D=1.48, E=0.5)$ and the long term bond has identical coupons at $t+2$ and $t+5(D=8.8, E=10.9)$. For the short bond the exact and approximated means are both equal to 0.045 and the exact and approximated volatilities are both equal to 0.015 (skewness 0.089, kurtosis 3.03). For the long bond we obtain 0.028 and 0.027 for the means, and 0.086 and 0.088 for the volatilities (skewness 0.73, kurtosis 4.05).

FIGURE 2: Exact(solid line) and approximated(dotted lined) PDF
of the return of a bond.
Short term bond

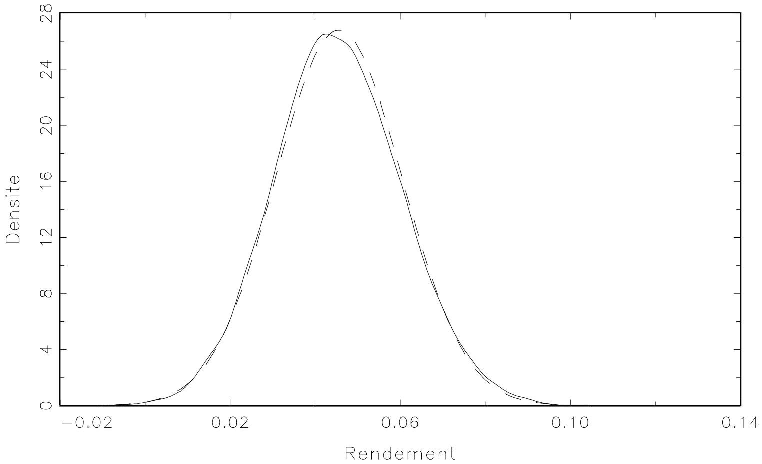
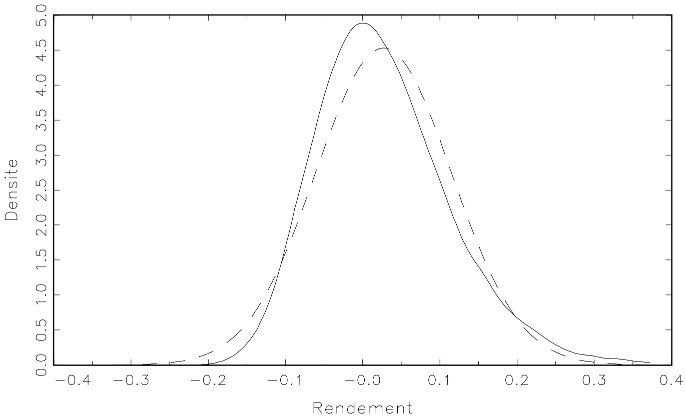


FIGURE 3: Exact(solid line) and approximated(dotted lined) PDF
of the return of a bond.
Long term bond



3 Modelling the asset

3.1 Return of the asset, mean, volatility, quantiles

We assume that the asset is a portfolio of stocks and bonds. We denote by w the share of stocks (and $1 - w$ the share of bonds). We denote by D and E, the duration and the extent of the bond component of the portfolio. We assume that the return r_e (e like equity) of the stocks is gaussian, with mean m_e and volatility σ_e and that its correlation with $r_b^{D,E}$ is denoted by ρ; note that this correlation is the opposite of the correlation between r_e and ε_{t+1} and, therefore, does not depend on D and E. The return of the asset is :

$$r_a = wr_e + (1 - w)r_b^{D,E} \tag{12}$$

and the distribution of r_a is $N(m_a, \sigma_a)$ with :

$$m_a = wm_e + (1-w)\left[R(t,1) + 2\alpha D + \beta(3D^2 - D + 3E^2) + \psi\frac{\sigma^2}{2}(D^2 + D + E^2)\right] \tag{13}$$

$$\sigma_a^2 = w^2\sigma_e^2 + (1 - w)^2 D^2\sigma^2 + 2w(1 - w)\sigma_e\rho\sigma D \tag{14}$$

For instance, we give the surfaces $m_a(w, D)$ and $\sigma_a(w, D)$, E fixed (see figures 4 and 5) for the numerical values :
$R(t, 1) = 4.10^{-2}, \alpha = 2.10^{-3}, \beta = -10^{-4}, \sigma = 10^{-2}, E = 5, m_e = 8.10^{-2}, \sigma_e = 12.10^{-2}, \rho = 0, 3.$

FIGURE 4:Mean of the asset return as a function of the stock share and of the bond duration

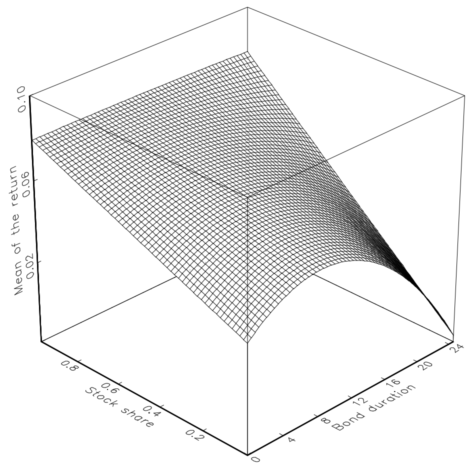
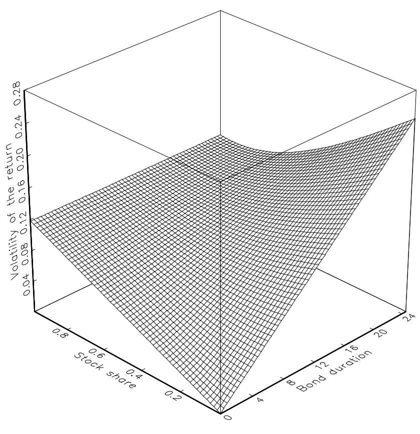


FIGURE 5:Volatility of the asset return as a function of the stock share and of the bond duration



We can also consider the 5% quantile q_a of r_a defined by :

$$P(r_a < q_a) = 5.10^{-2} \tag{15}$$

Since equation (15) can also be written :

$$P\left(\frac{r_a - m_a}{\sigma_a} < \frac{q_a - m_a}{\sigma_a}\right) = 5.10^{-2}$$

$$\text{we have :} \quad \frac{q_a - m_a}{\sigma_a} = \Phi(0.05) = -1.65$$

(Φ being the cumulative distribution function (cdf) of $N(0,1)$) and therefore :

$$q_a = m_a - 1,65\sigma_a \tag{16}$$

Figure 6 shows the surface $q_a(w, D)$, for E fixed and figure 7 gives contours of this surface.

FIGURE 6:5% Quantile of the asset return as a function of the stock share and of the bond duration

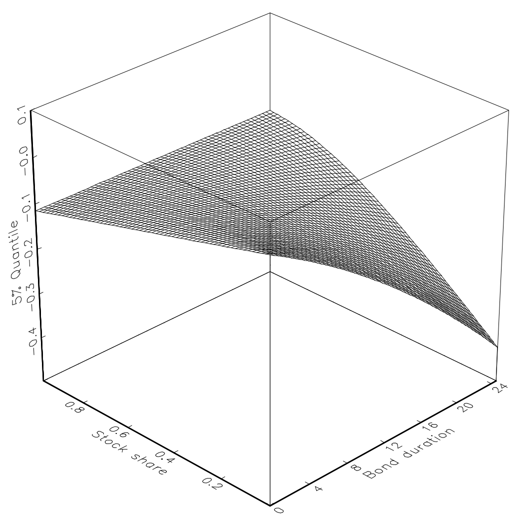
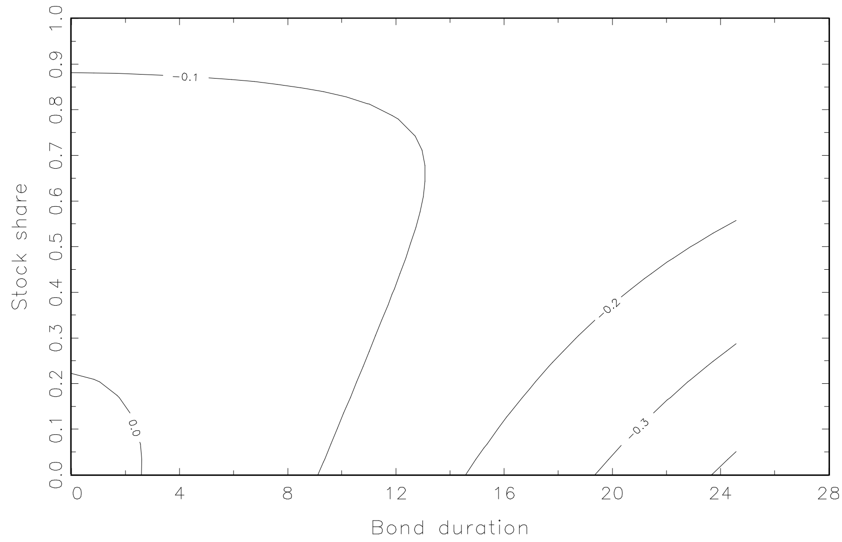


FIGURE 7:Contours of the 5% quantile of the asset return as functions of the stock share and of the bond duration



3.2 VaR constraint on the asset

Denoting by A_t the value of the asset at t, the VaR of risk level 5% is defined by :

$$P(A_{t+1} - A_t < -VaR_a) = 0.05 \tag{17}$$

or

$$P(A_t - A_{t+1} < VaR_a) = 0.95 \tag{18}$$

In other words the VaR is such that the loss $A_t - A_{t+1}$, between t and $t+1$, is smaller than the VaR with probability 0.95, and therefore larger than the VaR with probability 0.05.

Since $r_a = \dfrac{A_{t+1} - A_t}{A_t}$ we have :

$$P\left(r_a < -\frac{VaR_a}{A_t}\right) = 0.05$$

and, therefore :

$$VaR_a = -q_a A_t \tag{19}$$

If we impose that VaR_a is smaller than $v_a (VaR_a < v_a)$, this is equivalent to :

$$P(A_t - A_{t+1} < v_a) > 0.95 \tag{20}$$

or :

$$q_a > -\frac{v_a}{A_t} = u_a \tag{21}$$

So, for E given, the pairs (w, D) satisfying constraint (21) are those corresponding to a point of the surface, given in figure 6, above u_a, or to the points of figure 7 inside the contour corresponding to u_a.

3.3 An example

Let us assume that we take $u_a = -4.10^{-2}$ for the lower bound of q_a. In other words we impose that the VaR_a of risk level 5% is smaller than 4% of the asset, or, equivalently that the loss in the asset is smaller 4% of the initial asset with probability larger than 95%. This constraint can be visualized in the volatility-mean plan by the region above the line :

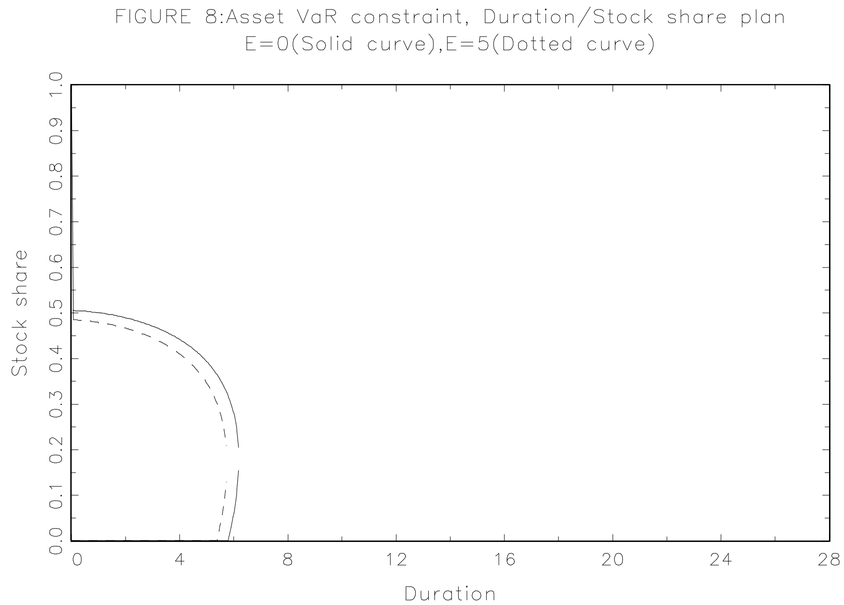
$$m_a - 1.65\sigma_a = -0.04 \tag{22}$$

and in the plan (D, w), for E fixed, this constraint becomes :

$$\begin{aligned} wm_e + (1-w)\left[R(t,1) + 2\alpha D + \beta(3D^2 - D + 3E^2 + \psi\frac{\sigma^2}{2}(D^2 + D + E^2)\right] \\ > 1.65[w^2\sigma_e^2 + (1-w)^2 D^2\sigma^2 + 2w(1-w)\sigma_e\rho\sigma D - 0,04 \end{aligned} \tag{23}$$

Figure 8 shows two regions of this kind. The region inside the solid line corresponds to $E = 0$, and the region inside the dotted line corresponds to $E = 5$.

The extent $E = 5$ is more constraining because of the shape of the rate curve (negative beta) penalizing the return of long term bonds.



4 Modelling the liability and the surplus

4.1 Surplus return, immunized portfolio, mean, volatility, quantiles.

The liability L_t is assimilated to a short position on a coupon bond (see section 5 for an extension). Denoting by D_L et E_L its duration and its extent, the return of the liability is :

$$r_L = r_b^{D_L,E_L} = R(t,1)+2\alpha D_L+\beta(3D_L^2-D_L+3E_L^2)+\psi\frac{\sigma^2}{2}(D_L^2+D_L+E_L^2)-D_L\varepsilon_{t+1} \quad (24)$$

The surplus at t is :

$$S_t = A_t - L_t \tag{25}$$

Taking the same definition of the return of the surplus as in Leibowitz-Bader-Kogelman (1996), we have :

$$r_s = \frac{S_{t+1} - S_t}{L_t} \tag{26}$$

Denoting by $F_t = \dfrac{A_t}{L_t}$ initial funding ratio we get :

$$\begin{aligned} S_{t+1} - S_t &= A_{t+1} - A_t - L_{t+1} + L_t \\ &= A_t r_a - L_t r_L \\ &= L_t (F_t r_a - r_L) \end{aligned} \tag{27}$$

and :

$$r_s = F_t r_a - r_L \tag{28}$$

We obtain that r_s follows a gaussian distribution with mean m_s and volatility σ_s defined by :

$$\begin{aligned} m_s &= F_t m_a - m_L \\ &= F_t \left\{ w m_e + (1-w) \left[R(t,1) + 2\alpha D + \beta(3D^2 - D + 3E^2) + \psi \frac{\sigma^2}{2}(D^2 + D + E^2) \right] \right\} \\ &- [R(t,1) + 2\alpha D_L + \beta(3D_L^2 - D_L + 3E_L^2) + \psi \frac{\sigma^2}{2}(D_L^2 + D_L + E_L^2)] \end{aligned} \tag{29}$$

$$\begin{aligned} \sigma_s^2 &= F_t^2 \left[w^2 \sigma_e^2 + (1-w)^2 D^2 \sigma^2 + 2w(1-w)\sigma_e \rho \sigma D \right] \\ &+ D_L^2 \sigma^2 - 2 F_t D_L \sigma [w \sigma_e \rho + (1-w) D \sigma] \end{aligned} \tag{30}$$

In particular, considering the case where the asset is made only of bonds ($w = 0$), we get :

$$\sigma_s^2 = (F_t D - D_L)^2 \sigma^2 \tag{31}$$

$$\text{so} \quad \sigma_s = (F_t D - D_L)\sigma, \text{ if } D \geq \frac{D_L}{F_t}$$

$$\sigma_s = -(F_t D - D_L)\sigma, \text{ if } D < \frac{D_L}{F_t}$$

The volatility of the surplus return is piecewise linear and is equal to zero for $D = \dfrac{D_L}{F_t}$. It vanishes for an "immunized" portfolio corresponding to a coupon bond with duration $\dfrac{D_L}{F_t}$. The surplus is then non random and equal to :

$$(F_t - 1)R(t,1) + 3\beta\left(\frac{D_L^2}{F_t} - D_L^2 + F_t E^2 - E_L^2\right) + \psi\frac{\sigma^2}{2}\left(\frac{D_L^2}{F_t} - D_L^2 + F_t E^2 - E_L^2\right) \tag{32}$$

This value is equal to zero if $F_t = 1$ and $E = E_L$. Figures 9, 10 show the surfaces $m_s(w, D)$ and $s_s(w, D)$ with E fixed, for the numerical values, $R(t,1) = 4.10^{-2}, \alpha = 8.10^{-3}, \beta = -10^{-4}, \sigma = 10^{-2}, E = 5, m_e = 8.10^{-2}, \sigma_e = 12.10^{-2}, \rho = 0.3, D_L = 10, E_L = 5, F_t = 1.1$.

We observe a "valley" shape for the volatility of the surplus return with a section at $w = 0$ made of two half-lines intersecting at the immunized portfolio.

FIGURE 9:Mean of the surplus return as a function of the stock share and of the bond duration

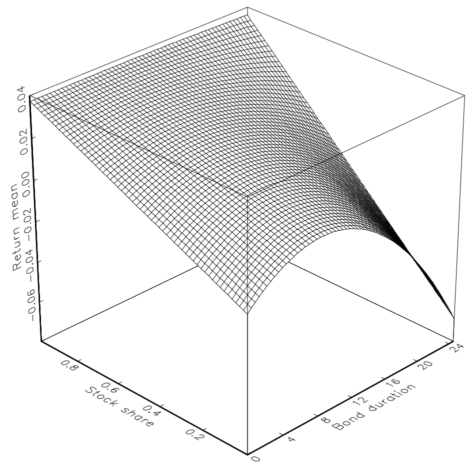
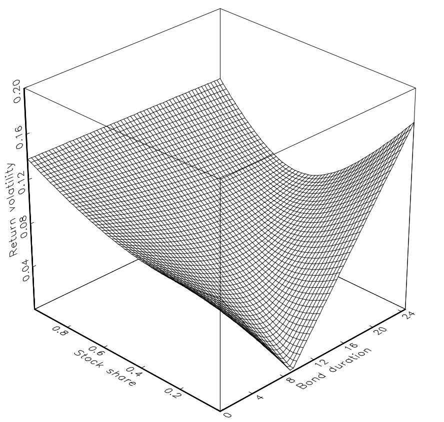


FIGURE 10:Volatility of the surplus return as a function of the stock share and of the bond duration



It is also interesting to consider the 5% quantile of the surplus return defined by :

$$q_s = m_s - 1.65\sigma_s \tag{33}$$

Figures 11 and 12 respectively show the surface $q_s(w, D)$, with E fixed and the contours of this surface.

Figure 11 exhibits an "inverted" shape, compared to that of the volatility.

FIGURE 11:5% Quantile of the surplus return as a function of the stock share and of the bond duration

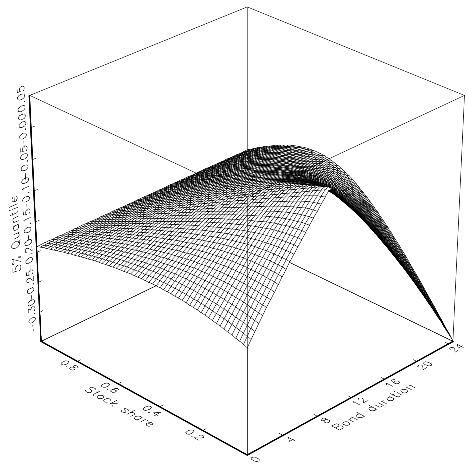
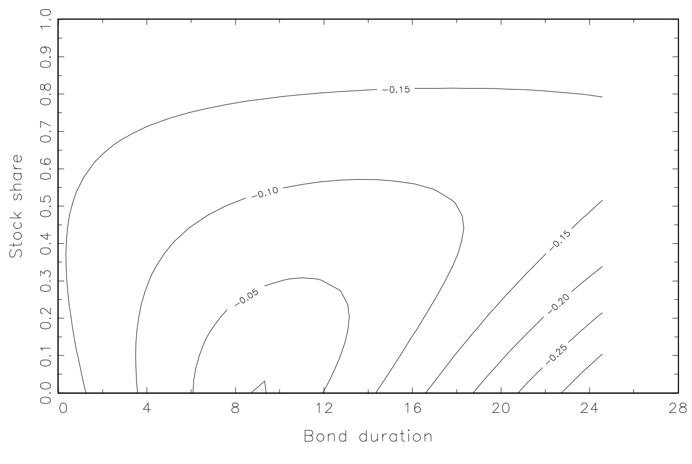


FIGURE 12:Contours of the 5% quantile of the surplus return as functions of the stock share and of the bond duration



4.2 VaR constraint on the surplus

The surplus VaR of risk level 5% is defined by :

$$P(S_{t+1} - S_t < -VaR_s) = 0.05$$
$$\text{or} \quad P\left(\frac{S_{t+1} - S_t}{L_t} < -\frac{VaR_s}{L_t}\right) = 0.05$$

$$\begin{aligned} \text{hence : } q_s &= -\frac{VaR_s}{L_t} \\ \text{or : } VaR_s &= -L_t q_s \end{aligned} \tag{34}$$

Therefore, a constraint $VaR_s < v_s$ is equivalent to :

$$q_s > -\frac{v_s}{L_t} = u_s \tag{35}$$

In particular if $v_s = S_t$ the condition becomes $q_s > 1 - F_t$ and means that $P(S_{t+1} < 0) < 0.05$.

4.3 Example

We consider the same numerical values as in 4.1 and we take $v_s = 0$. The quantile of the surplus return must satisfy $q_s > 1 - F_t = -0.1$.

This condition can also be written :

$$m_s - 1.65\sigma_s > -0.1 \tag{36}$$

or

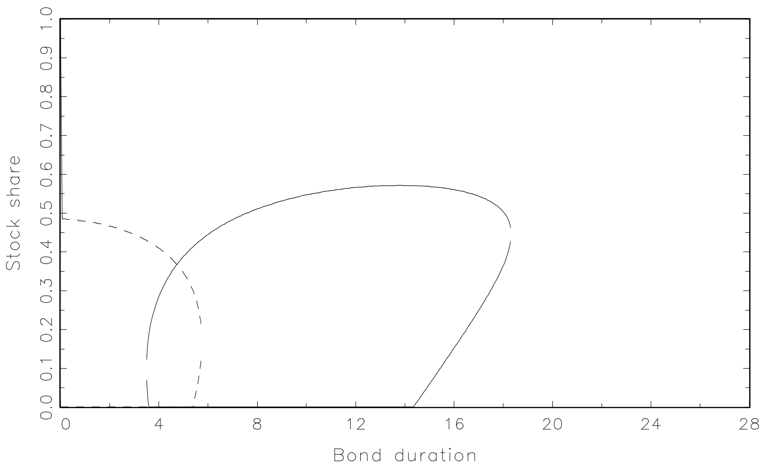
$$\begin{aligned} &F_t\{wm_e + (1-w)[R(t,1) + 2\alpha D + \beta(3D^2 - D + 3E^2) + \psi\frac{\sigma^2}{2}(D^2 + D + E^2)]\} \\ &-[R(t,1) + 2\alpha D_L + \beta(3D_L^2 - D_L + 3E_L^2) + \psi\frac{\sigma^2}{2}(D_L^2 + D_L + E_L^2)] \\ &-1.65\{F_t[w^2\sigma_e^2 + (1-w)^2 D^2\sigma^2 + 2w(1-w)\sigma_e\rho\sigma D] \\ &+D_L^2\sigma^2 - 2F_t D_L\sigma[w\sigma_e\rho + (1-w)D\sigma]\}^{1/2} > -0.1 \end{aligned} \tag{37}$$

Figure (13) illustrates this constraint and the asset VaR constraint in the plan (D, w), with E fixed. The surplus VaR constraint is satisfied inside the solid curve and the asset VaR constraint is satisfied inside the dotted curve. Both constraints are satisfied in the intersection of the two regions. The point of this intersection corresponding to a maximal w has the following characteristics :

$$w = 0.37, D = 4.75, m_a = 0.059, \sigma_a = 0.060$$

These VaR regions can be transposed in the volatility-mean plan using formulas (13) and (14). In figure (14) the region satisfying the asset VaR constraint is above the dotted line, and the region satisfying the surplus VaR constraint is inside the solid curves. The latter is delimited below by the parabola of the bonds with a given extent ($E = 5$). We have also shown on this figure the portfolios containing only stocks and cash (solid line); in this example such portfolios cannot satisfy simultaneously the two VaR constraints. We find again $m_a = 0.058$ and $\sigma_a = 0.059$ at the intersection of the dotted line and of the upper solid curve.

FIGURE 13:Asset and surplus VaR constraints,
Duration/Stock share Plan,
Solid line:Surplus VaR,Dotted line:Asset VaR



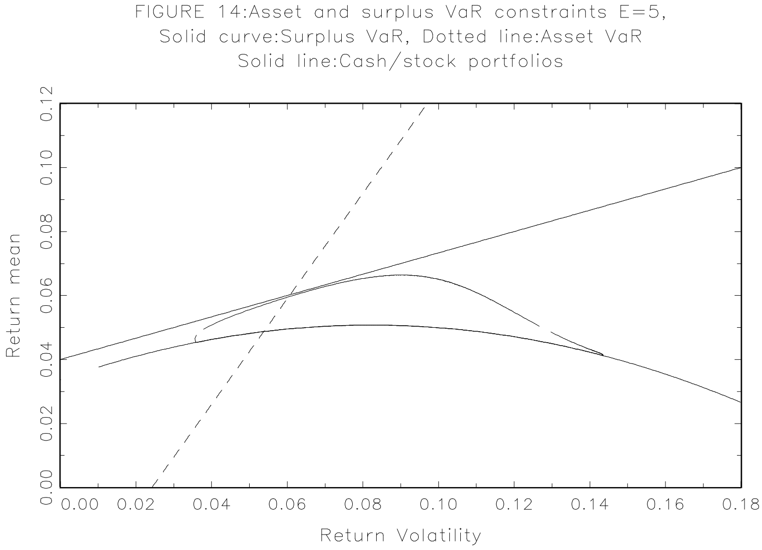


Figure 15 is similar to the previous one, with $E = 0$. We see that, in this case the solid line stock- cash portfolios crosses the region satisfying the VaR constraints. Moreover, the optimal admissible portfolio, in terms of average return, is better than the previous one, it corresponds to : $w = 0.41, D = 4.8, m_a = 0.065, \sigma_a = 0.063$. We obtain a gain of 0.6% in terms of average return by taking $E = 0$, that is by choosing a zero coupon bond as the bound of the asset. Moreover the optimal zero coupon has a duration of 4.8 much smaller than the duration of the liability $D_L = 10$. This is a consequence of the concavity of the rate curve. An alternative interpretation is obtained in the plan duration stock percentage (see figure 16) : we see that $E = 0$ is less constraining than $E = 5$ in terms of asset VaR as well as in terms of surplus VaR.

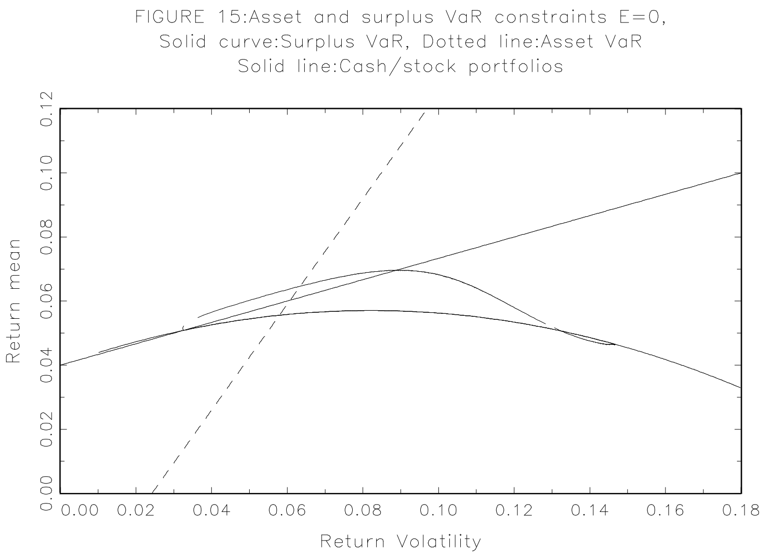


FIGURE 15:Asset and surplus VaR constraints E=0,
Solid curve:Surplus VaR, Dotted line:Asset VaR
Solid line:Cash/stock portfolios
Return mean
Return Volatility
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.00 0.02 0.04 0.06 0.08 0.10 0.12 0.14 0.16 0.18

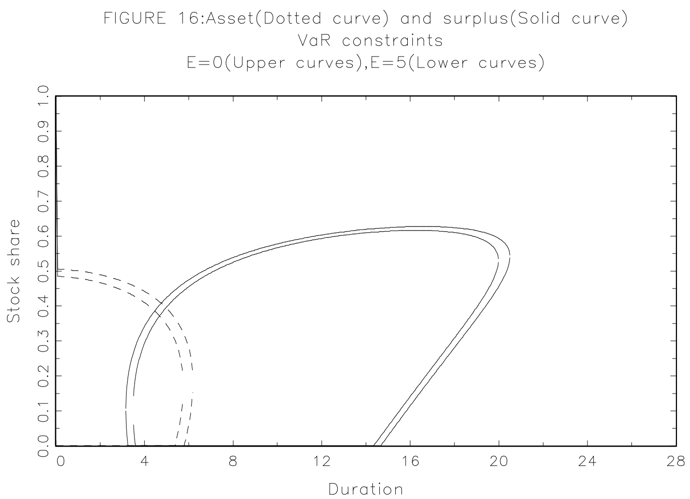


FIGURE 16:Asset(Dotted curve) and surplus(Solid curve)
VaR constraints
E=0(Upper curves),E=5(Lower curves)
Stock share
Duration
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
0 4 8 12 16 20 24 28

5 Extensions

5.1 Taking into account distorsions of the term structure curve

Among the possible extensions preserving the tractability of the model, one of the more promising is the generalization of the dynamics of the term structure of interest rates. Indeed we could extend (1) to :

$$R(t,h) = [1 + \tilde{\gamma}(h-1)]R(t,1) + \tilde{\alpha}(h-1) + \beta(h-1)^2 \tag{38}$$

and, keep the same dynamics for the short rate $R(t,1)$:

$$R(t+1,1) = R(t,1) + \sigma\varepsilon_{t+1},\ \varepsilon_{t+1} \sim N(0,1)$$

$R(t,h)$ and $R(t+1,h)$ can be written [with $\alpha = \tilde{\alpha} + \tilde{\gamma}R(t,1)$ depending on t, $\gamma = \sigma, \delta = \sigma\tilde{\gamma}$].

$$R(t,h) = R(t,1) + \alpha(h-1) + \beta(h-1)^2 \tag{39}$$

$$R(t+1,h) = R(t,1) + \alpha(h-1) + \beta(h-1)^2 + [\gamma + \delta(h-1)]\varepsilon_{t+1} \tag{40}$$

In this model, a shock on ε_{t+1} will have different impacts on $R(t+1,h)$ for different maturities.

The formulas of this article are easily generalized to this new model replacing (3) by :

$$\begin{aligned} r_b = R(t,1) \quad & +(2\alpha - \beta)D + 3\beta D_2 + \psi(\gamma-\delta)^2\tfrac{D}{2} \\ & +\psi(\gamma^2 - \delta^2)\tfrac{D_2}{2} + \psi\delta\left(\gamma - \tfrac{\delta}{2}\right)D_3 + \psi\tfrac{\delta^2}{2}D_4 \\ & -[(\gamma-\delta) + \delta D_2]\,\varepsilon_{t+1} \end{aligned} \tag{41}$$

D_3 and D_4 being the third and fourth order empirical moments of the maturity of the bond appearing in the asset (see appendix 1).

5.2 Taking into account liabilities linked with the stock market

Up to now the liability has been considered as a short position on a coupon bond taking into account the interest rate risk. In some contexts it is also useful to introduce a component of the liability return which is linked to the stock market. We can propose the model :

$$r_{L,t+1} = \lambda + \mu r_{b,t+1}^{D_L,E_L} + \nu r_{e,t+1} + \omega \xi_{t+1} \tag{42}$$

where $\xi_{t+1} \sim N(0,1)$ independent of $r_{b,t+1}^{D_L,E_L}$ and $r_{e,t+1}$.

Note that (44) can be written equivalently by introducing the return of an observable zero-coupon instead of $r_{b,t+1}^{D_L,E_L}$ which may be not observable if D_L and E_L have to be estimated . Let us consider for instance of a zero-coupon bond with maturity 2 :

$$r_{z,t+1} = R(t,1) + 2(\alpha + \beta) + \psi\sigma^2 - \sigma\varepsilon_{t+1} \tag{43}$$

This return is linked to $r_{b,t+1}^{D_L,E_L}$ since :

$$r_{b,t+1}^{D_L,E_L} = R(t,1)+(2\alpha-\beta)D_L+3\beta(D_L^2+E_L^2)+\psi\frac{\sigma^2}{2}(D_L^2+D_L+E_L^2)-\sigma D_L\varepsilon_{t+1} \tag{44}$$

Eliminating ε_{t+1} we get an equation of the form :

$$r_{L,t+1} = \lambda + \mu a[R(t,1), D_L, E_L] + \mu D_L r_{z,t+1} + \nu r_{e,t+1} + \omega\xi_{t+1} \tag{45}$$

Equation (47) contains in its right hand side a function of parameters and observable variables, perturbated by a noise. Therefore it can participate to the estimation of $\lambda, \mu, \nu, \omega, E_L, D_L$.

The special case of the previous sections is obtained for $\lambda = \nu = \omega = 0$ and $\mu = 1$.

The previous results can be extended to this new framework (see appendix 2).

Note that equation (44) contains two important special cases :

- the case "noisy bond" : $\lambda = -\nu m_e, \mu = 1$
- the case "noisy portfolio" : $\lambda = 0, \mu + \nu = 1$

6 Conclusion

The model proposed in this paper is flexible and tractable. It provides a simple framework for analyzing analytically or graphically the problem of optimal asset allocation under asset and surplus constraints. Since it is conveniently parameterized many sensitivity analyses could be easily performed. Moreover, the model is modular, in the sense that it would be possible to add satellite models formalizing other environments, like international environments (by modelling exchange rates, foreign stock markets and yield curves) or benchmark environments (by introducing VaR constraints on differential returns).

APPENDIX 1

Return of a coupon bond

$$r_b = \frac{p_{t+1}}{p_t} - 1 = \frac{p_{t+1} - p_t}{p_t}$$

$$\text{with } p_{t+1} = \sum_{h=1}^{H} \frac{C_{t+h}}{[1 + R(t+1, h-1)]^{h-1}}$$

$$p_t = \sum_{h=1}^{H} \frac{C_{t+h}}{[1 + R(t, h)]^{h}}$$

Neglecting the possible change of $\alpha, \beta, \gamma, \delta$ between t and $t+1$ we get :

$$R(t, h) = R(t, 1) + \alpha(h-1) + \beta(h-1)$$

$$R(t+1, h) = R(t, 1) + \alpha(h-1) + \beta(h-1)^2 + [\gamma + \delta(h-1)]\varepsilon_{t+1}$$

So r_b can be written :

$$r_b = \frac{1}{p_t} \sum_{h=1}^{H} \frac{C_{t+h}}{[1 + R(t,1) + \alpha(h-1) + \beta(h-1)^2]^{h-1}} (A_h - B_h)$$

$$\text{with } A_h = \frac{1}{\left[\frac{1 + R(t,1) + \alpha(h-2) + \beta(h-2)^2 + [\gamma + \delta(h-2)]\varepsilon_{t+1}}{1 + R(t,1) + \alpha(h-1) + \beta(h-1)^2}\right]^{h-1}}$$

$$B_h = \frac{1}{1 + 2R(t,1) + \alpha(h-1) + \beta(h-1)^2}$$

A_h can be written :

$$A_h = \frac{1}{\left[1 + \frac{[\gamma + \delta(h-2)]\varepsilon_{t+1} - \alpha + \beta[(h-2)^2 - (h-1)^2]}{1 + R(t,1) + \alpha(h-1) + \beta(h-1)^2}\right]^{h-1}}$$

Considering a first order expansion around $\alpha = \beta = \gamma = \delta = 0$, and taking into account the second order term the random component $[\gamma + \delta(h-2)]\,\varepsilon_{t+1}$ we get :

$$A_h \simeq 1 \quad - \frac{(h-1\{[\gamma+\delta(h-2)]\varepsilon_{t+1} - \alpha - \beta(2h-3)\}}{1+R(t,1)+\alpha(h-1)+\beta(h-1)^2}$$

$$+\frac{h(h-1)[\gamma+\delta(h-2)]^2}{2[(1+R(t,1)+\alpha(h-1)+\beta(h-1)^2]^2}$$

We approximate $[1+R(t,1)+\alpha(h-1)+\beta(h-1)^2]^2$ by $\psi^{-1}[1+R(t,1)+\alpha(h-1)+\beta(h-1)^2]$, and we choose ψ such that $(1+x)^2 \simeq 1+2x = \psi^{-1}(1+x)$, that is $\psi = \frac{1+x}{1+2x} \simeq 1-x$, or $\psi = 0.95$ with $x = 0.05$.

We get :

$$A_h \simeq 1 + \frac{(h-1)\{[-\gamma-\delta(h-2)]\varepsilon_{t+1} + \alpha + \beta(2h-3)\} + \psi h(h-1)/2.[\gamma+\delta(h-2)]^2}{1+R(t,1)+\alpha(h-1)+\beta(h-1)^2}$$

and :

$$A_h - B_h = \frac{R(t,1) - (h-1)[\gamma-\delta(h-2)]\varepsilon_{t+1} + 2\alpha(h-1) + \beta[(h-1)^2+(h-1)(2h-3)] + \psi h(h-1)/2[(\gamma+\delta(h-2)]^2}{1+R(t,1)+\alpha(h-1)+\beta(h-1)^2}$$

and using the notation $h^\star = h-1$

$$A_h - B_h = \frac{R(t,1) - h^\star[\gamma+\delta(h^\star-1)]\varepsilon_{t+1} + 2\alpha h^\star + \beta[h^{\star 2} + h^\star(2h^\star-1)] + \psi\left(\frac{h^\star+1}{2}\right)h^\star[\gamma+\delta(h^\star-1)]^2}{1+R(t,1)+\alpha h^\star + \beta h^{\star 2}}$$

The term $\frac{1}{2}(h^\star+1)h^\star[\gamma+\delta(h^\star-1)]^2$ can be written :

$$\begin{aligned}
& \tfrac{1}{2}(h^\star + h^{\star 2})(\gamma^2 + 2\gamma\delta(h^\star-1) + \delta^2(h^\star-1)^2) \\
= \; & \tfrac{1}{2}(h^\star + h^{\star 2})[(\gamma-\delta)^2 + 2\gamma\delta h^\star + \delta^2 h^{\star 2} - 2\delta^2 h^\star] \\
= \; & \tfrac{h^\star}{2}(\gamma-\delta)^2 + h^{\star 2}\left(\tfrac{(\gamma-\delta)^2}{2} + \gamma\delta - \delta^2\right) \\
+ \; & h^{\star 3}\left(\gamma\delta - \delta^2 + \tfrac{\delta^2}{2}\right) \\
+ \; & h^{\star 4}\left(\tfrac{\delta^2}{2}\right) \\
= \; & \tfrac{h^\star}{2}(\gamma-\delta)^2 + \tfrac{h^{\star 2}}{2}(\gamma^2-\delta^2) + h^{\star 3}\delta\left(\gamma - \tfrac{\delta}{2}\right) + h^{\star 4}\tfrac{\delta^2}{2}
\end{aligned}$$

Therefore :

$$r_b = R(t,1) \quad -[(\gamma-\delta)D + \delta D_2]\varepsilon_{t+1} + (2\alpha-\beta)D + 3\beta D_2$$

$$+\psi(\gamma-\delta)^2\tfrac{D}{2} + \psi(\gamma^2-\delta^2)\tfrac{D_2}{2} + \psi\delta\left(\gamma - \tfrac{\delta}{2}\right)D_3 + \psi\tfrac{\delta^2}{2}D_4$$

$$
\begin{aligned}
\text{with} \quad D &= \frac{\sum_{h=1}^{H}(h-1)C^*_{t+h}}{\sum_{h=1}^{H}C^*_{t+h}} \\
D_j &= \frac{\sum_{h=1}^{H}(h-1)^j C^*_{t+h}}{\sum_{h=1}^{H}C^*_{t+h}} \quad j = 2,3,4 \\
C^*_{t+h} &= \frac{C_{t+h}}{[1+R(t,1)+\alpha(h-1)+\beta(h-1)^2]} \\
E^2 &= D_2 - D^2
\end{aligned}
$$

Formula (3) is obtained by taking $\delta = 0, \gamma = \sigma$.

APPENDIX 2

Taking into account a stock component in the liability return

The return of the liability is :

$$r_{L,t+1} = \lambda + \mu r_{b,t+1}^{D_L,E_L} + \nu r_{e,t+1} + \omega \xi_{t+1} \tag{46}$$

where $\xi_{t+1} \sim N(0,1)$ is independent of $r_{b,t+1}^{D_L,E_L}$ and $r_{e,t+1}$

We use formula (3) for the bond return.
The return of the zero-coupon bond with maturity 2 $(D = 1, E = 0)$ is :

$$r_{z,t+1} = R(t,1) + 2(\alpha + \beta) + \psi\sigma^2 - \sigma\varepsilon_{t+1} \tag{47}$$

$\varepsilon_{t+1} \sim N(0,1)$.

Moreover :

$$r_{b,t+1}^{D_L,E_L} = R(t,1) + (2\alpha - \beta)D_L + 3\beta(D_L^2 + E_L^2) + \psi\frac{\sigma^2}{2}(D_L^2 + D_L + E_L^2) - \sigma D_L \varepsilon_{t+1} \tag{48}$$

Eliminating ε_{t+1} between (49) and (50) we get :

$$\begin{aligned} r_{b,t+1}^{D_L,E_L} &= R(t,1) + (2\alpha - \beta)D_L + 3\beta(D_L^2 + E_L^2) + \psi\tfrac{\sigma^2}{2}(D_L^2 + D_L + E_L^2) \\ &+ D_L[r_{z,t+1} - R(t,1) - 2(\alpha+\beta) - \psi\sigma^2] \end{aligned}$$

and using (48) :

$$r_{L,t+1} = \lambda + \mu a[R(t,1), D_L, E_L] + \mu D_L r_{z,t+1} + \nu r_{e,t+1} + \omega\xi_{t+1} \tag{49}$$

with :

$$\begin{aligned} a[R(t,1), D_L, E_L] &= R(t,1) + (2\alpha - \beta)D_L + 3\beta(D_L^2 + E_L^2) + \psi\tfrac{\sigma^2}{2}(D_L^2 + D_L + E_L^2) \\ &- D_L[R(t,1) + 2(\alpha+\beta) + \psi\sigma^2] \end{aligned}$$

Using the variability of $\alpha, \beta, \gamma, \delta$, equation (51) allows, in theory, to estimate $\lambda, \mu, \nu, \omega, D_L, E_L$. An exogenous estimation of D_L, et E_L may be also useful. We can also impose that $r_{p,t+1}$ has the same mean as $r_{t,t+1}^{D_L,E_L}, \forall t$, and equation (48) becomes ("noisy" bond case) :

$$r_{L,t+1} = r_{b,t+1}^{D_L,E_L} + \nu(r_{e,t+1} - m_{e,t}) + \omega\xi_{t+1} \tag{50}$$

and only depends on two parameters ν and ω.

We can also impose $\lambda = 0, \mu + \nu = 1$ ("noisy" portfolio case).

Formulas (29) and (30) giving m_s and σ_s^2 must be modified. Let us first compute the mean m_L and the variance σ_L^2 of the liability return. We have :

$$\begin{aligned} m_L &= \lambda + \mu[R(t,1) + (2\alpha - \beta)D_L + 3\beta(D_L^2 + E_L^2) + \psi\frac{\sigma^2}{2}(D_L^2 + D_L + E_L^2) + \nu m_e \\ \sigma_L^2 &= \omega^2 + \mu^2\sigma^2 D_L^2 + \nu^2\sigma_e^2 + 2\mu\nu D_L \rho\sigma\sigma_e \end{aligned}$$

(ρ is the correlation between $r_{e,t+1}$ and $r_{b,t+1}$ or the opposite of the correlation between $r_{e,t+1}$ and ε_{t+1})

Moreover, since $r_{s,t+1} = F_t r_{a,t+1} - r_{L,t+1}$

and $r_{a,t+1} = w r_{e,t+1} + (1-w) r_{b,t+1}^{D,E}$

the mean and the variance of $r_{s,t+1}$ are :

$$\begin{aligned} m_s &= F_t \left\{ w m_e + (1-w)[R(t,1) + (2\alpha - \beta)D + 3\beta(D^2 + E^2) + \psi\frac{\sigma^2}{2}(D_L^2 + D_L + E_L^2)] \right\} \\ &- m_p \\ \sigma_s^2 &= F_t^2 \left\{ w^2\sigma_e^2 + (1-w)^2\sigma^2 D^2 + 2w(1-w)\rho\sigma_e\sigma D \right\} \\ &+ \sigma_L^2 - 2F_t \left\{ w\sigma_{Le} + (1-w)\sigma_{Lb} \right\} \end{aligned}$$

where σ_{Le} is the covariance between $r_{L,t+1}$ and $r_{e,t+1}$ that is :

$$\sigma_{Le} = \mu\rho\sigma_e D_L \sigma + \nu\sigma^2$$

and where σ_{Lb} is the covariance between $r_{L,t+1}$ and $r_{b,t+1}^{D,E}$ that is :

$$\mu\sigma^2 D_L D + \nu\rho\sigma_e D\sigma$$

The surplus VaR constraint is :

$$m_s - 1,65\sigma_s > u_s$$

R E F E R E N C E S

Amenc N., L. Martellini, P. Foulquier and S. Sender (2006) : "The Impact of IFRS and Solvency II on Asset-Liability Management and Asset Management in Insurance Companies", EDHEC Discussion paper.

Cox, J.C, J.E. Ingersoll and S.A. Ross (1987) : "A Theory of the Term Structure of Interest Rates", Econometrica, 53 (2) 1943-1978.

Diebold F.X et C. Li (2006) : "Forecasting the Term Structure of Government Bond Yields", Journal of Econometrics, 130, 337-364.

Duffie, D. and R. Kan (1986) : "A Yield-Factor Model of Interest Rates", Mathematical Finance, 6, 379-406.

Gourieroux, C. and A. Monfort (2007) : "Econometric Specifications of Stochastic Discount Factor Models", Journal of Econometrics, 136, 509-530.

Gourieroux, C., A. Monfort and V. Polimenis (2002) : "Discrete Time Affine Term Structure Models", CREST Discussion paper.

Leibowitz M.L, L.N. Bader and S. Kogelman (1996) : "Return Targets and Shortfull Risks", Mc Graw Hill.

Monfort A. and F. Pegoraro (2006) : "Switching VARMA Term Structure Models", Journal of Financial Econometrics, 5, 1, 105-153.

Vasicek, O. (1977) : "An Equilibrium Characterization of the Term Structure", Journal of Financial Economics, 5, 177-188.