A Note on the Smith-Wilson Family

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Abstract

The Smith-Wilson approach is the benchmark for the extrapolation of yield curves in Solvency 2. We show in this note that this standard modelling is not compatible with the market consistency required in Solvency 2, that is it allows for perfect arbitrage opportunities.

Keywords : Yield Curve, Arbitrage Opportunity, Ultimate Forward Rate Smith-Wilson, Solvency 2.

1 Introduction

Smith, Wilson (2000) [see also Thomas, Mare (2007)] proposed a class of models for the prices of zero-coupon bonds, with special emphazis on two parameters that are a long term forward rate, called ultimate forward rate (UFR), and another parameter controlling the speed of convergence to the UFR, that is the degree of smoothness of the exprapolation.

The zero-coupon bond pricing formula is :

$$B(t,h) = e^{-f_\infty h} + \sum_{j=1}^{J} F_{j,t} e^{-f_\infty (h+u_j)} K_j(h), \tag{1.1}$$

$$\text{where} \quad K_j(h) = \alpha \min(h,u_j) - e^{-\alpha \max(h,u_j)} sinh[\alpha \min(h,u_j)], \tag{1.2}$$

(K_j is called the Wilson function) and $sinh(x) = \dfrac{e^x - e^{-x}}{2}$ denotes the hyperbolic sine function. In the formula above $B(t,h)$ denotes the price at date t of the zero-coupon bond with time-to-maturity h, f_∞ the UFR, α the smoothing parameter, J the number of liquid zero-coupon prices used for calibration, and $u_j, j = 1,\ldots,J$ their time-to-maturity. The formula (1.1)-(1.2) is a nonlinear spline approximation of the term structure of zero-coupon prices with nodes at $u_j, j = 1,\ldots,J$. When the coefficients $F_{j,t}$ are nonnegative and the UFR sufficiently large, it ensures a decreasing term structure of prices (see Appendix 1).

In the Solvency 2 practice [see e.g. European Commission (2010) a,b], the parameters f_∞ and α are assumed time independent, fixed by the supervision : typically the UFR is set to 4.2 % for the Eurozone, 5.2% for Brazil, Mexico, India, to 3.2% for Japan and Switzerland [QIS5 (2010), Appendix A], whereas α is set to 0.1 following the analysis in Thomas, Mare (2007). Then the coefficients $F_{j,t}, j = 1,\ldots,J$ are calibrated date by date by applying the pricing formula (1.1)-(1.2) to the observed prices of the set of liquid zero-coupon bonds, that is to $h = u_j, j = 1,\ldots,J$.[2]

The advantages of this approach are clear [see Q1S 5 (2010), Section 6.3] : this is a simple, linear, mechanized approach which can be implemented on

[2]The formula can also be applied with a set of liquid zero-coupon bonds varying in time, or with coupon bonds and swaps.

excel. It provides a perfect fit for the liquid zero-coupon bonds used in the calibration step and can be used uniformly among countries and insurance companies. Finally it gives a role to the macroeconomic supervision by the way the UFR and smoothing parameter are set.

However, contrary to a common belief [see e.g. Thomas, Mare (2007), p16 l4], the Smith-Wilson modelling is not consistent with the absence of arbitrage although this condition is required in Solvency 2 [see e.g. the document of the European Commission (2010), TP 2.96].[3] The aim of this note is to show this lack of consistency. In this respect our analysis is parallel to the analysis by Filipovic (1999), who proved the lack of consistency of the standard Nelson, Siegel extrapolation method [Nelson, Siegel (1987)]. [4]

The lack of consistency is discussed in Section 2 and the proof is given in an Appendix.

2 The lack of market consistency of the Smith-Wilson approach.

The Smith-Wilson model is a special case of the term structure models, which are affine in zero-coupon bond prices, that are such that :

$$B(t,h) = \sum_{j=0}^{J} F_{j,t} b_j(h) \equiv F_t' b(h). \tag{2.1}$$

In such models the term structure of prices at date t is written as a combination of baseline term structures $b_j(.), j = 0, \ldots, J$, with time dependent coefficients. Such affine models in prices have been analyzed in Gourieroux, Monfort (2013), and have to be distinguished from the affine models written on interest rates [see e.g. Duffie, Kan (1996), Duffee (2002), Gourieroux, Monfort, Polimenis (2006)].

[3] More generally several components of Solvency 2 are not time consistent, not only the term structure modelling [see e.g. Pelsser (2011), Gourieroux, Monfort (2015)].

[4] Note that despite its title the arbitrage free Nelson-Siegel model in Christensen, Diebold, Rudebusch (2009) is not arbitrage free. Indeed the long run interest rate tends to $-\infty$ (see footnote 6 in their paper).

2.1 A necessary consistency condition

A perfect fit to liquid zero-coupon prices is a necessary condition for no arbitrage. However it is not sufficient since it does not account for the possibility of dynamic arbitrages. Let us now consider the simple arbitrages between holding a zero-coupon up to maturity h and investing in a short term bond to buy next date a zero-coupon with time-to-maturity $h-1$. The no arbitrage opportunity implies :

$$\begin{aligned} B(t,h) &= E_t[B(t,1)B(t+1,h-1)] \\ &= B(t,1)E_t[B(t+1,h-1)], \forall h \geq 2, \end{aligned} \tag{2.2}$$

where E_t denotes the risk-neutral expectation conditional on the information available at date t and we use the fact that $B(t,1)$ is known at date t.

Thus for an affine model in price, we get :

$$\begin{aligned} & F_t'b(h) = F_t'b(1)E_t[F_{t+1}'b(h-1)] \\ \iff \quad & F_t'b(h) = F_t'b(1)(E_tF_{t+1})'b(h-1) \\ \iff \quad & (E_tF_{t+1})'b(h-1) = F_t'b(h)/F_t'b(1). \end{aligned} \tag{2.3}$$

This relation has to be satisfied for any term $h \geq 2$, but also for any environment F_t.

This large system of equations imply joint restrictions on the baseline functions $b_j, j = 1, \ldots, J$, and on the risk-neutral dynamics of the coefficients (factors) F_t. The implications of these restrictions are well-illustrated in the single factor case where $J = 1, F_{0,t} = 1$, which includes the Smith-Wilson model with $J = 1$. In this framework it is always possible to change the definition of factor F_t in (2.1) to get $B(t,1) = F_{1,t}$, since the factor is defined up to an affine invertible transformation. Then condition (2.3) becomes :

$$\begin{aligned} & b_0(h-1) + E_t(F_{1,t+1})b_1(h-1) = [b_0(h) + F_{1,t}b_1(h)]/F_{1,t}, \forall h, F_{1,t}, \\ \iff \quad & E_t(F_{1,t+1}) = \frac{1}{b_1(h-1)}[\frac{b_0(h)}{F_{1,t}} + b_1(h) - b_0(h-1)], \forall h, F_{1t}. \end{aligned}$$

Whenever $F_{1,t}$ can take at least two possible values, we deduce that there exist two constants λ and μ such that :

$$b_0(h)/b_1(h-1) = \lambda, [b_1(h) - b_0(h-1)]/b_1(h-1) = \mu, \forall h \geq 2,$$

$$\Leftrightarrow \begin{cases} b_0(h) - \lambda b_1(h-1) = 0 \\ \\ b_1(h) - b_0(h-1) - \mu b_1(h-1) = 0, \forall h \geq 2. \end{cases}$$

This system is easily solved. For instance, we see that the baseline term structure $b_1(.)$ satisfies the linear recursive equation of order 2 :

$$b_1(h) - \mu b_1(h-1) + \lambda b_1(h-2) = 0, \forall h \geq 3,$$

with the initial condition $b_1(1) = 1$. Thus it is a combination of two exponential functions associated with the roots of the second-order polynomial $x^2 - \mu x + \lambda = 0$. The form of the other baseline $b_0(.)$ follows. It is immediately checked that the function $e^{-f_\infty(h+u_1)} K_1(h)$ is not of this form.

This result is easily extended to any number J of liquid zero-coupon bonds. Indeed the condition for no arbitrage opportunity implies the existence of a (J, J) matrix C such that (see Gourieroux, Monfort (2013), Proposition 1, or appendix 2) :

$$b(h)' = b(1)' C^{h-1}.$$

In particular each baseline term structure b_j is a combination of exponential functions, possibly multiplied by polynomials. As already noted the Wilson function (multiplied by the appropriate exponential) is not of this form.

This lack of time consistency of the Smith-Wilson approach is a consequence of the way it has been derived. More precisely let us denote $\tilde{B}(t,h) = e^{f_\infty h} B(t,h)$, the zero-coupon prices adjusted by the UFR. Then the Smith-Wilson formula has been derived as the solution of the minimization of a smoothness criterion : $\frac{1}{2\alpha^3} \int_0^\infty \frac{\partial^2}{\partial h^2} \tilde{B}(t,h) dh + \frac{1}{2\alpha} \int_0^\infty [\frac{\partial}{\partial h} \tilde{B}(t,h)]^2 dh$, function of the slope and curvature at date t, submitted to the constraints of exact pricing of the J liquid zero-coupon bonds at date t. This optimization problem is static, that is specific to date t, and cannot incorporate the notion of absence of dynamic arbitrage opportunity.

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Appendix 1

Decreasing baseline term structures

In the Smith-Wilson model the baseline term structure on zero-coupon prices is :

$$g(h) = e^{-f_\infty (h+u)}[\alpha \min(h,u) - e^{-\alpha \max(h,u)} \frac{e^{\alpha \min(h,u)} - e^{-\alpha \min(h,u)}}{2}]$$

This baseline function has the following expressions in the two possible regimes :

If $h \leq u$,

$$g_1(h) = e^{-f_\infty (h+u)}[\alpha h - e^{-\alpha u} \frac{e^{\alpha h} - e^{-\alpha h}}{2}].$$

If $h \geq u$,

$$g_2(h) = e^{-f_\infty (h+u)}[\alpha u - e^{-\alpha h} \frac{e^{\alpha u} - e^{-\alpha u}}{2}].$$

The associated derivatives are :

$$\begin{aligned}
\frac{dg_1(h)}{dh} &= e^{-f_\infty (h+u)}[-f_\infty(\alpha h - e^{-\alpha u} \frac{e^{\alpha h} - e^{-\alpha h}}{2}) \\
&+ \alpha - \alpha e^{-\alpha u} \frac{e^{\alpha h} + e^{-\alpha h}}{2}], \text{if } h \leq u. \\
\frac{dg_2(h)}{dh} &= e^{-f_\infty (h+u)}[-f_\infty(\alpha u - e^{-\alpha h} \frac{e^{\alpha u} - e^{-\alpha u}}{2}) \\
&+ \alpha e^{-\alpha h} \frac{e^{\alpha u} - e^{-\alpha u}}{2}], \text{if } h \leq u.
\end{aligned}$$

Since $\alpha > 0, u > 0, f_\infty > 0$, it is easily checked that $\frac{dg_1(h)}{dh} < 0$, if $h \leq u$, and $\frac{dg_2(h)}{dh} < 0$, if $h \geq u$, when the UFR f_∞ is sufficiently large.

Thus if the UFR is sufficiently large and the coefficients $F_{j,t}$ nonnegative, the Smith-Wilson approach ensures nonnegative zero-coupon prices decreasing in h. Thus the frequent remark on the possibility to get partly increasing zero-coupon prices [see e.g. QIS 5, (2010) p15, Bouwman et al. (2012)] is likely due to a too small standard UFR set by the supervision.

Appendix 2

The arbitrage free term structures

Let us consider a portfolio of zero-coupon bonds with allocations $a_h, h = 1, \ldots, H$. Its value at date t is :

$$P_t(a) = \sum_{h=1}^{H} a_h B(t,h),$$

whereas its value at $t+1$ is :

$$P_{t+1}(a) = \sum_{h=1}^{H} a_h B(t+1, h-1), \text{with } B(t+1,0) = 1.$$

This portfolio is riskfree if and only if,

$$\sum_{h=2}^{H} a_h b(h-1) = 0, \tag{a.1}$$

and its future value is $P_{t+1}(a) = a_1$.

When it is riskfree, we get by no arbitrage the condition :

$$\begin{aligned} & P_{t+1}(a) B(t,1) = P_t(a) \\ \iff & \sum_{h=2}^{H} a_h B(t,h) = 0 \\ \iff & \sum_{h=2}^{H} a_h b(h) = 0. \qquad \text{(a.2)} \end{aligned}$$

Thus any allocation satisfying Eq.(a.1) satisfies Eq (a.2). By Farkas' Lemma [see e.g. Rockafellar (1979), p200], there exist a matrix C of Lagrange multipliers such that :

$$\begin{aligned}
\begin{bmatrix} b(2)' \\ \vdots \\ b(H)' \end{bmatrix} &= \begin{bmatrix} b(1)' \\ \vdots \\ b(H-1)' \end{bmatrix} C \\
\iff b(h)' &= b(h-1)'C, \forall h, \\
\iff b(h)' &= b(1)'C^{h-1}, \forall h.
\end{aligned}$$