

# Optimal incentives for labor force participation

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## Abstract

Optimal taxation is analyzed under a Rawlsian criterion in an economy where the only decision of the agents is to participate, or not, to the labor force. The model allows for heterogeneity both in the agent's productivities and aversions to work. At a first best optimal schedule, the marginal agent who decides to work pockets all of her productivity, while being just compensated for her work aversion. When the planner does not observe work aversion, financial compensation for work is lower than productivity. Theory puts little restrictions on the shape of the optimal tax schedules. The usual first order conditions involving the elasticities of participation only apply for sufficiently regular economies. We qualitatively show how the optimal incentives schemes depend on the underlying structure of the preferences: 100% marginal tax rates or subsidies to work are related to specific features of the economies.

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## Introduction

The equity efficiency dilemma is as old as economics. But it has been the object of renewed attention in the recent years in developed economies, confronted with the widening of the wage distribution, sometimes attributed to technical progress biased against low skilled workers or to trade globalization. Policy issues center around the type of benefits that should be given to people that are out of the labor force, and to the disincentives from work that such benefits might create.

Each country has a social minimum or safety net of its own. Some basic income support (sometimes also called negative income tax) is provided to all households in many European countries (the *Revenu Minimum d'Insertion* in France is an example). These programs often involve high marginal tax rates (100% for the RMI) in the range of incomes where the subsidy is phased out. To 'make work pay', some countries have implemented earning subsidies, which involve negative marginal tax rates: the Earned Income Tax Credit in the US and the Working Families Tax Credit in the UK are examples of such programs.

A number of descriptive studies undertake to measure the relevant participation elasticities and to provide a cost benefit analysis of the social minima. But, unfortunately, the normative approach has not been very fruitful. Indeed, the relevant framework of optimal taxation, which goes back to the seminal paper of Mirrlees (1971), seems too far from the tax-benefit systems observed in practice to be a useful guide for policy. Sadka (1976b) studies the shape of the income distributions that follow from the theory. When effort depends on financial incentives, at the *intensive* margin, the standard result has a zero marginal tax rate on the rich (which goes contrary to the common idea of equity, and is not observed, even in the US) and a zero marginal tax rate on the very poor, at least when they do work. The marginal tax rate is always non negative, which rules out pushing people to work through an earning subsidy, such as in the EITC.

It would be useful if the theory were more easily reconciled with the facts and one of the aims of the paper is to bring them closer. A number of researchers have worked in the same direction. Diamond (1998) and Salanié (1998) show that the 'don't tax the rich' result does not hold when the distribution of wages has a fat enough upper tail. Piketty (1997) studies the optimal taxation program in the intensive framework with a Rawlsian criterion. The interesting work of Saez (2001) uses the available empirical evidence on the shape of the wage distribution and labor supply elasticities to compute optimal tax schedules in an intensive model *à la* Mirrlees. Although the schedules are not too far from what we see in practice, they do always have non negative marginal tax rates.

More to the point that we are interested in, following the seminal example of Diamond (1980), Saez (2002) and Beaudry and Blackorby (1997) have worked on models of optimal income taxation with an *extensive* margin in order to look for properties of the optimal taxation schemes when agents may choose to participate or to stay out of the labor force. One purpose of Saez (2002) is to see whether one of the features of the EITC and of the WFTC, namely the local negative marginal tax rates which give large incentives to work to the concerned individuals, is compatible with optimal taxation in an extensive setup<sup>1</sup>.

In the present work, we focus on the participation decision and work on a model where the only choice of the agents is to work or to stay out of the labor force. We further assume that the financial compensation that makes an agent indifferent

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<sup>1</sup>Besley and Coate (1995) address a similar issue, but in a quite different setup. They assume that the government aims for a minimum consumption level, not accounting for the disutility of work. This makes their results difficult to compare with the standard optimal taxation literature.

between working and not working, her *work aversion*, is nondecreasing with her utility level: the richer she is, the more money she asks to go to work. In contrast with the previous literature, we mainly study the optimal taxation scheme under the Rawlsian criterion, which undertakes to maximize the welfare of the less favored agents in the economy. We discuss in the last section how some of our results carry over to the utilitarian case: the Rawlsian case turns out to be a natural benchmark for a utilitarian government.

In this setup, we first study the properties of the first best allocations. Given the social utility level, an agent can be on the dole, receive the social minimum and produce nothing, or work, generate an output equal to her productivity, and get the financial compensation necessary to be indifferent between working or not working. The social planner then decides to put her to work whenever her productivity exceeds her work aversion. When an agent crosses the border line between unemployment and employment, her income is discontinuous and increases by an amount equal to her productivity, which just compensates her for the penibility of work: her utility does not change.

The income schedules implemented in practice, even the ones most favorable to work incentives, such as the EITC in the US, do not exhibit such upwards discontinuities. We turn to second best situations, where the government is unable to observe the work aversions of the agents, to see whether this can get us closer to the observations. A crucial feature of the economy, which is central in the analysis, then turns out to be the distribution of work aversions of the agents, conditional on their productivity levels, when productivity varies. In the special case where this distribution is constant, independent of the productivity level, we completely characterize the second best allocations. The income schedule is an increasing function of productivity; the financial compensations granted to the agents are always strictly smaller than their productivities; the unemployment rate decreases with productivity.

One important result of the paper is the converse. Assume that productivities are bounded in the economy. Take any after tax income schedule which is a nondecreasing function of productivity, such that the financial incentive to work is always strictly smaller than productivity. Then there exists an economy with a suitable distribution of work aversions, which can be chosen independent of productivity, whose Rawlsian optimal tax scheme coincides with the given schedule. Theory by itself puts little restrictions on the shape of the tax schemes.

The paper is organized as follows. The model and notations are in Section 1. First best allocations are described in Section 2, followed by the characterization of second best allocations in Section 3. Section 4 presents a qualitative and graphical analysis of the associated income tax schedules. Section 5 shows that theory puts little restrictions on the properties of these schedules. Section 6 partially extends our results on the Rawlsian case to a utilitarian government. All the proofs are in an Appendix available on our homepage on the web.

## 1 The model

We consider an economy made of a continuum of agents. A typical agent is described by a set of exogenous characteristics, denoted by  $a = (w, x)$ , which include her tastes and (non negative) productivity  $w$ .

The agents do not differ only by their working abilities  $w$ , but also by their (lack of) taste for work. The utility function of a typical agent is a function of the non negative quantity  $c$  of commodity which she receives, and depends on the participation decision. The function  $v(c)$  represents the utility of the non participating agent, while  $u(c; a)$  is her utility when working.

We assume that the utility function of the unemployed persons is fixed, common to everyone in the economy. This simplifying assumption is unimportant as far as the participation decision of the agents is concerned (if  $v$  depends on  $a$ , it is always possible to renormalize the couple  $(u, v)$  while keeping labor supply unchanged). But it does play a role when formulating the government objectives. Our assumption means that the social planner does not make any distinction between the unemployed persons. All the unemployed individuals have the same contribution to social welfare. This is a value judgment which makes the analysis much simpler.

The functions  $v(\cdot)$  and  $u(\cdot; a)$ , for all  $a$ , are continuously differentiable with strictly positive derivatives, and they go to  $+\infty$  with their argument. The functions  $u(c; \cdot)$  and  $u'_c(c; \cdot)$  are continuous. The cumulative distribution of individual characteristics  $a = (w, x)$  is denoted  $F$ . We assume that the aggregate resources in the economy are finite, i.e. that  $w$  is integrable.

The only choice of the agent in our model is whether to participate, or not, in the work force. The participation status of agent  $a$  is described with a function  $s(a)$ , where  $s(a)$  is equal to 0 or 1. When agent  $a$  participates ( $s(a) = 1$ ), she produces  $w$  units of commodity, while she does not produce anything when she does not participate ( $s(a) = 0$ ).

The analysis relies heavily on a measure of the disutility of work, which we call *work aversion*, and note  $\Delta(c; a)$ . The work aversion is the minimum (possibly negative) income supplement which makes agent  $a$  indifferent between working or living on resources  $c$  without working, i.e. the unique solution in  $\Delta$  of the equation

$$u(c + \Delta; a) = v(c),$$

when such a solution exists. Otherwise we define it as  $+\infty$ , when the agent does not want to work, whatever the wage ( $\lim_{x \rightarrow +\infty} u(x; a) < v(c)$ ), or to  $-c$  when she always wants to work ( $u(0; a) > v(c)$ ). Note that, by the implicit function theorem, the function  $\Delta$  is continuously differentiable with respect to  $c$  when it is finite larger than  $-c$ . We postulate

**Assumption 1** *Whenever defined,  $\Delta(c; a)$  is a nondecreasing function of  $c$ . It is continuously differentiable on its domain whenever larger than  $-c$ .*

The larger income when unemployed, the larger the required income *supplement* to make it worthwhile to take a job. Assumption (1) is the translation in our setup of the idea that leisure is a normal good: the supply of labor is a decreasing function of the level  $c$  of income when not working. Indeed, given a gross income at work  $c + D$ , the agent's labor supply is equal to zero when  $D$  is smaller than  $\Delta(c; a)$ , and equal to one otherwise. Then the fact that  $\Delta(c; a)$  increases with  $c$  implies that labor supply decreases with  $c$ .

An allocation describes the employment status and the income of all the agents in the economy. Formally, it is defined as a pair of integrable functions  $s(a)$  and  $c(a)$  with values respectively in  $\{0, 1\}$  and  $\mathbb{R}_+$ . An allocation  $(s(\cdot), c(\cdot))$  is feasible when total consumption is equal to total production, i.e.:

$$\int c(a) dF(a) = \int_{s(a)=1} w dF(a). \quad (1)$$

At the *laissez-faire* allocation, an agent decides to work when her productivity makes it worthwhile, in comparison with a zero income when non participating, i.e. when

$$u(w; a) \geq v(0),$$

with indifference when there is equality.

Such an allocation can be very unequal, and it is of interest to look at redistribution schemes that tax the rich workers, with high  $w$ 's, and give the proceeds to the unemployed. Such a redistribution scheme typically reduces the incentives to work. Indeed if  $R(a)$ ,  $R(a) \leq w$ , is the income given to worker  $a$ , and  $r$ ,  $r \geq 0$ , the subsistence level attributed to the unemployed, the decision to work under the redistribution scheme is associated with the inequality

$$u(R(a); a) \geq v(r),$$

which is always more stringent than at the market allocation. The purpose of the paper is to look at the tradeoff between equity (more equal utility levels) and efficiency (loss of output due to non participation generated by redistribution) depending on the government objective and to see whether the optimal taxation schemes exhibit some general properties.

There are a number of possible ways to represent society's preferences among equity-efficiency tradeoffs. The one most used in the optimal taxation literature, following the seminal work of Mirrlees (1971), is close to utilitarianism. There is an increasing concave function  $\Psi$ , whose concavity is an indicator of society's desire for equality, such that, when  $c(a)$  is allocated to an agent of type  $a$ , welfare can be written as

$$W_U(c, s) = \int_{s(a)=1} \Psi[u(c(a); a)] dF(a) + \int_{s(a)=0} \Psi[v(c(a))] dF(a).$$

Diamond (1980) presents an example of an optimal utilitarian tax schedule with fixed hours of work. We shall mainly focus in this paper<sup>2</sup> on the Rawlsian criterion, which considers the utility of the worse off agents in the economy. Here it is equal to the lower bound of the support of the distribution of utilities in the population, i.e. its essential infimum:

$$W_R(c, s) = \text{ess inf} \{u(c(a); a)\mathbf{1}_{s(a)=1} + v(c(a))\mathbf{1}_{s(a)=0}\}.$$

Following tradition, we study the social planner choice in stages, starting with the case of complete information of the planner (first best), following with the situation where the planner only observes the productivity  $w$  of the workers (second best).

## 2 First best allocations

The first best allocations are obtained when the planner observes the agents' types  $a$ . The planner decides whether an agent of type  $a$  works or not and attributes her an income  $c(a)$  under the feasibility condition.

With a Rawlsian criterion, all efforts are made so that everybody gets the same level of utility. It then is worthwhile putting someone to work if and only if her productivity is larger than the extra income necessary to compensate her for the penibility of work.

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<sup>2</sup>The last section presents some extensions of our results to the utilitarian setup.

The first best Rawlsian allocation  $(c(a), s(a))$  can be derived in two steps. First, we take the value of the welfare  $W_R$  as a parameter and derive properties of an optimal allocation  $(c(a), s(a))$ . In a second step, left for the Appendix, the endogenous value of  $W_R$  is obtained through the government budget constraint.

The qualitative properties of an optimal allocation follow from the simple argument below:

- For each agent, we define  $R(a)$  (resp.  $r$ ) as the *minimum* nonnegative income that ensures that agent  $a$ 's utility is at least as large as  $W_R$  when she works (resp. does not work). Since all transfers between agents are possible, at the optimum, agent  $a$  collects  $c(a) = R(a)$  when she works and  $c(a) = r$  when she does not;
- It must be the case that the government cannot obtain a higher revenue without deteriorating the welfare. The working rule  $s(a)$  therefore maximizes the government revenue. The problem is simply to compare the government revenue when agent  $a$  works ( $w - R(a)$ ) and when she does not ( $-r$ ).

Formally, this is summarized in the following<sup>3</sup>:

**Theorem 1** *Suppose that the optimal Rawlsian allocation  $(c(a), s(a))$  leads to a social utility level  $W_R$ ,  $W_R \geq v(0)$ .*

*Let  $r$  be such that  $v(r) = W_R$ . Let  $R(a)$  be equal to zero if  $u(0; a) \geq W_R$  (or  $\Delta(r; a) = -r$ ) and to the unique solution of the equation  $u(R(a); a) = W_R$  otherwise. Then the incomes  $c(a)$  of the agents are equal to  $R(a)$  when they work and  $r$  when they do not work. Furthermore their employment status  $s(a)$  is given by*

$$\begin{aligned} w > \Delta(r; a) &\implies s(a) = 1 \\ w < \Delta(r; a) &\implies s(a) = 0. \end{aligned} \tag{2}$$

*The work status and allocation are indeterminate on the border line, when  $w = \Delta(r; a)$ . Then society is indifferent between having agent  $a$  working with income  $R(a)$  or not working with income  $r$ .*

The workers are exactly compensated for the penibility of their labor and receive an income  $R(a) = r + \Delta(r; a)$ . All agents such that  $w > \Delta(r; a)$  are working at the first best Rawlsian optimum, while all agents such that  $w < \Delta(r; a)$  are not working and receiving  $r$ . On the other hand, income is disconnected from productivity away from the pivot, being equal either to the subsistence level, or to the reservation wage. Note that taxes will be progressive when aversion for work does not depend on income: in fact all income above work aversion is taxed away at the optimum. But, if aversion for work is increasing with income, with a derivative larger than one, the optimal tax is regressive<sup>4</sup>.

When work aversion does not depend on the maintenance income  $r$ , the first best and laissez faire employment status in the economy coincide. When  $r$  is positive and  $\Delta$  strictly increases with  $r$ , the first best employment rate is typically smaller than the market rate.

It is of interest to look more explicitly at how the first best employment status of the agents depends on their types. Fix  $x$ , and let productivity  $w$  vary. The inequalities (2) determine who is employed and who is not. Since in general the work

<sup>3</sup>The elementary proof is left to the reader. Proposition A.1 in the Appendix shows that there exists a unique first best Rawlsian optimum.

<sup>4</sup>The standard study on the progressivity of the optimal income tax is in Sadka (1976a), when all agents have the same utility. Here attitudes towards labor differ across individuals.

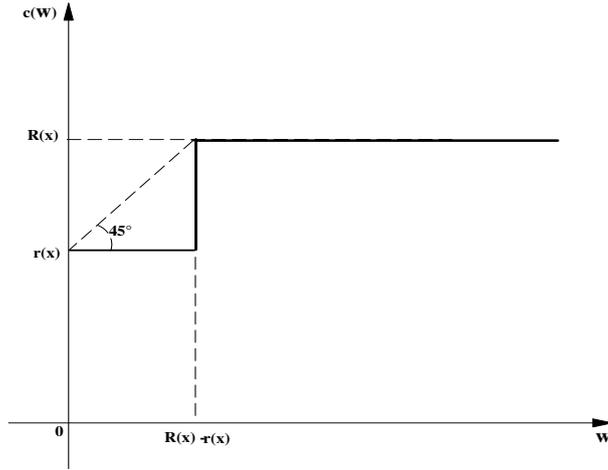


Figure 1: First best allocation:  $R - r = w$  for the pivotal agent

aversion varies with  $w$ , these inequalities are implicit in  $w$ : the set of productivities of the working agents is not necessarily a half line, of the form  $w \geq \omega$ , but can be a union of intervals. At the optimum, around any pivotal agent  $\omega$ , income is discontinuous (but, of course, utility is constant, equal to  $W_R$ ). If the agents do not work for  $w$  smaller than  $\omega$ , they receive  $c(a) = r$ , while the agents who work with  $w$  larger than  $\omega$  get  $c(a) = r + \Delta(r, a) = R(\omega_+, x)$ . The inequality (2) expresses the fact that the discontinuity  $R(\omega_+, x) - r = \Delta(r; a)$  is equal to the productivity  $\omega$  of the pivotal agent.

Consider the particular case where the utility functions  $u$  and  $v$  do not depend on the productivity  $w$ , but only on  $x$ . Then the work aversion  $\Delta$  only depends on  $x$ . Also the net resource  $R$ , as defined in Theorem 1, depends on  $x$ , but not on  $w$ . In that case, inequality (2) gives explicitly the set of workers: the workers are all the agents with productivity  $w$  higher than the threshold  $\Delta(r; x)$ . Figure 1 represents the income collected by agent  $(w, x)$  as a function of  $w$ , for a fixed  $x$  (recall that, in the first best situation, the government observes  $x$  and can condition the schedules on  $x$ ). In the picture, we have assumed that agents with productivity lower than  $R(x) - r(x)$  do not work, while those with a larger productivity are in the labor force<sup>5</sup>.

When everyone has the same (constant) disutility  $\Delta$  of work and the only information unknown to the government is the individual productivities, the first best result translates directly into an income schedule with a shape similar to that of Figure 1. After tax income is equal to  $r$  for all before tax income smaller than  $\Delta$ , to  $r + \Delta$  when before tax income is larger than  $\Delta$ . This amounts to a marginal tax rate equal to  $-\infty$  at the switching point, a large downward tax discontinuity. Such negative income taxes are not seen in practice. Quite the contrary, in France, apart from a temporary subsidy, there is a 100% marginal tax rate on earnings when one takes a job; in the US, a similar feature was associated with the Aid to Families with Dependent Children program before the 1996 reform. It was partially mitigated by the Earned Income Tax Credit. The EITC can amount to 40% of earnings, i.e. each dollar earned yields 1.4 dollar for the wage earner. After the welfare reform and the replacement of AFDC with TANF, there may exist some income schedules in some states with a zone of negative marginal tax rates. Still, this is far from the kind of discontinuities described above. But of course in practice work aversions are

<sup>5</sup>Another example in a two-type economy is described in detail below in section 4.4.

heterogeneous in the population and unobserved by the fiscal authorities: this fact is likely to smooth the shape of the optimal subsidy scheme, as will be seen below.

### 3 Characterization of second best Rawlsian allocations

We assume that agent  $a$ 's productivity  $w$  is observed by the government *only when agent  $a$  works*, but that no other individual characteristics of the agent can be used to base the tax-subsidy scheme. The government, however, knows the (typically non degenerate) distribution of individual characteristics in the economy. Here, the assumption that the government considers as identical the welfare of all the non working agents is crucial. The analysis extends without much difficulty to situations where the government observes the utility levels of these agents (see Laroque (2002)); it seems to be much harder otherwise.

#### 3.1 From incentive compatibility to schedules

In the game between the government and the agents, the government acts as a Stackelberg leader (the government is the 'principal'). We characterize the incentive compatible allocations  $(c(a), s(a))$  in that game, given the informational structure. By definition, a Rawlsian second best optimum maximizes  $W_R$  among all feasible incentive compatible allocations.

It is convenient to introduce a function  $U(c, s; a)$ , defined by

$$U(c, 1; a) = u(c; a) \quad \text{and} \quad U(c, 0; a) = v(c).$$

Then an allocation  $(c(a), s(a))$  is *incentive compatible* if

1.  $c(a)$  depends only on  $w$  when  $s(a) = 1$ ,
2.  $c(a)$  is a constant  $c_0$ , independent of  $a$ , when  $s(a) = 0$ ,
3. for all  $a'$  such that either  $s(a') = 0$  or  $w(a') = w(a)$

$$U(c(a), s(a); a) \geq U(c(a'), s(a'); a). \quad (3)$$

Equation (3) expresses the fact that agent  $a$  can mimic any agent  $a'$  who either does not work or has the same productivity as  $a$ .

Suppose that the government posts a menu  $(r, R(w))$ ,  $r \geq 0$ , of income schedules, respectively  $R(w)$  for the (potential) workers and  $r$  for the (potential) unemployed. Facing such a menu, an agent  $a$  chooses either to work and receive  $R(w)$  or not to work and receive  $r$ . This gives rise to an allocation  $(c(a), s(a))$ . It is very easy to check that this allocation is incentive compatible<sup>6</sup>. Let us say that an allocation  $(c(a), s(a))$  is implementable when it derives from a schedule  $(r, R(w))$ . Lemma B.1 in the Appendix states the following result: Given any incentive compatible feasible allocation  $(c(a), s(a))$ , there exists an implementable feasible allocation  $(c'(a), s'(a))$  which Pareto dominates  $(c(a), s(a))$ . Furthermore the corresponding subsistence income  $r$  can be chosen so that  $v(r) = \text{ess inf } U(c(a), s(a); a)$ .

Therefore, without loss of generality, we can work with schedules  $(r, R(w))$  rather than with allocations  $(c(a), s(a))$ . Actually, it will turn out to be more convenient to use the quantity  $D(w) = R(w) - r$ , which measures the financial incentives to

<sup>6</sup>Note that the value of the Rawlsian welfare for the allocation  $(c(a), s(a))$  is at least as large as  $v(r)$ . It is equal to  $v(r)$  when at least one agent does not work.

work, rather than the gross income  $R(w)$ . Since  $R(w) \geq 0$ , the incentive  $D(w)$  is larger than  $-r$ . When she is in front of such a menu  $(r, D(w))$ , agent  $a$  decides to work when  $\Delta(r, a) \leq D(w)$ , with indifference in case of equality.

We note  $G_{r,w}$  the c.d.f. of the distribution of work aversions  $\Delta(r, a)$  conditional on the agent productivity  $w$

$$G_{r,w}(D) = \Pr (\Delta(r; a) \leq D \mid w).$$

The normality of leisure (Assumption 1) implies that  $G_{r,w}(D)$  is a nonincreasing function of  $r$ . Suppose now that the government posts a schedule  $(r, D(w))$ . Then the probability that an agent with productivity  $w$  works when she faces this schedule is  $G_{r,w}(D(w))$ . The government revenue under the scheme  $(r, D(w))$  can be written as

$$\begin{aligned} T(r, D(\cdot)) &= \int [w - D(w)] \mathbf{1}_{\Delta(r;a) \leq D(w)} dF(a) - r \\ &= \int [w - D(w)] G_{r,w}(D(w)) d\tilde{F}(w) - r, \end{aligned} \quad (4)$$

where  $\tilde{F}$  is the distribution of productivities in the population. The pair  $(r, D(w))$  is feasible when it satisfies the budget constraint

$$T(r, D(\cdot)) = 0. \quad (5)$$

To characterize the second best allocations, we follow the same path as in the first best case. We first take the value of the welfare  $W_R = v(r)$  as given and determine the function  $D(w)$ . Then the value of  $r$  is chosen so that the budget constraint is binding: the government revenue must be zero at the optimum.

### 3.2 Characterization of the schedule $D(w)$

Suppose that the government tries to reach a Rawlsian utility level equal to  $W_R$ . We let  $\bar{d}(r|w)$  (possibly equal to  $+\infty$ ) denote the maximum of the support of  $G_{r,w}$ .

The following lemma provides a characterization of the least costly incentive compatible allocation that guarantees a welfare level equal to  $W_R$ .

**Theorem 2** *If  $(r, D(w))$  is an optimal schedule, then  $D(w)$  is an element of*

$$\operatorname{argmax}_D (w - D) G_{r,w}(D). \quad (6)$$

*As a consequence, we have, for all  $w$ ,  $D(w) \leq w$  and  $D(w) \leq \bar{d}(r|w)$ .*

**Proof** Suppose that changing the incentive scheme from  $D(w)$  to  $D_1(w)$  increases government revenue by some  $\alpha > 0$ . In that case, it would be possible to redistribute this gain to all the agents (replacing  $r$  with  $r + \alpha$ ) and to increase the value of the Rawlsian welfare. We conclude that the schedule  $D(w)$  must maximize the government revenue. The result follows directly from equation (4). ■

The first order condition associated with (6) can be written

$$\frac{w - D}{D} = \frac{1}{\eta_{r,w}(D)}$$

where  $\eta_{r,w} = D/G_{r,w}(D) \partial G_{r,w}(D)/\partial D$  is the elasticity of participation with respect to financial incentives for agents of skill level  $w$ . This is a similar formula

to that obtained in Saez (2002). As we will see later, generically the first order condition does not characterize the solution. However, when the problem is well behaved, given  $r$ , the above equation implicitly gives the optimal tax schedule.

An interesting property of the optimal allocation is that, for each  $w \geq 0$ , there exists an agent with productivity  $w$  whose utility is equal to  $v(r)$ . This is a consequence of Theorem 2:

- if there exists an unemployed agent with productivity  $w$ , we know that this agent's utility is  $v(r)$  ;
- otherwise<sup>7</sup>, all agents of productivity  $w$  work, so that  $G_{r,w}(D(w)) = 1$ . Thus  $D(w) \geq \bar{d}(r|w)$ . Theorem 2 gives the inequality in the other direction. Therefore an agent  $a$  with productivity  $w$  and work aversion  $\bar{d}(r|w)$  is just indifferent between working and not working: she consumes  $c(a) = r + D(w)$  and has utility  $u(c(a); a) = v(r)$ .

All the workers with productivity  $w$  and work aversion strictly lower than  $D(w)$  get a rent, i.e. their utilities are strictly larger than the social norm. In the second best environment, the government cannot extract all the rent from the agents. Hereafter, we denote  $K_r(w)$  the value of the maximum

$$K_r(w) = \max_{D \leq w} (w - D)G_{r,w}(D). \quad (7)$$

The quantity  $K_r(w)$  is related (but not equal) to the tax collected by the government on workers (recall  $w - D(w) = w - R(w) + r$ ). It can be interpreted as the share of the total surplus  $w$  collected by the government on agents with productivity  $w$  (the government leaves  $D(w)$  to the workers as an incentive to work). We can now complete the characterization by determining the value of  $r$ .

### 3.3 Determination of the subsistence revenue $r$

Theorem 2 gives a procedure to construct a Rawlsian optimum. For any  $r$  in  $[0, \int w d\tilde{F}(w)]$ , we note  $T(r)$  (with a slight abuse of notation) the government net revenue

$$T(r) = \int_w K_r(w) d\tilde{F}(w) - r. \quad (8)$$

Under Assumption 1, given productivity  $w$ , the surplus  $K_r(w)$  is a continuous and nonincreasing function of  $r$ . The proof of this assertion is in the Appendix. The fact that  $K_r(w)$  decreases in  $r$  is a direct consequence of Assumption 1. The continuity of  $K_r(w)$  holds even though the c.d.f.  $G_{r,w}$  may be discontinuous, when the distribution of work aversions has some mass points. The government revenue  $T(r)$  given by (8) then is a continuous and decreasing function of  $r$ . The value of the subsistence revenue  $r^*$  at the second best Rawlsian optimum is the unique solution in  $r$  to the equation  $T(r) = 0$ . The incentives to work  $D(w) = R(w) - r$  are then given by Theorem 2.

Some simple comparative statics results follow. From (7), we see that the function  $K_r$  decreases when the distribution of work aversions first order stochastically

<sup>7</sup>Note that it may happen that all agents work in the second best optimal allocation. A necessary condition for no unemployment at the optimum is  $w \geq \bar{d}(r|w)$  for all  $w$  (this requires in particular that  $\bar{d}(r|w) < +\infty$  for all  $w$ ). Welfare  $W_R$  is then equal to  $v(r)$ , where  $r$  is the unique solution to the government budget constraint, which writes, in that case

$$\int w d\tilde{F}(w) = r + \int \bar{d}(r|w) d\tilde{F}(w).$$

increases ( $G_{r,w}$  decreases). Therefore the government revenue also decreases for all  $r$  (see equation (8)). It follows that the optimal value of  $r$  (solution to  $T(r) = 0$ ) also decreases. We have shown

*Under Assumption 1, the second best Rawlsian optimum utility level  $W_R$  decreases when the distribution of work aversions  $G_{r,w}$  first order stochastically increases for all  $w$ .*

We can go further in the analysis of the problem when the distribution of work aversions is independent of the productivity of the agents, or equivalently, when the elasticity of participation with respect to financial incentives  $D$  does not depend on the skill level  $w$ .

**Assumption 2** *The conditional distribution of work aversions  $G_{r,w}(\cdot)$  is independent of  $w$ .*

From (7),  $K_r(w)$  is a nondecreasing function of  $w$ . It follows that the government revenue  $T$  (and consequently the optimum  $r$ ) increases when the distribution of  $w$  stochastically increases ( $G_r$  being fixed and thus also the function  $K_r$ ). We therefore have

*Under Assumptions 1 and 2, the second best Rawlsian optimum utility level  $W_R$  increases when the distribution of productivities  $\tilde{F}(w)$  first order stochastically increases.*

## 4 Qualitative analysis

The Rawlsian problem exhibits a particular structure, which makes many qualitative properties of optimal schedules easy to derive for a large class of distributions of work aversions. We first describe the fundamental structure of the problem and give a geometric representation. We then look at some particular cases, in particular at the circumstances under which it is optimal to use negative marginal tax rates.

### 4.1 The basic structure of the problem

The optimization problem has two important features: the objective is linear with respect to productivity  $w$  and it depends in a simple way on the distribution of work aversions.

**Theorem 3** *Consider a second best Rawlsian allocation. Under Assumptions 1 and 2, we have*

1. *The surplus  $K_r(w) = (w - D(w))G_r(D(w))$  raised by the government at the optimum is a non decreasing convex positive function of  $w$ , of slope at most equal to 1.*
2.  *$D(w)$  is a nondecreasing function of  $w$ , with  $D(w) \leq w$ . The proportion of agents of productivity  $w$  at work,  $G_r(D(w))$ , is also nondecreasing in  $w$ .*

**Proof** From Theorem 2,  $K(w)$  is the supremum of the set of linear mappings  $(w - d)G_r(d)$ , where  $d$  is any real number. It is positive ( $d = w$  is possible), convex as the supremum of convex functions.  $G_r(D(w))$  is a subgradient of  $K(w)$ , whose slope cannot thus exceed 1. Convexity implies that the subgradient is nondecreasing, which implies that  $G_r(D(w))$  is nondecreasing in  $w$ , and  $D(w)$  as well. ■

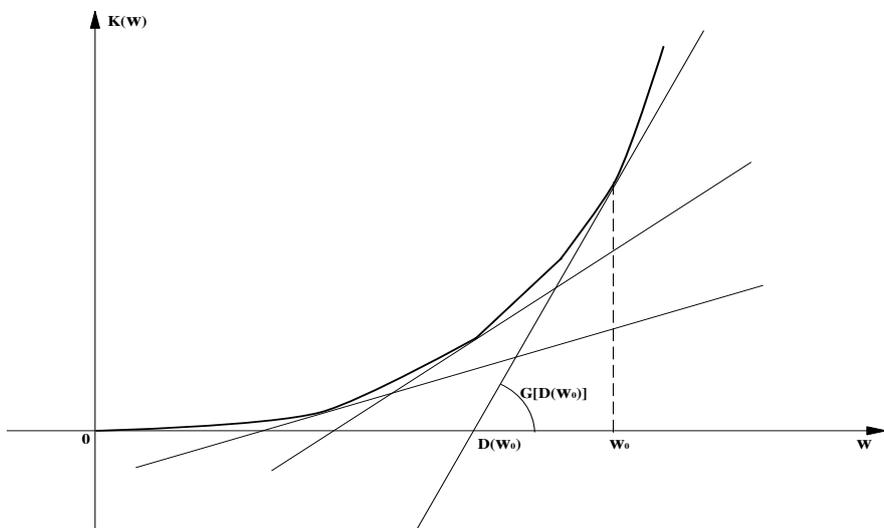
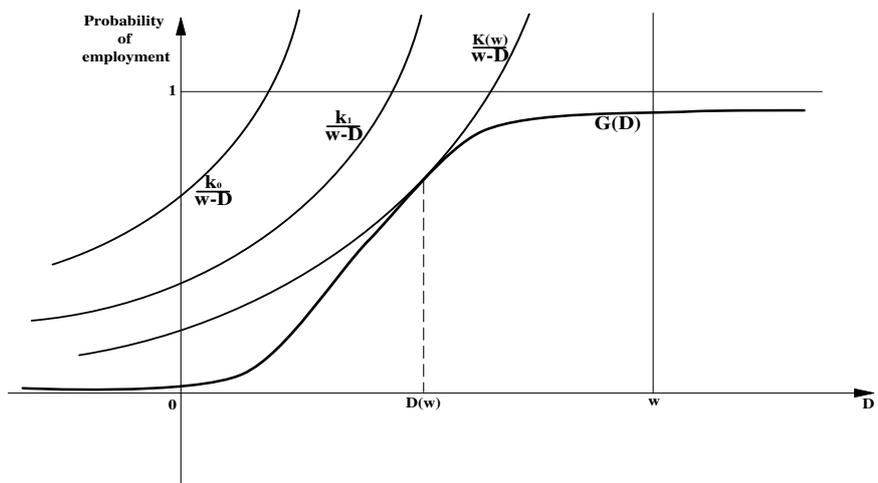


Figure 2: The optimization program

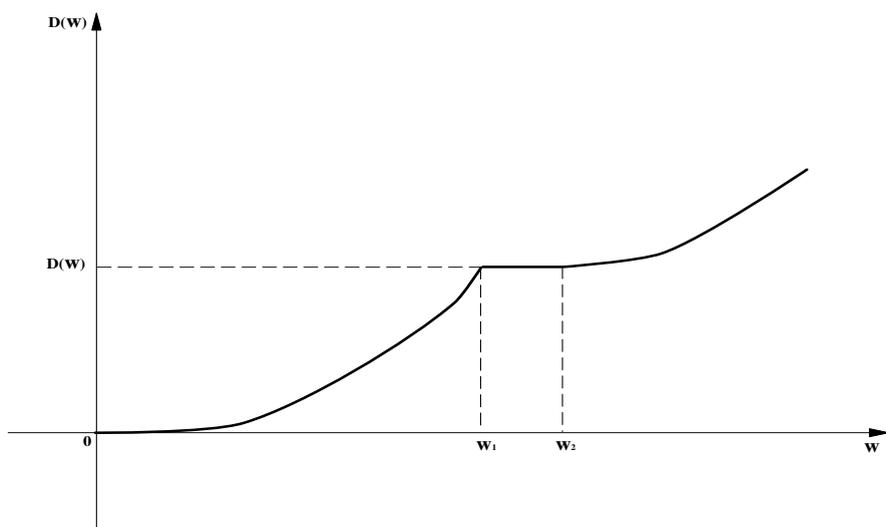
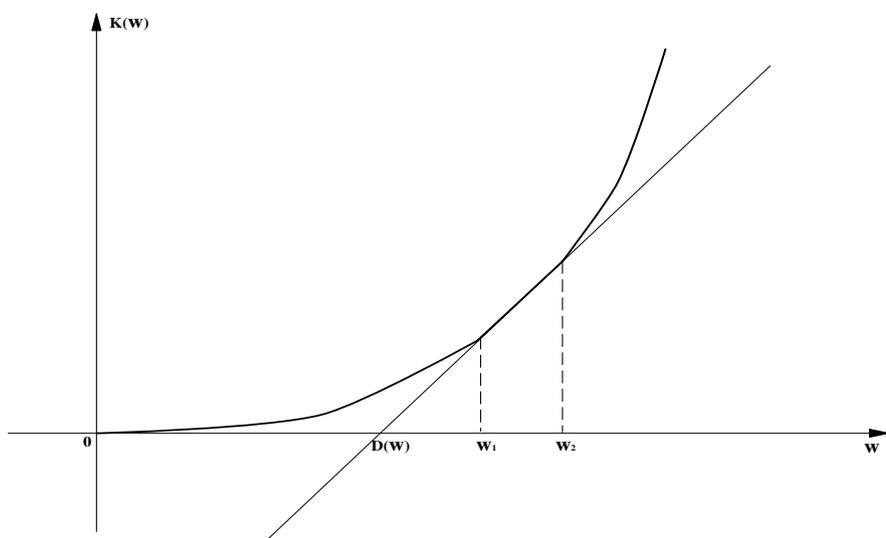
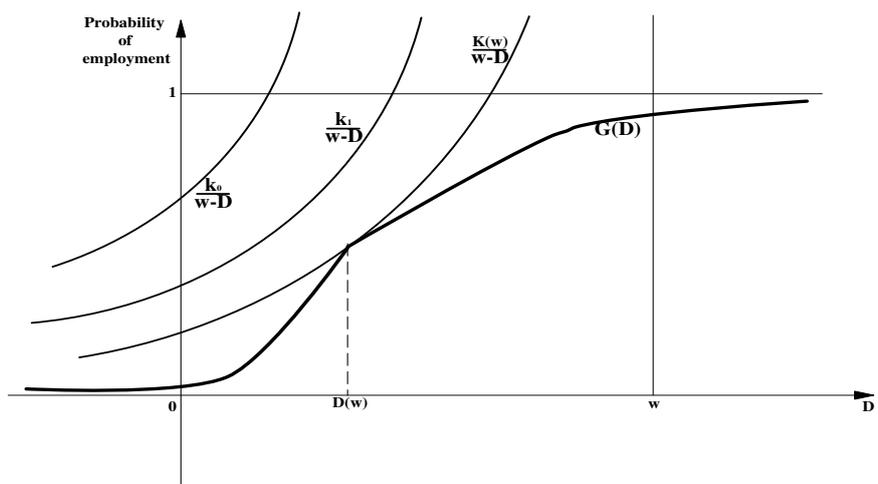


Figure 3: 100% marginal tax rate

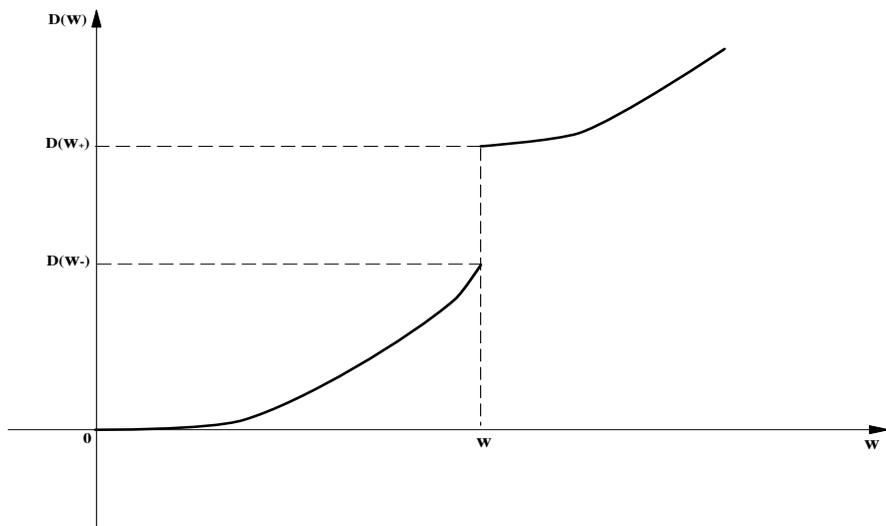
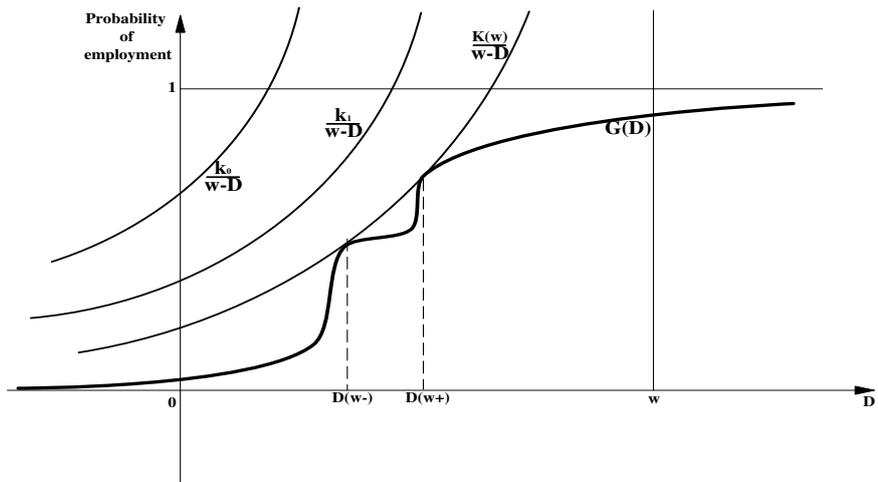


Figure 4: Discontinuity of the incentive scheme

The theorem shows that, under Assumption 2, the marginal tax rates  $1 - D'(w)$  are less than or equal to 1. The fact that  $D(w)$  is nondecreasing implies that it would not be in the interest of an agent to announce a productivity lower than the truth, if this were allowed. The tax schedule is incentive proof to the mimicking of agents with lower productivities. This result depends crucially on Assumption 2. As suggested by a referee, in the absence of this assumption, it would be natural to impose monotonicity of the incentive scheme.

A graphical representation helps to understand the structure of the problem. On the top panel of Figure 2, the c.d.f.  $G_r(D)$  is plotted: if  $D$  is selected by the government,  $G_r(D)$  is the proportion of agents that are willing to work. For a given value of  $w$ , the problem (see the definition of  $K_r$  (7)) is to find the maximum value of  $k$  such that  $k/(w - D)$  intersects the graph of the c.d.f.. Therefore, for a given  $w$ , we draw a bunch of isoquants of the form  $k/(w - D)$ , all arcs of hyperbolas whose asymptotes are the negative  $D$  axis and the vertical line of abscissa  $w$ . The solution is at the highest isoquant which is tangent to the c.d.f.. When  $w$  increases, the hyperbolas translate to the right, so that both  $D(w)$  and  $K_r(w)$  increase.

As we will see later, the point-wise optimization program, for a specific value of  $w$ , needs not be well behaved. However, the overall optimization is simple, as shown on the bottom panel of Figure 2 drawn in the plan  $(w, K(w))$ . The maximization involves taking the upper envelope  $K(w)$  of a set of straight lines of equation  $(w - D)G_r(D)$ , when  $D$  varies. The typical line intersects the  $w$  axis at  $D$ , and has slope  $G_r(D)$ , a number between 0 and 1. The function  $K_r(w)$  is increasing convex (and therefore continuous), and has a slope everywhere smaller than 1.

## 4.2 The case of a 100% marginal tax rate

Figure 3 shows a case where a 100% marginal tax rate is optimal. This occurs when the optimum is at a kink of the graph of the c.d.f.(top panel):  $D(w)$  and  $G_r(D(w))$  stay constant on a range of productivities. Let  $w_2$  be such that the corresponding isoquant is tangent to the right piece of the graph of the c.d.f.:  $K(w_2)/(w_2 - D(w))^2 = G'_r(D(w)_+)$ . Similarly define  $w_1$  for a tangency from the left:  $K(w_1)/(w_1 - D(w))^2 = G'_r(D(w)_-)$ . All the agents  $a$  with the same work aversion  $\Delta(r; a) = D(w)$  and productivity  $w_1 \leq w \leq w_2$  receive the same income  $r + D(w_1)$ : the marginal tax rate is equal to 100%. This translates into a linear portion of  $K$  (middle panel of Figure 3), and yields a flat bit in the income schedule with a 100% marginal tax rate (bottom panel).

## 4.3 When are negative marginal tax rates optimal?

The top panel of Figure 4 shows a situation where there are two tangency points. For this particular value of  $w$ , both  $D_1(w)$  and  $D_2(w)$  yield the optimum  $K_r(w)$ . All the agents with work aversions between these two values are always treated in the same way, either being non employed (for productivities smaller than  $w$ ) or employed (for productivities larger than  $w$ )<sup>8</sup>. This results in an upward discontinuity of  $D(w)$  or, equivalently, an infinite negative marginal tax rate at  $w$ .

Two remarks are necessary at this stage. First, such discontinuities have nothing pathological and prevail in a generic set of economies: they will occur as soon as the

<sup>8</sup> The choice of agents with productivity *equal* to  $w$  and work aversion between  $D_1(w)$  and  $D_2(w)$  depends on whether the government posts  $D(w) = D_1(w)$  or  $D_2(w)$  (the government is indifferent between these two possibilities). Note that, for some  $w$ , the hyperbola and the c.d.f. graph of  $G$  could be tangent along a continuous portion of the hyperbola. In that case, the government would be indifferent between all the corresponding values of  $D$  for the productivity  $w$ .

c.d.f. has pieces that are flatter than the arc of hyperbola going through them, for instance for discrete distributions. Second, we chose to represent the extreme case of an infinite negative tax rate. This should not induce the reader to believe that *finite* negative marginal tax rates are impossible. Actually, almost every nondecreasing schedule is indeed optimal for some distribution of work aversion (see section 5).

To understand intuitively why negative marginal tax rates can be optimal, suppose there is an accumulation of agents with work aversion close to  $d$  ( $d$  being known to the planner). Recall that work aversion is unobserved in the second best environment: the only available screening variable is  $w$ . For small  $w$ , it is too costly to put these agents to work; and it is optimal to do so for large  $w$ . If the distribution of work aversion is very concentrated around  $d$ , the second best solution is such that the incentives strongly increase ( $D'(w) > 1$ ) precisely at the point  $w$  such that  $D(w) = d$ . While Diamond (1980) and Saez (2002) have investigated the possibilities of work subsidies at the bottom of the wage distribution, we emphasize that in theory this pattern is not specific to low wages.

There exists a simple regularity assumption on the distribution of work aversion guaranteeing that negative marginal tax rates are never optimal. In particular, this assumption rules out mass points in the distribution  $G$ .

**Theorem 4** *When  $G$  is log concave, the optimal marginal tax rate is everywhere nonnegative.*

**Proof** The problem (6) can be rewritten  $\max_{D \leq w} \ln(w - D) + H(D)$ , with  $H(D) = \ln G$ . Since  $H$  is concave, the function  $D \rightarrow \ln(w - D) + H(D)$  is strictly concave and has a unique maximum, characterized by the first order conditions<sup>9</sup>

$$H'(D) = \frac{1}{w - D} \quad \text{or} \quad \frac{w - D}{D} = \frac{1}{\eta},$$

where  $\eta$  is the elasticity of participation. Since  $D$  is nondecreasing and  $H'$  is non-increasing, it follows that  $w - D(w)$  increases in  $w$ , which gives the result. ■

As mentioned in the introduction, one feature of the intensive model à la Mirrlees is that marginal tax rates are nonnegative at the optimum. We recover this property in our setup *provided the distribution of work aversions is log-concave*. In this circumstance, the use of negative tax rates (like with EITC in the US or WFTC in the UK) is not justified by incentive purposes under a Rawlsian optimality criterion.

To sum up, we have found two polar cases: the log-concave case, where the problem  $\max(w - D)G(D)$  is concave for all  $w$  and the marginal tax rates are nonnegative on the one hand, the case represented in Figure 4 where the problem  $\max(w - D)G(D)$  has distinct solutions and infinite negative marginal tax rates can happen, on the other hand. Between these polar cases, virtually every intermediate pattern is possible, including finite marginal negative tax rates (see section 5).

## 4.4 Examples

We present two examples where work aversion does not depend on productivity (Assumption 2 is satisfied) nor on the subsistence income  $c$ . We begin with a case where the distribution of work aversions is discrete with two mass points. The work aversion  $\Delta$  takes two values  $\Delta_j$ , with probability  $p_j$ ,  $j = 1, 2$ ,  $p_1 + p_2 = 1$ .

<sup>9</sup>When  $G$  has a kink, the first order condition is that 0 is in the subgradient of  $\ln(w - D) + H(D)$ .

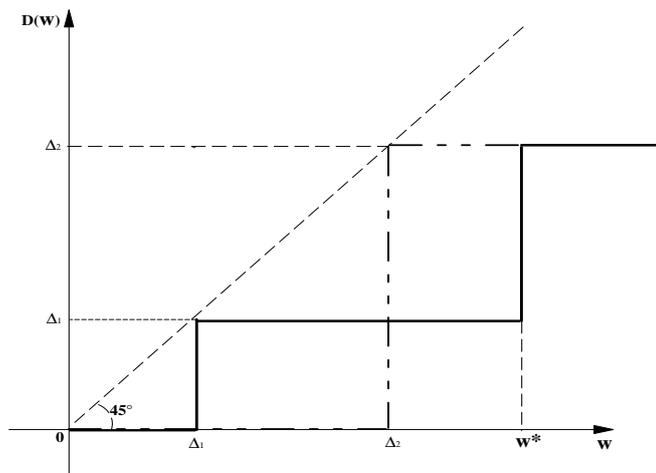
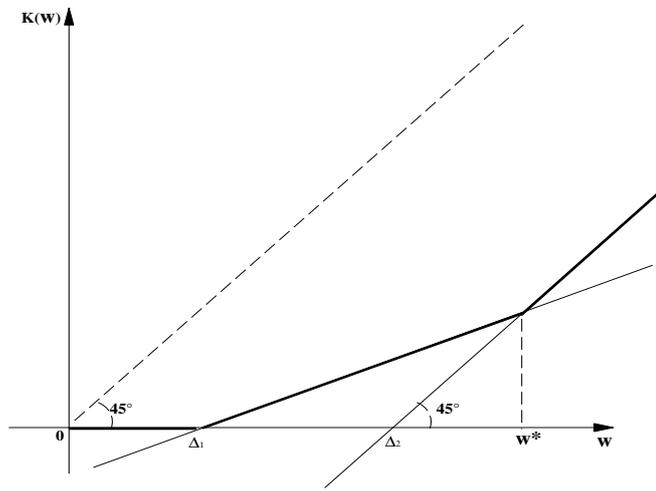
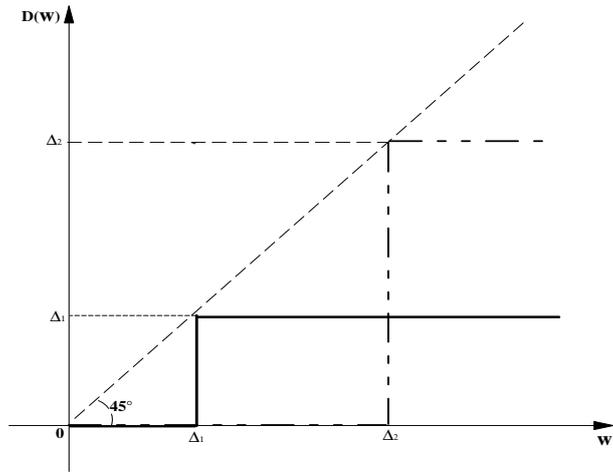


Figure 5: Example. First best and second best

In the first best situation, the government observes  $w$  and  $j$ . The income schedule differs with  $j$  and is given by  $D_j(w) = \Delta_j \mathbf{1}_{w \geq \Delta_j}$ , as shown on the upper panel of Figure 5. The level of  $r$  is determined by government revenue

$$r^{\text{FB}} = \int [p_1(w - \Delta_1) \mathbf{1}_{w \geq \Delta_1} + p_2(w - \Delta_2) \mathbf{1}_{w \geq \Delta_2}] d\tilde{F}(w).$$

In the second best situation, the government does not observe  $j$  and the schedule can only depend on  $w$ . The optimal Rawlsian allocation derives from solving

$$K(w) = \max(w - D)G(D) = \max[0, p_1(w - \Delta_1), w - \Delta_2],$$

as shown in the middle panel of Figure 5. The income schedule, represented on the lower panel of Figure 5, is given by

$$D(w) = \begin{cases} 0 & \text{if } w \leq \Delta_1 \\ \Delta_1 & \text{if } \Delta_1 \leq w \leq w^* \\ \Delta_2 & \text{if } w^* \leq w, \end{cases}$$

where  $w^* = (\Delta_2 - p_1 \Delta_1)/(1 - p_1)$ . When  $w$  is smaller than  $\Delta_1$ , any value of  $D(w)$  smaller than  $\Delta_1$  is acceptable and leads to the same allocation. The fraction of employed agents of productivity  $w$  is 0,  $p_1$  or 1, depending on the position of  $w$  with respect to the thresholds  $\Delta_1$  and  $w^*$ .

Note that  $w^* \geq \Delta_2$ . It follows that work averse agents (type 2 agents) with productivity  $w \in [\Delta_2, w^*]$  do not work at the second best optimum, while they would work in the first best if the government knew their type: labor supply is downwards distorted. Furthermore, compared with the first best, type 1 agents with high productivity ( $w > w^*$ ) get the rent  $\Delta_2 - \Delta_1$ .

The subsistence level under the second best is given by

$$r^{\text{SB}} = \int [p_1(w - \Delta_1) \mathbf{1}_{\Delta_1 \leq w \leq w^*} + (w - \Delta_2) \mathbf{1}_{w \geq w^*}] d\tilde{F}(w).$$

Both the distortion on types 2's labor supply and the rent of type 1 agents lower government revenue. Therefore the subsistence income and welfare are higher in the first best than in the second best:  $r^{\text{SB}} \leq r^{\text{FB}}$ .

Consider now a continuous case where the distribution of work aversions is logit,  $G(D) = \exp(D)/(1 + \exp(D))$ . This is a log-concave distribution, so that Theorem 4 applies. The first order condition can be written

$$w = 1 + D + \exp(D),$$

and the optimal schedule is implicitly defined by this equation.

## 5 The inverse problem

We now turn to one of the main results of the paper: how does theory restrict the type of income schedules that might prevail in an economy which implements a Rawlsian optimum? We take as given a schedule  $(r, D(w))$ , with  $D$  defined on  $\mathbb{R}_+$ , and the marginal distribution of productivities  $\tilde{F}$ . Under which conditions is it possible to find an economy (i.e. utility functions  $u$  and  $v$  and distributions of work aversion  $G_{r,w}$ ) such that  $(r, D(w))$  is a Rawlsian second best optimal scheme for some level  $T$  of government income? To be more in line with real life circumstances, we want to give little advanced knowledge to the planner: we restrict our attention

to economies where the distribution of work aversions is independent<sup>10</sup> of  $w$ , as in Assumption 2. Even then, Theorem 5 shows that theory by itself imposes few restrictions on the tax schedule: essentially any increasing function  $D(w)$ , with  $D(\bar{w}) < \bar{w}$  if the productivity distribution has an upper bound  $\bar{w}$  or  $D(w) < w^\alpha$  for some  $0 < \alpha < 1$  if the distribution of wages is unbounded, is the incentive schedule of a well chosen economy.

**Theorem 5** *Let  $\tilde{F}$  be a distribution of productivity. Let  $(r, D(w))$  be a scheme such that  $D(w)$  is nondecreasing on  $[0, +\infty[$ , satisfies  $D(w) \leq w$  for all  $w$  and  $D(\bar{w})$  is finite. Pick a utility function for the unemployed agents  $v(c)$ , which is increasing and differentiable on  $\mathbb{R}^+$ .*

*Then there exists an economy with utility functions  $u(c; a)$  for workers satisfying Assumption 1, and a distribution  $G$  of work aversions independent of  $r$  and  $w$ , such that the scheme  $(r, D(w))$  implements its second best Rawlsian optimum, provided that the government has resources  $T$  equal to*

$$r - \int (w - D(w))G(D(w))d\tilde{F}(w)$$

*to finance the redistribution scheme.*

Theorem 5 shows that there is a large number of degrees of freedom in the construction of the economy. In particular, we can take as given, besides the schedule itself, the utility function of nonworkers. Basically we define the utility functions of workers through  $u(c + x; x) = v(c)$  where  $x$  is distributed according to  $G$  (independently of  $w$ ). It follows that, in the constructed economy, the distribution of work aversions does not depend on  $w$  nor on  $r$ . The proof is in the Appendix.

## The shape of the tax schedule at infinity

In practice, Theorem 5 holds not only when  $D(\bar{w})$  is finite, but also provided that  $D(w)/w$  tends to 0 as  $w \rightarrow +\infty$ : in other words, we can only rationalize tax schedules whose average tax rate tends to 1 as productivity goes to infinity. This restriction is not too surprising since the social criterion is Rawlsian and we consider only the extensive margin. Indeed when the distribution of work aversion is uniformly bounded, once all the agents are put to work, it is optimal to use a 100% marginal tax rate, which provides maximum receipts to the government without inducing any disincentive effect on labor supply.

Even if one sticks to the framework developed here, we know from footnote 10 that we can dispense with the condition on the asymptotic behavior of the tax scheme if we do not insist on having a distribution of work aversions independent of productivity<sup>11</sup> (Assumption 2). A couple of examples, less extreme than the very special situation of footnote 10, are worth developing.

First, suppose that the work aversions increase dollar per dollar with productivity. Formally, say above a given productivity threshold  $w_*$ ,  $G_r(D|w) = \tilde{G}_r(D - (w - w_*))$ , for some fixed distribution  $\tilde{G}$ . Then, the optimal value of

<sup>10</sup>The problem has an easy, economically uninteresting, solution, if one allows for distributions of work aversions that depend on  $w$ . Indeed, consider the economy where all the agents with productivity  $w$  have the same work aversion, equal to  $D(w)$ : formally  $G_w$  is the Dirac mass at the aversion level  $D(w)$ . Then its optimal tax scheme is  $D(w)$ . It puts everyone to work, which maximizes the government revenue to be distributed equally through  $r$  among the population. We want to restrict ourselves to more realistic cases.

<sup>11</sup>It seems plausible that an agent who has accumulated human capital wants to receive a high compensation for her educational effort.

$D(w)$  increases in line with  $w$  for  $w \geq w_*$ :  $D(w) = w - w_* + C$  for some constant  $C$ . The marginal tax rate is identically equal to zero above  $w_*$  and the average tax rate tends to 0 as  $w \rightarrow +\infty$ .

Second, maybe more realistically, consider a multiplicative case. Assume that for high productivity agents, the distribution of work aversions is given by  $G_r(D|w) = \tilde{G}_r(Dw_*/w)$ . Then, for  $w \geq w_*$ ,  $D(w) = cw$ , for some constant  $c$  smaller than 1 which depends on  $\tilde{G}$  and can be chosen arbitrarily close to zero. The marginal tax rate is equal to  $(1 - c)$  for wages above  $w_*$  and the average tax rate tends to  $1 - c$  asymptotically.

## 6 The utilitarian case revisited

The utilitarian case has been explored by Diamond (1980). Although it has the same fundamental structure as the Rawlsian problem, additional terms in the objective function make it difficult to carry out the same qualitative analysis. These additional terms come from the fact that the planner now values the utility levels of all the agents: the disutility incurred by the workers explicitly enters the Lagrangian. This contrasts with the Rawlsian case, where the planner needs only consider the labor supply *behavior* of the agents.

Utilitarianism takes into account society's desire for equality through an increasing concave function  $\Psi$ , which is applied to the agents utilities. The utilitarian welfare criterion then can be written as

$$W_U = \int \{\Psi[u(r + D(w); a)] \mathbf{1}_{\Delta(r,a) \leq D(w)} + \Psi[v(r)] \mathbf{1}_{\Delta(r,a) > D(w)}\} dF(a).$$

The Lagrangian associated with the optimization problem is

$$\begin{aligned} \mathcal{L}_U &= \int \{\Psi[u(r + D(w); a)] - \lambda(r + D(w) - w)\} \mathbf{1}_{\Delta(r,a) \leq D(w)} dF(a) \\ &+ \int \{\Psi[v(r)] - \lambda r\} \mathbf{1}_{\Delta(r,a) > D(w)} dF(a). \end{aligned}$$

Let  $H(a|w)$  be the c.d.f. of the distribution of  $a$  in the population, conditional on the productivity level  $w$ . One can substitute  $dF(a)$  with  $dH(a|w)d\tilde{F}(w)$  and rewrite the objective function conditionally on  $w$ . It follows that for each  $w$ , the optimal financial incentive to work  $D(w)$  is solution to

$$K_r(w) = \max_D \{(w - D + \rho_{r,w}(D))G_{r,w}(D)\} \quad (9)$$

where  $\rho_{r,w}(D)$  denotes the average rent of the employed agents with productivity  $w$

$$\rho_{r,w}(D) = \frac{1}{\lambda} \frac{1}{G_{r,w}(D)} \int \{\Psi[u(r + D, a)] - \Psi[v(r)]\} \mathbf{1}_{\Delta(r,a) \leq D} dH(a|w)$$

The function  $\rho$ , however, cannot be expressed in terms of the c.d.f.  $G_r$  only, so that the simple geometric interpretation given in the Rawlsian context does not carry over to the utilitarian case.

A utilitarian version of Theorem 3 may, however, be stated. Under a stronger independence condition, so that the distribution of  $u(c; a)$  and  $\Delta(r; a)$  are both independent of  $w$ , the function  $K_r(\cdot)$  defined by (9) is convex with respect to  $w$ , and its derivative is equal to the employment rate in the productivity class  $w$ ; that rate, therefore, is increasing with productivity.

It is possible to compare the optimal second best Rawlsian and utilitarian allocations. First, as we have seen, the Rawlsian criterion amounts to maximize the subsistence income level  $r$ . It follows that *the subsistence income  $r_U^{\text{SB}}$  at a second best utilitarian optimum is smaller than the corresponding Rawlsian subsistence income  $r_R^{\text{SB}}$* . The comparison of the financial incentives in the two cases is not as straightforward. Given  $r$ , the programs which determine the financial incentives, respectively (9) and (7), yield always a level  $D_{U,r}(w)$  in the utilitarian situation that is larger than the Rawlsian outcome  $D_{R,r}(w)$ . This is natural, since the utilitarian planner puts weight on the rents enjoyed by the workers. However, as we have just seen, the subsistence level is smaller at the utilitarian optimum, which reduces the need for financial incentives under Assumption 1. The two effects play in opposite directions, which makes it impossible to conclude in general. However, we may state in the case where  $r$  does not modify the work aversion

**Theorem 6** *When the distribution of work aversions is independent of the subsistence level  $r$ , the financial incentives  $D_R(w)$  at the Rawlsian optimum are smaller than at the utilitarian optimum  $D_U(w)$ .*

**Proof** The result follows from the fact that  $\rho_{r,w}(D)G_{r,w}(D)$  is an increasing function of  $D$ . Therefore, for all  $D$  smaller than  $D_R$ ,  $\rho_{r,w}(D_R)G_{r,w}(D_R)$  is larger than  $\rho_{r,w}(D)G_{r,w}(D)$ . This implies that the maximum of (9) is reached for some  $D$  larger than  $D_R$ . ■

The Cobb-Douglas example<sup>12</sup> constructed by Diamond satisfies our assumptions 1 and 2. Diamond's analysis relies crucially on the first order conditions of Problem 9. They have two solutions, one of which satisfies the second order conditions. That solution has to be compared to the other possible candidate, namely the corner solution where nobody works. Finite or infinite<sup>13</sup> negative marginal tax rates may prevail.

Following Diamond's example which shows that many different patterns are possible, we conjecture that the following property holds:

*Given any social utility function  $\Psi$ , any productivity distribution  $\tilde{F}(w)$  and any well behaved schedule  $(r, D(w))$ , it is possible to find a utility function  $v(r)$  for nonworkers, a family of utility functions  $u(c; a)$  for workers and a distribution  $H$  of heterogeneity  $a$ , such that the optimal second best utilitarian schedule is  $(r, D(w))$ .*

Theorem 5 shows this property when society is only interested in the welfare of the worse off agents in the population (Rawlsian criterion). It is easy to check that a similar result holds in the polar case where the planner has no redistributive objective at all<sup>14</sup>: the function  $\Psi$  is the identity mapping. A proof of the intermediate cases is left for further research.

<sup>12</sup>In the example, using our notations,  $a = (w, x)$  and  $\Delta(r; x) = rx/(M - x)$  with  $M > 3/2$  and  $x$  uniformly distributed on  $[1/2, 3/2]$ . Therefore  $\Delta$  increases with  $r$  and is independent of  $w$ . The utility function when not working  $v(r) = \ln r + \ln M$  does not depend on  $x$ .

<sup>13</sup>The discontinuity comes from a jump from the corner solution to the interior solution.

<sup>14</sup>Suppose  $\Psi$  is the identity map and set  $v(r) = r$  and  $u(R; x) = R - x$ . Then, we have:  $\Delta(r; x) = x$ . Because of the linearity of the objective with respect to  $r, R$  and  $x$ , the utilitarian problem boils down to the maximization of total production, net of the disutility incurred by the workers. For each  $w$ , it reduces to maximizing over  $D$  the quantity

$$\int (w - x) \mathbf{1}_{x \leq D} dG(x) = (w - D + 1)G(D).$$

Therefore the solution is the same as in the Rawlsian case modulo a translation and Theorem 5 applies: it is possible to choose the distribution of  $\Delta$  so as to obtain any given schedule.

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# Optimal incentives for labor force participation

## Supplementary material

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### A Existence of first best optimum

Let  $W$  be the unknown social utility level at the optimum.  $W$  is at least as large as  $v(0)$ : it is feasible to have everyone not working, with zero income. For  $W \geq v(0)$ , let  $\rho(W)$  be the unique positive real number which satisfies  $v(\rho(W)) = W$ . Then the government budget surplus, if one were to implement a Rawlsian allocation satisfying Theorem 1 for this level of  $W$ , is

$$T(W) = \int \{[w - \Delta(\rho(W); a)]\mathbf{1}_{w \geq \Delta(\rho(W); a)} - \rho(W)\} dF(a). \quad (\text{A.1})$$

**Proposition A.1** *The government net revenue  $T(W)$  is a continuous and decreasing function of  $W$ . There exists a unique value  $W^*$  such that  $T(W^*) = 0$ . This value determines the first best Rawlsian welfare. The corresponding allocation is given by Theorem 1 above, with  $W_R = W^*$ .*

**Proof** First note that the functions under the sign  $\int$  are continuous with respect to  $W$ . Furthermore, since  $\rho$  is increasing in  $W$  by construction and  $\Delta$  is increasing in its first argument by Assumption 1,  $T(W)$  is decreasing in  $W$ . For  $W = v(0)$ ,  $T(W) = \int \max[w - \Delta(0; a), 0] dF(a)$  is positive. For  $W = \text{ess sup } v(w)$ ,  $\rho(W)$  is larger than  $w$ , so that  $T(W)$  is negative. The Rawlsian optimum corresponds to the unique solution of the equation  $T(W) = 0$ . ■

### B Incentive compatibility

The following lemma shows that every incentive compatible allocation  $(c(a), s(a))$  is Pareto dominated by an implementable allocation  $(c'(a), s'(a))$ .

**Lemma B.1** *Let  $(c(a), s(a))$  be an incentive compatible allocation such that*

$$v(0) \leq \text{ess inf } U(c(a), s(a); a). \quad (\text{B.2})$$

*We let  $r$  be such that  $v(r) = \text{ess inf } U(c(a), s(a); a)$ . We note  $E_w$  the set of workers with productivity  $w$ :*

$$E_w = \{a, s(a) = 1 \text{ and } w(a) = w\}.$$

*For all  $w \geq 0$ , we define  $R(w)$  by*

$$R(w) = \begin{cases} c(a) & \text{for } a \in E_w \text{ if } E_w \neq \emptyset \\ 0 & \text{if } E_w = \emptyset. \end{cases} \quad (\text{B.3})$$

*Then the allocation  $(c'(a), s'(a))$  implemented by  $(r, R(w))$  coincides with  $(c(a), s(a))$  for agents  $a$  such  $s(a) = 1$  and satisfies, for all agents:*

$$U(c'(a), s'(a); a) \geq U(c(a), s(a); a).$$

*Furthermore, if  $(c(a), s(a))$  is feasible, then  $(c'(a), s'(a))$  is feasible.*

Obviously, condition (B.2) is a necessary condition for implementability. This condition, however, is not very strong. In the Rawlsian context, any allocation of interest satisfies (B.2) (the allocation  $(c(a), s(a)) = (0, 0)$  for all  $a$  is feasible and guarantees the Rawlsian welfare  $v(0)$ ).

**Proof of Lemma B.1** Consider first a worker  $a$  ( $s(a) = 1$ ). We have to show that this agent, when she faces the schedule  $(r, R(w))$ , is willing to work and collect  $c(a)$ . In case she chooses to work, she collects  $R(w) = c(a)$ . Then the problem is reduced to check that  $u(R(w); a) \geq v(r)$ . This inequality follows from the definition of  $r$ .

Now suppose  $a$  does not work ( $s(a) = 0$ ) and receives  $c_0$ . It follows from the incentive compatibility conditions that:  $v(c_0) = \text{ess inf } U(c(a), s(a); a)$ , therefore  $c_0 = r$ . In other words, all unemployed individuals (if there are any) receive utility  $\text{ess inf } U(c(a), s(a); a)$ . Suppose first that there exists a worker  $a'$  with productivity  $w$  ( $E_w \neq \emptyset$ ). Agent  $a$  could mimic  $a'$ . Then the incentive compatibility constraints (3) imply  $v(c_0) = v(r) \geq u(R(w); a)$ , so that  $s'(a) = s(a) = 0$ . In the other case ( $E_w = \emptyset$ ), it may happen that agent  $a$  chooses to work ( $s'(a) = 1$ ) and to receive  $R(w) = 0$ . In that case, her utility increases (since it is a voluntary choice) and the budget cost of the change from  $s(a) = 0$  to  $s'(a) = 1$  is negative. Therefore the prime allocation is feasible and all agents are better off under the schedule  $(r, R(w))$ . ■

## C Existence of second best optimum

**Lemma C.2** *Given productivity  $w$ , the surplus  $K_r(w)$  is a continuous and non-increasing function of  $r$ .*

We write  $K_r(w)$  as

$$K_r(w) = \max_{(D,p) \in B(r)} (w - D)p$$

where  $B(r)$  is the set

$$B(r) = \{(D, p) \mid 0 \leq p \leq G_{r,w}(D) \text{ and } -r \leq D \leq w\}.$$

We want to prove the continuity with respect to  $r$ . To simplify notations, we drop the index  $w$  in the expression  $G_{r,w}(D)$ . The proof follows from three preliminary properties.

Claim 1: For each  $r$ , the set  $B(r)$  is compact.

Claim 2: The graph of the correspondence  $r \rightarrow B(r)$  is closed, namely

$$(D_k, p_k, r_k) \rightarrow (D, p, r) \text{ and } (D_k, p_k) \in B(r_k) \implies (D, p) \in B(r).$$

or, to be more specific

$$(D_k, p_k, r_k) \rightarrow (D, p, r) \text{ and } p_k \leq G_{r_k}(D_k) \implies p \leq G_r(D). \quad (\text{C.4})$$

Claim 3: For each sequence  $r_k \rightarrow r$  and each  $(D, p) \in B(r)$  with  $D < w$ , there exists a sequence  $(D_k, p_k) \in B(r_k)$  such that  $(D_k, p_k) \rightarrow (D, p)$ .

Proof of the continuity of  $r \rightarrow K_r$ :

Pick a sequence  $r_k \rightarrow r$ . Take some  $(D, p) \in B(r)$  such that  $(w - D)p = K_r$ . We can assume that  $D < w$  (if  $D = w$ , then  $K_r = 0$  and we can take any  $D < w$

and  $p = 0$ ). From Claim 3, there exists  $(D_k, p_k) \in B(r_k) \rightarrow (D, p)$ . Passing to the limit in

$$(w - D_k)p_k \leq K_{r_k}$$

we get  $K_r \leq \liminf K_{r_k}$ .

Now consider another sequence  $(D_k, p_k)$  defined by

$$(D_k, p_k) \in B(r_k) \quad \text{and} \quad (w - D_k)p_k = K_{r_k}. \quad (\text{C.5})$$

This sequence  $(D_k, p_k)$  is clearly bounded, then we can pick a convergent subsequence. We denote  $(D, p)$  the limit. From Claim 2, we know that  $(D, p) \in B(r)$ . Passing to the limit in (C.5) gives  $\limsup K_{r_k} \leq K_r$ , which completes the proof of continuity.

Proof of Claim 1: Since  $D \rightarrow G_r(D)$  is nondecreasing and right continuous, the set  $B(r)$  is closed. The fact that it is bounded is obvious.

Proof of Claim 2: The claim is obvious when  $G$  is continuous. In the general case, we use the properties of  $G$ :  $G_r(D)$  is right continuous and nondecreasing with respect to  $D$ , and left continuous and nonincreasing with respect to  $r$  (since  $\Delta$  is continuous and nondecreasing).

Any sequence can be separated into (at most) four subsequences.

1. A subsequence of  $k$ 's such that  $D_k \leq D$  and  $r_k \geq r$ . Then  $p_k \leq G_{r_k}(D_k) \leq G_r(D)$  for  $k \geq G$ , which gives  $p \leq G_r(D)$ ;
2. A subsequence of  $k$ 's such that  $D_k > D$  and  $r_k < r$ . Then we obtain (C.4) by taking the limit when  $k \rightarrow +\infty$  in  $p_k \leq G_{r_k}(D_k)$ , using the right (resp. left) continuity of  $G_r(D)$  w.r.t.  $D$  (resp.  $r$ );
3. A subsequence of  $k$ 's such that  $r_k < r$  and  $D_k \leq D$ . Then  $p_k \leq G_{r_k}(D_k) \leq G_{r_k}(D)$ , which gives (C.4) by passing to the limit;
4. The last case is symmetric of case 3.

Proof of Claim 3: We have to check that for all  $r_k \rightarrow r$  and  $p \leq G_r(D)$  and each  $(D, p)$  such that  $D < w$ , there exists  $(D_k, p_k)$ , such that  $p_k \leq G_{r_k}(D_k)$  and  $(D_k, p_k) \rightarrow (D, p)$ .

We separate the  $r_k$ 's that are below  $r$  from those that are above. First, consider the set of  $k$ 's such that  $r_k \leq r$ . Then take  $D_k = D$  and  $p_k = \min(p, G_{r_k}(D))$ . We have:  $p_k \leq G_{r_k}(D_k)$ ,  $D_k \rightarrow D$  and  $p_k \rightarrow p$  thanks to the left continuity of  $G_r(D)$  w.r.t.  $r$ .

Now consider the remaining  $k$ 's, such that  $r_k > r$ . Let  $C_r$  be a finite upper bound for  $\Delta'_c(s, a)$ ,  $s$  in some interval  $[r, r + \varepsilon]$ , which exists under our regularity assumption. Take  $D_k = D + C_r|r_k - r|$  and  $p_k = p$ . Then  $D_k \leq w$  for  $k$  large enough. It is easy to check that, for all  $a$  and for  $k$  large enough,

$$\Delta(r, a) \leq D \Rightarrow \Delta(r_k, a) \leq D_k,$$

which implies  $p_k \leq G_r(D) \leq G_{r_k}(D_k)$ . It follows that  $(D_k, p_k) \in B(r_k)$ . The fact that  $(D_k, p_k) \rightarrow (D, p)$  is obvious.

**Theorem C.2** *The government revenue  $T(r)$  given by (8) is a continuous and decreasing function of  $r$ . The value of the subsistence revenue  $r^*$  at the second best Rawlsian optimum is the unique solution in  $r$  to the equation  $T(r) = 0$ . The incentives to work  $D(w) = R(w) - r$  are then given by Theorem 2.*

**Proof** The first assertion ( $T(\cdot)$  continuous and decreasing) follows directly from Lemma C.2). It is positive when  $r = 0$  and negative when  $r = \int w d\tilde{F}(w)$ . Its unique zero corresponds to the Rawlsian optimum. ■

## D The inverse problem

### D.1 Intuitive derivation of the results

The inverse problem essentially consists in building (or recovering) the distribution of work aversions  $G$  from the incentive schedule  $D(w)$ . In the ‘regular’ case of Theorem 4, the first order condition

$$\frac{g(D(w))}{G(D(w))} = \frac{1}{w - D(w)}$$

characterizes the optimal schedule. Suppose that  $D(w)$  is increasing continuously differentiable, such that  $D(w) \leq w$ . Then multiply the two sides of the first order condition by  $D'(w)$  and integrate between  $w$  and  $w_1$ . We get

$$\ln \left[ \frac{G(D(w_1))}{G(D(w))} \right] = \int_w^{w_1} \frac{D'(x)}{x - D(x)} dx.$$

We need to separate two cases. First when the productivity distribution is bounded, take  $w_1$  equal to its upper bound, and let  $G(D(w_1)) = 1$ . Then, provided that  $w_1 > D(w_1)$ , a mild requirement, there is a maximal interval  $(w_0, w_1]$ , not reduced to a point,  $w_0 \neq w_1$ , such that the right hand side integral is finite for all  $w$  in the interval. The equality defines  $G(D(w))$  on the interior of the interval, and we take  $G(D(w_0)) = 0$  at the boundary. It is easy to check that the function  $G$  built in this way is log-concave under the circumstances of Theorem 4 ( $D'(w) < 1$ ). It follows that the corresponding economy indeed has  $(r, D(w))$  as its Rawlsian income schedule provided that the government has resources  $T$  equal to

$$r - \int [w - D(w)] G(D(w)) d\tilde{F}(w)$$

to finance the redistribution program.

A similar construction works when the wage distribution is unbounded under a condition on the behavior of taxes at infinity. Take  $w_1$  equal to  $+\infty$  and  $G(D(+\infty)) = 1$ . If the integral converges at infinity, for instance if  $D(w) \approx w^\alpha$  for some  $\alpha < 1$ , again there is a non degenerate interval  $(w_0, +\infty)$  on which the integral defines a distribution  $G$ , whose logarithm is concave provided that  $D'(w) < 1$ .

The above intuitive argument works for positive marginal tax rates, i.e.  $D'(w) \leq 1$ . It turns out that it generalizes to all non decreasing functions  $D(w)$ , such that  $D(w) \leq w$  which satisfies an appropriate boundary condition: we can have  $D'(w) > 1$ , negative marginal tax rates or even discontinuous tax schedules. To state formally the result, we need to define the generalized inverse of a nondecreasing function  $D$ . For  $y \geq D(0)$ , we set<sup>15</sup>

$$D^{-1}(y) = \inf\{w \geq 0, D(w) \geq y\}.$$

The generalized inverse reduces to the usual inverse function when the function is continuous and strictly increasing and goes to infinity with  $w$ . In the two types

<sup>15</sup>If the set  $\{w \geq 0, D(w) \geq y\}$  is empty, we set:  $D^{-1}(y) = +\infty$ . It is easy to check that the function  $D^{-1}$  is nondecreasing. Therefore  $D^{-1}$  is measurable on its domain of definition.

example of section 4.4, the inverse  $D^{-1}$  takes four values: 0 at 0,  $\Delta_1$  on  $(0, \Delta_1]$ ,  $w^*$  on  $(\Delta_1, \Delta_2]$  and  $+\infty$  on  $(\Delta_2, +\infty[$ .

For any nondecreasing schedule, we can consider its inverse and the quantity

$$I(y) = \int_y^{D(\bar{w})} \frac{dx}{D^{-1}(x) - x}. \quad (\text{D.6})$$

This integral is positive, possibly equal to  $+\infty$ , which allows us to define a function  $G(d)$  by

$$G(d) = \exp(-I(d)). \quad (\text{D.7})$$

In Section D.2, we show that equations (D.6) and (D.7) indeed defines the c.d.f. of a probability distribution. Its support is the interval  $[w_0, D(\bar{w}))$ , where  $w_0$  is the largest value of  $w$  smaller than  $\bar{w}$  such that  $D(w) = w$  (with  $w_0 = D(0)$  if  $D(w) < w$  for all  $w$ ). We prove that Theorem 5 holds provided that, in case  $D(\bar{w}) = +\infty$ , there exists some finite  $y$  with  $I(y)$  finite.

The relationship between the shape of the tax scheme  $D(w)$  and the support of the distribution of work aversions  $G$  is worth emphasizing. When productivities are bounded and  $D(w)$  is uniformly away from  $w$ , the difference  $w - D(w)$  is larger than some strictly positive number, the support of  $G$  goes from  $D(0)$  to  $D(\bar{w})$  (its convex hull is equal to the whole  $[D(0), D(\bar{w})]$ ). Except for the most productive agents (fully employed) or the less productive (with zero productivity), the employment rates of agents with productivity  $w$ ,  $0 < w < \bar{w}$ , are typically strictly between 0 and 1. On the other hand, there may exist a positive wage level  $w_0$  such that  $D(w_0) = w_0$  and  $w > D(w)$  for all  $w > w_0$ . This is not a feature seen in practice, since the after tax income of the agents of productivity  $w_0$  is equal to  $r + w_0$ : they receive a subsidy equal to  $r$ , on top of their productivity, when they work. This hypothetical case may occur when everyone in the society has a work aversion larger than  $w_0$  ( $G(d) = 0$  for all  $d < w_0$ ), so that all the agents with productivity smaller than  $w_0$  are left out of work, and the shape of the tax scheme for these levels of wages is immaterial, provided that  $D(w) < w_0$ , whenever  $w < w_0$ .

It is interesting to check that  $G$  given by (D.7) is log-concave *if and only if* the marginal tax rate is nonnegative for all  $w$ . Indeed the function  $\ln G = -I(d)$  is concave if and only if  $D^{-1}(x) - x$  is a nondecreasing function of  $x$  or, equivalently,  $w - D(w)$  is a nondecreasing function of  $w$ .

In the two types example (with  $D(\bar{w}) = +\infty$ ), we can check that

$$I(\Delta_1) = \int_{\Delta_1}^{+\infty} \frac{dx}{D^{-1}(x) - x} = \int_{\Delta_1}^{\Delta_2} \frac{dx}{w^* - x} = \ln \frac{w^* - \Delta_1}{w^* - \Delta_2} = -\ln p_1.$$

so that  $p_1 = G(\Delta_1)$  can be recovered as an integral of a function of the schedule. For  $\Delta_1 \leq d \leq \Delta_2$ , equations (D.6) and (D.7) lead to an hyperbolic formula for  $G$ , namely  $G(d) = (w^* - \Delta_2)/(w^* - d)$ . In this discontinuous case, we do not recover the original discrete distribution. This is not surprising from Figure 2, since many distributions  $G$ , with graphs below the hyperbola, lead to the same schedule.

To illustrate a case with negative marginal taxes, consider the incentive scheme

$$D(w) = \log(w + 0.5).$$

The marginal tax rate is equal to -1 at  $w = 0$ . This schedule is optimal when the distribution of work aversions is

$$G(D) = \exp - \int_D^\infty \frac{dx}{\exp(x) - x - 0.5}.$$

## D.2 Formal proof of the inverse Theorem

We first derive some properties of the map  $D^{-1}$  and the integral  $I(y)$ . Then we check that formulas (D.6) and (D.7) define the c.d.f. of a probability distribution. Finally we prove a technical lemma, which gives Theorem 5. We assume throughout that  $D(w)$  is nondecreasing and that  $D(w) \leq w$  for all  $w$ .

Then it is easy to check that  $x \leq D^{-1}(x)$  for all  $x \geq D(0)$ . Therefore the function

$$x \rightarrow \frac{1}{D^{-1}(x) - x}$$

is nonnegative with values in  $[0, +\infty]$  and measurable on  $[D(0), +\infty[$ . It follows that for all  $y \geq D(0)$  the integral (D.6) can be unambiguously defined (see Rudin, 1966, Real and Complex Analysis, McGraw-Hill, Chapter 1). Its value is either a nonnegative real number or  $+\infty$ .

### Some properties of the map $D^{-1}$

Note that the map  $D^{-1}$  is nondecreasing. In fact a direct consequence of the definition is

$$y_1 > y_2 \text{ and } y_2 = D(w_2) \text{ implies } D^{-1}(y_1) \geq w_2. \quad (\text{D.8})$$

It follows that  $D^{-1}$  is left continuous. Indeed set  $w = D^{-1}(y)$  and let  $\varepsilon > 0$ . We want to show that there exists  $\eta > 0$  such that for all  $z$ ,  $y - \eta < z \leq y$  implies  $w - \varepsilon \leq D^{-1}(z) \leq w$ . Now take  $\eta = y - D(w - \varepsilon)$ . The desired property follows from (D.8), with  $y_1 = z$  and  $y_2 = y - \eta$ . The set

$$B = \{y \geq D(0), D^{-1}(y) \leq y\} = \{y \geq D(0), D^{-1}(y) = y\}$$

turns out to be important. The set  $B$  is closed. Indeed, any increasing sequence  $y_k$  belonging to  $B$ , converging to  $y$ , is such that  $D^{-1}(y) = y$ , since  $D^{-1}$  is left continuous. Consider a decreasing sequence  $y_k$  in  $B$ , also converging to  $y$ . By definition,

$$D^{-1}(y) \leq \inf\{y_k, D(y_k) = y_k \geq y\} = y,$$

so that  $D^{-1}(y) = y$ , which again shows that  $y$  belongs to  $B$ .

If  $B$  is not empty, we define  $w_0$  as the upper bound of the set  $B$ . If  $B$  is empty, we set  $w_0 = D(0)$ .

### Some properties of the integral $I(y)$

Suppose that there exists  $y_1 \geq D(0)$ ,  $y_1 < D(\bar{w})$ , such that  $I(y_1) < +\infty$ . Since  $D^{-1}$  is nondecreasing and left continuous, the integral  $I(y)$  is finite for all  $w_0 < y \leq D(\bar{w})$ . On the other hand, we have  $I(y) = +\infty$  for all  $y < w_0$ , if any. Indeed for  $w_0 > x > y$ , we have  $D^{-1}(x) - x \leq D^{-1}(w_0) - x \leq w_0 - x$ , hence  $\int_y^{w_0} 1/[D^{-1}(x) - x] dx \geq \int_y^{w_0} 1/[w_0 - x] dx = +\infty$ . Finally note that  $I(y)$  tends to zero as  $y$  goes to  $D(\bar{w})$  (use Lebesgue dominated-convergence theorem, see e.g. Rudin, 1966).

To sum up, the function  $y \rightarrow I(y)$  is nonincreasing on  $\mathbb{R}_+$ , equals  $+\infty$  for  $y < w_0$ , is finite on  $(w_0, D(\bar{w})[$  and tends to zero as  $y$  goes to  $D(\bar{w})$ .

### Definition and support of the distribution $G$

We define the function  $G(d)$  by equation (D.7) for  $d \geq w_0$  and we set  $G(d) = 0$  for  $d < w_0$ . It follows that the function  $G$  is nonnegative, nondecreasing and is equal to 1 when  $d = D(\bar{w})$ . It is the c.d.f. of a probability distribution. Its support is the interval  $[w_0, D(\bar{w})]$ . When  $I(w_0) < +\infty$ , the distribution has a mass point at  $w_0$ . When  $I(w_0) = +\infty$ , then  $G(w_0) = 0$ .

**Lemma D.3** Let  $D$  be a nondecreasing function on  $[0, +\infty)$  satisfying

- $D(w) \leq w$  for all  $w$ ;
- there exists  $y_1 \geq D(0)$  such that  $I(y_1) < +\infty$ , where  $I$  is given by (D.6).

Let  $G(d) = \exp(-I(d))$ . Then we have

$$\max_{d \geq D(0)} (w - d)G(d) = (w - D(w))G(D(w)) \quad (\text{D.9})$$

for all  $\bar{w} \geq w \geq 0$ . Furthermore, suppose that  $D$  is discontinuous at some point  $w$ . Then for every  $d_1, d_2$  in  $[D(w_-), D(w_+)]$ , we have

$$(w - d_1)G(d_1) = (w - d_2)G(d_2). \quad (\text{D.10})$$

**Proof** First suppose that  $w < w_0$ . Then  $D(w) \leq w < w_0$ ,  $G(D(w)) = 0$  and (D.9) is obvious.

We now check that equation (D.9) is satisfied for  $w \geq 0$  such that  $w_0 < w$ . Then  $D(w) < w$ . Let  $d \geq 0$ . Consider first the case  $d \leq D(w)$ . For all  $x \leq D(w)$ , we have  $D^{-1}(x) \leq w$ , and

$$\ln \frac{w - d}{w - D(w)} = \int_d^{D(w)} \frac{dx}{w - x} \leq \int_d^{D(w)} \frac{dx}{D^{-1}(x) - x} = \ln \frac{G(D(w))}{G(d)},$$

which is equivalent to

$$(w - d)G(d) \leq (w - D(w))G(D(w)).$$

We show in a similar way that the above inequality holds for  $d \geq D(w)$  as well, which completes the proof of (D.9).

The last case ( $w = w_0$ ) is left to the reader.

Finally consider  $w$  such that  $D(w_-) < D(w_+)$  and take two numbers  $d_1 < d_2$  in  $[D(w_-), D(w_+)]$ . The function  $D^{-1}$  is constant and equal to  $w$  on  $[D(w_-), D(w_+)]$ . We have

$$I(d_2) - I(d_1) = \int_{d_1}^{d_2} \frac{dx}{w - x} = \ln \frac{w - d_1}{w - d_2},$$

which yields (D.10). ■

The following theorem (a more detailed version of Theorem 5 in the text) immediately follows from the lemma.

**Theorem D.3** Let  $\tilde{F}$  be the productivity distribution on  $\mathbb{R}_+$ . Let  $(r, D(w))$  be a scheme such that  $D(w)$  is nondecreasing on  $[0, +\infty[$ , satisfies  $D(w) \leq w$  for all  $w$  and there exists some  $y$  with  $I(y)$  finite.

Let  $w_0$  be the smallest number  $w_0 \geq D(0)$ , such that  $I(y)$  is finite for  $y$  larger than  $w_0$ . Define the distribution  $G$ , with support in  $[w_0, +\infty)$  by (D.7).

Take any increasing concave differentiable function  $v(c)$  on  $\mathbb{R}_+$  and consider the following economy. Let the space of agents  $a = (w, x)$  be a subspace of  $\mathbb{R}^2$ , such that  $w$  and  $x$  are independently distributed with respective c.d.f.  $\tilde{F}$  and  $G$ . Define  $u(c; a)$  on  $[w_0, +\infty)$  through  $u(c + x; a) = v(c)$  ( $u(c; a)$  can be any number smaller than  $v(0)$  for  $c$  smaller than  $w_0$ ). Then the scheme  $(r, D(w))$  implements a second best Rawlsian optimum of this economy when government expenditure on welfare is given by

$$r - \int (w - D(w))G(D(w))d\tilde{F}(w).$$