Interpreting histories without common language: Coordination in Sender-Receiver Games

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Abstract

This paper studies the communication in repeated common interest Sender-Receiver games when no common language is available to the players. The history of the game (i.e. the succession of actions and messages chosen by the players) constitutes the unique source of structure that can be used by the players to communicate. We are interested in understanding how the receiver can interpret a history to play the optimal action. We define several axioms that a "good" interpretation should satisfy. These axioms explicitly use symmetries and asymmetries of a history to impose restrictions on the interpretation. Our set of axioms characterize a unique "good" interpretation. In the spirit of Crawford and Haller (1990), we relate our interpretation function to attainable strategies. If players use optimal attainable strategies and if they care enough about future gains, a common language comprising the two words "yes" and "no" emerges.

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1 Introduction

This paper studies the communication in coordination games when no common language is available to the players. Sender-Receiver games (Green and Stokey, 1980; Crawford and Sobel, 1982) provide a simple environment to analyze this problem. Communication is essential in these games to link the receiver's actions to the sender's private information. We study a repeated common interest game that unroll as follows. The type of the Sender is drawn once and for all. For each type, a unique action yields a payoff 1 for both players, and 0 otherwise. The receiver starts the game by playing an action. The sender responds by sending a message.[1] These two steps are infinitely repeated and payoffs are observed at the end of the game.[2] Since there is no common language, the history of the game (i.e. the succession of actions and messages chosen by the players) constitutes the unique source of structure that can be used by the players to communicate. We are interested in understanding how such a history conveys information about the type of the sender. More precisely, we want to understand how the receiver can interpret a history to play the unique optimal action associated with the sender's type. Formally, we define an interpretation as a function $f(.)$ which associates an action (or the empty set) to each history that has been played. $f(h) = a$ means that receiver believes, after history h, that the action a is the optimal action whereas $f(h) = \emptyset$ means that the receiver has not yet located the optimal action.

To characterize "good" interpretations, we pursue an axiomatic route. These axioms specify how players in our coordination game should react in the absence of a common communication device. In the spirit of Crawford and Haller (1990), symmetries and asymmetries that exist or emerge along the history of the game are crucial. Our axioms explicitly use these symmetries and asymmetries to impose restrictions on the interpretation. Consider for example, a history $(a, 0, b, 0, c, 1)$ in which actions a, b and c have been played by the receiver and in which messages 0, 0 and 1 have been sent. Because no signal possesses a meaning before the game, it should be the case that this history is similar to history $(a, 1, b, 1, c, 0)$ in terms of informa-

[1]In our setup, the receiver starts the game. This is in contrast with classic Sender-Receiver games.

[2]This is also in contrast with classic repeated Sender-Receiver games in which payoffs are revealed after each period.

tion that it conveys. A "good" interpretation function should thus satisfy $f((a, 0, b, 0, c, 1)) = f((a, 1, b, 1, c, 0))$. In this spirit, we propose seven axioms that we believe a "good" interpretation has to satisfy.

We show that our set of axioms define a unique "good" interpretation. We fully characterize this interpretation that appears to be both quite simple and intuitive. A consequence of this interpretation is that no signalling takes place in the two actions case. The idea behind this result is that the receiver can always interpret a history in two different ways. For any history, he cannot distinguish the optimal action from the non-optimal one. Finally, in the spirit of Crawford and Haller (1990), we relate our interpretation function to attainable strategies that should be consistent with the interpretation function. More precisely, the information revealed by a history, conditioning on players' strategies, should be similar to the information revealed by the history itself. We show that if players care enough about future gains, the optimal attainable strategy for the sender is to use two messages. One message, that follows the optimal action, has the meaning "yes" whereas the other message, sent after all other actions, possesses the meaning "no". This specific association between a meaning and a message is known only by the sender at the beginning of the game. The receiver plays all actions in turn until he understands the meaning of the messages. At the end of the game, both players share the understanding of these two messages. We may say that a language has emerged. If the players play the same game again, they will possess a common language comprising the two words "yes" and "no".

A strand of the game theory literature has studied how players can cooperate and tried to understand the role of language in games.[3] Crawford and Haller (1990) is the pioneering article for all this strand of research that tries to eliminate the existence of a commonly understood language in pure coordination games. They define attainable strategies that must respect the symmetries of the game up to that point in the history. As mentioned previously, we follow their route and alter their definition in order to fit our repeated Sender-Receiver framework. Andreas Blume has, in a series of papers, directly attacked this question of language focusing on Sender-Receiver games. In the traditional modelling of Sender-Receiver games, a common language is generally available to the players. In particular, the potential labels

[3]If a unique equilibrium could be guaranteed in every game with a given solution concept (Nash, for instance), this concept could constitute the basic element of a language that would become common to every agent involved in this strategic situation. As the refinement approach demonstrates, this is unfortunately not the case.

of the various types of the informed player are listed, known previously to the game and, more importantly, can be used as meaningful messages (see Farrell (1993), Matthews, Okuno-Fujiwara and Postlewaite (1991), Rabin (1990) among others). When this is not the case, the message acquires its meaning through the equilibrium strategies (Crawford-Sobel (1982), Blume (1994)). However, many such equilibria may coexist and a need for a common language (about which equilibrium will be played) still exists. In Blume (1994), the effect of having a "small" set of messages is analyzed, even though these messages are commonly understood by the sender and the receiver through the associated equilibrium. Blume (2000) directly concentrates on language creation; it is one of the few papers that directly addresses the precise question of the endogenous birth of a language in strategic situations. In his approach a language is a set of rankings of elements of a finite set of objects. His goal is to label all objects that were previously nameless. When all objects are labelled, there exists a common language. Blume shows how the existence of structure in the games under study, for instance in coordination situations, allows emergence of a language. The "good" interpretation we proposed is a way to define a "good" labelling rule that the players can use. However, our problem is solved when one action is signalled. Our interpretation does not label all objects but only the optimal action. Furthermore, his approach is more general and applies to various games whereas ours relies heavily on the specificities of the game that we study. Our analysis is also related to the experimental literature on the subject (Blume, DeJong, Kim, and Sprinkle (1998); Blume and Gneezy (2000); Blume, DeJong, Kim and Sprinkle (2001); Blume, Douglas, DeJong, Neumann and Savin (2002)). These papers study the evolution of message meanings in experimental Sender-Receiver games. Their analysis show that randomly matched players may be able to efficiently communicate in such games, even when no common language pre-exists. This fact is in line with results from evolutionary theories.[4]

The paper proceeds as follows. Section 2 presents the model. The axioms are described in section 3. We characterize the "good" interpretation in section 4 and define attainable strategies in section 5. Section 6 concludes.

[4]See also Blume, Kim, and Sobel (1993), Blume and Arnold (2004), among others, who look at Sender-Receiver games in an evolutionary framework.

2 The model

We study a repeated sender-receiver game played between an informed sender and an uninformed receiver. The sender is privately informed about his type $t \in \mathcal{T}$. The game is a succession of stages in which the receiver takes an action $a \in \mathcal{A}$ and the sender responds by sending a message $m \in \mathcal{M}$. The sets $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{T}$ are finite and types $t \in \mathcal{T}$ are equally likely.

More precisely, we consider a simple common-interest game. Both players get the same stage payoff equal to 1 when the unique optimal action that corresponds to each type of the sender is played. For all other actions, the stage payoff is equal to zero. The game is infinitely repeated and the payoffs are observed at the end. Table

	a_1	a_2
t_1	1,1	0,0
t_2	0,0	1,1

is an example to represent the stage payoffs. In this example, if the sender's type is t_1, the unique optimal action is a_1 whereas if the sender's type is t_2, the unique optimal action is a_2.

Even though the procedure and all the above elements are common knowledge, the messages or sequence of messages have no meaning for the receiver before the game is played. There is no common language on which the sender and the receiver can rely to coordinate. For instance "I am of type 1", even if it has a meaning for the sender, should not be understood by the receiver at the beginning of the game. Similarly, I send the message "yes" if you play the optimal action has no meaning for the receiver since it presupposes the common understanding of at least one word, namely "yes", before the game started.

We are interested in understanding, in the context of this game, how the history i.e. the succession of actions and messages chosen by the players conveys information. In order to locate the optimal action, the receiver has to interpret the history that he observes. Formally, if we note $\mathcal{H}$ the set of all histories, we define an interpretation as a function $f : \mathcal{H} \longrightarrow \mathcal{A} \cup \{\emptyset\}$ which associates an action (or the empty set) to each history that has been played. $f(h) = a$ means that the receiver believes, after history h, that the action a is the optimal action whereas $f(h) = \emptyset$ means that the receiver has not yet located the optimal action.[5] By convention $f(\emptyset) = \emptyset$.

[5]It is also possible to define the interpretation function as a function $f : \mathcal{H} \longrightarrow 2^{\mathcal{A}}$ which associates a set of actions to each history. In that case, $f(h) = \{a, b, c\}$ means the

3 Interpreting Histories

We will argue in this part that an interpretation of a history must satisfy several properties. These properties or axioms reflect what we believe are some natural features for a "good" interpretation function.

3.1 The three basic axioms

The absence of a common language before the game implies that there is no common vocabulary i.e. there is no message that can be sent in isolation and have a meaning for the receiver. As already said, the message "Yes" if you play the optimal action and "No" for the other actions should not be understood before the game has started. Similarly, if the receiver thinks history $(a, 0, b, 0, c, 1)$ indicates c as the optimal action, then $(a, 1, b, 1, c, 0)$ should indicate it too. Indeed, messages 0 or 1 have no proper meaning and the interpretation must not change when we permute the messages. The information contained in history $(a, 0, b, 0, c, 1)$ is exactly the same as the information contained in history $(a, 1, b, 1, c, 0)$. Noting h^j a history where j actions and j messages have been played, we can state our first axiom:

Axiom 1. ***Label-free axiom***. *An interpretation function f satisfies the label-free axiom iff for every permutation ν of $\mathcal{M}$, for every j and for every history $h^j = (a^1, m^1, ..., a^j, m^j)$, $f(\nu(h^j)) = f(h^j)$ where $\nu(h^j) = (a^1, \nu(m^1), ..., a^j, \nu(m^j))$. By convention, $\nu(\emptyset) = \emptyset$.*

Our second axiom has the same intuitive content seen from the action side. For instance, if the receiver thinks history $(a, 0, b, 0, c, 1)$ indicates c as the optimal action, then $(c, 0, b, 0, a, 1)$ should indicate a is the optimal action. Indeed, the receiver has uniform beliefs on the optimality of each action at the beginning of the game. A permutation of the actions should thus induce a similar permutation on the interpretation. The action-free axiom captures this intuition.

Axiom 2. ***Action-free axiom***. *An interpretation function f satisfies the action-free axiom iff for every permutation ρ of $\mathcal{A}$, for every j and for every history $h^j = (a^1, m^1, ..., a^j, m^j)$, $f(\rho(h^j)) = \rho(f(h^j))$ where $\rho(h^j) = (\rho(a^1), m^1, ..., \rho(a^j), m^j)$. By convention, $\rho(\emptyset) = \emptyset$.*

receiver believes, after history h, that actions a, b and c are the only possible optimal actions.

Consider now the following strategies for the sender. The sender can decide to alternate between messages 0 and 1 until the receiver plays the optimal action, in which case the sender repeats her last message. On the contrary she can repeat the same message, say 0, until the optimal action is played, in which case she sends the message 1. Examining these strategies, we see that identical histories induce different interpretation for the receiver. For instance, under the first "grammar", the sequence $(a, 0, b, 1)$ indicates b is not the optimal action while, under the second one, it shows b must be the optimal one. We interpret this possibility as exhibiting common knowledge of a grammar since sentences – sequences of messages and actions – possess a common interpretation before the game is played. To get rid of such commonly known grammars, we introduce the following axiom.

Axiom 3. ***Grammar-free axiom.*** *An interpretation f satisfies the grammar-free axiom iff for every j, for every $i < j$ and for every history $h^j = (a^1, m^1, ..., a^i, m^i, a^{i+1}, m^{i+1}, ..., a^j, m^j)$, $f(\mu_i(h^j)) = f(h^j)$ where $\mu_i(h^j) = (a^1, m^1,$
$..., a^{i+1}, m^{i+1}, a^i, m^i, ..., a^j, m^j)$ when $a^i \neq a^{i+1}$ and $\mu_i(h^j) = h^j$ otherwise.*

This axiom breaks grammar rules by breaking the sequence of messages. It differs from the previous ones since it imposes symmetry on objects that are distinguished by temporal order. A direct consequence of this axiom is that the receiver only cares about the sequence of messages associated with each action. For example, when interpreting the history $(a, 0, b, 0, a, 2, b, 1)$, the receiver compares a that receives the messages 0 and 2 with b that receives the messages 0 and 1. In terms of interpretation, this history is similar to histories $(a, 0, a, 2, b, 0, b, 1)$, $(b, 0, b, 1, a, 0, a, 2)$, $(a, 0, b, 0, b, 1, a, 2)$, $(b, 0, a, 0, b, 1, a, 2)$ and $(b, 0, a, 0, a, 2, b, 1)$.

These three axioms (label, action-free and grammar axioms) are the basis to define a "good" interpretation function. To better understand their role, let consider the following example where the receiver believes history $(a, 0, b, 1)$ shows action b is optimal. Can such an interpretation satisfy the three basic axioms? If yes, then by the label-free axiom, $(a, 1, b, 0)$ would show action b is optimal. By the action-free axiom, $(b, 1, a, 0)$ would show action a is optimal. Finally, the grammar-free axiom would allow us to conclude that the receiver believes history $(a, 0, b, 1)$ shows action a is optimal, a direct contradiction with the fact that the receiver believes b is optimal. To conclude, an interpretation function f that satisfies the three basic axioms cannot satisfy $f((a, 0, b, 1)) = b$.

On the contrary, one can easily verify that it is possible to have an interpretation function that satisfies the three basic axioms and such that the receiver believes history $(a, 0, b, 0, c, 1)$ signals action c as the optimal one. However, this belief imposes some constraints on other histories. The receiver should for example believe histories $(a, 1, b, 1, c, 0)$, $(a, 0, c, 1, b, 0)$, $(a, 1, c, 0, b, 1)$ show action c is optimal, histories $(c, 0, b, 0, a, 1)$, $(a, 1, b, 0, c, 0)$, $(a, 0, b, 1, c, 1)$ show action a is optimal,...

A nice interpretation of these three axioms is obtained when one uses the notion of group acting on a set[6]. As already explained, because of the grammar-free axiom, the receiver only cares about the sequence of messages associated with each action. Formally, the receiver interprets a history h by examining the ordered subset $M(a, h)$ of all messages that were sent just after action a in history h. It is thus possible to define a history h as an application from the set of actions $\mathcal{A}$ to the set of finite sequence of messages $\mathcal{M}^{(IN)}$ by $h(a) = M(a, h)$. We now turn to the label-free axiom. The symmetric group S_n (the group of permutations) where n is the cardinal of $\mathcal{M}$ acts on $\mathcal{M}^{(IN)}$ by the application $\nu.(m_1, ..., m_j) = (\nu(m_1), ..., \nu(m_j))$. As a direct consequence, S_n also acts on the set $\mathcal{H}$ of histories: $\nu.h$ is the function that associates $\nu.h(a)$ to a. If we note $\overline{h}$ the orbit of h under this action, the label-free axiom is equivalent to say that f must be constant on the orbit $\overline{h}$. Similarly, we can define $\mu.h$ for a permutation μ on action-message pairs and note $\widetilde{h}$ the orbit of h under this action. The grammar free axiom is equivalent to say that f must be constant on the orbits $\widetilde{h}$. We can also define the action $\rho.h$ by $\rho.h = h \circ \rho^{-1}$ i.e $(\rho.h)(a) = h(\rho^{-1}(a))$. The action free axiom says that f is known on the orbits $\underline{h}$ of h when it is known on h and is defined by $f(\rho.h) = \rho(f(h))$. Finally if we define the action of (ν, ρ, μ) by $(\nu, \rho, \mu).h = \nu.(\rho.(\mu.h))$, we obtain that if $\underline{\overline{\widetilde{h}}}$ is the orbit of h, f is known on $\underline{\overline{\widetilde{h}}}$ if it is known on h and is defined by $f((\nu, \rho, \mu).h) = \rho(f(h))$.

In the previous example, we have seen that if the receiver believes history $(a, 0, b, 0, c, 1)$ signals action c is the optimal one, it imposes some constraints on other histories. The receiver should believe histories $(a, 1, b, 1, c, 0)$, $(a, 0, c, 1, b, 0)$, $(a, 1, c, 0, b, 1)$ show action c is optimal,... Another way to express this idea is to state that, by imposing these axioms, the beliefs of the receiver are defined on the orbit $\underline{\overline{\widetilde{h}}}$ once they are defined for history $h = (a, 0, b, 0, c, 1)$. This idea will play a key role for the comprehension of the "good" interpretation

[6]See Blume (2000) for a clear presentation and the basic definitions in group theory

function.

It is quite clear that these axioms are crucial to compare histories in which actions have been played the same number of times. However they are not sufficient and one needs two other axioms to interpret other histories.

3.2 Two other axioms

The first axiom assumes that playing an action that has already been played more than the others will reveal no further information. Suppose for example the receiver believes history $(a, 1, b, 0, c, 0, a, 2)$ signals that a is the optimal action. The idea is that it was not necessary to play a a second time to believe this. When trying to understand if b (or c) is the optimal action, the receiver compares the message b received (0) with the message c received (0) and the first message associated with a (1). When trying to understand if a is the optimal action, the receiver compares only the first message a received with messages associated with actions b and c. The second message (2) that a received cannot be compared to anything and is therefore useless when interpreting history $(a, 1, b, 0, c, 0, a, 2)$. The receiver should already believe that a was the optimal action after history $(a, 1, b, 0, c, 0)$. The comparability axiom captures this idea that the last message associated with an action played strictly more than the others has no interpretable meaning as it cannot be compared to anything else.

Axiom 4. ***Comparability axiom***. *For every j and for every history $h^j = (a^1, m^1, ...a^{j-1}, m^{j-1}, a^j, m^j)$, if the last action a^j is the unique most played action in h then $f(h^j) = f((a^1, m^1, ...a^{j-1}, m^{j-1}))$.*

The second axiom ensures a coherence between the interpretation of a history and its continuations. It states that if a history indicates that a is the optimal action then there exists at least one continuation of this history for each possible action that confirms this choice. Moreover the repetition of the optimal action cannot induce a change in the interpretation. To illustrate this axiom, let us consider the history $h = (a, 1, b, 0, c, 0, c, 0, d, 0, d, 0)$ and suppose h is interpreted as designating action a as the optimal action. The coherence axiom states that the receiver cannot change her mind if she plays action a again i.e. for every $m \in \mathcal{M}$, $f((a, 1, b, 0, c, 0, c, 0, d, 0, d, 0, a, m)) = f(h) = a$. Furthermore, to support her interpretation, it must exist a continuation history where the receiver plays b and does not change her belief. In our example, one can suppose the receiver interprets the continuation history $(a, 1, b, 0, c, 0, c, 0, d, 0, d, 0, b, 0)$ (in which b is played a second time and receives the message 0) as still designating action a as the optimal one. The general idea of this axiom is that the receiver should interpret a history only if she is "sure" of her interpretation. She will never change her mind if she plays the action she thinks is optimal and she will not change her mind for sure if she plays another action.

Before stating our coherence axiom, we need an additional notation: for every j, every history $h^j = (a^1, m^1, ..., a^j, m^j)$, every message $m \in \mathcal{M}$ and every action $a \in \mathcal{A}$, we note $h^j \cup \{a, m\}$ the history $(a^1, m^1, ..., a^j, m^j, a, m)$.

Axiom 5. ***Coherence axiom.***

- *For every history h in which no action is played strictly more than any other action*[7]*, $f(h) = a$ implies that if there exists another action $b \in \mathcal{A}$ played more than a in history h then for every $m \in \mathcal{M}$, $f(h \cup \{a, m\}) = a$.*

- *For every history h in which no action is played strictly more than any other action, $f(h) = a$ implies that for every action $b \in \mathcal{A}$, $b \neq a$, if b is not the mostly played action then there exists a message $m \in \mathcal{M}$ such that $f(h \cup \{b, m\}) = a$.*

The comparability and the coherence axioms will allow us to extend the "good" interpretation function from complete histories to incomplete ones where each actions has potentially been played a different number of times.

3.3 The last two axioms

These axioms will ensure the uniqueness of a "good" interpretation function. Both have straightforward interpretations. On one hand, the non-ambiguity axiom states that a history cannot be interpreted by the receiver whenever two different interpretations leading to different beliefs on the optimal action exist. An action is signalled by a history only if there is no doubt about the interpretation of this history. On the other hand, the efficiency axiom ensures that the receiver does not discard an interpretation when one is available.

To present our last axioms, we need to introduce some further notations. We say a history is complete when all actions $a \in \mathcal{A}$ have been played exactly

[7]We restrict our attention to these histories because of the comparability axiom.

the same number of times. We also say an action $a \in \mathcal{A}$ is in excess in history h when this action is played strictly more than all other actions. For each action $a \in \mathcal{A}$, $card(a, h)$ represents the number of times action a has been played in history h. We denote by $c_a(h) = \max_{b \in \mathcal{A}} card(b, h) - card(a, h)$ the number of times action a has to be played in an attempt to complete history h. Finally, we denote by $c(h) = \sum_{a \in \mathcal{A}} c_a(h)$ the number of actions to be played to complete history h.

Axiom 6. ***No-ambiguity axiom***. *An interpretation function f is said* (0) *unambiguous iff*

- *f satisfies the previous axioms*
- *for every complete history h, $f(h) = a$ implies that there does not exist an interpretation f' that satisfies the previous axioms and an action $a' \in \mathcal{A}$, $a' \neq a$ such that $f^{'}(h) = a'$*

For every $k \geq 1$, an interpretation f is said (k) unambiguous iff

- *f is $(k-1)$ unambiguous*
- *for every history h in which no action is in excess and such that $c(h) = k$, $f(h) = a$ implies that there does not exist a $(k-1)$ unambiguous interpretation f' and an action $a' \in \mathcal{A}$, $a' \neq a$ such that $f^{'}(h) = a'$*

An interpretation function f satisfies the no-ambiguity axiom iff f is (k) unambiguous for every k.

As previously mentioned, this axiom states that a history cannot be interpreted by the receiver whenever another interpretation leading to a different choice of action exists. The no-ambiguity axiom states that, as soon as an action is signalled by a history, interpretation of this history must be unique.

To ensure that the receiver will interpret a history as soon as he has the possibility to do so, we need to add another axiom that we called the efficiency axiom.

Axiom 7. ***Efficiency axiom*** . *An interpretation function f satisfies the efficiency axiom iff*

- *f satisfies the previous axioms*

- *for every k and every history h in which no action has been played more than any other action and such that $c(h) = k$, $f(h) = \emptyset$ implies that there does not exist an action $a \in \mathcal{A}$ and a (k) unambiguous interpretation f' such that $f'(h) = a$*

Because of this axiom, the receiver cannot discard an interpretation when one is available.

4 The "good" interpretation function f^*

The seven axioms we impose on the interpretation function allow us to define what we believe to be a "good" interpretation function in the context of the sender-receiver game we study. To define this "good" interpretation function f^*, we denote by ν a permutation on the set $\mathcal{M}$ of messages, by ρ a permutation on the set $\mathcal{A}$ of actions and for every i, by μ_i the transformation defined in the grammar-free axiom.

Definition 1. 1. *Consider first a history h in which no action $a \in \mathcal{A}$ is in excess. f^* is defined by backward induction starting from complete histories:*

   - *for every action $a \in \mathcal{A}$, $f^*(h) = a$ iff a is the unique action that verifies:*
     - *there does not exist any (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $b \in \mathcal{A}$, $M(b, h) = M(b, \nu \circ \rho(h))$*
     - *for every message $m \in \mathcal{M}$, $c_a(h) > 0$ implies $f^*(h \cup \{a, m\}) = a$*
     - *for every action $b \in \mathcal{A}$, $b \neq a$, $c_b(h) > 0$, implies that there exists a message $m \in \mathcal{M}$ such that $f^*(h \cup \{b, m\}) = a$*
   - *$f^*(h) = \emptyset$ otherwise. By convention $f^*(\emptyset) = \emptyset$.*

2. *Consider then a history h in which action $a \in \mathcal{A}$ is in excess. f^* is defined by recursion starting from histories in which no action is in excess: $f^*(h) = f^*(h')$ where h' is the history constructed from h by erasing the last action-message pair that contains a.*

The definition of this "good" interpretation function comprises two parts. The second part focuses on histories where one action has been played strictly more than all other actions. The definition comes directly from the comparability axiom. The first part of the definition is more complex. An action a is signalled if and only if it satisfies three properties and is the unique action to do so. The uniqueness of such an action a is imposed by the no-ambiguity and the efficiency axioms. Among the three properties, the second and third heavily rely on the coherence axiom. The first property however relies on the label-free, action-free and grammar-free axioms.

In order to better understand the definition of the interpretation function f^*, we consider several examples of complete histories. This is convenient as, in this case, only the first property has to be analyzed.

- We already explained why an interpretation function f that satisfies the label-free, action-free and grammar-free axioms should satisfy $f((a, 0, b, 1)) = \emptyset$ ($\mathcal{A} = \{a, b\}$ and $\mathcal{M} = \{0, 1\}$). The reason is that one can transform this history using permutations over actions, messages and action-message pairs to find a contradiction. More precisely, there exists a permutation ρ over actions: $\rho(a) = b$, $\rho(b) = a$ and a permutation ν over messages: $\nu(0) = 1$, $\nu(1) = 0$ that leads to a contradiction. Histories $\nu \circ \rho(h)$ and h are similar in the sense that $M(a, h) = M(a, \nu \circ \rho(h)) = 0$ and $M(b, h) = M(b, \nu \circ \rho(h)) = 1$ but lead to different conclusions. The first property in the definition of f^* deals with this problem. It controls that no contradiction can be found when a history signals an action. If this is not the case, it insures that no action is signalled. In our example, $f^*((a, 0, b, 1)) = \emptyset$.

- Consider now history $(a, 0, b, 0, c, 1)$ when $\mathcal{A} = \{a, b, c\}$ and $\mathcal{M} = \{0, 1\}$. Using the previous reasoning, it is clear that no interpretation function can satisfy the action-free and the grammar-free axioms and signals actions a or b as the optimal one. On the contrary, as already mentioned, there is no incompatibility between these axioms, and an interpretation function that would signal action c as the optimal one. c is thus, without ambiguity, the unique action that could be interpreted as the optimal one. Because of the efficiency axiom, the receiver must interpret this history as signaling action c and we have $f^*((a, 0, b, 0, c, 1)) = c$.

- Consider finally history $(a, 0, a, 0, b, 1, b, 0)$ when $\mathcal{A} = \{a, b\}$ and $\mathcal{M} = \{0, 1\}$. One can easily verify that there is no incompatibility between the label-free, action-free and grammar-free axioms, and an interpretation function that would signal either a or b. Here, there is an ambiguity on which action is the optimal one. The receiver should conclude, because of the non-ambiguity axiom, that no action has been signaled and we have $f^*((a,0,a,0,b,1,b,0)) = \emptyset$.

It may also be useful to analyze an example of an incomplete history. This example shows how to construct by induction the interpretation of history $h = (a,0,b,1,c,1,c,1,d,1,d,1)$ when $\mathcal{A} = \{a,b,c,d\}$ and $\mathcal{M} = \{0,1\}$. By definition, to interpret this history, we should first interpret its continuations that complete h. Table 2 presents these histories and their interpretation.

Table 2. Interpretation of h and its continuation histories

$f^*(h) = a$	$f^*(h \cup \{a,0\}) = a$	$f^*(h \cup \{a,0\} \cup \{b,0\}) = \emptyset$
		$f^*(h \cup \{a,0\} \cup \{b,1\}) = a$
	$f^*(h \cup \{a,1\}) = a$	$f^*(h \cup \{a,1\} \cup \{b,0\}) = \emptyset$
		$f^*(h \cup \{a,1\} \cup \{b,1\}) = a$
	$f^*(h \cup \{b,0\}) = \emptyset$	$f^*(h \cup \{b,0\} \cup \{a,0\}) = \emptyset$
		$f^*(h \cup \{b,0\} \cup \{a,1\}) = \emptyset$
	$f^*(h \cup \{b,1\}) = a$	$f^*(h \cup \{b,1\} \cup \{a,0\}) = a$
		$f^*(h \cup \{b,1\} \cup \{a,1\}) = a$

To fill this table, we begin by complete histories (third column). For example $f^*(h \cup \{a,0\} \cup \{b,1\}) = a$ because there is no incompatibility between such an interpretation and the three basic axioms. However $f^*(h \cup \{a,0\} \cup \{b,0\}) = \emptyset$ because it is ambiguous to know if action a or b is the optimal one.

Once all complete histories have been interpreted, we can focus on incomplete histories described in column 2. To interpret these histories, we should verify the three points of the first part of the definition. Consider for example histories $h \cup \{a,0\}$ and $h \cup \{a,1\}$.

- a satisfies the three properties. (i) No contradiction is found when applying the three basic axioms. (ii) The second property is here irrelevant as $c_a(h) = 0$. (iii) There exists a continuation where b is played and that still signals action a as the optimal one.

- b does not satisfy the second property. The continuation histories where b is played do not always signal b as the optimal one.

- c and d do not satisfy the first property.

a is thus the only action such that the three properties of the definition are satisfied and we conclude that $f^*(h \cup \{a, 0\}) = a$ and $f^*(h \cup \{a, 1\}) = a$. Similarly, one can check when examining history $h \cup \{b, 1\}$ that a is the only action such that the three properties of the definition are satisfied. In particular, in all continuation histories where a is played, regardless of the message played by the sender, a is always signalled as the optimal action. This property is not true anymore when looking at history $h \cup \{b, 0\}$ and no action is signalled in this history. The idea here is that the receiver has a doubt between actions a and b.

We can finally turn to the history h we want to interpret. a is again the unique action that satisfies the three properties. In particular, when a is played, regardless of the message sent by the sender, a is always signalled as the optimal action. Furthermore, when b is played, there exists a message and a history, namely $h \cup \{b, 1\}$, that signals a as the optimal action. In conclusion, h can be interpreted as signalling action a as the optimal one: $f^*(h) = a$.

We believe the previous examples show that the interpretation function f^* corresponds to an intuitive interpretation of histories. The next proposition shows that this intuitive interpretation function is the "good" interpretation that we were looking for.

Proposition 1. *The interpretation function f^* is the only interpretation function that satisfies our axioms.*

Proof. See Appendix 7.1 □

The above proposition shows that, with simple and intuitive axioms, it is possible to characterize the way the receiver should interpret histories. This "good" interpretation function defines the way the receiver analyzes a history in order to locate the optimal action.

Another way to understand what this interpretation function represents is to use the group approach presented in section 3.1. This approach is useful only for complete histories. In group theory, the property that there does not exist any (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $b \in \mathcal{A}$, $M(b, h) = M(b, \nu \circ \rho(h))$ is equivalent to the proposition that if (ν, ρ, μ) stabilizes h, then we must have $\rho(f(h)) = f(h)$. $f^*(h) = a$ is thus equivalent to say that a is the unique action stabilized by the stabilizer of h. This group approach is related to Blume(2000)'s paper on coordination with partial language. Blume uses a group approach to label the objects given a labelling rule. Our interpretation function can be viewed as a way to define a good labelling rule. This labelling rule consists in the association of the unique action stabilized by the stabilizer of h with the term "good action" and of the others actions with the term "wrong action". Note that in our context, we do not need to have a full common language i.e. a different term for every action because only one action is different from the others: the good one.

One consequence of the "good" interpretation function is that a history in which only two actions are played or only one message is sent can not be interpreted.

Proposition 2. *A history h in which only two actions are played or only one message is sent is such that $f^*(h) = \emptyset$.*

If $card(\mathcal{A}) = 2$ or $card(\mathcal{M}) = 1$, for every history h, $f^*(h) = \emptyset$.

Proof. See Appendix 7.2 □

These results are quite natural. If only one message is sent, there is of course no way to signal the optimal action. Moreover, when only two actions are played, it is always possible to interpret each history in two opposite ways, one signalling an action, the other signalling the other action. The receiver will always have a doubt and will thus never locate the optimal action.

The interpretation function f^* can also be used to reduce the set of strategies available for the sender and the receiver. The players should play attainable strategies that are in accordance with this interpretation function.

5 Attainable strategies

The notion of attainable strategies and of optimal attainable strategy combination were first introduced by Crawford and Haller (1990). This notion was used in order to reduce the set of strategies that players can use and was based on some specific properties of the game, in particular symmetries in histories.

We first define, in the context of our model, attainable strategies for the sender. A strategy for the sender is a function that associates a message to an action played by the receiver, depending on the past history. For every j, we denote by h_s^j the history (resp. one of the histories) $(a^1, m_s^1, ..., a^j, m_s^j)$ obtained when the receiver plays actions $(a^1, ..., a^j)$ and when the sender uses a pure (resp. mixed) strategy s. We also denote, for every action $a \in \mathcal{A}$, by $P(a = a^*|s, h_s^j)$ the probability that action a is the optimal action a^* given a history h_s^j and the fact that the sender uses the strategy s. Using these notations, we can state our definition of an attainable strategy for the sender.

Definition 2. *A strategy s is an attainable strategy for the sender if it is such that :*

- *for every j, every history h_s^j and every action $a \in \mathcal{A}$ played in h_s^j, $P(a = a^*|s, h_s^j) = 1$ if and only if $f^*(h_s^j) = a$*
- *for every j and every history h_s^j, $f^*(h_s^j) = \emptyset$ implies that for every action $a \in \mathcal{A}$ played in h_s^j, there exists another action $b \in \mathcal{A}$ such that $P(a = a^*|s, h_s^j) = P(b = a^*|s, h_s^j)$*
- *for every j and every history h_s^j, two actions that have not been played in history h_s^j, have the same probability to be the optimal action.*

The definition of an attainable strategy is based on the interpretation function. If, given the strategy of the sender, the history reveals that a is the optimal action then the interpretation of this history must reveal the same thing and conversely. Furthermore, we have seen that a history is not interpretable if there is some doubt about the optimal action (no-ambiguity axiom). The fact the receiver does not interpret a history signals that each action a cannot be differentiated from at least another action. Because both actions have the same probability to be the optimal action at the beginning of the game and because the history has not allowed to differentiate between them, the second point of the definition imposes that both actions must still have the same probability of being the optimal one. Finally, two actions that have not been played cannot be differentiated and should therefore have the same probability to be the optimal action.

We now turn to the definition of an attainable strategy for the receiver.

Definition 3. *A strategy r is an attainable strategy for the receiver if it is such that actions that have not yet been played have the same probability to be played.*

This definition allows us to capture the fact that when the receiver decides to play a "new" action, he does not know which action he should play. Because all actions have the same probability to be the optimal action, we assume the receiver is indifferent between all such actions. For example, the strategy "I play each action in turn" is attainable whereas the strategy "I first play action a, then action b,..." is not. The idea here is that the sender does not know in which order the receiver will play different actions, even if she knows that each action is played in turn.

As in Crawford and Haller, we define an optimal attainable strategy combination as an attainable strategy for the receiver and an attainable strategy for the sender that maximize both players' payoffs over the set of attainable strategies.

We denote by s^* the strategy of the sender that consists in sending the same message after every non-optimal action and a different message after the optimal action. r^{1*} designates the strategy of the receiver that consists in playing all the actions in turn until the history is interpretable. Then the receiver plays forever the action that has been signalled as the optimal one. We denote by r^{2*} the strategy of the receiver that consists in playing all the actions in turn until two actions receive two different messages. If only two actions have been played when he receives two different messages, then the receiver plays randomly one of these two actions until the end. Otherwise, he plays the action associated with the message that differs from the others forever i.e. he plays the action that has been signalled as the optimal one.

The next proposition shows that these strategies can form an optimal attainable strategy combination. The result depends on the common discount factor δ used by the players to discount their payoffs.

Proposition 3. *If $\delta > 1/2$, the strategies s^* and r^{1*} form an optimal attainable strategy combination.*

If $\delta < 1/2$, the strategies s^* and r^{2*} form an optimal attainable strategy combination.

Proof. See Appendix 7.3 □

This proposition shows that when players care about future gains ($\delta > 1/2$), the optimal strategy for the sender is to use two messages. One message, that follows the optimal action, has the meaning "yes" whereas the other message, sent after all other actions, means "no". This specific association between a meaning and a message is known only by the sender at the beginning of the game. The receiver plays all actions in turn until he understands the meaning of the messages. This meaning is revealed because only one action, the optimal one, is associated with the word "yes" whereas several actions are associated with the word "no". This means in particular (see proposition 2) that the receiver should be able to play at least three different actions. If this is not the case, the receiver will never know what the optimal action is. With more than two actions, both players will share the understanding of these two messages at the end of the game. We may say that a language has emerged. If the players play the same game again, they will possess a common language comprising the two words "yes" and "no". However, this result does not hold when players care less about the future ($\delta < 1/2$). The receiver prefers to randomize between two actions when she knows that one of these actions is optimal but ignores which one is the good one. The idea is that it is too costly to play a third action that is not optimal to discover the truth for sure. The information that the optimal action is one of these two actions is sufficient for the receiver. In such a case, no language emerges and the significance of the messages is not revealed to the receiver.

6 Conclusion

This article examined a repeated Sender-Receiver coordination game without common language. We were interested in understanding how the history of the game reveals information to the receiver. The receiver has to interpret histories to understand which action is optimal. We proposed several axioms that a "good" interpretation function should satisfy. These axioms define a unique interpretation function which is both simple and intuitive. Finally, based on this interpretation function, we defined attainable strategies à la Crawford and Haller (1990) and exhibited an optimal attainable strategy combination. The sender uses two messages: one after the optimal action, another after all other actions. If the receiver cares enough about future payoffs, she plays all actions in turn until one is signalled i.e. until she understands the significance of the messages. A common language that includes the words Yes and No has emerged.

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7 Appendix

In this appendix, ν denotes a permutation on the set of messages, ρ a permutation on the set of actions and for every i, μ_i is the transformation defined in the grammar-free axiom.

Before proving proposition 1, we need the following lemma.

Lemma 1. ν, ρ, μ_i *are commutative permutations.*

Proof.

- For every j and every history $h^j = (a^1, m^1, ..., a^j, m^j)$,

$$\nu \circ \rho(h^j) = (\rho(a^1), \nu(m^1), ..., \rho(a^j), \nu(m^j)) = \rho \circ \nu(h^j)$$

- For every j, every $i < j$ and every history $h^j = (a^1, m^1, ..., a^j, m^j)$ such that $a^i \neq a^{i+1}$ (a similar proof applies if $a^i = a^{i+1}$),

$$\begin{aligned}
\mu_i \circ \rho(h^j) &= \mu_i((\rho(a^1), m^1, ..., \rho(a^i), m^i, \rho(a^{i+1}), m^{i+1}, ..., \rho(a^j), m^j)) \\
&= (\rho(a^1), m^1, ..., \rho(a^{i+1}), m^{i+1}, \rho(a^i), m^i, ..., \rho(a^j), m^j) \\
&= \rho((a^1, m^1, ..., a^{i+1}, m^{i+1}, a^i, m^i, ..., a^j, m^j)) \\
&= \rho \circ \mu_i(h^j)
\end{aligned}$$

- For every j, every $i < j$ and every history $h^j = (a^1, m^1, ..., a^j, m^j)$ such that $a^i \neq a^{i+1}$ (a similar proof applies if $a^i = a^{i+1}$),

$$\begin{aligned}
\mu_i \circ \nu(h^j) &= \mu_i((a^1, \nu(m^1), ..., a^i, \nu(m^i), a^{i+1}, \nu(m^{i+1}), ..., a^j, \nu(m^j))) \\
&= (a^1, \nu(m^1), ..., a^{i+1}, \nu(m^{i+1}), a^i, \nu(m^i), ..., a^j, \nu(m^j)) \\
&= \nu((a^1, m^1, ..., a^{i+1}, m^{i+1}, a^i, m^i, ..., a^j, m^j)) \\
&= \nu \circ \mu_i(h^j)
\end{aligned}$$

□

7.1 Proof of proposition 1

Before proving that f^* satisfies our axioms, we need to show that f^* is correctly defined. f^* is well defined for complete histories in which all actions have been played the same number of times. Indeed, in such cases, the interpretation is defined by the first property of the first part of the definition and does not depend on other histories. Once the interpretation for complete histories is known, the first part of the definition can be applied to interpret histories where only one action has been played one time less than other actions. By recursion, the first part of the definition extends the interpretation function to all histories in which no action has been played more than any other action. Finally, the second part of the definition is used to extend the interpretation function to all histories in which one action is in excess.

We now prove successively that f^* satisfies the seven axioms.

1. *The label free axiom*

   Let ν' be a permutation on the messages. We want to prove that for every history h, $f^*(\nu'(h)) = f^*(h)$.

   - Consider first a complete history h ($c(h) = 0$).

     By definition, $c(\nu'(h)) = c(h) = 0$ and history $\nu'(h)$ is also a complete history. As a consequence, $f^*(\nu'(h)) = a$, $a \in \mathcal{A}$, if and only if a is the unique action that satisfies the following property. There does not exist any (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $b \in \mathcal{A}$, $M(b, \nu'(h)) = M(b, \nu \circ \rho(\nu'(h)))$. Because permutations are commutative, for every action $b \in \mathcal{A}$, the equality $M(b, \nu'(h)) = M(b, \nu \circ \rho(\nu'(h)))$ is equivalent to $M(b, \nu'(h)) = M(b, \nu'(\nu \circ \rho(h)))$, which in turn is equivalent to $M(b, h) = M(b, \nu \circ \rho(h))$. Thus, $f^*(\nu'(h)) = a$ if and only if a is the unique action that satisfies the property that there does not exist any (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $b \in \mathcal{A}$, $M(b, h) = M(b, \nu \circ \rho(h))$ i.e. if and only if $f^*(h) = a$.

     This proves that for every complete history h, $f^*(\nu'(h)) = f^*(h)$.

   - Suppose now that $f^*(\nu'(\widetilde{h})) = f^*(\widetilde{h})$ is true for every history $\widetilde{h}$ where no action is in excess and such that $c(\widetilde{h}) = k$. We consider here a history h in which no action is in excess and such that $c(h) = k + 1$.

     By definition, $f^*(\nu'(h)) = a$, $a \in \mathcal{A}$ iff a is the unique action that satisfies:

     - there does not exist any (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $b \in \mathcal{A}$, $M(b, \nu'(h)) = M(b, \nu \circ \rho(\nu'(h)))$
     - for every message $m \in \mathcal{M}$, $c_a(\nu'(h)) > 0$ implies $f^*(\nu'(h) \cup \{a, m\}) = a$

- for every action $b \in \mathcal{A}$, $b \neq a$, $c_b(\nu'(h)) > 0$, implies that there exists a message $m \in \mathcal{M}$ such that $f^*(\nu'(h) \cup \{b, m\}) = a$

For every action $d \in \mathcal{A}$, $c_d(\nu'(h)) = c_d(h)$. Furthermore, for every action $d \in \mathcal{A}$ such that $c_d(\nu'(h)) > 0$ and every message $m \in \mathcal{M}$, $f^*(\nu'(h) \cup \{d, m\}) = f^*(\nu'(h \cup \{d, \nu'^{-1}(m)\}))$. Because $c(h \cup \{d, \nu'^{-1}(m)\}) = c(h) - 1 = k$, we can apply the recursive hypothesis and obtain that $f^*(\nu'(h) \cup \{d, m\}) = f^*(h \cup \{d, \nu'^{-1}(m)\})$. These results and a similar reasoning to the one we made for complete histories lead to the conclusion that $f^*(\nu'(h)) = a$, $a \in \mathcal{A}$ iff a is the unique action that verifies:

- there does not exist any (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $b \in \mathcal{A}$, $M(b, h) = M(b, \nu \circ \rho(h))$
- for every message $m' = \nu'^{-1}(m) \in \mathcal{M}$, $c_a(h) > 0$ implies $f^*(h \cup \{a, m'\}) = a$
- for every action $b \in \mathcal{A}$, $b \neq a$, $c_b(h) > 0$, implies that there exists a message $m' = \nu'^{-1}(m) \in \mathcal{M}$ such that $f^*(h \cup \{b, m'\}) = a$

i.e. $f^*(\nu'(h)) = a$ if and only if $f^*(h) = a$.

In conclusion, by a recursive argument, we obtain that for every history h in which no action has been played strictly more than all the others, $f^*(\nu'(h)) = f^*(h)$.

- Consider finally a history h in which an action $a \in \mathcal{A}$ is in excess. We note $\widetilde{h}$ the history constructed from h where the last $|\min_{b \neq a} c_b(h)|$ action-message pairs that contain a have been erased. In $\widetilde{h}$, no action is in excess. The second part of the definition of f^* leads to $f^*(\widetilde{h}) = f^*(h)$. Similarly we note $\widetilde{\nu'(h)}$ the history constructed from $\nu'(h)$ where the last $|\min_{b \neq a} c_b(\nu'(h))| = |\min_{b \neq a} c_b(h)|$ action-message pairs that contain a have been erased. The second part of the definition of f^* leads to $f^*(\widetilde{\nu'(h)}) = f^*(\nu'(h))$. Furthermore, it is clear that $\widetilde{\nu'(h)} = \nu'(\widetilde{h})$. Because no action has been played strictly more than the others in histories $\widetilde{h}$ and $\nu'(\widetilde{h})$, one can apply the previous result to conclude that $f^*(\nu'(\widetilde{h})) = f^*(\widetilde{h})$. Finally, we proved that $f^*(\nu'(h)) = f^*(\widetilde{\nu'(h)}) = f^*(\nu'(\widetilde{h})) = f^*(\widetilde{h}) = f^*(h)$.

This ends the proof for the label free axiom. For every history h, $f^*(\nu'(h)) = f^*(h)$.

2. *The action free axiom*

   Let ρ' be a permutation on the actions. We want to prove that for every history h, $f^*(\rho'(h)) = \rho(f^*(h))$. The proof is very similar to the proof developed for the label free axiom and is therefore not developed here.

3. *The grammar free axiom*

   Let $h^j = (a^1, m^1, ..., a^j, m^j)$ and μ_i, $i < j$ be a history and a permutation on action-message pairs as defined in the grammar free axiom. We want to prove that, $f^*(\mu_i(h^j)) = f^*(h^j)$. This is true if $a^i = a^{i+1}$ as $\mu_i(h^j) = h^j$. We will thus suppose in the rest of this proof that $a^i \neq a^{i+1}$.

   - Consider first a complete history h^j.

     The permutation μ_i changes the order of action-message pairs. Such a permutation has thus no effect on the messages received by each action. For every action $b \in \mathcal{A}$, $M(b, \mu_i(h^j)) = M(b, h^j)$ and $M(b, \nu \circ \rho(\mu_i(h^j))) = M(b, \nu \circ \rho(h^j))$. As a consequence, for every action $a \in \mathcal{A}$, $f^*(\mu_i(h^j)) = a$ iff a is the unique action that satisfies that there does not exist any (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $b \in \mathcal{A}$, $M(b, h^j) = M(b, \nu \circ \rho(h^j))$ i.e. $f^*(\mu_i(h^j)) = a$ iff $f^*(h^j) = a$.

     This proves that $f^*(\mu_i(h^j)) = f^*(h^j)$ for every complete history h^j.

   - Suppose now that $f^*(\mu_i(\widetilde{h^j})) = f^*(\widetilde{h^j})$ is true for every history $\widetilde{h^j}$ in which no action is in excess and such that $c(\widetilde{h^j}) = k$. We consider here a history h^j in which no action is in excess and such that $c(h^j) = k + 1$.

     For every action $a \in \mathcal{A}$, $c_a(\mu_i(h^j)) = c_a(h^j)$. Furthermore, for every action $a \in \mathcal{A}$ with $c_a(\mu_i(h^j)) > 0$ and every message $m \in \mathcal{M}$, $\mu_i(h^j) \cup \{a, m\} = \mu_i(h^j \cup \{a, m\})$. Because $c(h^j \cup \{a, m\}) = c(h^j) - 1 = k$, we can apply the recursive hypothesis and obtain that $f^*(\mu_i(h) \cup \{a, m\}) = f^*(h \cup \{a, m\})$. These results and a similar reasoning to the one we made for complete histories lead to the conclusion that for every $a \in \mathcal{A}$, $f^*(\mu_i(h)) = a$ iff $f^*(h) = a$.

This recursive argument proves that for every history h^j where no action has been played strictly more than all the others, $f^*(\mu_i(h^j)) = f^*(h^j)$.

- Consider finally a history h^j in which action a, $a \in \mathcal{A}$, is in excess. We note $\widetilde{h^j}$ the history constructed from h^j where the last $|\min_{b\neq a} c_b(h^j)|$ action-message pairs that contain a have been erased. In $\widetilde{h^j}$, no action is in excess. The second part of the definition of f^* leads to $f^*(\widetilde{h^j}) = f^*(h^j)$. Similarly we note $\widetilde{\mu_i(h^j)}$ the history constructed from $\mu_i(h^j)$ where the last $|\min_{b\neq a} c_b(\mu_i(h^j))| = |\min_{b\neq a} c_b(h^j)|$ action-message pairs that contain a have been erased. The second part of the definition of f^* leads to the conclusion that $f^*(\widetilde{\mu_i(h^j)}) = f^*(\mu_i(h^j))$. Finally, no action has been played more than any other action in histories $\widetilde{h^j}$ and $\widetilde{\mu_i(h^j)}$, in which all actions received exactly the same messages. Applying the previous results, we can thus conclude that $f^*(\widetilde{\mu_i(h^j)}) = f^*(\widetilde{h^j})$. Finally, combining these equations leads to the desired result: $f^*(\mu_i(h^j)) = f^*(\widetilde{\mu_i(h^j)}) = f^*(\widetilde{h^j}) = f^*(h^j)$.

This ends the proof of the grammar-free axiom. For every history h^j, $f^*(\mu_i(h^j)) = f^*(h^j)$.

4. *The comparability axiom*

   The comparability axiom is satisfied by definition. It corresponds to the second part of the definition of f^*.

5. *The coherence axiom*

   For every history h in which no action is in excess, every action $a \in \mathcal{A}$, $f^*(h) = a$ implies that

   - if $c_a(h) > 0$ then for every message $m \in \mathcal{M}$ $f^*(h \cup \{a, m\}) = a$
   - if $c_b(h) > 0$ then for every action $b \in \mathcal{A}$, $b \neq a$, there exists a message $m \in \mathcal{M}$ such that $f^*(h \cup \{b, m\}) = a$.

   f^* satisfies thus by definition the coherence axiom.

6. *The no-ambiguity and the efficiency axioms*

We prove simultaneously that f^* satisfies both the no-ambiguity and the efficiency axioms. The proof is again by recursion starting from complete histories.

- Consider first a complete history h.
  - Suppose that there exists an action $a \in \mathcal{A}$ such that $f^*(h) = a$. Suppose also that there exists another interpretation function f' that satisfies the previous axioms and an action $d \in \mathcal{A}$, $d \neq a$, such that $f'(h) = d$. To prove that f^* is (0) unambiguous, we will show that these hypotheses lead to a contradiction.

    By definition $f^*(h) = a$ implies that there exists (ν, ρ) such that $\rho(d) \neq d$ and such that for every action $b \in \mathcal{A}$, $M(b, h) = M(b, \nu \circ \rho(h))$. Histories h and $\nu \circ \rho(h)$ are thus similar and because f' satisfies the grammar-free axiom, we must have $f'(h) = f'(\nu \circ \rho(h)) = d$. Using the label-free and the action-free axioms, we however conclude that $f'(\nu \circ \rho(h)) = \rho(f^*(h)) = \rho(d)$. This contradicts the fact that $\rho(d) \neq d$.
    To summarize, there does not exist an interpretation function that satisfies the previous axioms and interprets history h differently than f^*. f^* is (0) unambiguous.

  - Suppose that $f^*(h) = \emptyset$. Suppose also that there exists a (0) unambiguous interpretation function f' and an action $a \in \mathcal{A}$ such that $f'(h) = a$. To prove that f^* satisfies the efficient axiom for complete histories, we will show that these hypotheses lead to a contradiction. Two subcases must be distinguished.
    (a) In the first subcase, $f^*(h) \neq a$ because there exists a couple (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $b \in \mathcal{A}$, $M(b, h) = M(b, \nu \circ \rho(h))$. In this case, one can prove again that f' cannot satisfy simultaneously the label-free, action-free and grammar-free axioms.
    (b) In the second subcase, $f^*(h) \neq a$ because there exists another action $c \in \mathcal{A}$, $c \neq a$ such that there does not exist any (ν, ρ) that satisfies $\rho(c) \neq c$ and such that for every action $b \in \mathcal{A}$, $M(b, h) = M(b, \nu \circ \rho(h))$. In this

case, one can construct a third interpretation function f'' that satisfies all the previous axioms and such that $f''(h) = c$. This contradicts the hypothesis that f' is (0) unambiguous.
$f''(.)$ is defined in the following way: for every permutations ρ and ν, for every n and every permutations $\mu_{r_1}, ..., \mu_{r_n}$, $f''(\nu \circ \rho \circ \mu_{r_1} \circ ... \circ \mu_{r_n}(h)) = \rho(c)$. Furthermore, for every action $b \in \mathcal{A}$, every K and every set of messages $(m^1, ..., m^K)$, $f''(\nu \circ \rho \circ \mu_{r_1} \circ ... \circ \mu_{r_n}(h \cup \{b, m^1\} \cup ... \cup \{b, m^K\})) = f''(\nu \circ \rho \circ \mu_{r_1} \circ ... \circ \mu_{r_n}(h)) = \rho(c)$. Finally, $f'' = \emptyset$ on all other histories.
Because of the property satisfied by action c, this interpretation is well defined. It is easy to prove that by definition it satisfies the label-free, action-free, grammar-free, comparability and coherence axioms. Because $f''(h) = c \neq a$, f' cannot be (0) unambiguous.

To conclude, f^* satisfies the efficiency axiom for complete histories.

- Suppose now that f^* is (k) unambiguous and efficient for every history $\widetilde{h}$ in which no action is in excess and such that $c(\widetilde{h}) = k$. We consider here a history h in which no action is in excess and such that $c(h) = k + 1$.
  - Suppose that there exists an action $a \in \mathcal{A}$ such that $f^*(h) = a$. Suppose also that there exists a (k) unambiguous interpretation function f' and an action $d \in \mathcal{A}$, $d \neq a$ such that $f'(h) = d$. To prove that f^* is $(k + 1)$ unambiguous, we will show that these hypotheses lead to a contradiction. Two subcases must be distinguished.
    (a) If $c_d(h) > 0$, there exists, by definition of f^*, a message $m \in \mathcal{M}$ such that $f^*(h \cup \{d, m\}) = a$. On the contrary, by the coherence axiom, f' must satisfy that $f'(h \cup \{d, m\}) = d$. As $c(h \cup \{d, m\}) = k$, this result contradicts the recursive hypothesis.
    (b) If $c_d(h) = 0$, an action c exists such that for every message $m \in \mathcal{M}$, $f^*(h \cup \{c, m\}) \neq d$. Indeed if this was not the case, a would not be the unique action that satisfies the three required properties in the first part of the definition

of f^*. d would also satisfy these properties[8] and this would contradict the fact that $f^*(h) = a$. Furthermore, by the coherence axiom, there exists a message $m' \in \mathcal{A}$ such that $f'(h \cup \{c, m'\}) = d$. Because $c(h \cup \{c, m'\}) = k$, these results contradicts the recursive hypothesis. It is indeed not possible that $f'(h \cup \{c, m'\}) = d$ and that simultaneously $f^*(h \cup \{c, m'\}) \neq d$. If $f^*(h \cup \{c, m'\}) \neq \emptyset$, we have a contradiction with the hypothesis that f^* is (k) unambiguous. If $f^*(h \cup \{c, m'\}) = \emptyset$, we have a contradiction with the hypothesis that f^* is efficient for histories $\widetilde{h}$ such that $c(\widetilde{h}) = k$.

This proves that f^* is $(k+1)$ unambiguous.

- Suppose now that $f^*(h) = \emptyset$. Suppose also that there exists a $(k+1)$ unambiguous interpretation function f' and an action $a \in \mathcal{A}$ such that $f'(h) = a$. To prove that f^* satisfies the efficient axiom, we will show that these hypotheses lead to a contradiction. Two subcases must be distinguished.

  (a) In the first subcase, $f^*(h) \neq a$ because a does not satisfy one of the three required properties in the definition of f^*.

  As usual, if the first property is not true, it is not possible to have $f'(h) = a$ and f' that satisfies the three basic axioms.

  If the second property is not true, there exists a message $m \in \mathcal{M}$ such that $f^*(h \cup \{a, m\}) \neq a$. On the contrary, by the coherence axiom, $f'(h \cup \{a, m\}) = a$, in contradiction with the recursive hypothesis. Indeed, if $f^*(h \cup \{a, m\}) \neq \emptyset$, we have a contradiction with the hypothesis that f^* is (k) unambiguous. If $f^*(h \cup \{a, m\}) = \emptyset$, we have a contradiction with the hypothesis that f^* is efficient for histories $\widetilde{h}$ such that $c(\widetilde{h}) = k$.

  Finally, if the third property is not true, it exists an action $b \in \mathcal{A}$, $b \neq a$, such that for every message $m \in \mathcal{M}$, $f^*(h \cup \{b, m\}) \neq a$. However, because f' satisfies the

[8] Because $f'(h) = d$, the first property in the definition is satisfied for d. The second property is not relevant here as $c_d(h) = 0$.

coherence axiom, it exists a message $m' \in \mathcal{M}$ such that $f'(h \cup \{b, m'\}) = a$. This constitutes again a contradiction with the hypothesis f^* is (k) unambiguous and efficient for histories $\widetilde{h}$ such that $c(\widetilde{h}) = k$.

(b) In the second subcase, $f^*(h) \neq a$ because it exists another action $b \in \mathcal{A}$, $b \neq a$ that satisfies the three properties. In this case, we can construct a (k) unambiguous interpretation function f'' such that $f''(h) = b$. This contradicts the fact that f' is $(k+1)$ unambiguous.
f'' is defined in the following way. For every permutations ρ and ν, for every n and for every permutations $\mu_{r_1}, ..., \mu_{r_n}$, $f''(\nu \circ \rho \circ \mu_{r_1} \circ ... \circ \mu_{r_n}(h)) = \rho(b)$. Furthermore, for every action $c \in \mathcal{A}$ such that $c_c(h) = 0$, every K and every set of messages $(m^1, ..., m^K)$, $f''(\nu \circ \rho \circ \mu_{r_1} \circ ... \circ \mu_{r_n}(h \cup \{c, m^1\} \cup ... \cup \{c, m^K\})) = f''(\nu \circ \rho \circ \mu_{r_1} \circ ... \circ \mu_{r_n}(h)) = \rho(b)$. Finally, $f'' = f^*$ on all other histories. This interpretation is well defined because of the properties satisfied by action b. It is also easy to prove that by definition it satisfies the label-free, action-free, grammar-free, comparability and coherence axioms. Furthermore f'' is (k) unambiguous because f^* is. Finally $f''(b) = b \neq a$, and we obtain a contradiction with the hypothesis that f' is $(k+1)$ unambiguous.

We proved here that f^* satisfies the efficiency axiom for histories h such that $c(h) = k + 1$.

By recursion, f^* satisfies both the no-ambiguity and the efficiency axioms.

In conclusion, f^* satisfies all our axioms. It is the unique interpretation function satisfying them. The uniqueness of such an interpretation is indeed ensured by the no-ambiguity and the efficiency axioms for histories in which no action is in excess and by the comparability axiom for all other histories i.e. for histories in which one action has been played more than all other actions.

7.2 Proof of proposition 2

Consider first a history h in which only two actions $a \in \mathcal{A}$ and $b \in \mathcal{A}$ have been played. Suppose, for example, that $f^*(h) = a$. Using the second part of the definition of f^*, we can suppose without loss of generality that no action is in excess. Then, by the first part of the definition, there exists (ν, ρ) such that $\rho(b) \neq b$ and such that for every action $c \in \mathcal{A}$, $M(c, h) = M(c, \nu \circ \rho(h))$. Because only two actions have been played in h, we must have $\rho(b) = a$ and $\rho(a) = b$. Thus there exists (ν, ρ) such that $\rho(a) \neq a$ and such that for every action $c \in \mathcal{A}$, $M(c, h) = M(c, \nu \circ \rho(h))$. This contradicts $f^*(h) = a$. Consequently, if only two actions are played in h, $f^*(h) = \emptyset$.

Consider now a history h in which only one message $m \in \mathcal{M}$ has been sent and suppose that there exists an action $a \in \mathcal{A}$ such that $f^*(h) = a$. If a was the unique action played in h, the comparability axiom would lead to the conclusion that $f^*(h) = f^*(\emptyset) = \emptyset$ and would contradict our hypothesis. Consequently, another action $b \in \mathcal{A}$, $b \neq a$ has been played in history h. Applying the comparability axiom, we can suppose that no action is in excess. Without loss of generality, we can also suppose that b is one of the most played action. Two cases should be considered. In the first case, both actions have been played the same number of times. Because the same message has always been sent, we have $M(b, h) = M(a, h)$. Using the permutation ρ which exchanges a and b, we conclude that there exists $(\nu = id, \rho)$ such that $\rho(a) \neq a$ and such that for every action $c \in \mathcal{A}$, $M(c, h) = M(c, \nu \circ \rho(h))$. This contradicts $f^*(h) = a$. In the second case, a has been played strictly less than b. Using the coherence axiom, we obtain a similar contradiction for the continuation history of h in which a is played and always received the message m. To conclude, if only one message is sent in h, $f^*(h) = \emptyset$.

7.3 Proof of proposition 3

It is easy to check that s^* is an attainable strategy for the sender whereas r^{1*} and r^{2*} are attainable strategies for the receiver.

To prove proposition 3, it is thus sufficient to prove that for every attainable strategy combination, the expected gains of the players are inferior to the gains obtained when they use these strategies. Equivalently, what we have to prove, is that the information revealed during the game is "maximal" when the sender and the receiver play these strategies.

- At the beginning of the game, no action has been played and no information is revealed. If $card(\mathcal{A}) = n$, each action in $\mathcal{A}$ has a probability $1/n$ to be optimal.
- Consider now a history in which only one action has been played. We know that such a history cannot be interpreted by the receiver. Consequently, because the sender uses an attainable strategy, there exists a non played action that has the same probability to be optimal than the played action. Furthermore, every action not played in the history has the same probability to be optimal. Each action has thus the same probability, $1/n$, to be the optimal one.
- Consider then a history in which only two actions have been played. Using Proposition 2, we know this history cannot be interpreted. Because the sender uses an attainable strategy, both actions have the same probability P to be optimal. We have to distinguish two possibilities.
  - If one of the two played actions is effectively the optimal one, the information that an attainable strategy reveals is maximal when it leads to the conclusion that probability P is equal to 1/2.
  - If not, the information that an attainable strategy reveals is maximal when it leads to the conclusion that probability P is equal to 0.

  Both attainable strategy combinations (s^*, r^{1*}) and (s^*, r^{2*}) transmit this maximal amount of information.
- Let us finally consider a history in which at least three different actions have been played. If one of these actions is the optimal one, the maximal information attainable strategies can reveal is to designate the optimal action. If not, the maximal information attainable strategies can reveal is that these actions are not optimal.

  Once more, both attainable strategy combinations (s^*, r^{1*}) and (s^*, r^{2*}) transmit this maximal amount of information.

To summarize, the "maximal" amount of information is revealed when players use attainable strategies of the following type. The sender use two messages, one for the optimal action, one for other actions. The receiver plays randomly the first action. She learns nothing. She then plays randomly another action and learns if one of these two actions is optimal. If these actions are not optimal, she continues to play different actions until she learns which action is the good one. If one of first two actions is optimal, she has two possibilities. The first one consists in playing another action which she knows is non-optimal to learn which one of the two first actions is the optimal one. The second strategy consists in playing one of the first two actions indefinitely.

The attainable strategy combinations (s^*, r^{1*}) and (s^*, r^{2*}) correspond to these two cases. The gains of both the receiver and the sender associated to these strategies are:

$$EG_{(s^*,r^{1*})} = \frac{1}{n}\left[1 + \delta + \delta^2 + 3\delta^3 + \cdots + (n-1)\delta^{n-2}\right] + \frac{\delta^{n-1}}{1-\delta}$$

and

$$EG_{(s^*,r^{2*})} = \frac{1}{n}\left[1 + \delta + 2\delta^2 + 2\delta^3 + \cdots + (n-2)\delta^{n-2}\right] + \frac{n-1}{n}\frac{\delta^{n-1}}{1-\delta}$$

A straightforward comparison of the expected gains leads to the desired result. If $\delta > 1/2$, (s^*, r^{1*}) form an optimal attainable strategy combination. If $\delta < 1/2$, (s^*, r^{2*}) form an optimal attainable strategy combination.