Dynamic frailties and credit portfolio modelling

Martin Delloye, **Jean-David Fermanian** and **Mohammed Sbai** *estimate and discuss a reduced-form credit portfolio model in a proportional hazard framework. They propose an innovative method of generating flexible amounts of dependence between underlying defaults by introducing unobservable dynamic common explanatory variables, called dynamic frailties*

Credit portfolio models are key tools for portfolio credit risk management, for economic capital calculations and for providing relevant inputs for Basel II regulatory requirements. Moreover, they can be used as pricing models for collateralised debt obligations (CDOs) and basket derivatives.

Intuitively, it is well understood that default probabilities and rating transitions are influenced by macroeconomic variables. Our goal is to model every rating transition simultaneously in a consistent way in a reduced-form model. These transitions are assumed to be independent conditionally on macroeconomic factors. These factors are explicit and easily observable. Such a model can be considered as a particular Cox process.

To be specific, we seek to calibrate a model using past default events and rating transitions to predict credit events under the physical probability. Nonetheless, we incorporate current market or macroeconomic factors in our framework, by assuming that past events can explain risks in the future (the usual risk measurement approach). Such a proportional hazard model can be valuable to price and/or hedge credit-sensitive instruments, especially complex credit derivatives (CDOs, baskets, *n*th to default, etc). Then, it is necessary to define another set for calibration, typically credit spread.

The basic model

First, we postulate a competing risk model. At every time t, any firm i is faced with $p - 1$ underlying risks, which are the risks of rating changes (including default). In this study, we have worked with the usual aggregated rating scale: eight underlying credit states (AAA, AA, A, BBB, BB, B, CCC and the default state D, with the Standard & Poor's (S&P) scale). Thus, p equals eight, but may be larger. The associated time durations are assumed to be independent conditionally on some exogenous macroeconomic process and on the fixed and time-dependent idiosyncratic firm characteristics. All these variables are recorded in a firm-specific exogenous vector z. Formally, let $\alpha_{hji}(t)$ be the intensity of the transition from h to j and for the firm i. As in Kavvathas (2000) or Couderc & Renault (2005), we set the traditional proportional hazards assumption. For every time t and every transition (h, j), $h \neq j$:

$$\alpha_{hji}(t|z) = \alpha_{hj0}(t)\exp\left(\beta'_{hj} z_{hji}(t)\right) \quad (1)$$

where α_{hj0} is an unknown deterministic function (the baseline hazard function), β_{hj} is an unknown parameter and $z_{hji}(t)$ is the value of i's covariate at time t. In this basic model case, the covariate is the same for every transition (h, j). This assumption can easily be relaxed.

Denote by $N_{hji}(t)$ the number of transitions from h towards j, for the firm i between the dates zero and t. Thus, $dN_{hji}(t)$ is zero except when i makes a transition between ratings h and j at time t. Let $Y_{hi}(t)$ be an indicator variable valued one when i is rated h just before the time t, and zero otherwise. The estimation of the unknown coefficients β_{hj} and α_{hj0} have been led by a full maximum likelihood procedure, as detailed in Andersen *et al* (1993). With n firms, the log-likelihood can be written:

$$\ln \mathcal{L} = \sum_{j \neq h} \left\{ \sum_t \sum_{i=1}^n dN_{hji}(t)\left[\ln \alpha_{hj0}(t) + \beta'_{hj} z_{hji}(t)\right] - \sum_{i=1}^n \int_0^\infty Y_{hi}(u)\alpha_{hj0}(u)\exp\left(\beta'_{hj} z_{hji}(u)\right)du \right\}$$

Note that this likelihood can be split into a sum over every couple (h, j). Thus, we can estimate the parameters for any transition separately. This is very useful for practical implementations. Such likelihoods can be optimised with standard routines. For convenience, we assume that α_{hj0} are constant functions. This assumption can be relaxed, particularly with the methodology proposed by Couderc & Renault (2005), but this would add some complexity into the framework.

A right-censoring process disrupts our data set. This is due to the end of the observation period (December 31, 2004 in our case) or caused by the entry into the category 'not rated'. The censoring variables have been assumed to be independent of the underlying risks. Though standard, this assumption could be questionable, especially for 'not rated' outcomes. Nonetheless, there is no clear evidence of an empirical link between the intensity towards the 'not rated' category and some macroeconomic variables (for instance, a stock index). Moreover, the defaults that occur just after a 'not rated' rating have been recorded in the database and taken into account in our likelihood. The left truncation can be neglected: by estimating the basic model only on the firms that enter the sample after 1980, we get almost the same estimates as with all the firms.

In terms of performance, the difference between credit portfolio models is mainly in calibration issues. To be more specific, the need for a historical database is crucial. Here, we have worked with the CreditPro historical database from S&P. This database contains individual firm rating histories recorded from December 31, 1980 to December 31, 2004. The exact date of every transition is put into the likelihood. This is an important difference from the usual fits, which rely on annual or even quarterly transition matrices. Actually, such procedures lack much information concerning quick (infra-annual) rating transitions, especially just before default. This will not be true in our case, and the superiority of 'continuous time' fits is now widely recognised (see Lando & Skødeberg, 2002). To get statistically significant estimates, we group together some transitions of the same type, under the assumption that the explanatory variables act similarly on all of them. Otherwise, β_{hj} can be set to zero when the number of transitions is too low (typically far from the main diagonal, for example, when $h = AAA$ and $j = CCC$).

Obviously, it is necessary to identify the main explanatory variables that drive rating transitions and defaults. Some of the previously cited authors achieved a similar study in the past. Particularly, see an extensive analysis in Couderc & Renault (2005), but concerning only the transitions towards default. For convenience, we have chosen a simple, perhaps crude, method: we focus only on transitions to default by assuming that the relevant drivers to default will also be relevant for the other rating transitions. Clearly, this assumption could be dropped by leading a particular econometric analysis for every rating transition. This will surely improve the empirical results and is left for further developments. To be specific, we have explained the monthly default rates for speculative firms in the US between July 1986 and July 2002 as provided by the Federal Reserve Bank using a simple linear regression. We have retained four explanatory variables:

- the annual variation rate of the index of the industrial production in the US (denoted by 'IPI' below)
- the annual variation rate of the S&P 500 index (denoted by 'SP')
- the difference between the short-term (three months) and the long-term (10 years) US government rates (denoted by 'Slope')
- the three-month US government rate (denoted by 'Short rate').

All these variables have been lagged. Indeed, the current values are not necessarily the most relevant ones to explain default. A change in the macroeconomic environment may take some time before affecting the observed default rates. These lags have been identified by a simple cross-correlation study. The 'in-sample fit' (by a simple ordinary least squares regression) is excellent, with $R^2 = 87\%$.

Additional firm-specific variables have been added as dummies: to be a firm that is based outside western Europe, US or Canada (denoted by 'not US-EU' below) and to be a financial firm (denoted 'bank'). Moreover, following some authors (Lando & Skødeberg, 2002), we take into account the firm-specific tendency towards upgrade or downgrade. Thus, at any time t, any firm will be in the state 'up' (respectively, 'down') if it has been upgraded (respectivey, downgraded) within the 12 months before t. Therefore, we have attributed two time-dependent dummies to every firm, which try to capture some non-Markovian features of the rating process.

We show the estimated coefficients β_{hj} for some groups of transitions (for brevity, we have not provided the coefficients that are related to the constants). For instance, the monthly intensity at time t for a transition from rating BBB to BB is:

$$\alpha_{BBB\to BB}(t) = \exp\big(-5.3422 - 2.838 \times IPI_t - 1.177 \times S\&P_t - 0.0043 \times slope_t + 0.0243 \times short\ rate_t + 0.0243 \times short\ rate_t + 0.128 \times \mathbf{1}(neither\ US, Canada, west\ Europe) - 0.574 \times \mathbf{1}(upgraded\ \ last\ \ year)\big)$$

The equation can be read as follows: at the first order, when the explanatory variable 'variation rate of IPI' goes up 1% from the previous month, the transition intensity from BBB to BB goes down around 2.8%. The slope of the interest rates and the short rate are recorded as percentages. Therefore, when the short rate increases from 3% to 4%, the transition intensity goes up approximately 2.4% (see table A).

To our knowledge, this article is the first that provides such estimates in a full competing risk framework. Sometimes, some components of these coefficients do not appear because they are not significant (p-values larger than 1%), by usual univariate Wald tests. These tests are asymptotic and based on the empirical Hessian matrix. Then, we have estimated the full model again by

A. Estimation of model (1): main results

Transition from CCC to D		
Parameter	Estimation	Standard deviation
SP	−1.0696	0.1833
Slope	−0.1853	0.0347
Short rate	−0.0843	0.0187
Down	1.3134	0.0667
Transitions from BB to B and from B to CCC		
Parameter	Estimation	Standard deviation
IPI	−7.2104	0.5858
Slope	−0.1563	0.0200
Short rate	−0.0622	0.0098
Not US-EU	0.1102	0.0601
Down	1.1474	0.0499
Up	−0.8694	0.1908
Transitions from AA to A, from A to BBB and from BBB to BB		
Parameter	Estimation	Standard deviation
IPI	−2.8389	0.5489
SP	−1.1774	0.1178
Slope	−0.0043	0.0176
Short rate	0.0243	0.0084
Not US-EU	0.1285	0.0578
Up	−0.5749	0.1273
One notch upgrades		
Parameter	Estimation	Standard deviation
IPI	3.6968	0.5936
Slope	0.1131	0.0187
Short rate	0.0816	0.0086
Not US-EU	−0.1278	0.0711
Bank	0.4349	0.0552
Down	−0.5187	0.0992
Up	−0.4994	0.1369

B. Yearly transition matrix 1981–2004 as provided by our model (%)

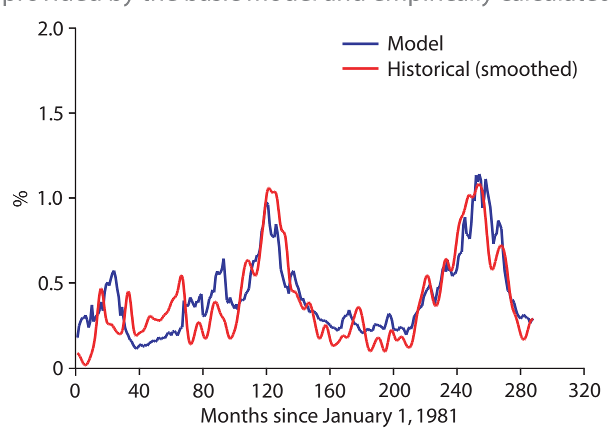
	AAA	AA	A	BBB	BB	B	CCC	D
AAA	91.76	7.39	0.68	0.11	0.06	0.00	0.00	0.00
AA	0.61	90.76	7.91	0.58	0.08	0.05	0.01	0.02
A	0.06	1.97	91.70	5.61	0.45	0.19	0.02	0.07
BBB	0.03	0.23	3.85	89.88	4.99	0.85	0.09	0.11
BB	0.03	0.09	0.48	5.13	83.81	8.58	1.29	0.69
B	0.00	0.06	0.19	0.46	5.01	83.07	7.28	4.58
CCC	0.04	0.01	0.22	0.31	0.70	6.28	57.81	36.53

C. Yearly transition matrix 1981–2004 as provided by S&P (%)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	91.83	7.50	0.48	0.12	0.06	0.00	0.00	0.00
AA	0.65	90.24	8.30	0.62	0.05	0.12	0.02	0.01
A	0.05	2.20	91.05	5.98	0.46	0.18	0.04	0.05
BBB	0.03	0.24	4.26	89.04	5.01	0.87	0.22	0.33
BB	0.03	0.09	0.39	5.91	82.84	8.26	1.12	1.36
B	0.00	0.08	0.24	0.33	5.67	82.08	4.97	6.63
CCC	0.10	0.00	0.30	0.50	1.59	10.43	53.03	34.06

Source: CreditPro 7.0

1 Monthly default rates for US-EU speculative grade, as provided by the basic model and empirically calculated



retaining only the significant variables. This is a standard empirical procedure for maximum likelihood estimation. Briefly, we can analyse these results qualitatively.

- For almost every transition, when the IPI or the S&P indexes increase, the probability of downgrades (including default) goes down. This is particularly the case for the transition to default.
- The explanatory power of our variables is stronger for downgrades than for upgrades, even if an increase in the IPI rate has a significant impact on the latter.
- When the rates go up, the macroeconomic environment is better. Thus, speculative-grade firms are downgraded less frequently. Surprisingly, the opposite effect occurs with investment-grade firms. This may show their tendency to take more risk in such an environment.
- Banks are more easily upgraded than other firms but can downgrade just as easily as any other sector.
- The rating process seems to be harder for firms located outside the US and western Europe. Such firms are more often downgraded and less often upgraded, although their default rates are not higher when they are rated CCC.
- When a firm has been downgraded for less than a year, its downgrade (including default) probability increases significantly. Such an effect is less sensitive when the firm has been upgraded for less than a year. Surprisingly, when a firm has been upgraded by one notch, its current probability of additional upgrades falls. In other words, it is easier to be downgraded quickly than upgraded quickly.

After having estimated the matrices of transition intensities for firm i by $I_i(t) = [\alpha_{hji}(t)]_{1 \le h, j \le p}$, we deduce an estimate of monthly transition matrices at time t by $\hat{P}_i(t, t+1) = Id_p + \hat{I}_i(t)$. Obviously, we set $\alpha_{hhi} = -\Sigma_{j \ne h}\alpha_{hji}(t)$. To calculate transition matrices between two arbitrary dates t_1 and t_2 (in months), we multiply successive monthly transition matrices. This procedure is justified in the usual counting process theory (see Andersen *et al*, 1993).

To assess the quality of the model, we calculate the mean annual transition matrix 1981–2004 as computed by our model, using the past values of the relevant macroeconomic explanatory variables. We compare this matrix with the S&P annual transition matrix as provided by CreditPro (see tables B and C). The fit is good, especially concerning the main diagonal and the 'one-notch' downgrades or upgrades. Nonetheless, the yearly default rates are too low, especially for the categories BBB and BB. This is due to a strong non-Markovian feature of these particular rating transitions. (In other words, it has been observed in the past as 'too high' a number of such transitions with respect to the model.) Note that, due to the continuous time estimation, all the transition probabilities in table C are non-zero (even if they are very low, sometimes).

Moreover, we have focused on the monthly default rates themselves. They are compared with those obtained empirically from the S&P database. We obtain figure 1. After smoothing the empirically observed default rates, the fit is very good. Thus, the model is able to recover the fluctuations of the historical default rates that are due to the economic cycle.

Extension to frailty models

In the statistical literature, it is well recognised that some unobservable or missing random variables can affect the underlying endogenous process. Thus, we extend the basic model by introducing a so-called 'frailty' variable γ_{hjit}, to take into account and summarise the effect of all the (systemic or specific) variables that were forgotten in the basic model. Frailty models have been used for a long time in survival analysis and have been applied in many fields. In the finance area, they have been recently proposed for credit risk modelling (see Metayer, 2004, and Schönbucher, 2003). Metayer introduces frailties as static multiplicative factors in a Cox model, allowing the calculation of closed-form likelihood criteria. Schönbucher assumes a strong contagion effect between firms whose (static) frailties are strongly correlated. In Fermanian & Sbai (2006), it is shown that dealing with frailties in intensity-based credit risk models induces the same dependence levels between default events as in the usual Merton-style models. Such type of conclusions appeared in Schönbucher (2003).

To be more specific, we will differ from the current literature by considering a frailty process rather than 'static' frailties. For every firm i, every time t and every transition (h, j), $h \ne j$, we now assume:

$$\alpha_{hji}(t|z) = \gamma_{hijt}\alpha_{hj0}(t)\exp\left(\beta'_{hj}z_{hji}(t)\right) \quad (2)$$

Thus, the 'conditional' likelihood (the likelihood of the transitions knowing the frailty values) is the product of n 'firm-specific conditional' likelihoods:

$$\mathcal{L}_i^c = \left\{\prod_t \prod_{j\neq h}\left(\gamma_{hjit} e^{\beta_{hj0}+\beta_{hji}^T z_{hji}(t)}\right)^{dN_{hji}(t)}\right\} \times e^{-\sum_{j\neq h}\int_0^\infty Y_{hi}(u)\gamma_{hji,u} e^{\beta_{hj0}+\beta_{hji}^T z_{hji}(u)}du} \quad (3)$$

The full likelihood is the expectation of this conditional likelihood with respect to the law of the frailties. We assume the frailties do not depend on i. To lighten the notations, we consider a single group of transitions (for example, only the one-notch downgrades). Therefore, the sub-indexes h, j can be removed. That is why the frailty process is denoted now simply γ_t. We get the full likelihood:

$$\mathcal{L} = \mathbb{E}_\gamma\left(\mathcal{L}^c\right) = C(\beta)\mathbb{E}_\gamma\left(\prod_{t=1}^{T_0}\gamma_t^{\sum_{i=1}^n\sum_{j\neq h}\Delta N_{hji}(t)} \times e^{-\gamma_t\sum_{i=1}^n\sum_{j\neq h}\int_{t-1}^t Y_{hi}(u)e^{\beta_{hj0}+\beta_{hji}^T z_{hji}(u)}du}\right) \quad (4)$$

where:

$$C(\beta) = \prod_{i=1}^n\prod_t\prod_{j\neq h} e^{\left(\beta_{hj0}+\beta_{hji}^T z_{hji}(t)\right)dN_{hji}(t)}$$

Here, we have chosen an annual time unit. It means that the frailties are constant during a whole year. The time T_0 denotes the historical length of our database, that is, $T_0 = 23$. Because of the dynamic feature of the frailties, the previous equation (4) cannot be simplified: no closed-form formulas exist, except in the special case of constant frailties.

Dynamic frailties are intuitively natural: as with the observed macro factors z, the unobserved factors that drive the credit risk should be time-dependent. This is not a detail. Indeed, in our multi-period framework, it is the only way to increase significantly the dependence levels between rating changes with respect to the basic model. Actually, by drawing independently different frailties from one period of time to another, a 'law of large numbers' implies some type of compensation between successive periods: what we get in hectic conjuncture (in terms of losses) can be lost hereafter during quiet periods. Therefore, it is important to introduce some inertia in the frailty process $(\gamma_{hijt})_{t=1,\dots,T}$. We choose the following specification :

$$\gamma_{hij1} = \tilde{\gamma}_{hij1} \quad \gamma_{hijt} = \gamma_{hij,t-1}\tilde{\gamma}_{hij,t} \quad \text{for} \quad t = 2,\dots,T$$

The random quantities $\tilde{\gamma}_{hij,t}$ are drawn independently for every t. The variables $\tilde{\gamma}_{hij,t}$ follow a gamma law, whose single parameter is denoted by α. We also impose that $\mathbb{E}[\tilde{\gamma}_{hij,t}] = 1$ and $\mathrm{Var}(\tilde{\gamma}_{hij,t}) = 1/\alpha$. Thus, the frailties add some additional variability on the hazard rates, by preserving their means. Such dynamic frailty models are seldom found in the literature. Nonetheless, Yue & Chan (1997) have studied the same type of autoregressive process as ours. Notably, Koopman, Lucas & Monteiro (2005) have proposed the same family of models as ours independently. Nonetheless, we differ concerning the estimation procedure (based on the Kalman filter in their case and the Monte Carlo Markov chain (MCMC) here). In such models, the main difficulty lies in inference and the lack of tractable formulas. Particularly, no more closed-form formulas are available for the maximum likelihood criterion, contrary to constant (static) gamma frailty components.

Denote by p the density of the frailties vector $\gamma^{T_0} = (\gamma_1, \dots, \gamma_{T_0})'$. Clearly:

$$p\left(d\gamma^{T_0}\right) = \prod_{t=1}^{T_0} g\left(\alpha, \frac{\alpha}{\gamma_{t-1}}\right)(\gamma_t)\,d\gamma_1\cdots d\gamma_{T_0}$$

where $g(\alpha, \beta)$ denotes the density of a gamma random variable $\mathcal{G}(\alpha, \beta)$:

$$g(\alpha,\beta)(x) = \frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)}e^{-\beta x}\mathbf{1}_{\mathbb{R}^+}(x)$$

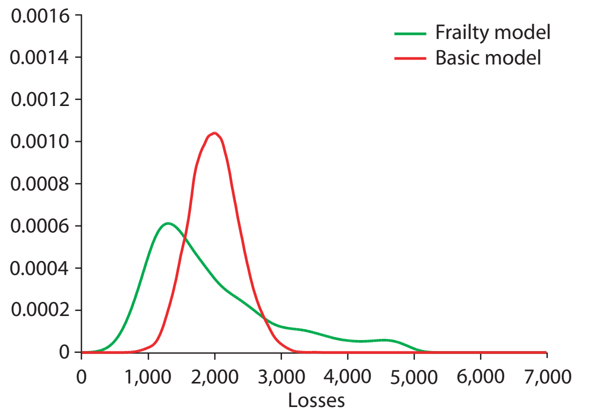
Moreover, we have set $\gamma_0 = 1$ as a convention. Coming back to the likelihood (4), we can go one step further:

$$\mathcal{L}(\theta) = \int_{\mathbb{R}^{T_0}}\bar{\mathcal{L}}(\theta)\,p\left(d\gamma^{T_0}\right)$$

where:

$$\bar{\mathcal{L}}(\theta) = C(\beta)\prod_{t=1}^{T_0}\gamma_t^{\sum_{i=1}^n\sum_{j\neq h}\Delta N_{hji}(t)} \times e^{-\gamma_t\sum_{i=1}^n\sum_{j\neq h}\int_{t-1}^t Y_{hi}(u)e^{\beta_{hj0}+\beta_{hji}^T z_{hji}(u)}du}$$

2 Loss distributions given by the basic model and the frailty model



$\bar{\mathcal{L}}(\cdot)$ is the so-called 'complete' likelihood (the likelihood associated with all the rating transitions and frailties, if the frailties were observable). It is a function of θ, the vector of all the underlying parameters. The vector θ contains the unknown α but also the coefficients β_{hj}.

The estimation of the model implies the full maximisation of $\mathcal{L}$ with respect to θ. This is clearly challenging. To tackle this issue, we propose an 'expectation-maximisation' inference method based on MCMC simulation techniques. The EM algorithm is a popular and efficient approach to maximum likelihood estimation for incomplete data. It induces an iterative optimisation method that generates a sequence of estimated parameters (θ_k). Starting from an 'arbitrary' θ_0, and assuming we have found $\theta_1, \ldots, \theta_k$, we need to maximise the following $Q(\cdot|\theta_k)$ criterion defined as:

$$Q(\theta|\theta_k) = \int_{\mathbb{R}^{T_0}} \ln\left(\bar{\mathcal{L}}(\theta)\right) p\left(d\gamma^{T_0} | Y, \theta_k\right) \quad (5)$$

where $p(\cdot|Y, \theta_k)$ denotes the density of the frailties vector γ^{T_0} knowing all the observations Y (all the rating transitions) and assuming the value of our parameter is θ_k. At each step, we approximate the integral in equation (5) by a sum, by a usual Monte Carlo procedure. This is the usual simulated EM approach, called SEM (Celeux & Diebolt, 1985). Therefore, we need to draw in the conditional law $p(\cdot|Y, \theta_k)$.

This last step can be done by MCMC techniques, here a Hastings-Metropolis algorithm with random walk. Knowing $\gamma_s^{T_0}$:

- Generate $y_s \sim q(\cdot - \gamma_s^{T_0})$.
- Generate:

$$\gamma_{s+1}^{T_0} = \begin{cases} y_s & \text{with prob } \rho = \min\left(1, p\left(y_s | Y, \theta_k\right) / p\left(\gamma_s^{T_0} | Y, \theta_k\right)\right) \\ \gamma_s^{T_0} & \text{with prob } 1 - \rho \end{cases}$$

The instrumental distribution q has been chosen to be uniform on $[-0.1, 0.1]^{T_0}$. Thus, we simulate a Markov chain $(\gamma_t^{T_0})_{t=1,\ldots,S}$ whose stationary law is $p(\cdot|Y, \theta_k)$. By doing the same procedure S times, we approximate the criterion $Q(\theta|\theta_k)$ by:

$$\tilde{Q}(\theta|\theta_k) = \frac{1}{S}\sum_{s=1}^{S} \ln\left(\bar{\mathcal{L}}(\theta)\right)\left(\gamma_s^{T_0}\right)$$

that we maximise with respect to the parameters θ to get θ_{k+1}. This idea (MCMC random draws inside a SEM procedure) has already been proposed in the statistical literature (see Diebolt & Ip, 1996, for instance).

This frailty model extends the previous 'basic model'. To simplify the analysis and to avoid a type of dilution of the frailty effects among the different group of rating transitions, we have reduced their number. We have kept only two groups: one-notch downgrades and one-notch upgrades. The estimated coefficients β_{hj} are qualitatively closed to those obtained in the previous section, so they are not recorded again here. Our estimated α is 23.6 for the downgrades and 52.0 for the upgrades. Such values seem to be large and generate small $\tilde{\gamma}_t$ variances. Nonetheless, in the long run, the process γ_t can reach relatively high values. For instance, the standard deviation of γ_t for $t = 10$, 20 and 30 are respectively 0.33, 0.46 and 0.58. For $t = 10$ and larger, the γ_t quantile at 95% is larger than 1.61.

By introducing dynamic frailties, the dependence between rating transitions and default events can be significantly increased, especially in the long run. This is particularly relevant for multi-period economic capital calculations. As an illustration, we have considered a simple portfolio (50 firms, different ratings from AAA to CCC), with long-term (30 years) constant equal exposures. We have simulated 1,000 random paths for the observable explanatory random variable (assuming they follow a one-order vectorial autoregressive process $VAR(1)$). For every previous random path, 10 frailty paths are drawn. Then, we calculate the losses due to the default events between today and the time horizon, along the 10,000 drawn scenarios. We obtain figure 2. Clearly, the tails are potentially a lot fatter when dynamic frailties are introduced. For the basic model, the shape of the loss distribution appears Gaussian (even if this is not really the case), which is surprising. Actually, we have noticed that these shapes depend strongly on the z_t process we choose in practice: mean reversion or not, the degree q in $VAR(q)$, etc. So, by choosing other econometric processes, the tails can become fatter, even for the basic model.

D. Default rates provided by the 'basic' model, the frailty model (1,000 trials, standard deviations in brackets) and S&P statistics (CreditPro 7.0) (%)

	One year				Two years				Five years			
	Basic	Frailty		S&P	Basic	Frailty		S&P	Basic	Frailty		S&P
AAA	0.000	0.000	(10^{-4})	0.00	0.001	0.001	(0.001)	0.000	0.016	0.017	(0.024)	0.098
AA	0.002	0.002	(0.001)	0.01	0.010	0.011	(0.012)	0.041	0.080	0.104	(0.267)	0.319
A	0.007	0.007	(0.006)	0.04	0.037	0.043	(0.066)	0.135	0.279	0.335	(0.681)	0.654
BBB	0.107	0.112	(0.057)	0.33	0.335	0.379	(0.390)	0.903	1.641	1.704	(1.917)	3.266
BB	0.686	0.706	(0.560)	1.27	2.212	2.174	(1.918)	3.824	7.806	7.052	(4.978)	12.10
B	4.588	4.503	(2.321)	6.06	10.46	9.720	(4.984)	13.22	24.08	21.12	(8.999)	26.40
CCC	36.53	28.49	(12.90)	30.5	48.11	41.91	(14.62)	39.50	67.44	59.17	(14.13)	52.68

E. Correlations between default events at several horizons as provided by our models and by S&P (%)									
	One year			Two years			Five years		
	IG	SG	Crossed	IG	SG	Crossed	IG	SG	Crossed
S&P	0.10	1.37	0.26	0.28	2.13	0.56	0.31	2.68	0.67
Basic	0.003	1.02	0.12	0.025	1.628	0.447	0.010	0.183	–0.147
Frailty									
Mean	0.0104	1.208	0.175	0.089	2.130	0.708	0.174	1.340	0.472
*q*5%	0.0012	0.388	–0.011	0.005	0.563	–0.033	0.006	0.078	–0.361
*q*25%	0.0015	0.572	0.024	0.007	0.938	0.0440	0.021	0.340	–0.266
*q*50%	0.0021	0.798	0.062	0.0115	1.343	0.165	0.037	0.805	–0.203
*q*75%	0.0047	1.423	0.157	0.0357	2.415	0.577	0.071	1.381	–0.008
*q*95%	0.0296	2.533	0.562	0.261	5.001	2.475	0.271	3.441	1.460

Note: in the frailty case, we get a sample of correlations (one for each frailty random path). We have detailed the means and the quantiles of these samples

For the sake of comparison, we have calculated the mean default rates at several horizons as provided by our basic model, the frailty model and S&P (historical rates) (see table D). Globally, our models tend to underestimate the default rates for investment-grade firms. This is probably due to some non-Markovian features in the historical rating processes. A simple way to tackle this issue would be to increase the time unit: instead of estimating monthly transition matrices, consider quarterly or even semi-annual matrices. But we would then lose the nice features of the time-continuous description.

Finally, we have compared the correlations between the default events (see table E). For the sake of comparison, we have followed the same methodology as S&P. By simulating 10,000 rating transitions over one, two and five years for every cohort, we find a coherence with the S&P correlation levels, particularly with the frailty model and for short-term horizon. The historical correlation levels are relatively well recovered, which was *a priori* a difficult task.

It is important to note that the extra variability induced by the frailty processes allows us to obtain a significant range of default rates and correlation levels. Thus, for a significant number of trajectories, these quantities are equal or even larger than those observed historically. This is important in a credit VAR or economic capital perspective. The frailty model will be more conservative than the basic one.

Finally, note that, for all these empirical comparisons, we have controlled the sample effects: for every cohort, we have considered the same number of firms as in the S&P database in terms of ratings and other idiosyncratic characteristics (bank or not, US-EU or not, up or down). As S&P does, we calculate our statistical indicators by some weighted means with respect to the sizes of these cohorts.

Conclusion

We have explained how to define and estimate a reduced-form credit portfolio model in a full competing risk approach. Extensions towards dynamic frailty models allow us to take into account unobservable explanatory variables and generate potentially a large range of dependence levels between rating changes. The 'in-sample' fit with the S&P historical database is satisfying, especially concerning the correlation levels between default events. For economic capital calculations, this model can be applied by assuming the explanatory random variables follow a particular random process, for instance vectorial autoregressive. We have observed the performances of the model can be relatively sensitive for different choices of the latter process. But, even if such a point is important in practice, it does not jeopardise the relevance of our approach. Moreover, our model can be used straightforwardly as a predictor of the credit market, under some stress-test assumptions or some macroeconomic forecasts. ■

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