A top-down approach for Asset-Backed securities: a consistent way of managing prepayment, default and interest rate risks.

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Abstract

We define a new approach to manage prepayment, default and interest rate risks simultaneously in some standard asset-backed securities structures. We propose a parsimonious top-down approach, by modeling directly the portfolio loss process and the amortization process. Both are correlated to interest rates. The methodology is specified for sequential- and pro-rata pay bonds (ABS, CMO, CDO of ABS), cash or synthetic. We prove analytical formulas to price all tranches, under and without the simplifying assumption that amortization occurs in the most senior tranche only. The model behavior is illustrated through the empirical analysis of an actual synthetic ABS trade.

Key words: Asset-backed securities, top-down models, default risk, prepayment.

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1 Introduction

The residential mortgage market is traditionally the largest debt sector in most countries. The development of a strong housing finance market requires the participation of investors who are willing to hold residential mortgage loans as either whole loans, or in the form of a security. The best developed housing finance market in the world is that of the United States. It is partly due to the ability of investment bankers (in conjunction with government agencies) to create securities backed by a pool of residential mortgage loans that are more appealing for institutional investors to hold rather whole loans. These securities, referred to as mortgage-backed securities (MBS), were issued in the 70's. The process of creating these securities is referred to as securitization. More specifically, the first MBS issued were mortgage pass-through securities, securities where the cash flow of the pool of residential mortgage loans (amortization including credit losses, prepayments, interest after servicing expenses and any guarantee fees) is prorated among the certificate holders. At those dates, the uncertainty about the cash flow to investors in a mortgage pass-through security was due to prepayments mainly, i.e. payments different from the (originally scheduled) principal amortization. In the early 80's, investment bankers created a different type of MBS. Instead of distributing the cash flow from the pool of mortgage loans on a pro rata basis, it is now distributed according to rules for principal (both amortization and prepayments) and interest to different bond classes (tranches) in the structure. This type of MBS is called a collateralized mortgage obligation (CMO). With a typical bond class with sequential payments, investors are protected from prepayment and default by the lower bond classes in the capital structure. By the tranching process, investors who buy such a bond can benefit from some credit enhancements, depending on the risk/return profile they wish. Recently, another floor of securitization has appeared, with CDOs of asset-backed securities (CDOs of ABS). Strictly speaking, they are securitization of pools of CMO tranches, i.e. "securitization of securitization", creating highly sophisticated securities and new modeling challenges.

All the previous products are particular cases of "Asset-backed securities (ABS)". ABS [1] are commonly traded in the USA and now in Europe. These financial products are built from large pools of asset classes: mortgages, home equity loans, commercial loans, student loans, credit cards etc. In that paper, the two main risks that will be associated with ABS are

- a prepayment risk : some underlying assets can be repaid quicker/slower than expected,

[1] In that article, the terms MBS and ABS will be used equivalently. Even if our developments are particularly relevant to pools of credit risky mortgages (subprime loans, for instance), the results can be used and adapted easily to cover other asset classes.

inducing a (marked-to-market) loss for investors possibly.

- a default risk : some borrowers may be unable to reimburse some coupons or the principal of their loans fully.

Beside these typical risks, a more standard (in the Fixed Income world) interest rate risk applies [2]. It is important to note that interest rates have an impact on prepayment and credit risk too. Indeed, lower interest rates induce some incentives to prepay current loans to enter into new ones under better financial conditions. Alternatively, when interest rates rise, the weight of periodic reimbursements will become heavier for weak floating-rate borrowers, pushing them to bankruptcy. Moreover, it is recognized that Home Price Indices (HPIs) are negatively correlated with the level of interest rates (see Levin and Davidson, 2008). Thus, when rates go up, HPIs go down and then Loan-To-Values (LTV) of current loans increase, inducing larger likelihoods of future default events.

Following some seminal papers in the end of the 80's (Richard and Roll 1989, Schwartz and Torous 1989), a significant stream of mortgage-related papers has appeared in the academic and the professional areas. Surprisingly, this literature has increased largely independently from the main stream of asset pricing papers. "To some degree derivatives pricing and securitization pricing evolve as two different trade practices" (Lou, 2007). This is partly due to the particular features and risks of ABS markets. One one side, some authors try to explain borrower's behaviors under a theoretical point of view and to value the underlying prepayment and default options that convey mortgages (the "optional" approach). On the other side, others have proposed to explain the duration of mortgages directly by econometric models (the so-called "reduced-form" approach): Deng (1997), Kariya and Kobayashi (2000), Kariya et al. (2002), among others.

In the literature, most of the authors have tackled prepayment risk and default risk separately. And the former has retained the attention more frequently than the latter, partly because of the implicit governmental guarantee of agency mortgages. Nonetheless, it has been recognized for a long time that prepayment and default are not independent processes. Thus, some authors have investigated the interaction of prepayment and default decisions: among others, Schwartz and Torous (1993), Kau, Keenan, Muller and Epperson (1995) under the reduced-form approach, more recently Sharp et al. (2008), Dunsky and Ho (2007) under the optional point of view.

When dealing with default and prepayment risks simultaneously, the framework of competing

[2]For instance, the interest rate risk for a fixed coupon MBS is that the price drops as rates increase because of the differential between the coupon being paid in the MBS and the market rates

risk models appears naturally: see Deng et al. (2000), Kau et al. (2006) for instance. Therefore, the mortgage-related literature has applied models and concepts from Survival Analysis. More recently, the development of credit risk models and stochastic intensities approaches have influenced this stream of the financial literature, as in Lou (2007) or Goncharov (2005). Indeed, as the latter author said: "After all, from a mathematical point of view nothing precludes one from interpreting prepayment as a "default" in the intensity-based approach to pricing credit risk." In our opinion, the previous credit risk intuition is far from being fully exploited in the ABS world. Indeed, among the thousands of papers in Credit Risk, only a few have been revisited or adapted in light of mortgages.

That article is an attempt to fill this gap partly. We propose to borrow some theoretical concepts from the asset pricing literature in general, and from credit derivatives in particular. We choose the relatively recent "top-down" approach for the pricing of structured products, CDOs in particular: Bennani (2005), Schönbucher (2005), Andersen et al. (2008), Giesecke et al. (2010), among others. Here, the basic idea is to deal with aggregated loss processes instead of trying to detail numerous individual loss intensities and their dependencies (see Bielecki and Rutkowski 2002, for a complete overview). This seems to be particularly relevant in the case of mortgage pools that put together thousands of underlying loans. We will state our results in a continuous time framework. It is rather unusual in the mortgage literature, but is a lot more standard in asset pricing.

Actually, some authors have already shared our idea of drawing inspiration from credit derivatives models: Lou (2007), Jäckel (2008), Garcia and Goossens (2008a, 2008b)... But all of them have adapted the classical Gaussian copula model of CDOs to price ABS structures, whose shortcomings are well-known (Lipton and Rennie, 2007). Therefore, they propose a "static" bottom-up approach, contrary to our "dynamic" top-down model. And none of the authors have obtained the equivalent of our analytic pricing formulas until now [3].

Therefore, the main advantage of our framework is to state nice closed-form formulas, what is very rare in the ABS world. Thus, we avoid computationally intensive simulation-based pricing methods. Of course, the achievement of closed-form solutions requires some restrictions about the underlying assumptions. Notably, Rom-Poulsen (2007), Collin-Dufresne and Harding (1999) provide semi-analytical intensity-based formulas, but they deal with prepayment only in a one- or two factor model respectively. On our side, we propose a parsimonious and consistent way of

[3] except under the "infinitely" granular assumption, as in Vasicek (2002)

dealing with the main underlying risks together. It is a new approach in the mortgage/ABS world, to the best of our knowledge, even if some authors have already understood the benefit of dealing with mortgage portfolio losses directly. For instance, Gautier (2003) extracts a risk-neutral loss distribution from market quotes, but no underlying dynamics. Levin and Davidson (2008) propose a dynamic factor models driven by Home Price Indices and interest rates, but a simulation pricing scheme only.

Actually, to value CMOs, there exists an alternative to simulations. Indeed, McConnell and Singh (1993) solve PDEs, by adapting the two-factor methodology of Schwartz and Torous (1989), under the "rational" prepayment approach (Dunn and McConnell 1981a, 1981b; Stanton 1995, e.g.). Their techniques are extended and adapted to credit-risky multiclass CMBS by Childs et al. (1996). Instead of relying on numerical algorithms to solves PDEs, we promote rather analytical formulas and a reduced-form model in this article. In our opinion, our approach relies less on borrower behavior assumptions than the "optional" point of view, and is a lot more efficient numerically (see the discussion in Lou 2007, or Sharp et al. 2008).

As a consequence, we will not try to exhibit an optimal mortgagor behavior in terms of prepayment. Paydown risk is aggregated at the portfolio level, and only its dependence on the global factors in the economy is relevant for us. A proxy of these factors is given by the yield curve itself, or the risk-free bond prices. Contrary to a large part of the literature that deals with conforming mortgages [4], default events are the second main source of risk in the structures we consider. It is particularly the case for structures that contain a large proportion of floating rate notes or that integrate a lot of low quality debtors (subprimes). Moreover, it is also relevant for continental European structures where prepayment incentives are a lot weaker than in the USA, partly due to associated penalties, and where loans are not guaranteed most of the time.

We illustrate our methodology first with the pricing of simple sequential-pay CMOs: credit losses will be recorded from junior to senior tranches. Payments of principal will follow the opposite order. Even if overcollateralization can be summarized in the most junior tranche, we have simplified reality. A lot of complexities that can occur in real deals are not modeled explicitly: excess spread mechanisms, OC tests, credit triggers, more or less exotic PAC (planned amortization classes) etc. Here, the idea is rather to show to what extent closed-form formulas can be obtained in a reasonable framework. Actually, such a scheme is not far from the features of standard synthetic ABCDS or of the first CDOs of ABS in the market. And we are able to price easily

[4] implicitly insured by Government Sponsored Enterprises in the USA

standardized coupon-bearing bonds. Note that our model specification is independent from the nature or the characteristics of the underlyings. These features will be taken into account through some aggregated quantities only: expected losses, outstanding principals, expected prepayment and default rates.

In section 2, we introduce the main ideas and the main equations. They describe the behavior of the portfolio expected loss process and the random amortization profile. Then, we provide the relevant closed-form pricing formulas by some change of numeraire techniques, in the case of a (simple) sequential-pay synthetic structure (section 3) and of a (simple) pass-through cash structure (section 4). We provide some empirical results and a sensitivity analysis of the model with respect to (w.r.t.) its parameters in section 5. These results will be stated under the assumption that the most senior tranche of the capital structure only will be hit by amortized and prepaid amounts (assumption (A)). The latter assumption, that was reasonable before the "subprime" crisis, can be questioned from that time. Therefore, we generalize significantly our results by removing the assumption (A) in the appendix B. There, we prove semi-analytical formulas for sequential or pro-rata pay ABS structures, cash or synthetic.

2 A top-down pricing model

In a top-down framework, we do not try to fully use loan-per-loan information: age and gender of borrowers, geographical area, Loan-To-Value, maturity and size of loans, financial strengths as provided by some scores (FICO)... We prefer to restrict the potentially huge information set to a few "information summaries" that will be sufficient to price the deal. Particularly, the expected amortization profile is a crucial element for an ABS deal. Then, a random process is built around the latter profile to take into account the uncertainty of principal paydowns. Similarly, the underlying default risk will be tackled by the expected loss random process of the whole portfolio. Both processes will be correlated with each other, and with the yield curve.

Therefore, the pricing procedure will be based on three random processes. In practical terms, our approach will be a lot cheaper than managing thousands of individual loan descriptions and their interdependencies inside micro-econometric models. These models have been put in place in the last years among the most advanced dealers. Some of them are offered for sale but some specialized boutiques [5]. Unfortunately, this family of models requires lengthly simulations and

[5] "Andrew Davidson & Co, Inc" e.g.

suffers from the “double stochastic” curse: simulation of factors (yield curve paths, Home Price Appreciation forecasts), and, conditionally on these realizations, simulation of every borrower behavior in terms of prepayment and default. In fact, in usual MBS/ABS, a loan-per-loan risk management is clearly costly, due to calculation time and IT constraints particularly, and it is exposed to a significant model risk. Thus, in that article, we assume the diversification in the underlying pool is so high that a few number of macro-factors is sufficient to price and risk-manage the structures we are considering. A consensus among practitioners converges towards two or three factors [6].

Without a lack of generality, we assume the total principal amount of our pool of assets is one (million of USD, for instance) at origination. These assets can be any type of ABS that may suffer from default and/or prepayment, traditionally mortgages. They are pooled and cash flows are “tranched” as in CMOs. Traditionally, loans are the basic (first level) assets in securitizations, and ABS bonds constitute the (second level) tranche products. Here, we cover broader situations, like CDOs of ABS, where ABS bonds themselves are the basic units. To fix the ideas, the tranching process is related to several detachment points $K_0 < K_1 < \ldots < K_p$. We set $K_0 = 0$ and $K_p = 1$. The bond with the attachment point K_{j-1} and the detachment point K_j will be called the $[K_{j-1}, K_j]$ tranche.

As explained in the introduction, the principal amounts of these tranches can be reduced due to different effects :

- The “natural” amortization process. It is deterministic for every underlying name and deduced from contractual terms.
- the prepayment process. It can be seen as a randomization of the previous amortization profile.
- the loss process. It is due to failures to pay some remaining coupons or principal.

Potentially, all previous effects can reduce a given tranche simultaneously, at least from a certain time on. For the sake of simplicity, we would like to illustrate our ideas first from an over-simplified structure. It will be synthetic (no stream of coupons), as in CDS on ABS tranches (so-called ABCDS). The credit losses will be sequential “from below”. When a loan is liquidated, the recovered amount is lower than the remaining balance, most of the time. Then, the corresponding

[6] for example, the slope of the interest curve, an Home Price index, and possibly the unemployment rate, as in Patruno et al., 2006.

loss reduces the most junior tranche $[0, K_1]$ first. When the latter has gone fully through, credit losses attack the subsequent tranche $[K_1, K_2]$, etc. Finally, the principal paydown (due to "normal" amortization, prepayment or loss recovery) will be recorded sequentially "from the top": it reduces the most senior tranche $[K_p, 1]$ first. When this tranche has disappeared, subsequent payments reduce the tranche $[K_{p-1}, K_p]$ and so on. Independently, all these assumptions are reasonable and are met in reality in some ABS-type securities. Nonetheless, it is rare to meet all of them into a single security. Strictly speaking, the products we have detailed can be called "synthetic sequential-pay CMO tranches". We think they will constitue a good example of the pricing technique we promote. Further extensions will be provided afterwards: coupon-bearing structures (section 4), pro-rata payments (appendix B). We are convinced our methodology is able to manage a lot more products, even if closed-form formulas can become involved technically.

For the moment, we do not have to take care about coupon payments, and complex waterfall rules more generally. Note that these payment rules may correspond to the simplest usual specification of a synthetic CDO of ABS. In such a product, the underlyings are some ABCDS. No initial fund is necessary to invest in such a structure. The cash flows are coming only from principal paydowns and defaults. The main price driver is here default risk, likewise usual synthetic corporate CDOs. The price of CDOs of ABS is very difficult to evaluate. These exotic products have been put under the sunspots during the credit crisis in $2007-2010$. They have concentrated criticism, and appeared as icons of the excesses in the exotic credit derivative business.

Now, let us specify our model. At every time t, the outstanding principal of the whole portfolio will be denoted by $O(t)$ and the outstanding principal of the tranche $[0, K]$ by $O_K(t)$. Obviously, these quantities are random. Let $RP_{t,K}$ and $DL_{t,K}$ be respectively the risky principal and the default leg that are associated with the "equity" (or junior) tranche $[0, K]$ at time t. The former quantity is a credit risky duration, but weighted by a fraction of principal. The latter is the mean amount of credit losses that will be recorded for this particular equity tranche. This terminology is standard in Credit Derivatives, for the pricing of CDS or synthetic CDOs for instance (see O'Kane, 2008). For instance, the t-spread $s_{t,j}$ of a synthetic CDO tranche $[K_{j-1}, K_j]$ is defined by

$$s_{t,j}\{RP_{t,K_j} - RP_{t,K_{j-1}}\} = DL_{t,K_j} - DL_{t,K_{j-1}}, \tag{1}$$

for every t and every $j = 1, \ldots, p$. The same logic applies with ABCDS or CDOs of ABS, for example. Therefore, the first goal of our model will be to evaluate these risky principals and these default legs in closed-form. Indeed, they are sufficient to value the products we consider in this article.

Let us denote by T^* the maturity of our structure. In ABS securitization, the legal final maturity date T_∞ is the last maturity of its pooled mortgages, and $T^* \leq T_\infty$ obviously. Actual life of a securitization is much shorter in practice due to prepayment, early redemption and clean-up call features [7]. Moreover, the maturity associated to securitization of securitization is often different from the legal maturity. Then, in a pool of 30 year subprime loans, the actual weighted average life may be 7 years, and T^*, the maturity of the structure, may be 5 years. Finally, deals often embed options to call partially or fully outstanding liabilities [8]. We will not value such embedded options, but consider simply that the deal maturity will be T^* for sure.

By the standard arbitrage pricing theory and under the equivalent martingale measure Q, we have

$$RP_{t,K} = E\left[\int_t^{T^*} \exp\left(- \int_t^s r_u \, du \right) O_K(s) \, ds | \mathcal{F}_t \right], \tag{2}$$

by denoting (r_s) the usual short interest rate process. We will denote by $E_t[\cdot]$ expectations conditionally on the market information $\mathcal{F}_t$ at time t and under the risk-neutral measure Q. The filtration $(\mathcal{F}_t)$ records all of the past and current relevant information concerning the description of the cash flows and the underlyings: past payments, contractual features, past and current interest rates, recorded losses etc.

To fix the ideas, let us denote by $A(s)$ the portfolio amortized amount at time s. Moreover, let $A_K(s)$ be the same amount, but related to the tranche $[0, K]$. In other words, $A_K(s) = [A(s) - (1-K)]^+$ in the case of our pure sequential-pay schedule. The latter quantity is the amount of money the tranche $[0, K]$ has been reduced "from above", due to the amortization process only. Actually, since this tranche is reduced potentially "from below" by the default events, the s-outstanding principal of this tranche is $O_K(s) = [K - L(s) - A_K(s)]^+$. Here, we have introduced $L(s)$, the loss of the whole portfolio at time s. It is simply the accumulated amount that is due to default events from inception up to s. The same quantity, but related to the tranche $[0, K]$ is denoted by $L_K(s)$. Note that the latter quantity depends on the outstanding principal process of this tranche, and then on the amortization profile too. This feature complicates the asset pricing

[7] when the sponsor liquidates the remaining collateral and pay off the notes after pool balance has dropped significantly, say to 10% of the original principal

[8] To stimulate the call option, "there is usually a significant step-up in the coupon rate in the event the call is not exercised. As a result, ABS were commonly priced assuming the call is exercised, as the option is expected to be deeply in the money at the call date. With the subprime crisis going forward, this assumption has started to be stressed, however." (Pénasse, 2008)

formulas significantly. Moreover, note that the portfolio outstanding principal $O(s)$ is related to the other quantities by the relation $O(s) = 1 - L(s) - A(s)$.

The loss process that refers to the tranche $[0, K]$ can be rewritten $L_K(s) = L(s).\mathbf{1}(L(s) \leq K - A_K(s))$, when the tranche $[0, K]$ has not been fully paid down. Otherwise, the amount of losses is fixed, and keeps its last value (just before this tranche has been fully fed). Thus, we can write the default leg of the tranche $[0, K]$ as seen at time t:

$$
\begin{aligned}
DL_{t,K} &= E_t \left[\int_t^{T^*} \exp \left(- \int_t^s r_u \, du \right) L_K(ds) \right] \\
&= E_t \left[\int_t^{T^*} \exp \left(- \int_t^s r_u \, du \right) \mathbf{1}(L(s) + A_K(s) \leq K) L(ds) \right]. \qquad (3)
\end{aligned}
$$

In practice, we need to evaluate the latter integral with some grid of dates $T_0 = t, T_1, \ldots, T_p = T^*$. Therefore, as it is done in practice, we consider that [9]

$$
DL_{t,K} \simeq \sum_{i=1}^{p} E_t \left[\exp \left(- \int_t^{T_i} r_u \, du \right) \mathbf{1}(L(T_i) + A_K(T_i) \leq K) \left(L(T_i) - L(T_{i-1}) \right) \right],
$$

with a reasonable accuracy. To evaluate the functions $RP_{t,K}$ and $EL_{t,K}$ and thanks to some elementary algebraic operations, it is sufficient to calculate the expectations

$$
\mathcal{E}_1(s) = E_t \left[\exp \left(- \int_t^s r_u \, du \right) \left[K - L(s) - [A(s) - 1 + K]^+ \right]^+ \right], \qquad (4)
$$

$$
\mathcal{E}_2(s, \bar{s}) = E_t \left[\exp \left(- \int_t^s r_u \, du \right) \mathbf{1}\{L(s) + [A(s) - 1 + K]^+ \leq K\} L(\bar{s}) \right], \qquad (5)
$$

for every couple $(s, \bar{s})$, $t \leq \bar{s} \leq s \leq T^*$. In the sequel, we will often use s and $\bar{s}$ as two generic dates chosen from the set $\{T_1, \ldots, T_p\}$. Particularly, couples of successive dates are necessary, because increments of losses/amortized amounts have to be considered in our pricing formulas.

Now, we concentrate our efforts on the evaluation of the previous expectations $\mathcal{E}_1(s)$ and $\mathcal{E}_2(s, \bar{s})$. Unfortunately, these expressions involve some tricky double indicator functions. For the moment and to lighten the presentation, we assume

Assumption (A): The amortization process and the prepayment process will reduce the most senior tranche only.

In other words, under (A), we consider only trajectories where the amortization process is

[9] We neglect accrued payments due to defaults between two successive dates.

stopped (or the structure is repaid) before the most senior tranche is fully paid back from above. This assumption implies that $A_K(t, T_i) = 0$ for all dates $T_i \leq T^*$ and for all detachment points $K_j < 1$.

Even strong, the assumption (A) can provide a realistic approximation. Indeed, in a lot of ABS structures, the most junior tranches are often a lot thinner than the most senior tranche. For example, it is not unusual the latter one is related to more than 90% of total initial portfolio nominal. Moreover, in practice, these structures are called when the amortization process has reduced a large part of the pool, typically 90% for clean-up calls. In every case, the assumption (A) will be removed and more general semi-analytical formulas will be provided in appendix B.

Under (A), instead of $\mathcal{E}_1(s)$ and $\mathcal{E}_2(s, \bar{s})$, we need to evaluate the simpler quantities

$$E_1(s) = E_t \left[\exp\left(-\int_t^s r_u \, du \right) [K - L(s)]^+ \right],$$

and

$$E_2(s, \bar{s}) = E_t \left[\exp\left(-\int_t^s r_u \, du \right) \mathbf{1}\{L(s) \leq K\} L(\bar{s}) \right],$$

for all tranches except the most senior one. Concerning the most senior tranche, we have to calculate

$$E_1^*(s) = E_t \left[\exp\left(-\int_t^s r_u \, du \right) [1 - L(s) - A(s)]^+ \right],$$

and

$$E_2^*(s, \bar{s}) = E_t \left[\exp\left(-\int_t^s r_u \, du \right) L(\bar{s}) \right].$$

The previous expectations E_1 and E_2 can be deduced from the value of some options that are written on the loss process $L(.)$. Basically, it is more relevant to work in terms of the (not discounted) Expected Loss process itself, which is defined by

$$EL(t, T) := E[L(T) | \mathcal{F}_t] = E_t[L(T)].$$

Indeed, this process is taking into account all forecasts in the market continuously. So, it is more acceptable to set some diffusion processes on the expected losses rather than on the credit losses directly. Moreover, this allows to specify a large variety of behaviors depending on the remaining time to maturity in a consistent way. The effect of time and maturities is strong for MBS/ABS: at the beginning of a pool, default/prepayment rates are low, due to borrowers selection processes (the so-called "seasoning effect"). Then, these rates are usually upward sloping and after some years, they are decreasing. Indeed, the borrowers with "bad" individual characteristics have already defaulted/prepaid, and only the relatively safest borrowers stay in the pool (the "burnout effect"). At last, when the process L is essentially increasing, there is no such constraint on EL, which is a desirable property in terms of model specification.

Thus, these previous expectations can be rewritten as functions of the Expected Losses themselves by noting that $L(s) = EL(s,s)$. For instance,

$$E_1(s) = E_t\left[\exp\left(-\int_t^s r_u\,du\right)[K - EL(s,s)]^+\right], \tag{6}$$

and

$$E_2(s,\bar{s}) = E_t\left[\exp\left(-\int_t^s r_u\,du\right)\mathbf{1}\{EL(s,s) \leq K\}EL(\bar{s},\bar{s})\right]. \tag{7}$$

The latter expectations will be calculated as functions of the spot curve $EL(t,\cdot)$ and the model parameters only. From now on, we will consider the Expected Loss process as our underlying. Similarly, we define the Expected Amortized amount process $A(t,T)$ by $A(t,T) := E_t[A(T)]$.

Moreover, to evaluate the functions $RP_{t,K}$ and $DL_{t,K}$, we have to take into account the randomness of interest rates. Indeed, it has been observed for a long time that the price of mortgage-backed securities depends strongly on the interest rate curves (e.g. see Boudoukh et al. 1997), at least through prepayment incentives. But we go further. It is intuitively clear that default rates themselves depend on interest rates, as explained in the introduction.

Now, we assume that the Expected Loss process is lognormal. Since $(EL(\cdot,T))$ is a Q-martingale by definition, its drift is zero. Thus, we do the

Assumption (E): For every times t and T, $EL(dt,T) = EL(t,T)\sigma(t,T)dW_t$.

Obviously, $(W_t)_{t\in[0,T]}$ is an $\mathcal{F}$-adapted Brownian motion under Q. For the moment, we assume we know the quantities $EL(t,\cdot)$ at the current time t, as if they were observed in the market. Note that $EL(t,T)$ is the sum of the (already) realized credit losses between the inception date [10] and t, i.e. $L(t)$, plus the expectation of all future losses between t and T. Now, let us deal with the interest rate process.

[10] In practice, the underlying loans may have different origination dates for a given bond. This discrepancy is generally smaller then six months. Thus, we neglect the defaults that could occur between the loan originations and the bond origination.

Assumption (IR): Every discount factor $B(t,T)$ follows the dynamics

$$\frac{B(dt,T)}{B(t,T)} = (...)dt + \bar{\sigma}(t,T)d\bar{W}_t,$$

for every T and $t \in [0,T]$. Here, $(\bar{W}_t)$ is an $\mathcal{F}$-adapted Brownian motion under Q, and $E[dW_t.d\bar{W}_t] = \rho dt$.

Thus, a single factor is describing the whole interest curve dynamics. Even restrictive, this assumption is sufficient in this "toy" model, and it could be weakened easily. There exist several ways of specifying the previous volatility functions. Since the underlying borrowers are numerous, we are closed to the infinitely granular hypothesis and the EL paths will be considered as continuous. So, when the time t tends to any horizon T, the Expected Loss $EL(t,T)$ trajectories will remain continuous and $\sigma(t,T)$ tends towards a nonzero constant. This implies that the volatility $\sigma(t,T)$ is surely a decreasing function of t. In practice, we could assume that $\sigma(t,T) = \sigma_0(T-t)^\alpha$, for some unknown constant $\alpha \geq 0$. Alternatively, we could decide to set $\sigma(t,T) = \sigma_0 \left[\exp(\lambda(T-t)) - 1\right]$, with some positive unknown parameters σ_0 and λ, as in Gaussian HJM-type models.

Obviously, we could assume the same type of specifications for the volatility of the discount factors themselves, for instance $\bar{\sigma}(t,T) = \bar{\sigma}_0 \left[\exp(\bar{\lambda}(T-t)) - 1\right]$.

Similarly, we can tackle the amortization/prepayment process $A(t,T)$, that will be assumed lognormal.

Assumption (AM): For every times t and T, $A(dt,T) = A(t,T)\tau(t,T)d\tilde{W}_t$, where $(\tilde{W}_t)$ is an $\mathcal{F}$-adapted Brownian motion under Q, and $E[d\tilde{W}_t.d\bar{W}_t] = \tilde{\rho}dt$.

The volatility behavior of the amortization process $A(t,T)$ can be dealt as the one of the Expected Loss. Thus, for example, we could state $\tau(t,T) = \tilde{\sigma}_0 \left[\exp(-\tilde{\lambda}(T-t)) - 1\right]$, for some positive constants $\tilde{\sigma}_0$ and $\tilde{\lambda}$. Moreover, the lognormal specification does not prohibit some surprising features like $A(t,s) > 1$ or $EL(t,s) > 1$ for some dates (s,t) and with some (small) probability. Theoretically, such events are unlikely, even if excess spread mechanisms could generate paths with negative amortization in practice. Actually, this apparently annoying feature does not induce issues to price any base tranche with detachment point $K < 1$. Indeed, in our pricing formulas, the two processes EL and A will be truncated above by one: see equations (4) and (5). This truncation is taken into account formally in the appendix B. This is even true under the assumption (A) (see equations (6) and (7)), except for the evaluation of the whole portfolio or a super senior tranche (theorem 3.2, for which there will be a warning). Therefore, for us, the single consequence of the Geometric Brownian Motion assumptions (E) and (AM) is the strictly positive probability that expected losses and/or amortization are capped at one for some dates in the future, clearly a rather weak constraint.

It should come as no surprise that the Expected Loss process $EL(\cdot, T)$ may decrease over time, for a given time horizon T. Indeed, this process is partly related to some expectations of future losses and partly to the current realized losses. Future expected losses could become smaller tomorrow if the market participants become more confident concerning the financial strength of the borrowers in the pool. Moreover, in the ABS world, it is even possible to recover some realized losses in the future. Indeed, in some exotic structures, these losses can be temporary, because they are based on some statistical models and projected cash flows. Therefore, "marked-to-model" losses can be recorded one day and recovered at least partly afterwards. These temporary losses are clearly a source of difference with corporate CDOs.

More generally, since we are working with expected quantities, (conditionally on current information sets), there is no theoretical reason for imposing any ordering between $EL(t, T)$ and $EL(t', T')$, $t \neq t'$ and $T \neq T'$, and similarly between $A(t, T)$ and $A(t', T')$. In practice, such random quantities will fluctuate around the current corresponding expected curves $EL(t, \cdot)$ and $A(t, \cdot)$. Even if the latter curves are increasing w.r.t. T for instance, it is not guaranteed that such a property will remain for future dates $t' > t$ in general and any random path. Nonetheless, due to the definition of these processes, it has not to be considered as a weakness of the model.

3 Pricing formulas of a synthetic ABS

Now, we are able to evaluate the risky principals and the default legs of our simple ABS, with the previous toy model. Recall that the value of a synthetic ABCDS, a tranche $[K_{j-1}, K_j]$, is given [11] by

$$PV_t = s_{t,j}\{RP_{t,K_j} - RP_{t,K_{j-1}}\} - DL_{t,K_j} + DL_{t,K_{j-1}}.$$

As explained above, it is sufficient to evaluate the previous expectations E_1 and E_2. Instead of brute-force calculations, it is simpler to invoke change of numeraire techniques: for every time s, we will be working under the underlying s-Forward neutral probability Q_s. Let us introduce the quantity $\nu_{t,\bar{s},Q_s} = \rho\sigma(t, \bar{s})\bar{\sigma}(t, s)$, that will appear several times in the sequel. We prove in the appendix A.1:

[11] see Pénasse (2008), for instance

Theorem 3.1 *Under the assumptions (A), (E) and (IR), for every couple $(s,\bar{s})$, $t \leq \bar{s} \leq s$, we have*

$$E_1(s) = B(t,s)\exp\left(\int_t^s \rho\bar{\sigma}(u,s)\sigma(u,s)\,du\right) Put\left(EL(t,s), K_s^*, \sigma(.,s), s-t\right),$$

where $Put(Fwd, Strike, Vol, Maturity)$ denotes the usual Black-Scholes formula of an European Put (zero interest rates), and $K_s^* = K\exp\left(-\int_t^s \nu_{u,s,Q_s}\,du\right)$. Moreover,

$$E_2(s,\bar{s}) = B(t,s)EL(t,\bar{s})\exp\left(\int_t^{\bar{s}} \rho\bar{\sigma}(u,s)\sigma(u,\bar{s})\,du\right)$$
$$\cdot \quad \Phi\left(\left[\ln(\frac{K}{EL(t,s)}) - \int_t^{\bar{s}} \sigma(u,s)\sigma(u,\bar{s})\,du + \frac{1}{2}\int_t^s \sigma^2(u,s)\,du\right] / (\int_t^s \sigma^2(u,s)\,du)^{1/2}\right).$$

Then, the risky principal of the tranche $[0,K]$ with $K < 1$, is $RP_{t,K} = \int_t^{T^*} E_1(s)\,ds$, and its default leg is $DL_{t,K} \simeq \sum_{i=1}^p \left[E_2(T_i,T_i) - E_2(T_i,T_{i-1})\right]$.

Note that the integrals from t and $\bar{s}$ above can be considered equivalently from t and s. Indeed, by assumption, $\bar{s} \leq s$ and, obviously, $\sigma(u,\bar{s}) = 0$ if $u > \bar{s}$.

Now, we have to price the most senior tranche. It is sufficient to evaluate the quantities

$$E_1^*(s) = B(t,s)E_{t,Q_s}\left[(1 - EL(s,s) - A(s,s))^+\right],$$

and

$$E_2^*(s,\bar{s}) = B(t,s)E_{t,Q_s}\left[\mathbf{1}\{EL(s,s) + A(s,s) \leq 1\}EL(\bar{s},\bar{s})\right].$$

Formally, these expressions will be calculated in appendix B, in a semi-analytical way. By neglecting the likelihood of the event $\{EL(s,s) + A(s,s) > 1\}$ [12], we can find simple closed-form formulas.

Theorem 3.2 *Under the assumptions (A), (E), (IR) and (AM), for every couple $(s,\bar{s})$, $t \leq \bar{s} \leq s$, we have*

$$E_1^*(s) \simeq B(t,s)\left[1 - A(t,s)\exp\left(\tilde{\rho}\int_t^s \tau(u,s)\bar{\sigma}(u,s)\,du\right) - EL(t,s)\exp\left(\rho\int_t^s \sigma(u,s)\bar{\sigma}(u,s)\,du\right)\right], \tag{8}$$

$$E_2^*(s,\bar{s}) = B(t,s)EL(t,\bar{s})\exp\left(\rho\int_t^{\bar{s}} \sigma(u,\bar{s})\bar{\sigma}(u,s)\,du\right). \tag{9}$$

[12]This approximation may be poor. For instance, consider the following realistic choice of parameters: $T =$ 5 years, $\sigma(t,T) = \nu(t,T) = 50\%$, $EL(0,T) = 30\%$, $A(0,T) = 50\%$, $\tilde{\rho} = 50\%$. Then, the probability that $EL(T,T) + A(T,T)$ is larger than one lies around 23%. In this case, we expect biases in the evaluation of $E_1^*(T)$ and $E_1^*(T,T)$, and we advise to use the formulas stated in the appendix B instead.

Then, the risky principal of the tranche $0 - 100\%$ *(the whole portfolio) is* $RP_{t,1} = \int_t^{T^*} E_1^*(s)\, ds$, *and its default leg is* $DL_{t,1} \simeq \sum_{i=1}^{p} \left[E_2^*(T_i, T_i) - E_2^*(T_i, T_{i-1}) \right]$.

See the proof in the appendix A.2. Thus, playing with different risky principals and default legs, we can apply theorem 3.2 to price a super-senior tranche, under the assumption that the amortization process will never reduce entirely this tranche before maturity. The price of the tranche $[K_{p-1}, 1]$ is then

$$PV_t = s_0\{RP_{t,1} - RP_{t,K_{p-1}}\} - DL_{t,1} + DL_{t,K_{p-1}} \tag{10}$$

where s_0 denotes the spread of this tranche at inception. See also more general formulas in the appendix B.

Now, to obtain t-prices of tranches and apply the previous results, it is sufficient to evaluate the t-spot Expected Losses $EL(t,T)$ and, for the most senior tranche, the Expected amortized amount $A(t,T)$ (theorems 3.1 and 3.2). Since they are not observed directly in the market, we have to make additional assumptions concerning the shape of the t-current profile $T \mapsto E(t,T)$. For example, we could state that, with a constant rate $\theta_t > 0$, we have

$$EL(t, dT) = \theta_t . E_t[O(T)]dT.$$

This constant θ_t has the status of a constant default rate, even if it is related to some expectations of losses. Note that the previous relation induces a feedback of losses towards the amortization process $A(.,.)$ through $O(.,.)$.

Formally, we could deal with the amortization process as with the Expected Loss process itself. For the moment, we have just to evaluate $A(t,T)$ knowing the information at time t. We will assume that

$$A(t, dT) = [\xi_{t,T} + b_t].E_t[O(T)]dT,$$

where $\xi_{t,T}$ is the theoretical amortization rate at time t for the T maturity, and b_t is a constant risk premium. The former quantity is the time T rate of the portfolio principal, assuming there will be no prepayments between T and $T+dT$. It may be deduced from the planned amortization schedule of the surviving assets in the pool at time T, but as seen at t. The latter quantity is the global risk premium associated with the amortization process. If $\xi_{t,.}$ were a constant, then $(\xi_{t,.} + b_t)$ would be the t-amortization rate, and b_t could be seen as the so-called "Constant Prepayment

Rate" (evaluated at time t). The constancy of b_t and θ_t has just been made for convenience. It is straightforward to extend our results to deal with time-varying but deterministic term structures of prepayment and default rates.

It is well-known that prepayment incentives increase when interest rates fall. Such a feature can be taken into account in two ways. First, through the correlation level $\tilde{\rho}$ between the moves of $B(\cdot, T)$ and $A(\cdot, T)$ (see the assumptions (IR) and (AM)), our specification induces quicker amortization when rates go up ($\tilde{\rho} \geq 0$). It is true for any random trajectory. This is even true when we update our (spot) amortization profiles after a sudden move of the current discount factors. To be specific, if $\Delta \bar{W}_t = W_{t+\Delta t} - W_t > 0$ and $\Delta t << 1$, then it is reasonable to state that the expected amortization profile is given now by $A(t + \Delta t, T) = A(t, T)(1 + \tau(t, T))\tilde{\rho}\Delta \bar{W}_t$ for every T. Therefore, it is possible to estimate the sensitivities of our prices (or our spreads) w.r.t. changes in the discount factors. This can be done analytically when the process A appears in the pricing formulas [13]. Second, as we said, the amortization speed b_t is similar to a constant prepayment rate. When rates go up, this trader input can be updated. Even if such a "manual update" is not model-driven, this is in line with the current practice in the market.

Note that our model takes into account the full description of theoretical cash flows through the knowledge of $A(t, v)$, $v \geq t$. It means we do not make any simplifying assumptions concerning the shape of the amortization, contrary to a lot of papers in the literature. It is classical to price "simple exposure" fixed-rate path-through securities, as in Kau et al. (1995), Kariya and Kobayashi (2000) or Nakamura (2001). In the more complex case of CMOs, McConnell and Singh (1994) tackle exponentially decreasing exposures. And to price CDOs of ABS, the most complex products in the ABS-type family, Garcia and Goossens (2008a, 2008b) assume some ad-hoc monotonic shapes.

Since $E_t[O(T)] = 1 - A(t, T) - EL(t, T)$, we deduce

$$E_t[O(dT)] = -A(t, dT) - EL(t, dT) = -(\theta_t + b_t + \xi_{t,T})E_t[O(T)]dT,$$

and therefore

$$E_t[O(T)] = O(t) \exp\left(-(\theta_t + b_t).(T - t) - \int_t^T \xi_{t,u}\, du\right).$$

[13] Under the assumption (A), it is the case for the most senior tranche only. See the formulas in theorem 8, or in the appendix.

Finally, we get

$$EL(t,T) = EL(t,t) + O(t)\theta_t \int_t^T \exp\left(-(\theta_t + b_t).(u-t) - \int_t^u \xi_{t,v}\, dv \right) du. \tag{11}$$

Thus, at the current time t, the Expected Loss depends only on the "no-default, no-prepayment" amortization profile and on some constants θ_t and b_t. We have replaced a whole unknown spot curve $EL(t,\cdot)$ by a parametrization of this curve, given by (11). Obviously, other choices of spot EL curves are possible, and could reflect trader's views on the future trends in the market largely. Note that equation (11) is not contradictory with the assumption (E).

Similarly, we deduce the spot amortization profile, i.e. the curve $A(t,\cdot)$:

$$A(t,T) = A(t,t) + O(t) \int_t^T (\xi_{t,u} + b_t) \exp\left(-(\theta_t + b_t).(u-t) - \int_t^u \xi_{t,v}\, dv \right) du.$$

Therefore, we have obtained all elements to apply the previous theorems, and then to price standard synthetic tranches.

4 Pricing of a cash MBS/ABS tranche

To illustrate the potentialities of the method, we apply our model to price coupon-bearing structures. Consider, as previously, a tranched ABS structure. But now, we have to manage the additional cash flows that are related to coupon payment rules. Broadly speaking, such a cash structure differs from the synthetic one in the previous section like a cash CDO differs from a synthetic CDO. We simplify the picture by assuming that, at some pre specified payment dates, a fixed or floating coupon rate will be applied to the current outstanding principal of any base tranche. Then, under this assumption, we are able to calculate tranche prices, by the expected cash flow method. Formally, cash ABS tranches are similar to amortizing bonds with random schedules [14].

Consider a given base tranche $[0,K]$. As previously, principal payments are sequential and we assume now that coupon payments are "pass-through": they are split across all tranches and proportionally to the tranche sizes. Thus, no cash is diverted from some tranche to repay quicker another one. Since the cash flows are due to coupons payments or principal paydowns, the "ABS

[14] As a consequence, we do not cover some path-dependent features, for instance credit triggers that would change the order of priority of coupon payments among tranche, or potentially complex excess spread mechanisms.

bond" price at time t is given by

$$P_t = E_t \left[\sum_{i \le p, T_i \ge t} \exp(-\int_t^{T_i} r_u \, du) \, \{C_{T_i} \Delta_i O_K(T_{i-1}) \right.$$
$$+ \quad \left. \mathbf{1}(K \ge L(T_{i-1}) + A_K(T_{i-1})).[A_K(T_i) - A_K(T_{i-1})]\} + \exp(-\int_t^{T_p} r_u \, du) O_K(T_p) \right],$$

where C_{T_i} is the fixed or floating coupon associated with that tranche, whose value is known at the fixing date just before i.e. at T_{i-1}, and Δ_i is the coverage between T_{i-1} and T_i.

First, let us deal with the case of a fixed coupon in a given tranche.

Assumption (FC): For every date T_i, the related coupon rate is the same constant C_0. In other words, $C_{T_i} = C_0$ for all indices i.

We consider the case of fixed coupon bonds (or tranches) that are not impacted by amortization during the whole life of the deal, except at maturity (assumption (A)). To evaluate the bond price, it is sufficient to calculate the expectations

$$F_1(s, \bar{s}) = E_t \left[\exp\left(- \int_t^s r_u \, du \right) O_K(\bar{s}) \right],$$

for every times $(s, \bar{s})$, $t \le \bar{s} \le s \le T^*$. Note that $F_1(s, s)$ is exactly the so-called $\mathcal{E}_1(s)$ (see equation (4)). With similar arguments as previously, we obtain easily the following result.

Theorem 4.1 *Under the assumptions (A), (E), (IR) and (FC), the cash bond price is given by*

$$P_t = \sum_{i \le p, T_i \ge t} C_0 \Delta_i F_1(T_i, T_{i-1}) + E_1(T_p),$$

where E_1 had been defined in theorem 3.1 and, for every couple $(s, \bar{s})$, $t \le \bar{s} \le s$,

$$F_1(s, \bar{s}) = B(t, s) \exp\left(\int_t^{\bar{s}} \nu_{u, \bar{s}, Q_s} \, du \right) .Put\left(EL(t, \bar{s}), K \exp\left(- \int_t^{\bar{s}} \nu_{u, \bar{s}, Q_s} \, du \right), \sigma(., \bar{s}), \bar{s} - t \right).$$

To deal with the most senior tranche, we have to take into account the amortization process too. To prove simple formulas, we assume in the next theorem that the likelihood of the events $\{EL(s, s) + A(s, s) > 1\}$ is negligible. Thus, by mimicking theorem 3.2, we get easily:

Theorem 4.2 *Under the assumptions (A), (E), (IR), (AM) and (FC), the cash bond price of the* *whole portfolio (i.e. the tranche* $[0, 100\%]$*) is given by*

$$P_t = \sum_{i, T_i \geq t, i \leq p} \{C_0 \Delta_i E_3^*(T_i, T_{i-1}) + A_1^*(T_i, T_i) - A_1^*(T_i, T_{i-1})\} + E_1(T_p),$$

where, for every couple $(s, \bar{s})$*,* $t \leq \bar{s} \leq s$*,*

$$E_3^*(s, \bar{s}) = B(t, s) \left[1 - A(t, \bar{s}) \exp\left(\tilde{\rho} \int_t^{\bar{s}} \tau(u, \bar{s}) \bar{\sigma}(u, s)\, du\right) - EL(t, \bar{s}) \exp\left(\rho \int_t^{\bar{s}} \sigma(u, \bar{s}) \bar{\sigma}(u, s)\, du\right)\right], \tag{12}$$

and

$$A_1^*(s, \bar{s}) = B(t, s) A(t, \bar{s}) \exp\left(\tilde{\rho} \int_t^{\bar{s}} \tau(u, \bar{s}) \bar{\sigma}(u, s)\, du\right). \tag{13}$$

The pricing formulas of theorems 4.1 and 4.2 but without assumption (A) are available in the appendix B (corollary 2). Let us now deal with the trickier case of floating rate bonds. To fix the ideas, let us assume that the coupon rate is a standard Libor rate. To add a constant margin to this rate would be straightforward by applying theorems 4.1 and 4.2.

Assumption (FlC): At every date T_i, the paid coupon rate is corresponding to the Libor rate at T_{i-1}, for the same periodicity . We denote $C_{T_i} = L(T_{i-1}, T_i)$.

Now, it is sufficient to evaluate expressions like

$$F_2(s, \bar{s}) = E_t \left[\exp\left(-\int_t^s r_u\, du\right) L(\bar{s}, s) O_K(\bar{s})\right],$$

where, as usual, $\bar{s} \leq s$. In the appendix A.3, we prove that:

Theorem 4.3 *Under the assumptions (A), (E), (IR) and (FlC), the cash bond price of the* $[0, K]$ *tranche is given by*

$$P_t = \sum_{i \leq p, T_i \geq t} \Delta_i F_2(T_i, T_{i-1}) + E_1(T_p),$$

where, for every couple $(s, \bar{s})$*,* $t \leq \bar{s} \leq s$*,* $F_2(s, \bar{s}) = F_1(\bar{s}, \bar{s}) - F_1(s, \bar{s})$*.*

Note that E_1 and F_1 have been defined in theorems 3.1 and 4.1 respectively. Moreover, by the same reasoning, we deal easily with the last tranche.

Theorem 4.4 *Under the assumptions (A), (E), (IR), (AM) and (FlC), the cash bond price of* *the whole portfolio (i.e. the tranche* $[0, 100\%]$*) is given by*

$$P_t = \sum_{i, T_i \geq t, i \leq p} \{\Delta_i F_2^*(T_i, T_{i-1}) + A_1^*(T_i, T_i) - A_1^*(T_i, T_{i-1})\} + E_1(T_p),$$

where, for every couple $(s, \bar{s})$*,* $t \leq \bar{s} \leq s$*, we have* $F_2^*(s, \bar{s}) = E_3^*(\bar{s}, \bar{s}) - E_3^*(s, \bar{s})$*.*

Remind that A_1^* and E_3^* have been defined in theorem 4.2. Therefore, we have obtained closed-form formulas to price simple cash ABS tranches. Note that the formulas above have been stated under the so-called assumption (A). To remove the latter assumption and to obtain semi-analytical formulas, see the appendix B.

This treatment of coupons is surely crude. Indeed, these amounts are coming directly from regularly paid coupons of the surviving loans in the pool. Since the structure of the pool is changing progressively due to defaults and prepayments, the effective coupon rates are random and dependent on the amortization process itself. We argue this effect can be captured partly by assuming a term structure of coupons (fixed or floating). For instance C_{T_i} may be a Libor rate between T_{i-1} and T_i, plus a deterministic spread (function of T_i). Therefore, if we tackle the two cases of fixed and floating coupon rates, we are able to implement the previous idea of spread term structures. In theory, it would be possible to model random coupon rates, dependent on all the other random processes, but this point is left for further research.

5 Illustration: the pricing of a real RMBS

For the sake of illustration, we consider now a real recent trade in light of our valuation model, under the assumption (A). It is a synthetic European RMBS with a total issued principal around 2 billion euros. The principal paydown is sequential without any interest payments. The maturity of the structure will be assumed 5 years, even if it is a call date. In other words, we do not try to evaluate the price of the embedded call option. The underlying loan portfolio has been tranched into six slices, with detachment points 1%, 3%, 5%, 7%, 10% and 100%. A very wide super senior tranche is typical of such structures. As usual in our framework, principals will be reimbursed from the top and default losses recorded from the bottom.

For the moment, assume we want to price and risk manage these tranches at inception. Thus, no loss has been recorded yet, and no amortization has occurred in the underlying pool. In practice, our model parameters should be calibrated to some observed tranche prices of current or similar structures. Since the liquidity of such deals is very limited, it is usual to invoke some trader inputs, that would reflect his/her expectations in light of historical data or recent trends. In our framework, we would like to guess which parameters are the most crucial ones in terms of calibration. For this purpose, we calculate the present value impacts of some parameter changes. To simplify, we have assumed that all volatilities we consider are constant over time. Moreover, to simplify, we have assumed a perfect correlation between the amortization and the interest rates processes, i.e. $\tilde{\rho} = 1$. We have led a sensitivity analysis of tranche prices (in terms of par spreads) with respect to some input parameters: volatility of the Expected Loss, default rate θ, prepayment rate b, correlation ρ and current loss amount at inception. Figures 1, 2, 3, 4, 5 and 6 summarize the results. The reference set of parameters is the following one: $\theta = 0.4\%$, $b = 0.5\%$, $\rho = 30\%$ and constant volatilities $\sigma_0 = 85\%$, $\bar{\sigma}_0 = 1\%$, $\tau = 25\%$.

The most junior part of the capital structure benefits from high Expected Loss volatilities (figure 1). Indeed, in a fully deterministic framework, such first thin tranche would entirely disappear before maturity, with our default rate assumption. Therefore, more uncertainty concerning realized losses is good news for the owners of such tranches, all other things being equal. Obviously, it is the opposite for the owners of the most senior tranches. Mezzanine tranche profiles are humped, meaning they behave rather like senior tranches under low expected loss volatilities, and rather like equity ones under higher volatilities.

As expected, par spreads are monotonically increasing functions of the default rates (figure 2). This relation is almost linear, at least for a wide range of realistic default rates, and even for the most junior tranches. The latter effect is due to the trade-off between risky principal and expected loss: with higher default rates, the tranche expected loss is capped when its risky principal will decrease, so increasing its par spread. For all tranches, par spreads decrease when prepayment rates increase: see figure 3. Indeed, the spot expected loss curves depend on these rates. The quicker the payment process is, the smaller the expected losses are in the whole portfolio. This phenomenon can be observed with all tranches. In the case of the most senior tranche, its risky principal is reduced significantly by increasing prepayment rates, canceling almost of the latter expected loss effect. That is why, apparently strangely, the super senior tranche is the less sensitive tranche w.r.t varying prepayment rates.

Moreover, except for the most senior tranche, the effect of the correlation between amortization/interest rates and default losses is very weak (figure 4). Indeed, under our assumption (A), these tranches are not hit by the amortization process. Thus, the latter correlation has an influence through the risk free discount factors only. Clearly, this induces a lot smaller effect on prices than a reduction of tranche principals. At the opposite, the most senior tranche par spread is an increasing function of the correlation significantly: high correlations increase the likelihood of higher tranche reduction through the joint effect of default losses and quicker amortization. An investor requires a higher premium to be covered against the latter risk.

Now, assume that some losses have already been recorded: see figure 5. Here, the most junior tranches have already been fully or partially reduced, but, since losses can be recovered [15], the processus can be reversed. Therefore, even when realized losses are higher than 20%, the model spread of the tranche $[0, 1\%]$ is not zero. Actually, it is around 2% of the remaining principal, that will be zero for most trajectories! This value can be interpreted as the probability of recovering some part of the most junior tranche before maturity. Note the humped shape behavior of the par spreads: for a given tranche, the associated par spreads reach their maxima when the realized losses hit that tranche. After that, as explained above, par spreads decrease but do not reach zero. Note that the super senior tranche does not exhibit such a profile, because only unrealistic large realized losses could possibly illustrate this phenomenon for this tranche.

Finally, we have analyzed the effect of the (not risky) interest rates on the tranche prices. The reference set of parameters is still as above. For convenience, we assume that interest rate curves are flat. Thus, the (current) short rate r_t is the single driver of these curves. In other words, for any dates t, T, $t \geq T$, $B(t,T) = \exp(-r_t(T-t))$. The reference short rate is 3%. In the figure 6, we observe that all tranche spreads are decreasing functions the current short rate, except the most risky one. This effect is almost linear and it remains reasonable, except for very high short rates (several times higher than the spot short rate). Even unlikely, such large changes are not impossible due to the long maturities of the structured products we consider (most often several decades). Note that this analysis has been done all other things being equal, i.e. without updating the amortizing profiles and/or the prepayment parameter b_t (see the discussion above in section 3).

Clearly, a true calibration exercise would be necessary to assess the performances of the model in practice. This is kept for future research. Nonetheless, some guidelines can be proposed. First, the key parameter seems to be θ, the "constant default rate", when default risk is the main driver of such structures (our assumption). And a wide range of prices can be obtained by playing

[15] this point is a particularity of the ABS sector w.r.t. corporate-based structures. And this point is integrated in the model through the diffusion specification (EL).

with this parameter. Thus, θ could play the same role as the implied correlation parameter in a standard corporate CDO: one θ per tranche is fitted, and this parameter should share the same status as a price. Second, when prepayment is important and default risk is low, the "constant prepayment rate" b becomes as important as θ, and both should be calibrated implicitly. Third, most of the other parameters (volatilities, correlations) could be fitted separately over a set of similar bonds, possibly historically by using proxies, or from some trader's views.

6 Conclusion

We have provided a theoretical framework to price and analyze a lot of ABS-type products (ABCDS, CMOs, CDOs of ABS, subprime bonds etc). By working in a parsimonious "top-down" model and inspired by this stream of the Credit Derivatives literature, we were able to provide closed form formulas, or at least semi-analytical formulas, in the case of (simplified) sequential or pro-rata pay structures. For most of structures that convey amortization, prepayment and default risks simultaneously, the related markets are poorly liquid. Thus, it would not be reasonable to propose top-down models with plenty of parameters. They would induce a high risk of "over-fitting" and a poor risk management. Our specification appears to be a good compromise. We have taken into account the correlation between the dynamics of the interest rates, prepayment and basket loss processes in a simple but realistic way. We are convinced that an assumption of independence between the credit events and interest rate moves, so usual in the credit area, is too strong here and we have proposed a tractable and parsimonious alternative. Closed-form formulas are highly valuable. They provide benchmarks without having to build huge IT infrastructure (to retrieve loan informations, simulate numerous random factors and revaluate portfolios thousands of times, or to solve complex PDEs). Moreover, they enable an investor to complete portfolio optimization analyses that would be infeasible otherwise. Indeed, "the most valuable applications of the closed-form formula lie in the area of portfolio management. For example, the question of how a portfolio investor such as a depository or a GSE should fund mortgage portfolios is difficult to answer using Monte Carlo simulation." (Collin-Dufresne and Harding 1999). Finally, we have tried to keep a balance between the likelihood of the hypothesis, the number of the underlying random factors and the calibration issues. We think our approach can be considered as a relative value tool for arbitrage purpose. It provides an original point of view to compare several tranches inside the same structure or between similar structures, beside other professional tools.

The current framework can be extended significantly towards several directions: alternative specifications in terms of the underlying processes or in terms of spot EL and A curves, addition of more random factors (Home Price Index), comparison between several default/prepayment intensities assumptions, inclusion of some triggers or excess spread mechanisms etc. Particularly, an avenue for further research would be to replace the randomness of the amortized amount A by the randomness of the prepayment rates b and of the "natural" amortization rate $\xi_{t,T}$. In the latter rate, it would be possible to integrate loan-per-loan information, and possibly some forecasts deduced from micro-econometric models.

Our assumptions concerning the waterfalls we consider could appear rather restrictive. Actually, we are convinced our framework and our formulas could be revisited to price deals with other principal-pay types and/or interest-pay types: Interest Only or Principal Only [16] bonds, Planned Amortization Classes (PAC), Targeted Amortization Classes (TAC), triggers that determine the switch between the sequential and the pro-rata amortization, Z-bonds, etc. Despite the fact that "the complexity of securitization products has been a major holdback of similar analytics advancement as in portfolio or structured credit derivatives." (Lou, 2007), we think a new stream of research is now open. Hopefully, a lot of (more and more) complex formulas will be proved, to integrate the multiple features of ABS structures, but by keeping or extending our framework. In other words, we have proved pricing formulas of some vanilla products, leaving open the way towards exotic product valuations.

[16] a PO bond can be seen as a vanilla option written on the process A

A Proofs under the assumption (A)

A.1 Proof of theorem 3.1.

Under the s-forward neutral probability Q_s, the discount factor $(B(t,s))_{t,t\leq s}$ process is the numeraire and we have, for all the tranches except the most senior one,

$$E_1(s) = B(t,s)E_{t,Q_s}\left[(K - EL(s,s))^+\right], \tag{14}$$

and

$$E_2(s,\bar{s}) = B(t,s)E_{t,Q_s}\left[\mathbf{1}\{EL(s,s) \leq K\}EL(\bar{s},\bar{s})\right].$$

A technical issue is coming from the fact we change one probability into another one every time. Formally, the processes $(EL(t,T))_t$ (for different T) have not the same laws under all these probabilities. Actually, only the drifts are changing. It is possible to state explicitly all these drifts. Remind that the drift of $(EL(\cdot,T))$ under the risk neutral measure Q is zero. By classical arguments (see Brigo and Mercurio 2001, e.g.), we obtain: Under the Q_s Forward measure, the drift of the process $EL(t,T)$ is given by $-\rho EL(t,T)\sigma(t,T)\left(\sigma_Q(t) - \bar{\sigma}(t,s)\right)$, where σ_Q is the volatility of the usual numeraire. Since this usual numeraire is the money market account, it has no volatility. Thus, $\sigma_Q(t) = 0$ and, for every t, T, the latter drift is

$$EL(t,T)\nu_{t,T,Q_s} = \rho EL(t,T)\sigma(t,T)\bar{\sigma}(t,s). \tag{15}$$

Therefore, under any Forward neutral probability Q_s, the Expected Loss processes are still lognormal: $EL(dt,T) = EL(t,T).\left(\nu_{t,T,Q_s}dt + \sigma(t,T)dW_t\right)$.

To evaluate the expectation (14), we can invoke the usual Black-Scholes formula:

$$\begin{aligned}
E_1(s) &= B(t,s)E_{t,Q_s}\left[(K - EL(s,s))^+\right] \\
&= B(t,s)E_{t,Q_s}\left[\left(K - EL(t,s)\exp\left(\int_t^s \nu_{u,s,Q_s}\,du - \int_t^s \sigma^2(u,s)\,du/2 + \int_t^s \sigma(u,s)dW_u\right)\right)^+\right] \\
&= B(t,s)\exp\left(\int_t^s \nu_{u,s,Q_s}\,du\right) \\
&\cdot E_{t,Q_s}\left[\left(K_s^* - EL(t,s)\exp\left(-\int_t^s \sigma^2(u,s)\,ds/2 + \int_t^s \sigma(u,s)dW_u\right)\right)^+\right].
\end{aligned}$$

So, we prove the formula for $E_1(s)$. To deal with $E_2(s,\bar{s})$, choose now the numeraire $EL(\cdot,\bar{s})$.

Under the probability $Q^E_{\bar{s}}$ that is induced by this new numeraire,

$$E_2(s,\bar{s}) = B(t,s)E_{t,Q_s}[EL(\bar{s},\bar{s})].E_{t,Q^E_{\bar{s}}}\left[\mathbf{1}\{EL(s,s) \leq K\}\right],$$

or equivalently

$$E_2(s,\bar{s}) = B(t,s)EL(t,\bar{s})\exp\left(\int_t^{\bar{s}} \nu_{u,\bar{s},Q_s}\, du\right).E_{t,Q^E_{\bar{s}}}\left[\mathbf{1}\{EL(s,s) \leq K\}\right].$$

But, under the probability $Q^E_{\bar{s}}$, the Expected Loss processes $EL(\cdot,s)$ follows the diffusion equation

$$EL(dt,s) = EL(t,s).\left(\sigma(t,s)\sigma(t,\bar{s})dt + \sigma(t,s)dW_t\right).$$

Then, we obtain an explicit expression for $E_2(s,\bar{s})$. □

A.2 Proof of theorem 3.2.

Under (A), we have

$$E_1^*(s) = B(t,s)E_{t,Q_s}\left[1 - EL(s,s) - A(s,s)\right], \text{ and } E_2^*(s,\bar{s}) = B(t,s)E_{t,Q_s}\left[EL(\bar{s},\bar{s})\right].$$

By our change of measure, the processes $(EL(t,T))$ and $(A(t,T))$ are no more martingales under the new measures. Concerning the Expected Loss process, we had already found the Q_s-drift change (see equation (15)). This implies

$$E_{t,Q_s}[EL(\bar{s},\bar{s})] = EL(t,\bar{s})\exp\left(\int_t^{\bar{s}} \nu_{u,\bar{s},Q_s}\, du\right) = EL(t,\bar{s})\exp\left(\rho\int_t^{\bar{s}} \sigma(u,\bar{s})\bar{\sigma}(u,s)\, du\right).$$

Similarly, we can deal with the amortization process $A(.,s)$ as with the Expected Loss process. Therefore, we have

$$E_{t,Q_s}\left[A(s,s)\right] = A(t,s)\exp\left(\tilde{\rho}\int_t^{s} \tau(u,s)\bar{\sigma}(u,s)\, du\right),$$

so the result. □

A.3 Proof of theorem 4.3.

Under the Q_s Forward measure, the drift of the process $EL(t,\bar{s})$ is given by $\nu_{t,\bar{s},Q_s}$, with our previous notations. By some standard conditional expectation arguments, we have

$$\begin{aligned}
F_2(s,\bar{s}) &= E_t\left[\exp\left(-\int_t^{\bar{s}} r_u\,du\right) B(\bar{s},s)L(\bar{s},s)\Delta(\bar{s},s)O_K(\bar{s})\right] \\
&= E_t\left[\exp\left(-\int_t^{\bar{s}} r_u\,du\right) B(\bar{s},s)\left\{\frac{1}{B(\bar{s},s)}-1\right\} O_K(\bar{s})\right] \\
&= E_t\left[\exp\left(-\int_t^{\bar{s}} r_u\,du\right) \{1-B(\bar{s},s)\}\, O_K(\bar{s})\right] \\
&= F_{1,1}(\bar{s},\bar{s}) - E_t\left[\exp\left(-\int_t^{\bar{s}} r_u\,du\right) E_{\bar{s}}\left[\exp(-\int_{\bar{s}}^{s} r_u\,du)\right] O_K(\bar{s})\right] \\
&= F_{1,1}(\bar{s},\bar{s}) - F_{1,1}(s,\bar{s}).\ \Box
\end{aligned}$$

B Semi-analytical formulas without the assumption (A)

In this appendix, we extend our formulas to remove the convenient previous assumption (A). Now, the amortization process can reduce any tranche, possibly the most senior one. Closed-form formulas are no longer available, but we can rely on semi-analytical formulas instead. Broadly speaking, the method is simple: conditionally on the value of the amortization process at some time horizon, the "base case" formulas apply, by shifting the relevant strikes. Then, an integration w.r.t. the law of the expected amortized amounts provide the results.

First, let us consider the previous synthetic structure and the evaluation of risky principals and default legs of all tranches (including the most senior one). Recall that the risky principals of the equity tranche $[0,K]$ are defined by $RP_{t,K} = E_t\left[\int_t^{T^*} \exp\left(-\int_t^s r_u\,du\right) O_K(s)\,ds\right]$, and its default legs are

$$DL_{t,K} = E_t\left[\int_t^{T^*} \exp\left(-\int_t^s r_u\,du\right) \mathbf{1}(L(s)+A_K(s)\le K)L(ds)\right].$$

As previously, we cover the case of sequential-pay bonds, for which $A_K(s) = [A(s)-(1-K)]^+$. But it should be noted that we deal with the case of pro-rata bonds too, for which $A_K(s)$ is a fixed proportion of $A(s)$, as in some stripped pass-through securities. To include these two reference situations explicitly [17], we assume the repaid principal of the tranche $[0,K]$ at time s is a deterministic function of the portfolio repaid principal $A(s)$ only: for every K and s, there

[17] and possibly others of the same type

exists a function ψ_K such that

$$A_K(s) := \psi_K(A(s)) = \psi_K(A(s,s)).$$

To price the tranche $[0,K]$, it is sufficient to evaluate all quantities like

$$\mathcal{E}_1(s) = E_t\left[\exp\left(-\int_t^s r_u\,du\right)(K - EL(s,s) - \psi_K(A(s,s)))^+\right],$$

and

$$\mathcal{E}_2(s,\bar{s}) = E_t\left[\exp\left(-\int_t^s r_u\,du\right)\mathbf{1}\{EL(s,s) + \psi_K(A(s,s)) \le K\}EL(\bar{s},\bar{s})\right],$$

for every couple $(s,\bar{s})$, $t \le \bar{s} \le s \le T^*$. Actually, we will calculate first the quantity

$$\mathcal{F}_1(s,\bar{s}) = E_t\left[\exp\left(-\int_t^s r_u\,du\right)(K - EL(\bar{s},\bar{s}) - \psi_K(A(\bar{s},\bar{s})))^+\right],$$

when $\bar{s} \le s$. Indeed, note that $\mathcal{E}_1(s) = \mathcal{F}_1(s,s)$. Moreover, $\mathcal{F}_1$ is the same as the so-called term F_1 that had been calculated in section 4 under the assumption (A). We will need $\mathcal{F}_1$ for pricing coupon-bearing securities hereafter.

To fix the ideas, at time t, the event $A(\bar{s},\bar{s}) = a$ will be identical to $\int_t^{\bar{s}} \tau(u,\bar{s})\,d\tilde{W}_u = w(a)$, for some value $w(a)$ that will depend on the spot curve $A(t,\cdot)$. Clearly,

$$\begin{aligned}
\mathcal{F}_1(s,\bar{s}) &= B(t,s)E_{t,Q_s}\left[(K - EL(\bar{s},\bar{s}) - \psi_K(A(\bar{s},\bar{s})))^+\right] \\
&= B(t,s)E_{t,Q_s}\left[E_{t,Q_s}[(K - EL(\bar{s},\bar{s}) - \psi_K(a))^+\,|A(\bar{s},\bar{s}) = a]\right],
\end{aligned}$$

and the conditional expectation can be evaluated easily. Here, the conditioning event is

$$a = A(t,\bar{s})\exp\left(\tilde{\rho}\int_t^{\bar{s}} \bar{\sigma}(u,s)\tau(u,\bar{s})\,du - \frac{1}{2}\int_t^{\bar{s}} \tau^2(u,\bar{s})\,du + \int_t^{\bar{s}} \tau(u,\bar{s})\,d\tilde{W}_u\right),$$

or equivalently

$$\int_t^{\bar{s}} \tau(u,\bar{s})\,d\tilde{W}_u = w(a).$$

But we can break down

$$\int_t^{\bar{s}} \sigma(u,\bar{s})\,dW_u = \xi_{\bar{s}}\int_t^{\bar{s}} \tau(u,\bar{s})\,d\tilde{W}_u + \varepsilon,$$

where

$$\xi_{\bar{s}} = \frac{\rho\tilde{\rho}\int_t^{\bar{s}} \sigma(u,\bar{s})\tau(u,\bar{s})\,du}{\int_t^{\bar{s}} \tau^2(u,\bar{s})\,du}, \tag{16}$$

and $\varepsilon \sim \mathcal{N}(0, \mu_{\bar{s}}^2)$, by setting

$$\mu_{\bar{s}}^2 = \int_t^{\bar{s}} \sigma^2(u,\bar{s})\,du - \xi_{\bar{s}}^2 \int_t^{\bar{s}} \tau^2(u,\bar{s})\,du. \tag{17}$$

Implicitly, note that the variance of ε depends on all the underlying volatility functions and arguments. Then, under Q_s and conditionally on $A(\bar{s},\bar{s}) = a$, the random variable $EL(\bar{s},\bar{s})$ can be written

$$EL(\bar{s},\bar{s}) = EL(t,\bar{s}) \exp\left(\int_t^{\bar{s}} \rho\sigma(u,\bar{s})\bar{\sigma}(u,s)\,du - \frac{1}{2}\int_t^{\bar{s}} \sigma^2(u,\bar{s})\,du + \xi_{\bar{s}} w(a) + \varepsilon \right),$$

and

$$\begin{aligned}
&E_{Q_s}[(K - EL(\bar{s},\bar{s}) - \psi_K(a))^+] \\
&= \; Put\left(EL(t,\bar{s}) \exp\left(\int_t^{\bar{s}} \rho\sigma(u,\bar{s})\bar{\sigma}(u,s)\,du - \frac{1}{2}\int_t^{\bar{s}} \sigma^2(u,\bar{s})\,du + \xi_{\bar{s}} w(a) + \frac{1}{2}\mu_{\bar{s}}^2 \right), \right. \\
&\quad \left. K - \psi_K(a), \mu_{\bar{s}}, \bar{s} - t \right) \cdot \mathbf{1}(K \geq \psi_K(a)).
\end{aligned}$$

It is sufficient to integrate the latter formula w.r.t. the r.v. $\int_t^{\bar{s}} \tau(u,\bar{s})\,d\bar{W}_u$ to prove the result.

Theorem B.1 *Under (E), (IR) and (AM),*

$$\mathcal{F}_1(s,\bar{s}) = B(t,s) \int Put\left(EL_w, K - \psi_K(a_1(w)), \mu_{\bar{s}}, \bar{s} - t\right) \phi\left(\frac{w}{(\int_t^{\bar{s}} \tau^2(u,\bar{s})\,du)^{1/2}} \right) \cdot \frac{\mathbf{1}(K \geq \psi_K(a_1(w)))}{(\int_t^{\bar{s}} \tau^2(u,\bar{s})\,du)^{1/2}} dw,$$

where

$$EL_w := EL(t,\bar{s}) \exp\left(\int_t^{\bar{s}} \rho\sigma(u,\bar{s})\bar{\sigma}(u,s)\,du - \frac{1}{2}\int_t^{\bar{s}} \sigma^2(u,\bar{s})\,du + \xi_{\bar{s}} w + \frac{1}{2}\mu_{\bar{s}}^2 \right).$$

Moreover $\xi_{\bar{s}}$ (resp. $\mu_{\bar{s}}$) is given by (16) (resp. (17)) and

$$a_1(w) := A(t,\bar{s}) \exp\left(\tilde{\rho} \int_t^{\bar{s}} \sigma(u,\bar{s})\tau(u,s)\,du - \frac{1}{2}\int_t^{\bar{s}} \tau^2(u,\bar{s})\,du + w \right).$$

Similarly and by leading the same changes of numeraire as in theorem 3.1, we obtain

$$\mathcal{E}_2(s,\bar{s}) = B(t,s)EL(t,\bar{s})\exp\left(\int_t^{\bar{s}} \nu_{u,\bar{s},Q_s}\,du\right).E_{t,Q^E_{\bar{s}}}\left[\mathbf{1}\{EL(s,s)+\psi_K(A(s,s))\leq K\}\right]$$
$$= B(t,s)EL(t,\bar{s})\exp\left(\int_t^{\bar{s}} \nu_{u,\bar{s},Q_s}\,du\right)$$
$$\cdot \quad E_{t,Q^E_{\bar{s}}}\left[E_{t,Q^E_{\bar{s}}}\left[\mathbf{1}\{EL(s,s)+\psi_K(a_2(w))\leq K\}|\int_t^s \tau(u,s)\,d\tilde{W}_u = w\right]\right]$$

where

$$a_2(w) = A(t,s)\exp\left(\rho\tilde{\rho}\int_t^{\bar{s}} \sigma(u,\bar{s})\tau(u,s)\,du - \frac{1}{2}\int_t^s \tau^2(u,s)\,du + w\right). \tag{18}$$

Note that the latter function is slightly different from the previous one $a_1(w)$. Indeed, under $Q^E_{\bar{s}}$, the instantaneous drift of $A(t,s)$ is now proportional to $\rho\tilde{\rho}\sigma(t,\bar{s})\tau(t,s)$. We deduce

$$E_{t,Q^E_{\bar{s}}}\left[\mathbf{1}\{EL(s,s)+\psi_K(a_2(w))\leq K\}|A(s,s)=a_2(w)\right]$$
$$= \mathbf{1}(K\geq\psi_K(a_2(w))).\Phi\left(\{\ln(K-\psi_K(a_2(w)))\right.$$
$$- \left.\ln EL(t,s) - \int_t^{\bar{s}} \sigma(u,s)\sigma(u,\bar{s})\,du + \frac{1}{2}\int_t^s \sigma^2(u,s)\,du - \xi_s w\}/\mu_s\right),$$

where ξ_s and μ_s^2 have been defined above.

Theorem B.2 *Under (E), (IR) and (AM), we have*

$$\mathcal{E}_2(s,\bar{s}) = B(t,s)EL(t,\bar{s})\exp\left(\int_t^{\bar{s}} \nu_{u,\bar{s},Q_s}\,du\right)$$
$$\cdot \int \Phi\left(\frac{1}{\mu_s}\{\ln(K-\psi_K(a_2(w))) - \ln EL(t,s) - \int_t^s \sigma(u,s)\sigma(u,\bar{s})\,du\right.$$
$$+ \left.\frac{1}{2}\int_t^s \sigma^2(u,s)\,du - \xi_s w\}\right)\cdot\phi\left(\frac{w}{(\int_t^s \tau^2(u,s)\,du)^{1/2}}\right)\frac{\mathbf{1}(K\geq\psi_K(a_2(w)))dw}{(\int_t^s \tau^2(u,s)\,du)^{1/2}},$$

where a_2, $\xi_.$ and $\mu_.$ are defined by the identities (18), (16) and (17) respectively.

Corollary 1 *Let us consider a base tranche $[0,K]$, $K\in[0,1]$, of a synthetic ABS structure. Under the assumptions of theorems B.1 and B.2, its risky principal is $RP_{t,K} = \int_t^{T^*}\mathcal{F}_1(s,s)\,ds$, and its default leg is $DL_{t,K}\simeq\sum_{i=1}^p[\mathcal{E}_2(T_i,T_i)-\mathcal{E}_2(T_i,T_{i-1})]$.*

To extend fully the results of the previous sections, it remains to tackle the case of cash structures. The next to last missing building block (to deal floating rate coupons) is

$$\mathcal{F}_2(s,\bar{s}) = E_t\left[\exp\left(-\int_t^s r_u\,du\right)L(\bar{s},s)O_K(\bar{s})\right],$$

But, invoking the same arguments as in the proof of theorem 4.3, we obtain easily

$$\mathcal{F}_2(s,\bar{s}) = B(t,\bar{s})E_{t,Q_{\bar{s}}}\left[O_K(\bar{s})\right] - B(t,s)E_{t,Q_s}\left[O_K(\bar{s})\right] = \mathcal{F}_1(\bar{s},\bar{s}) - \mathcal{F}_1(s,\bar{s}),$$

and it is a known quantity. Thus, to evaluate principal paydowns in this case, the two last missing building blocks are the evaluation of

$$\mathcal{A}_1(s,\bar{s}) = E_t\left[\exp\left(-\int_t^s r_u\,du\right)\mathbf{1}\{EL(\bar{s},\bar{s}) + \psi_K(A(\bar{s},\bar{s})) \leq K\}A_K(s,s)\right],$$

and

$$\mathcal{A}_2(s,\bar{s}) = E_t\left[\exp\left(-\int_t^s r_u\,du\right)\mathbf{1}\{EL(\bar{s},\bar{s}) + \psi_K(A(\bar{s},\bar{s})) \leq K\}A_K(\bar{s},\bar{s})\right],$$

for every couples $(s,\bar{s})$, $t \leq \bar{s} \leq s \leq T^*$. After some tedious calculations, it can be proved that:

Theorem B.3 *Under (E), (IR) and (AM), we have*

$$\begin{aligned}\mathcal{A}_1(s,\bar{s}) = B(t,s)\int \Phi\Bigg(&\frac{1}{\mu_{\mathcal{A}_1}}\{\ln(K-\psi_K(a_3(w))) - \ln EL(t,\bar{s}) - \int_t^{\bar{s}} \rho\sigma(u,\bar{s})\bar{\sigma}(u,s)\,du \\ +\ & \frac{1}{2}\int_t^{\bar{s}}\sigma^2(u,\bar{s})\,du - \xi_3 w - \xi_4\tilde{w}\}\Bigg)\cdot\phi_{\rho^*}\left(\frac{w}{v_{\bar{s}}},\frac{\tilde{w}}{v_s}\right) \\ \cdot\ & \psi_K(a_4(\tilde{w}))\frac{\mathbf{1}(K \geq \psi_K(a_3(w)))}{v_s v_{\bar{s}}}\,dw\,d\tilde{w},\end{aligned}$$

where ϕ_{ρ^*} denotes the density of a bivariate random vector of standard Gaussian r.v. with correlation parameter ρ^*, and where we have set

$$\xi_3 := \rho\tilde{\rho}\frac{\int_t^{\bar{s}}\sigma(.,\bar{s})\tau(.,\bar{s}).\int_t^s\tau^2(.,s) - \int_t^{\bar{s}}\sigma(.,\bar{s})\tau(.,s).\int_t^{\bar{s}}\tau(.,s)\tau(.,\bar{s})}{\int_t^{\bar{s}}\tau^2(.,\bar{s}).\int_t^s\tau^2(.,s) - \left(\int_t^{\bar{s}}\tau(.,s)\tau(.,\bar{s})\right)^2},$$

$$\xi_4 := \rho\tilde{\rho}\frac{\int_t^{\bar{s}}\sigma(.,\bar{s})\tau(.,s).\int_t^{\bar{s}}\tau^2(.,\bar{s}) - \int_t^{\bar{s}}\sigma(.,\bar{s})\tau(.,\bar{s}).\int_t^{\bar{s}}\tau(.,s)\tau(.,\bar{s})}{\int_t^{\bar{s}}\tau^2(.,\bar{s}).\int_t^s\tau^2(.,s) - \left(\int_t^{\bar{s}}\tau(.,s)\tau(.,\bar{s})\right)^2},$$

$$\mu^2_{\mathcal{A}_1} := \int_t^{\bar{s}}\sigma^2(.,\bar{s}) - \xi_3^2\int_t^{\bar{s}}\tau^2(.,\bar{s}) - \xi_4^2\int_t^s\tau^2(.,s) - 2\xi_3\xi_4\int_t^{\bar{s}}\tau(.,s)\tau(.,\bar{s}),$$

$$a_3(w) := A(t,\bar{s})\exp\left(\tilde{\rho}\int_t^{\bar{s}}\tau(u,\bar{s})\bar{\sigma}(u,s)\,du - \frac{1}{2}\int_t^{\bar{s}}\tau^2(u,\bar{s})\,du + w\right),$$

$$a_4(\tilde{w}) := A(t,s)\exp\left(\tilde{\rho}\int_t^s\tau(u,s)\bar{\sigma}(u,s)\,du - \frac{1}{2}\int_t^s\tau^2(u,s)\,du + \tilde{w}\right),$$

$$v_s^2 := \int_t^s \tau^2(u,s)\,du, \quad \rho^* := \frac{\int_t^{\bar{s}} \tau(.,s)\tau(.,\bar{s})}{\left(\int_t^s \tau^2(.,s).\int_t^{\bar{s}} \tau^2(.,\bar{s})\right)^{1/2}}.$$

Note that the previous term $\mathcal{A}_1$ involves a two-dimensional integration. At the opposite, the term $\mathcal{A}_2(s,\bar{s})$ is simpler and similar to $\mathcal{E}_2$. We prove easily

Theorem B.4 *Under (E), (IR) and (AM), we have*

$$\mathcal{A}_2(s,\bar{s}) = B(t,s)\int \Phi\left(\frac{1}{\mu_{\mathcal{A}_2}}\{\ln(K-\psi_K(a_5(w))) - \ln EL(t,\bar{s}) - \int_t^{\bar{s}} \rho\sigma(u,\bar{s})\bar{\sigma}(u,s)\,du \right.$$
$$\left. + \quad \frac{1}{2}\int_t^{\bar{s}} \sigma^2(u,\bar{s})\,du - \xi_{\bar{s}} w\}\right)\cdot\phi\left(\frac{w}{v_{\bar{s}}}\right)\cdot\psi_K(a_5(w))\frac{\mathbf{1}(K\geq\psi_K(a_5(w)))}{v_{\bar{s}}}\,dw,$$

where

$$\mu_{\mathcal{A}_2}^2 = \int_t^{\bar{s}} \sigma^2(.,\bar{s}) - \xi_{\bar{s}}^2\int_t^{\bar{s}} \tau^2(.,\bar{s}),$$

and

$$a_5(w) = A(t,\bar{s})\exp\left(\tilde{\rho}\int_t^{\bar{s}} \bar{\sigma}(.,s)\tau(.,\bar{s}) - \frac{1}{2}\int_t^{\bar{s}} \tau^2(.,\bar{s}) + w\right).$$

Thus, the previous theorems allow us to evaluate cash structures, as described in section 4.

Corollary 2 *Under the assumptions (E), (IR), (AM) and (FC) and with our previous notations, the cash bond price of section 4 is*

$$P_t = \sum_{i,T_i\geq t}\{C_0\Delta_i\mathcal{F}_1(T_i,T_{i-1}) + \mathcal{A}_1(T_i,T_{i-1}) - \mathcal{A}_2(T_i,T_{i-1})\} + \mathcal{E}_1(T_p).$$

Under the assumptions (E), (IR), (AM) and (FlC), the related cash bond price is

$$P_t = \sum_{i,T_i\geq t}\{\Delta_i\mathcal{F}_2(T_i,T_{i-1}) + \mathcal{A}_1(T_i,T_{i-1}) - \mathcal{A}_2(T_i,T_{i-1})\} + \mathcal{E}_1(T_p).$$

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References

[1] Andersen, L., Piterbarg, V. & Sidenius, J. (2008). A new framework for dynamic credit portfolio loss modeling. *Journal of Theoretical and Applied Finance*, *11*, 163 – 197.

[2] Bennani, N. (2005). The Forward loss model: A dynamic term structure approach for the pricing of portfolio credit derivatives. Royal Bank of Scotland.

[3] Bielecki, T. & Rutkowski, M. (2002). *Credit risk: modeling, valuation and hedging*. Springer.

[4] Brigo, D. & Mercurio, F. (2001). *Interest rate models. Theory and practice*. Springer.

[5] Boudoukh, J., Whitelaw, R.F., Richardson, M. & Stanton, R. (1997). Pricing mortgage-backed securities in a multifactor rate environment: a mutivariate density estimation approach. *Review of Financial Studies*, *10*, 405 – 446.

[6] Paul D. Childs, Steven H. Ott & Timothy J. Riddiough (1996). The Pricing of Multiclass Commercial Mortgage-Backed Securities. *The Journal of Financial and Quantitative Analysis*, *31*, 581 – 603.

[7] Collin-Dufresne, P. & Harding, J.P. (1999). A closed form formula for valuing mortgages. *Journal of Real Estate Finance and Economics*, *19*, 133 – 146.

[8] Deng, Y. (1997). Mortgage Termination: an Empirical Hazard Model with a Stochastic Term Structure. *The Journal of Real Estate Finance and Economics*, *14*, 309 – 331.

[9] Deng, Y., J.M. Quigley, & R. Van Order (2000). Mortgage Termination, Heterogeneity, and the Exercise of Mortgage Options. *Econometrica 68*, 275 – 307.

[10] Dunn, K. B. & J. J. McConnell (1981a). Valuation of GNMA mortgage-backed securities. *Journal of Finance*, *36*, 599 – 617.

[11] Dunn, K. B. & J. J. McConnell (1981b). A comparison of alternative models for pricing GNMA mortgage-backed securities. *Journal of Finance*, *36*, 375 – 392.

[12] Dunsky, R.M. & Ho T. (2007). Valuing Fixed Rate Mortgage Loans with Default and Prepayment Options. *Journal of Fixed Income*, *16*, 7 – 31.

[13] Garcia, J. & Goossens, S. (2008a). One Factor Models for the ABS Correlation Market Pricing TABX Tranches. http://www.sergeandjoao.com/CorrelationPapers/tabxonefactor.pdf

[14] Garcia, J. & Goossens, S. (2008b). One Credit Event Models for CDO of ABS. http://www.sergeandjoao.com/CorrelationPapers/abscdo.pdf

[15] Gautier, L. (2003). Market-Implied Losses and non-agency Subordinated MBS. *Journal of Fixed Income*, *13*, No 1, 49 − 74.

[16] Giesecke, K., Goldberg, L.R. & Ding, X. (2010). A Top-down Approach to Multi-name Credit. http://papers.ssrn.com/ id=1142152

[17] Goncharov, Y. (2005). An Intensity-Based Approach to Valuation of Mortgage Contracts subject to Prepayment Risk. *International Journal of Theoretical and Applied Finance*, *9*(6), 889 − 914.

[18] Kariya, T., & M. Kobayashi (2000). Pricing Mortgage Backed Securities (MBS). A Model Describing the Burnout Effect. *Asian Pacific Financial Markets*, *7*, 182 − 204.

[19] Kariya, T., S.R. Pliska, & F. Ushiyama (2002). A 3-Factor Valuation Model for Mortgage Backed Securities (MBS). *Kyoto Institute of Economic Research.* http://www.math.fsu.edu/ jfrade/PAPERS/MBS.pdf

[20] Kau, J. B., Keenan, D.C., Muller III, W. J., & Epperson, J. F. (1995). The valuation at origination of fixed-rate mortgages with default and prepayment. *The Journal of Real Estate Finance and Economics*, *11*, 5 − 36.

[21] Kau, J., Keenan D. & Smurov, A. (2006). Reduced-form mortgage pricing as an alternative to option-pricing models. *The Journal of Real Estate Finance Economics*, *33*, 183 − 196.

[22] Jäckel, P. (2008). Gamma Loss and Prepayment. *Risk,* September, 134 − 139. http://www.awdz65.dsl.pipex.com/TheGammaLossAndPrepaymentModel.pdf

[23] Levin, A. & Davidson, A. (2008). The concept of credit OAS in valuation of MBS. *Journal of Portfolio Management*, *34*, Spring, 41 − 55.

[24] Lipton, A. & Rennie, A. (2007). *Credit correlation: life after copulas.* World Scientific Public.

[25] Lou, W. (2007). Valuation of Subprime Mortgage-Backed Securities: a portfolio Credit Derivatives Approach. Available at http://louwujiang.com/

[26] McConnell, J.J. & Singh, M. (1993). Valuation and analysis of Collateralized Mortgage Obligations. *Management Science*, *39*, 692 − 709.

[27] McConnell, J.J. & Singh, M. (1994). Rational Prepayments and the Valuation of Collateralized Mortgage Obligations. *Journal of Finance*, *49*, 891 − 921.

[28] Nakamura, N. (2001). Valuation of mortgage-backed securities based upon a structural approach. *Asia-Pacific Financial Markets*, *8*, 259 − 289.

[29] O'Kane, D. (2008). *Modelling single-name and multi-name credit derivatives.* Wiley.

[30] Patruno, G.N., Ilinic, N. & Zhao, E. (2006). Introducing the unified Deutsche Bank Prepayment Model. Deutsche Bank.

[31] Pénasse, J. (2008). Cash Flow-Wise ABCDS pricing. http://mpra.ub.uni-muenchen.de/10853/

[32] F. Richard & R. Roll (1989). Prepayment on Fixed-Rate Mortgage-backed Securities. *Journal of Portfolio Management*, *15*, 73 − 82

[33] Rom-Poulsen, N. (2007). Semi-anaytical MBS pricing. *The Journal of Real Estate Finance and Economics*, *34*, 463 − 498.

[34] Schönbucher, P. (2005). Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives. *ETH, Zurich.* http://www.defaultrisk.com/

[35] Schwartz, E. & W.N. Torous (1989). Prepayment and the valuation of Mortgage-Backed-Securities. *Journal of Finance*, *44*, 375 − 392.

[36] Schwartz, E. & W.N. Torous (1993). Prepayment, Default, and the Valuation of Mortgage Pass-Through Securities. *Journal of Business*, *65*, 221 − 239

[37] Sharp, N.J., Duck, P. & Newton, N. (2008). An Improved Fixed-Rate Mortgage Valuation Methodology with Interacting Prepayment and Default Options. *The Journal of Real Estate Finance and Economics*, *36*, No 3, 307 − 342.

[38] Stanton, R. (1995). Rational prepayment and the value of mortgage-backed securities. *The Review of Financial Studies*, *8*, 677 − 708.

[39] Vasicek, O. (2002). The distribution of portfolio loan value. *Risk,* December, 130 − 133.

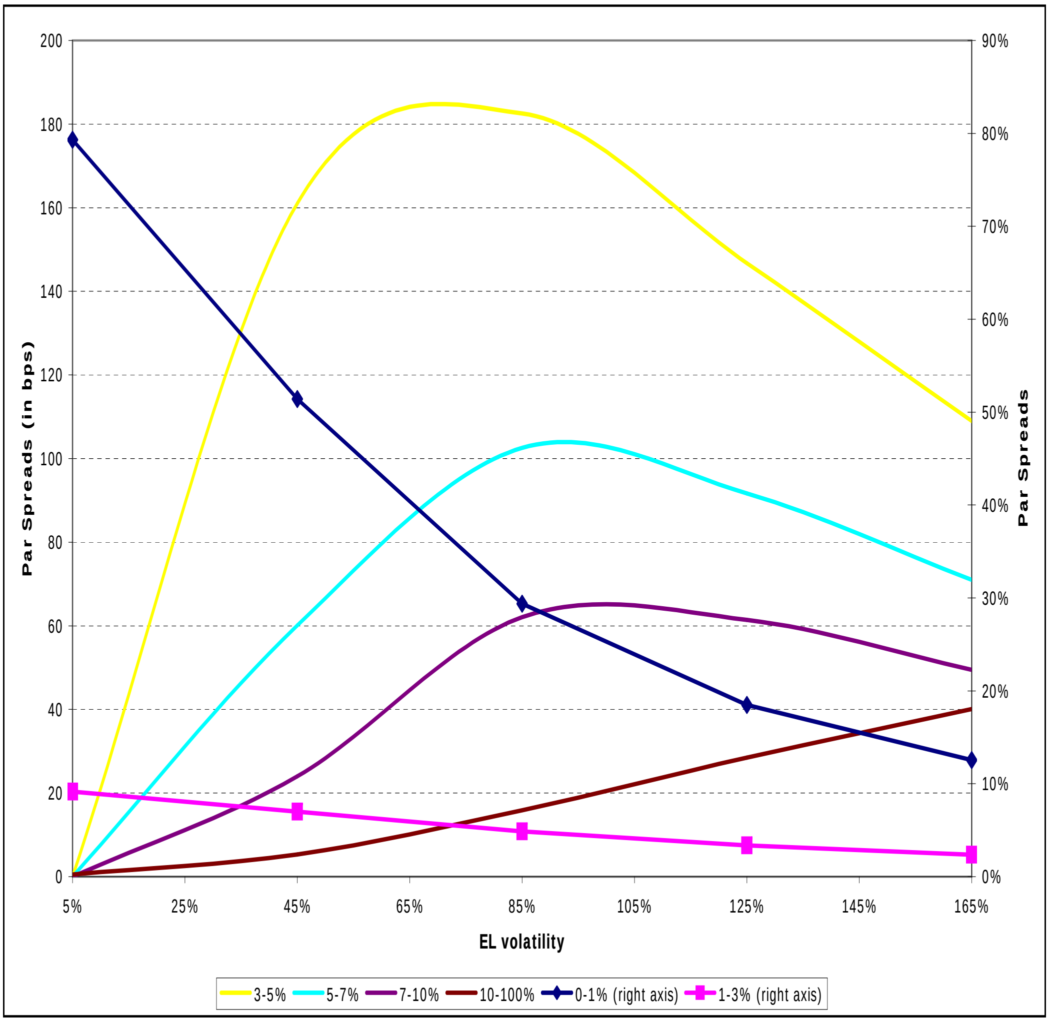

Figure 1: Par spreads of RMBS tranches as a function of Expected Loss volatility

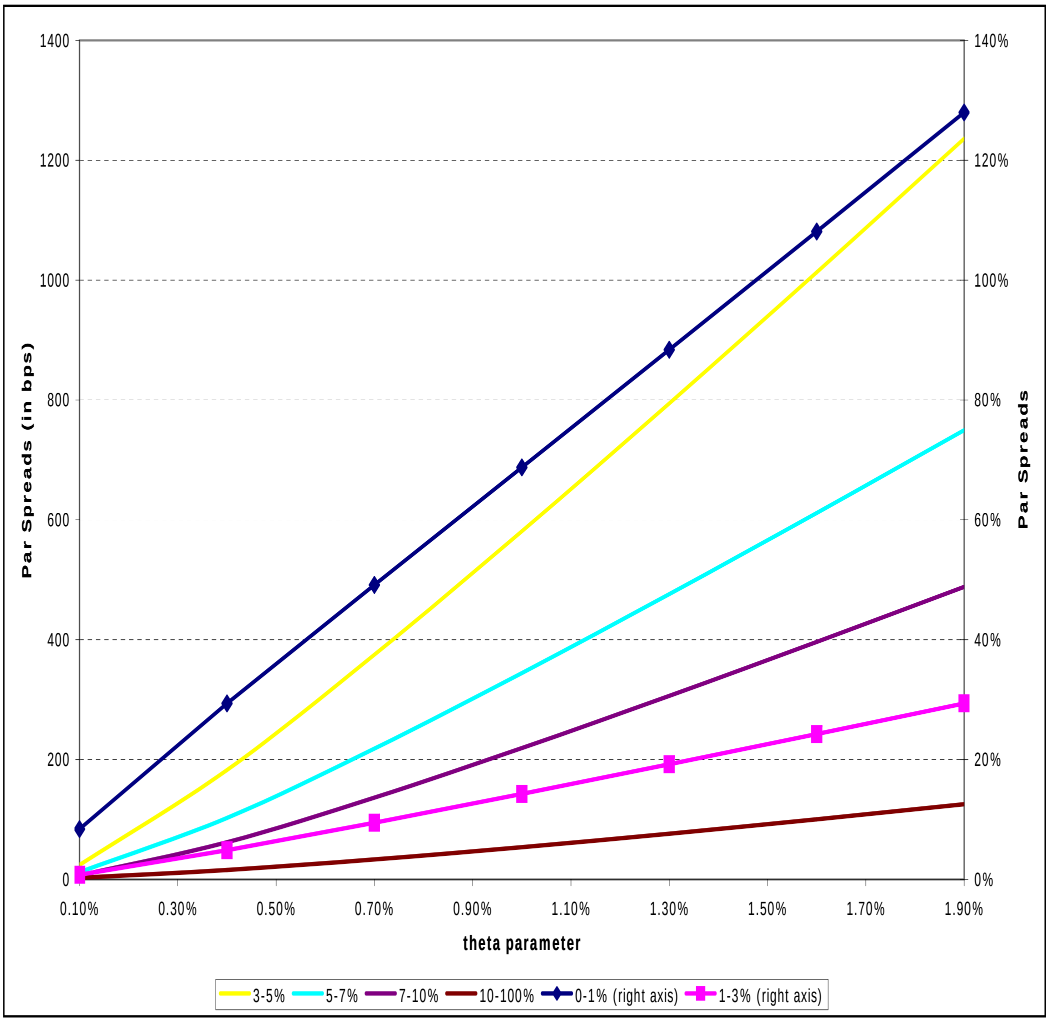

Figure 2: Par spreads of RMBS tranches as a function of the default rate θ

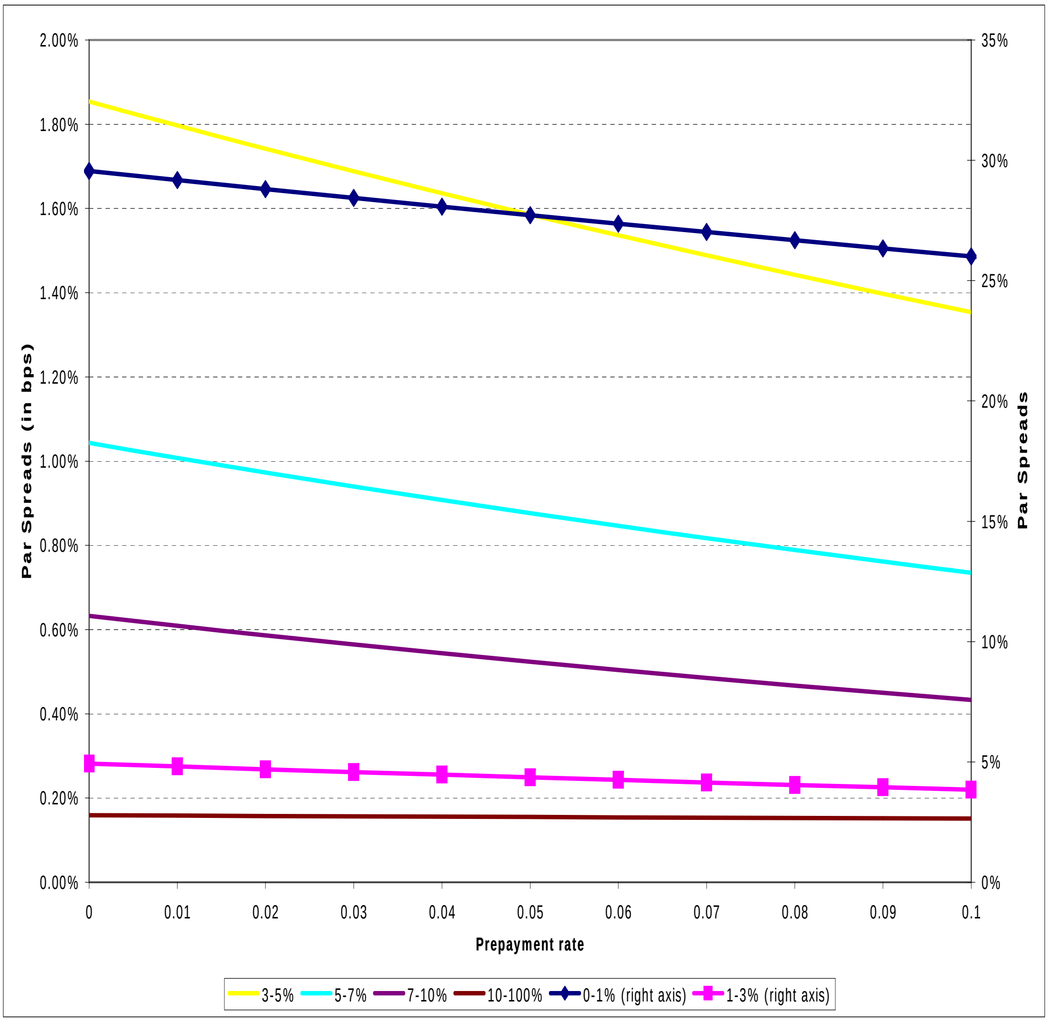

Figure 3: Par spreads of RMBS tranches as a function of the prepayment rate b

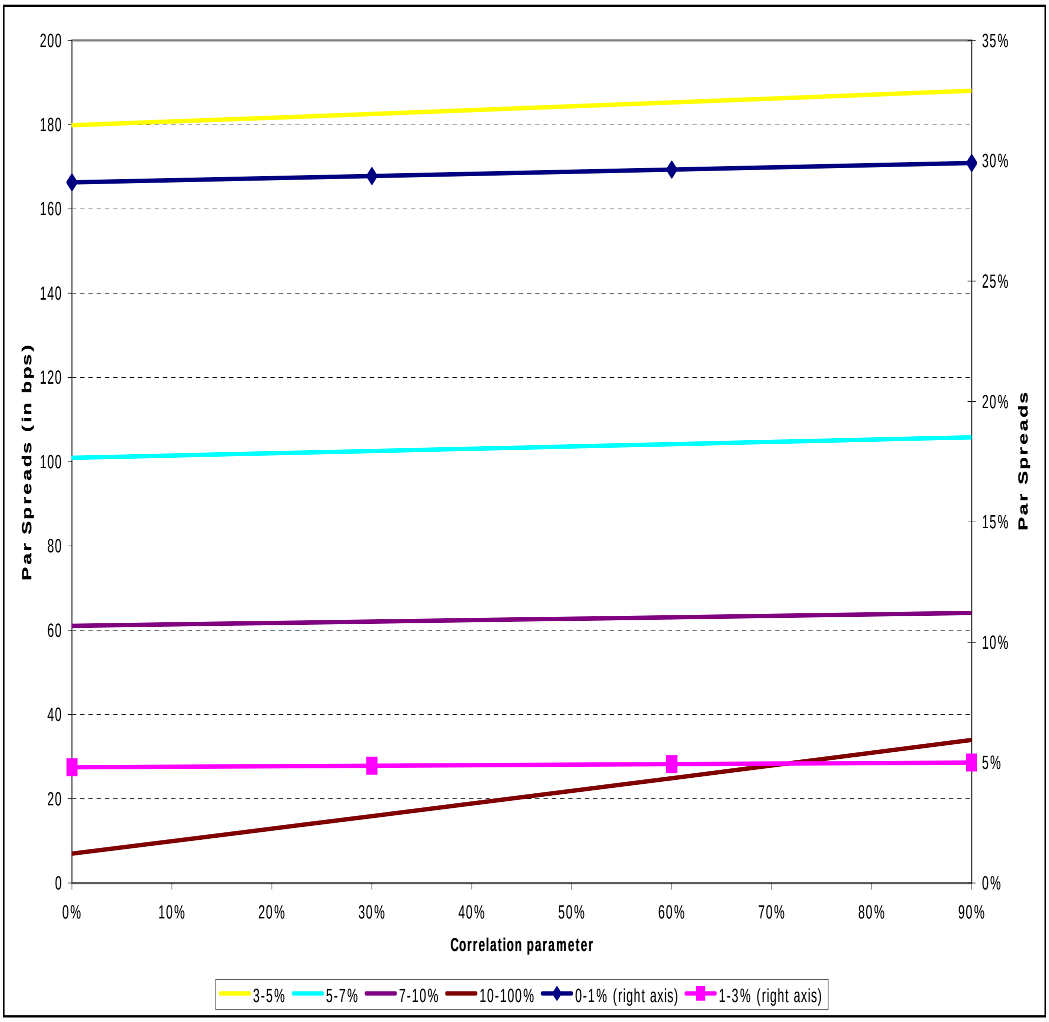

Figure 4: Par spreads of RMBS tranches as a function of correlation ρ

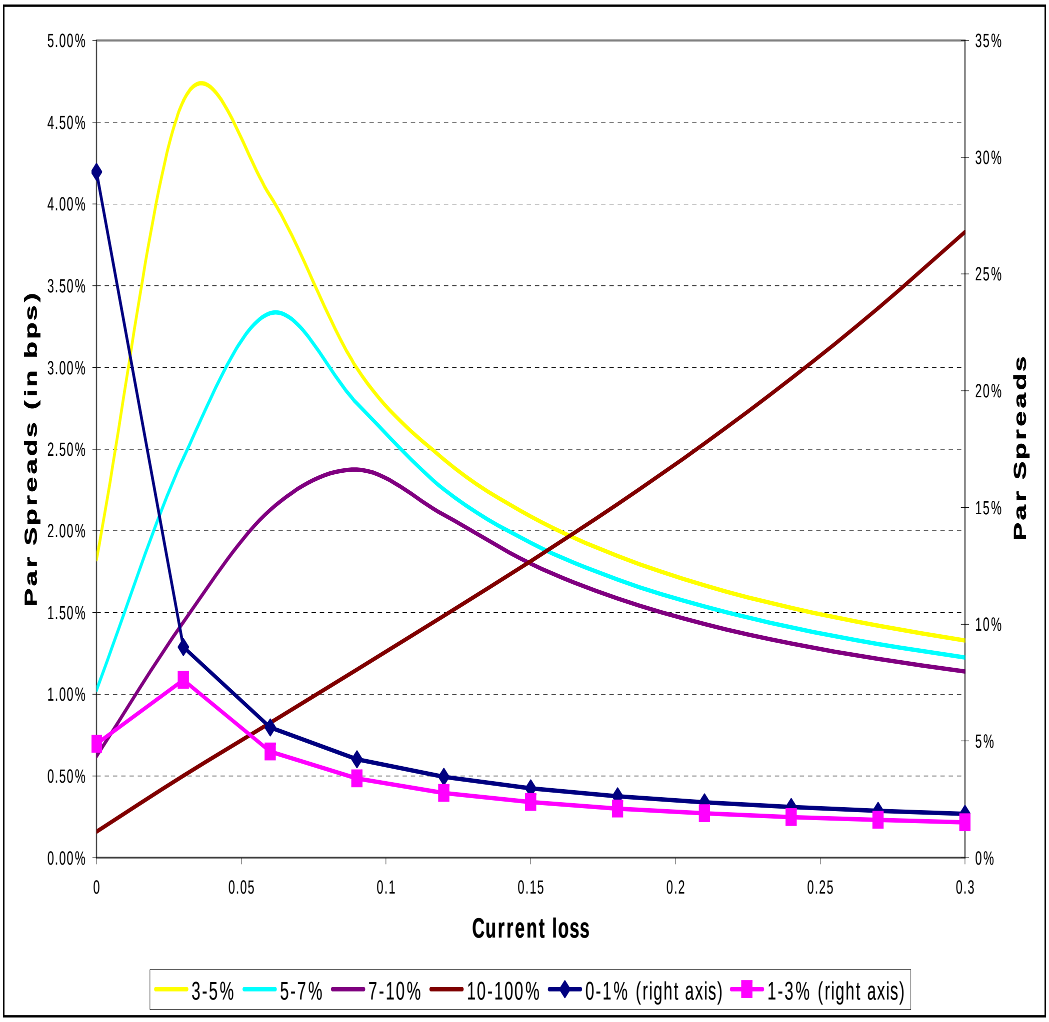

Figure 5: Par spreads of RMBS tranches as a function of the current loss

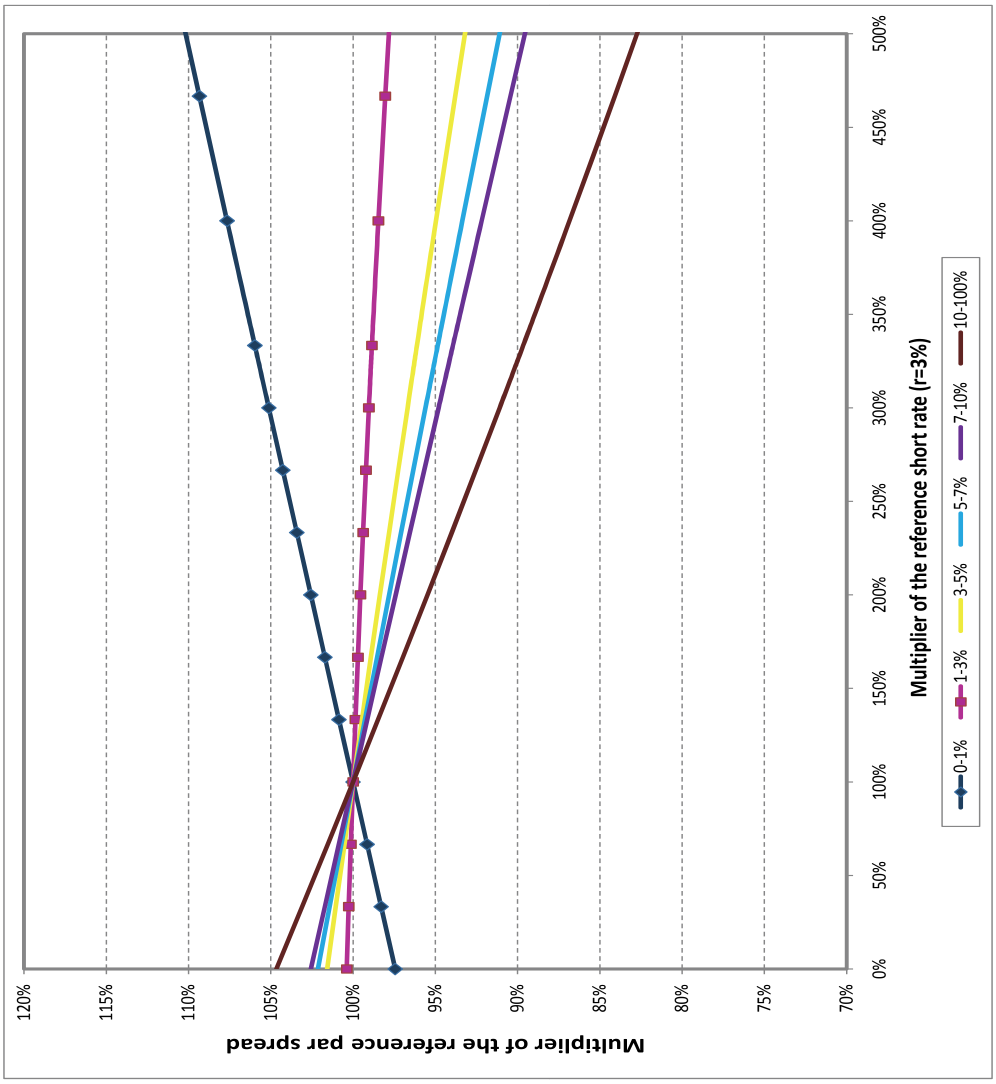

Figure 6: Par spreads of RMBS tranches as a function of the spot short rate (both dimensions are in relative terms)