A top-down approach for MBS, ABS and CDO of ABS: a consistent way to manage prepayment, default and interest rate risks.

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Abstract

We define a model for managing prepayment, default and interest rate risks simultaneously in standard ABS-type structures (ABS, MBS, CDO of ABS, cash or synthetic). We propose a parsimonious top-down approach: two random factors drive the main underlying risks (prepayment and default), through their current expectations at every time horizon and some volatility assumptions. We get closed-form formulas for pricing all tranches under the assumption that amortization occurs in the most senior tranche only. When the latter assumption is removed, semi-analytical formulas are obtained. The model behavior is illustrated through the empirical analysis of a real synthetic ABS trade.

Key words and phrases: Mortgage, top-down models, default risk, prepayment.

JEL Classification: G12, G13.
1 Introduction

Mortgage Based Securities, and ABS more generally, are commonly traded securities in the USA and now in Europe. They are related to some pools of assets that are sufficiently numerous so that the underlying portfolios can be considered as infinitely granular, at least as a first step. These pools are tranched so that investors can benefit from some credit enhancements, depending on the risk/return profile they wish. The risks that are associated with such structures are mainly:

- A prepayment risk: some underlying assets have a random maturity, and some of them can be repaid quicker/slower than expected. This uncertainty can induce a marked-to-market loss for investors. This loss can be seen as an opportunity loss: investors tend to be repaid when rates are lower and then it may become difficult for them to find alternative investment opportunities. Traditionally, this is the main risk that is associated with MBS.

- A default risk: some borrowers may be unable to reimburse the coupons or the principal of their loans fully. At the same time, the value of their collateral (their house, in the case of residential loans) may fall, due to unfavorable real-estate market moves. In the USA, such a risk is often guaranteed by some well-known Agencies (Fannie Mae, Freddie-Mac, Ginnie Mae), but it is not mandatory, as for subprime loans for instance. In Europe, loans are not guaranteed most of the time.

- An interest rate risk, that is strongly connected to the two previous ones. Indeed, lower interest rates induce some incentives to renegotiate current loans, or simply to fully prepay them to enter into new ones under better financial conditions. Note that, when interest rates increase, the weight of periodic reimbursements will become heavier for floating-rate borrowers. The proportion of individual bankruptcies would then go up. Thus, credit risk is strongly related to interest risk.

In the literature, most of the authors have tackled prepayment risk and default risk separately. And the former has retained the attention a lot more frequently than the latter, partly because of the assumed small credit risk of MBS until recently. Following some seminal papers in the end of the 80’s (Richard and Roll (1989), Schwartz and Torous (1989)), a significant stream of Mortgage-related papers has appeared in the academic and the professional areas. Most of them are rather empirical: the goal is to explain prepayments through econometric models. We can be surprised by the way this literature has increased largely independently from the remaining asset pricing literature. This is partly due to the particular features if the ABS markets and the risks.

1 mortgages, home equity loans, commercial loans, student loans, credit cards etc
2 The basic case is given by an agency MBS, for which there is no credit risk virtually. The presumed robustness of these Agencies is less a certainty in the current crisis.
they convey. In this paper, we propose to borrow some theoretical concepts from the asset pricing literature in general, and credit derivatives particularly. We will apply them to valuate MBS, ABS and even CDO of ABS. Moreover, we will be able to value some coupon-bearing cash-flows too. So, if waterfalls are not "too complicated", cash structures can be dealt within our framework.

Prepayments can be seen as (latent) fatal risks during the whole life of any mortgage loan. In the light of Reliability theory and following Schwartz and Torous (1989), some authors have proposed to explain the life duration of mortgages directly in a reduced-form approach: Deng (1997), Deng et al. (2000), Karya and Kobayashi (2000), Kariya et al. (2002), among others. Instead of trying to explain the (possibly optimal) behavior of mortgagors in terms of prepayment, a pragmatic econometric point of view is adopted. When dealing with default and prepayment risks simultaneously, the framework of competing risk models appears naturally: see Kau et al. (2006) for instance. More recently, the development of credit risk models and stochastic intensities approaches have influenced this stream of the financial literature: Goncharov (2002), particularly. Indeed, as the latter author said: "After all, from a mathematical point of view nothing precludes one from interpreting prepayment as a "default" in the intensity-based approach to pricing credit risk."

In our opinion, the credit risk inspiration has been far from being fully exploited in the ABS world. Among the thousands of papers in Credit Risk, only a few ones have been revisited or adapted in the light of mortgages. In this article, we will try to fill this gap partly. We will borrow the recent "top-down" approach for pricing structured products, CDOs particularly: Andersen et al. (2008), Bennani (2005), Schönbucher (2005) among others. The basic idea will be to deal with aggregated loss processes instead of trying to detail individual loss intensities and their dependencies. This seems to be particularly relevant in the case of mortgage pools that take together thousands of underlying loans. We will state our results in a continuous time framework. It is rather unusual in the mortgage literature, but is a lot more standard in asset pricing. It allow us to state nice closed formulas in particular.

Note that we will not try to exhibit an optimal mortgagor behavior in terms of prepayment. Repayment risk is aggregated at the portfolio level, and only its dependence on the global factors in the economy is relevant for us. A proxy for these factors is given by the yield curve itself, or the not-risky bond prices. Contrary to a large part of the literature that deals with conforming mortgages, default events are the second main source of risk in the structure we consider. It is particularly the case for structures that contain a large proportion of floating rate notes or that integrate a lot of low quality debtors (subprimes). More generally, it is often the case in continental European structures where prepayments incentives are a lot weaker than in the USA, partly due to associated penalties.

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by exhibiting some observable explanatory variables, by postulating the functional form of the underlying hazard rates and by calibrating parameters historically

so implicitly insured by quasi-governmental agencies
Our ways of modelling are probably more relevant for the simplest ABS structures in the market, for instance synthetic ones. Indeed, by nature, it will be difficult to take into account in an analytical framework all the options, special features or triggers that can be met in the real world. Nonetheless, beyond all the differences between those structures, it is of interest to build some "core model", as a general approach to price and risk manage them. And it is always possible to keep our model specifications (or to modify them slightly), but by relying on some simulation-based methodology, if we want to tackle some particular features.

We will adopt a rather "macro-economic" point of view: we do not try to fully use loan-per-loan information. We prefer to reduce the (potentially huge) information set by restricting ourselves to some relevant "information summaries": a mean amortization profile and some anticipated prepayment rates. Such information is standard for practitioners in the market. We will focus on the term structure of these curves, their changes in time and their dependencies. In practical terms, our approach is surely cheaper than managing thousands of individual loan descriptions and their interdependencies inside some econometric models. Potentially, a drawback would be a lack of accuracy in terms of fine-tuned description, inducing possibly a poor risk management. For instance, some practitioners could feel more comfortable to risk-manage on a "name-per-name basis" such structures. For usual MBS/ABS that involve thousand of names, it is clearly unrealistic. In the case of CDO of ABS, that may be built with one hundred of names, it can be targeted indeed. In this article, we assume the diversification in the underlying pool is sufficient so that a few number of macro-factors (mainly moves of the interest rate curve) are sufficient to price and risk-manage the structures we are dealing with. Nonetheless, such an assumption has to be tested for every family of deals.

To illustrate the idea, we know that most of the mortgage dealers have some tools to evaluate and forecast future amortization profiles and finally prices, under some assumptions related to interest rates, prepayment, default rates, household market prices etc. By randomizing a large number of scenarii and averaging, they get such prices, if parameters are chosen conveniently. It is our intuition too. But we assume the final process could be saved. Indeed, at every time t, we assume it is possible to infer easily a mean amortization profile and expected default rates. The model will assume some continuous-time dynamics for both quantities. We argue all these current "information summaries" and convenient dynamics should be sufficient for our purpose. Moreover, our approach can be considered as a relative value tool for arbitrage purpose. In a certain sense, it would provide an original point of view for comparing several

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5for instance: age and gender of borrowers, geographical area, Loan-To-Value, maturity and size of loans, financial strengths as provided by some scores (FICO)...

6simply because there exist no hedging instrument in the market to covers the purely idiosyncratic risk associated with any particular borrower

7but even in this case, it is a bit an illusion, because the underlying "ABS bonds" are themselves complex tranched products. In every case, it is necessary to reduce the potentially huge information set for pricing purpose.
tranches of the same structure or between similar structures, beside other tools like Intex, Bloomberg, AFT, Loan performance etc.

Therefore, we propose a parsimonious and consistent way of dealing with all the previous underlying risks together. We choose some diffusion-based framework that is usual in the Fixed Income world, but is really a new approach in the mortgage/ABS world, to the best of our knowledge. We will consider some aggregated quantities as portfolio credit losses (realized and expected), outstanding notionals and expected prepayments without going down to the bottom level of basic loans. Thus, we will specify term structures for such quantities and their changes in time.

Typically, in such structures, the most junior tranches are very thin w.r.t. the most senior tranches. It is not unusual the latter ones are related to more than 90% of total initial portfolio notionals. We assume that credit losses will be attributed to the most junior tranches first ("bottom-up"). At the opposite, amortized and prepaid amounts will reduce the size of the most senior tranches first ("top-down"). These are the most frequent specifications in the Mortgage market. Therefore, our formulas have to be adapted to tackle different features (for instance, a proportional reduction of all the tranche notionals when prepayments occur).

First, in section 2, we will introduce the main equations that describe the behaviors of the portfolio expected loss process and the random amortization profile. Then, we provide the relevant pricing formulas by some change of numeraire techniques, in the case of synthetic structures (section 3) and simple cash structures (section 4). At last, we provide some empirical results and a sensitivity analysis of the model w.r.t. its parameters in section 5.

The results will be stated under the following assumption: only the most senior tranche of the capital structure will be hit by amortized and prepaid amounts. This allows to find nice closed-form formulas. Nonetheless, without the latter assumption, semi-analytical pricing formulas will be detailed in appendix B.

## 2 A top-down pricing model

Without a lack of generality, we can consider the total notional amount of our pool of assets is one. The tranching process is related to several detachment points $K_1 < K_2 < \ldots < K_p$. We set $K_0 = 0$ and $K_p = 1$. At every time $t$, the outstanding notional of the whole portfolio will be denoted by $O(t)$ and the outstanding notional of the tranche $[0, K]$ by $O_K(t)$. Obviously, these quantities are random.

As explained in the introduction, the notional amounts of these tranches can be reduced due to three different effects:
• The "natural" amortization process, that is deterministic for every underlying name and deduced from contractual terms. The loans are amortized from the most senior tranches to the most junior ones ("top-down").

• the prepayment process, that deals with the most senior tranches first too. It can be seen as a randomization of the previous amortization profile.

• the default process (failure to pay remaining coupons or nominals). It is concerning the most junior tranches at first ("bottom-up").

Potentially, all these reduction effects can feed a given tranche simultaneously, at least from a certain time on.

For the sake of simplicity, we consider first a synthetic structure. In this case, we do not have to take care of coupon payments and waterfalls more generally. The case of cash structures will be detailed in a subsequent section. Typically, we can keep in mind a synthetic CDO of ABS, whose underlyings are some CDS on ABS tranches (so-called ABCDS). In this case, no initial fund is necessary to invest in such a structure. The cash-flows are coming only from notional repayments and defaults. The main price driver is here default risk, as for usual synthetic corporate CDOs.

Let $RL_{t,K}$ and $DL_{t,K}$ be respectively the time-$t$ risky level and the default leg that are associated with the tranche $[0,K]$. The latter default leg is related to default event losses only. The spread associated with the tranche $[K_{j-1},K_j]$ is denoted by $s_{t,j}$ and satisfies by definition

$$
st_{t,j} \{ RL_{t,K_j} - RL_{t,K_{j-1}} \} = DL_{t,K_j} - DL_{t,K_{j-1}}$$

(1)

for every time $t$ and every $j = 1, \ldots, p$. The approach is standard in Credit Derivatives, for pricing CDOs for instance. The first goal of our model will be to evaluate these risky levels and these default legs. This will be done in closed-form as far as possible.

Let us denote by $T^*$ the maturity of our structure. It may be seen as the largest maturity date of all the underlyings, or a potential ("almost" sure) call date of the structure. By definition,

$$RL_{t,K} = E \left[ \int_t^{T^*} \exp \left( - \int_t^s r_u du \right) O_K(s) \, ds \right]_{\mathcal{F}_t},$$

(2)

by denoting $(r_s)$ the usual short interest rate process. Concerning notations, we denote by $E_t[\cdot]$ the expectation conditionally on the market information $\mathcal{F}_t$ at time $t$ and under a risk-neutral measure $Q$. The market information $\mathcal{F}_t$ records all the past and current relevant information concerning the description of the cash-flows and the underlyings: past payments, contractual features, interest rates, recorded losses etc. Note that our so-called risky level definition is rather unusual because it is homogenous with a duration times a notional amount.
To fix the ideas, let us denote by $A(s)$ the portfolio amortized amount at time $s$, as a percentage of the initial amount. Moreover, let $A_K(s)$ be the latter amount but related to the tranche $[0, K]$, i.e.

$$A_K(s) = [A(s) - (1 - K)]^+.$$  

The latter quantity is the amount of money the tranche $[0, K]$ has been reduced "from above", due to the amortization process only. Actually, since this tranche is reduced potentially "from below" by the default events, we have

$$O_K(s) = [K - L(s) - A_K(s)]^+.$$  

We have introduced the loss $L(.)$ of the whole portfolio at time $s$. It is simply the accumulated amount that is due to default events. The same quantity, but related to the tranche $[0, K]$ is denoted by $L_K(s)$. Note that the latter quantity depends on the outstanding notional process of this tranche, so on the amortization profile too. This feature complicates the asset pricing formulas significantly. Moreover, note that the outstanding notional $O(s)$ is related to the other quantities by the relation

$$O(s) = 1 - L(s) - A(s).$$

The loss process that refers to the tranche $[0, K]$ can be rewritten

$$L_{K}(s) = L(s).1(L(s) \leq K - A_{K}(s)),$$

when the tranche $[0, K]$ has not been fully fed. Otherwise, the amount of losses is fixed, and keeps its last value (just before this tranche has been fully fed) most often, even if some recovery process can occur with some likelihood.

Thus, we can write the default leg of the tranche $[0, K]$ as seen at time $t$:

$$DL_{t,K} = E_t \left[ \int_t^{T^*} \exp \left( - \int_t^s r_u du \right) L_K(ds) \right]$$

$$= E_t \left[ \int_t^{T^*} \exp \left( - \int_t^s r_u du \right) 1(L(s) + A_{K}(s) \leq K) L(ds) \right].$$  \hfill (3)

In practice, we need to evaluate the latter integral with some grid of dates $T_0 = t, T_1, \ldots, T_p = T^*$. Thus, we consider that $8$

$$DL_{t,K} \simeq E_t \left[ \sum_{i=1}^{p} \exp \left( - \int_{T_i}^{T_{i+1}} r_u du \right) 1(L(T_i) + A_{K}(T_i) \leq K) (L(T_i) - L(T_{i-1})) \right],$$

\hfill 8We neglect accrued payments due to defaults between two successive dates.
with a reasonable accuracy. Thus, to evaluate the functions $RL_{t,K}$ and $EL_{t,K}$ and thanks to some elementary algebraic operations, it is sufficient to calculate the expectations

$$E(s) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) \left( K - L(s) - [A(s) - 1 + K]^+ \right) \right],$$

(4)

$$E(s, \tilde{s}) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) \mathbf{1} \left\{ L(s) + [A(s) - 1 + K]^+ \leq K \right\} L(\tilde{s}) \right],$$

(5)

for every couple $(s, \tilde{s})$, $t \leq \tilde{s} \leq s \leq T^*$. Now, we concentrate our efforts on the evaluation of the previous expectations. The latter expressions involve some tricky double indicator functions. To simplify the analysis and to get simpler closed-form formulas, we assume for the moment

**Assumption (A):** The amortization process and the prepayment process will reduce the most senior tranche only.

In other words, under (A), we consider only trajectories where the amortization process is stopped (or the structure is repaid) before the most senior tranche is fully fed from above. This assumption implies that $A_K(t, T_i) = 0$ for all dates $T_i \leq T^*$ and for all detachment points $K < 1$. That assumption will be removed and general semi-analytical formulas are provided in appendix B.

Assumption (A) is not unrealistic because the upper detachment points (less than one) are often very low, e.g. less than 10% in a lot of structures. Moreover, in practice, such structures are called when the amortization and prepayment process has fed a large part of the pool (typically 90%). Thus, this assumption is reasonable.

Under (A), we need to evaluate the simpler quantities

$$E_1(s) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) [1 - L(s)]^+ \right],$$

and

$$E_2(s, \tilde{s}) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) \mathbf{1} \left\{ L(s) \leq K \right\} L(\tilde{s}) \right],$$

for all the tranches except the most senior one. Concerning the most senior tranche, we have to calculate

$$E_1^*(s) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) [1 - L(s) - A(s)]^+ \right],$$

and

$$E_2^*(s, \tilde{s}) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) L(\tilde{s}) \right].$$
The previous expectations $E_1$ and $E_2$ can be deduced from the value of some options that are written on the loss process $L(.)$. Basically, it is more relevant to work in terms of the (not discounted) Expected Loss process itself, which is defined by

$$EL(t, T) := E[L(T)|F_t] = E_t[L(T)].$$

Indeed, this process takes into account all the forecasts in the market continuously. So, it is more acceptable to set some diffusion processes on the expected losses than on the credit losses directly. Moreover, it allows us to specify a large variety of behaviors depending on the remaining time to maturity in a consistent way. This effect is strong for MBS/ABS: at the beginning of the pool, default/prepayment rates are low, due to borrowers selection processes (the so-called "seasoning effect"). Then, typically, these rates are upward sloping and after some years, they are decreasing. Indeed, the borrowers that have "bad" individual characteristics have already defaulted/prepaid, and only the relatively "safest" borrowers stay in the pool (the "burnout effect"). At last, when the process $L$ is (essentially) increasing, there is no such constraint on $EL$, what is a desirable property in terms of model specification.

Thus, these previous expectations can be rewritten as functions of the Expected Losses themselves by noting that $L(s) = EL(s, s)$:

$$E_1(s) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) \left[ K - EL(s, s) \right]^+ \right],$$

and

$$E_2(s, \bar{s}) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) \mathbf{1}\{EL(s, s) \leq K\} EL(\bar{s}, \bar{s}) \right].$$

The latter expectations will be calculated as functions of $EL(t, s)$, $EL(t, \bar{s})$, and the model parameters only. From now on, we will consider the Expected Loss process only, that can be seen as our underlying.

Similarly, we will consider the Expected Amortized amount process $A(t, T)$ that is defined by

$$A(t, T) := E_t[A(T)].$$

Moreover, to evaluate the functions $RL_{t,K}$ and $DL_{t,K}$, we have to take into account the randomness of the interest rates. Generally speaking, it has been observed that the price of mortgage-backed securities depend strongly on the interest rate curves (e.g. see Boudoukh et al. (1997)), at least through prepayment.

On the other hand, the prepayment process is mainly due to to some exogenous factors: house moving, divorce, death, etc. These phenomena are diversified inside the pool (statistical prepayment), even if their behaviors has generated a lot of modelling efforts. But another source of varying prepayment rates is due to the moves of the interest rates curve itself. In the literature, it is widely recognized that interest rates moves are the most important drivers
of the prepayment randomness, because declining interest rates induce more prepayment incentives. Usually, some deterministic functions have been fitted to define the prepayment rate as a function of one or several interest rate based indicators (levels, slopes...). Thus, we will be deciding that the random factor of repayments will be the same as the random factors of the interest rates themselves.

But we go further. For us, it is intuitively clear that default rates themselves depend on the spot level of interest rates. Generally speaking, the higher the short rate, more numerous are the default events. Indeed, such increase induces financial difficulties for the debtors that are involved in floating rate loans. Obviously, the strength of this effect depends strongly on the proportion of fixed/floating-rate loans in our pool.

Now, we assume that the Expected Loss process is lognormal. Since \( (EL(t, T)) \) is a \( Q \)-martingale by construction, its drift is zero. Thus, we do the

**Assumption (E):** For every times \( t \) and \( T \),

\[
EL(dt, T) = EL(t, T)\sigma(t, T)dW_t.
\]

Equivalently, conditionally on the information at time 0, we assume that

\[
EL(t', T) = EL(t, T)\exp \left( -\int_t^{t'} \frac{\sigma^2(u, T)}{2} du + \int_t^{t'} \sigma(u, T) dW_u \right),
\]

(6)

for every times \( t < t' \).

Obviously, \((W_t)_{t \in [0, T]}\) is an \( \mathcal{F} \)-adapted Brownian motion under \( Q \). For the moment, we assume we know the quantities \( EL(t, \cdot) \) at the current time \( t \), as if they were observed in the market.

Basically, we could lead the same analysis with the Forward Expected Loss \( FEL(\cdot, \cdot) \) rather than the Expected Loss. The former does not take into account past losses when the latter do. Formally, it means that

\[
FEL(t, T) := E[L(T)|\mathcal{F}_t] - L(t).
\]

Actually, we can rewrite the whole model in terms of Forward Expected Losses. We have just to include the current realized loss \( L(t) \) into the strike levels. Particularly, this process could be assumed as lognormal. Nonetheless, in this case, its volatility has not the same behavior as the so-called previous volatility \( \sigma(\cdot, \cdot) \). Particularly, its dependence w.r.t. the remaining time to maturity can be dramatically different. For instance, the FEL volatility could be assumed to be a constant, when this assumption is highly questionable for \( \sigma(\cdot, \cdot) \). The choice between an "accumulated" Expected Loss approach and a Forward Expected Loss...
Loss one is largely formal. The complexity is exactly the same, even if the formulas change slightly. That is why rewriting our formulas under the Forward Expected Loss approach is left to the reader.

Now, let us deal with the interest rate process.

**Assumption (IR):** Every discount factor $B(t, T)$ follows the dynamics

$$\frac{B(dt, T)}{B(t, T)} = (...)dt + \tilde{\sigma}(t, T)d\tilde{W}_t,$$

for every $T$ and $t \in [0, T]$. Here, $(\tilde{W}_t)$ is an $\mathcal{F}$-adapted Brownian motion under $Q$, and $E[d\tilde{W}_td\tilde{W}_t] = \rho dt$.

There exist several ways to specify the previous volatility functions. Since the Expected Loss process records the realized losses from the inception date on, and since the underlying borrowers are numerous, we are closed to the infinitely granular hypothesis. So, when the time $t$ tends to any horizon $T$, the Expected Loss $EL(t, T)$ trajectories will remain continuous and $\sigma(t, T)$ tends to a nonzero constant. This implies that the volatility $\sigma(t, T)$ is a decreasing function of $(T - t)$. In practice, we could assume that

$$\sigma(t, T) = \sigma_0(T - t)^\alpha,$$

for some unknown constant $\alpha \geq 0$. Alternatively, we could decide to set

$$\sigma(t, T) = \sigma_0[\exp(\lambda(T - t)) - 1],$$

with some positive unknown parameters $\sigma_0$ and $\lambda$, as for pricing a bond in Gaussian HJM models.

Obviously, we could assume the same type of specifications for the volatility of the discount factors themselves, for instance

$$\tilde{\sigma}(t, T) = \tilde{\sigma}_0[\exp(\tilde{\lambda}(T - t)) - 1].$$

Similarly, we can tackle with the amortization process $A(t, T)$, that will be assumed lognormal.

**Assumption (AM):** For every times $t$ and $T$,

$$A(dt, T) = A(t, T)\tau(t, T)d\tilde{W}_t, \quad (7)$$

or equivalently

$$A(t', T) = A(t, T)\exp\left(-\int_t^{t'} \frac{\tau^2(u, T)}{2} du + \int_t^{t'} \tau(u, T)d\tilde{W}_u\right),$$

for all times $(t, t'), t \leq t' \leq T$. 

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As previously, the volatility behavior of the amortization process \( A(t, T) \) can be identified with the one of the Expected Loss. Thus, for instance, we could state
\[
\tau(t, T) = \tilde{\sigma}_0 \left[ \exp(-\tilde{\lambda}(T - t)) - 1 \right],
\]
for some positive constants \( \tilde{\sigma}_0 \) and \( \tilde{\lambda} \).

Note the perfect correlation between the amortization process and the interest rate process \(^{10}\). Moreover, the lognormal specification does not prohibit some strange features like \( A(t, s) > 1 \) or \( EL(t, s) > 1 \) for some dates \((s, t)\). Theoretically, such events cannot occur in practice. This can be seen as the price to be paid for having a simple specification, under assumption (A). Actually, this feature will no more be an issue when truncating these process above by one \(^{11}\). This will be done when we will extend our formulas by removing the assumption (A): see appendix B.

It should not be surprising that the Expected Loss \( EL(\cdot, T) \) process may decrease in time, for a given time horizon \( T \). Indeed, this process is partly related to some expectations of future losses and partly to the current realized losses. Future expected losses could become smaller tomorrow if the market participants become more confident concerning the financial strength of the borrowers in the pool. Moreover, in the ABS world, it is even possible to recover some "realized" losses in the future. Indeed, recording losses do not imply necessarily the closure of deals/tranches. These losses can be temporary, because they are based on some statistical models and projected cash-flows. Therefore, marked-To-market and marked-to-model losses can be recorded one day and recovered at least partly afterwards. These temporary losses is clearly a source of difference with corporate CDOs, for instance. To strengthen the latter effect, there exist sometimes some Excess Spread mechanisms, that play the role of an additional equity tranches. When some losses occur, they will be absorbed (at least partially) by the past and/or future Excess Spread amounts. For similar reasons, the outstanding nominal of the whole portfolio will not be a decreasing function in time necessarily.

3 Explicit pricing formulas via change of numeraire techniques

Instead of brute-force calculations of the expectations \( E_1 \) and \( E_2 \), it is simpler to invoke change of numeraire techniques: for every time \( s \), we will be working under the underlying \( s \)-Forward neutral probability \( Q_s \). Let us introduce the quantity
\[
\nu_{t, s, Q_s} = \rho \tilde{\sigma}(t, \tilde{s}) \tilde{\sigma}(t, s),
\]

\(^{10}\) This assumption could be weakened at the cost of additional technicalities.

\(^{11}\) at least with reasonable parameter values, and without pricing very high tranches into the capital structure.
that will appear several times in the sequel.

**Theorem 1** Under the assumptions (A), (E) and (IR), for every couple \((s, \bar{s})\), \(t \leq \bar{s} \leq s\), we have

\[
E_1(s) = B(t, s) \exp \left( \int_t^s \rho \tilde{\sigma}(u, s)\sigma(u, s)\, du \right) \text{Put} \,(EL(t, s), K^*_s, \sigma(., s), s - t),
\]

where \(\text{Put} \,(\text{Fwd, Strike, Vol, Maturity})\) denotes the usual Black-Scholes formula of an European Put (zero interest rates), and

\[
K^*_s = K \exp \left( - \int_t^s \nu_{u,s,Q} \, du \right).
\]

Moreover,

\[
E_2(s, \bar{s}) = B(t, s)EL(t, \bar{s}) \exp \left( \int_t^{\bar{s}} \rho \tilde{\sigma}(u, s)\sigma(u, \bar{s})\, du \right) \cdot \Phi \left( \left[ \ln \left( \frac{K}{EL(t, s)} \right) - \int_t^s \sigma(u, s)\sigma(u, \bar{s})\, du + \frac{1}{2} \int_t^s \sigma^2(u, s)\, du \right] / \left( \int_t^s \sigma^2(u, s)\, du \right)^{1/2} \right).
\]

Then, the risky level of the tranche \([0, K]\) with \(K < 1\), is

\[
RL_{t,K} = \int_t^T E_1(s)\, ds,
\]

and its default leg is

\[
DL_{t,K} \simeq \sum_{i=1}^P \left[ E_2(T_i, T_i) - E_2(T_i, T_{i-1}) \right].
\]

See the proof in appendix A. Note that the integrals from \(t\) and \(\bar{s}\) above can be considered equivalently from \(t\) and \(s\). Indeed, by assumption, \(\bar{s} \leq s\) and, obviously, \(\sigma(u, \bar{s}) = 0\) if \(u > \bar{s}\).

Now, we have to deal with the super-senior tranche. It is sufficient to evaluate the quantities

\[
E_1^*(s) = B(t, s)EL, \left[ (1 - EL(s, s) - A(s, s))^+ \right],
\]

and

\[
E_2^*(s, \bar{s}) = B(t, s)EL, \left[ 1 \{ EL(s, s) + A(s, s) \leq 1 \} \right] EL(\bar{s}, \bar{s})].
\]

Formally, these expressions will be calculated in appendix B, in a semi-analytical way. If we accept to neglect the likelihood of the event \(\{ EL(s, s) + A(s, s) > 1 \}\), then we can find true closed-form formulas.
Theorem 2 Under the assumptions (A), (E), (IR) and (AM), for every couple 
\((s, \bar{s}), t \leq \bar{s} \leq s\), we have 
\[
E^*_1(s) = B(t, s) \left[ 1 - A(t, s) \exp \left( \int_t^s \tau(u, s) \sigma(u, s) \, du \right) \right. \\
\left. - EL(t, s) \exp \left( \rho \int_t^s \sigma(u, s) \sigma(u, s) \, du \right) \right],
\]
(8)

\[
E^*_2(s, \bar{s}) = B(t, s) EL(t, \bar{s}) \exp \left( \rho \int_t^{\bar{s}} \sigma(u, \bar{s}) \sigma(u, s) \, du \right).
\]
(9)

Then, the risky level of the tranche 0 – 100% (the whole portfolio) is 
\[
RL_{t, 1} = \int_t^{T^*} E^*_1(s) \, ds,
\]
and its default leg is 
\[
DL_{t, 1} \simeq \sum_{i=1}^P \left[ E^*_2(T, T_i) - E^*_2(T, T_{i-1}) \right].
\]

See the proof in appendix A. Thus, for the particular case of the most senior tranche, we can apply theorem 2, particularly if we assume that the amortization process will never feed entirely this tranche before the considered maturity \(^{12}\). But we advise to use rather the formulas that will be expanded in appendix B, that do not imprison the amortized amounts in the most senior tranche.

Now, assume the current time is \(t\). For getting \(t\)-prices of tranches, it is just necessary to evaluate the \(t\)-spot Expected Losses \(EL(t, T)\) and, for the most senior tranche, the Expected amortized amount \(A(t, T)\). Since they are not observed directly in the market \(^{13}\), we have to make some additional assumptions concerning the shape of the \(t\)-current profile \(T \mapsto E(t, T)\). For instance, we could state that, for some constant rate \(\theta_t > 0\), we have 
\[
EL(t, dT) = \theta_t, E_t[O(T)]dT.
\]
This constant \(\theta_t\) is \(\mathcal{F}_t\)-measurable. It has the status of a constant default rate, even if it is related to some expectations of losses \(^{14}\). Note that the previous relation induces a feedback of losses towards the process \(A(., .)\).

Formally, we could deal with the amortization process as with the Expected Loss process itself. For the moment, we have just to evaluate \(A(t, T)\) knowing the information at time \(t\). We will assume that 
\[
A(t, dT) = [\xi_{t, T} + b_t], E_t[O(T)]dT,
\]
\(^{12}\)Indeed, the par spread of a tranche \([K, 1]\) will be given by equation (1).
\(^{13}\)they cannot be deduced from some market quotes directly
\(^{14}\)the constancy of \(\theta_t\) knowing \(\mathcal{F}_t\) can be discussed. More generally, we could imagine that \(\theta_t\) depends on \(t, T, O(T)\) and \(L(T)\).
where $\xi_{t,T}$ is the "theoretical" amortization rate at time $t$ for the $T$ maturity, and $b$ is a constant risk premium. The former quantity is the time $T$ rate of decrease in terms of notional, assuming there will be no prepayments. It can be inferred from a description of the cash-flows that are associated with the survival assets in the pool at time $t$. The latter quantity is the global risk premium associated with the prepayment process and the amortization process together. If $\xi_{t,u}$ were a constant, then $(\xi_{t,u} + b)$ would be the $t$-amortization rate, and $b$ could be seen as the Constant Prepayment Rate. The constancy of $b$ has just been made for convenience. It is straightforward to extend our results to deal with (deterministic) term structures of prepayment rates.

Since $E_t[O(T)] = 1 - A(t,T) - EL(t,T)$, we deduce

$$E_t[O(dT)] = -A(t,dT) - EL(t,dT) = -(\theta_t + b_t + \xi_{t,T})E_t[O(T)]dT,$$

and therefore

$$E_t[O(T)] = O(t) \exp \left( - (\theta_t + b_t)(T - t) - \int_t^T \xi_{t,u} \, du \right).$$

Finally, we get

$$EL(t,T) = EL(t,t) + O(t) \theta_t \int_t^T \exp \left( - (\theta_t + b_t)(u - t) - \int_t^u \xi_{t,v} \, dv \right) \, du.$$  \hspace{1cm} (10)

Thus, at the current time $t$, the Expected Loss depends only on the "no-default, no-prepayment" amortization profile and on some constants $\theta_t$ and $b_t$. We have replaced a whole unknown spot curve $EL(t,\cdot)$ by a parameterization of this curve, given by (10). Obviously, other choices of spot EL curves are possible, even a pure nonparametric point of view, if the data related to the pool are sufficiently rich. Note that equation (10) is not contradictory with (6).

Similarly, we deduce the spot amortization profile, i.e. the curve $A(t,\cdot)$:

$$A(t,T) = A(t,t) + O(t) \int_t^T (\xi_{t,u} + b_t) \exp \left( - (\theta_t + b_t)(u - t) - \int_t^u \xi_{t,v} \, dv \right) \, du.$$  \hspace{1cm} (10)

Thus, we have got all the building blocks to price standard tranches.

4 Pricing of a cash MBS/ABS tranche

Now, we are dealing with the most common case of MBS/ABS tranches. They are tranched products, but now, we have to manage additional cashflows that are related to coupon payments. Broadly speaking, they differ from the products in the previous section exactly like cash CDOs differ from synthetic CDOs. We

\footnote{Just replace the constant $b$ by a function $b(t,T)$ and consider integrals of $b(t,\cdot)$ instead of the product of $b_t$ times the remaining maturities.}
will assume that waterfalls can be assimilated to usual coupon-bearing cashflows. In other words, at some pre-specified payment dates, a fixed or floating coupon rate will be applied to the current outstanding notional of the related base tranche \([0, K]\). Then, under this assumption, we are able to calculate tranche prices by the expected cash flow method. Formally, MBS/ABS tranches will be assimilated to amortized bonds with random schedules \(^{16}\).

Consider a tranche \([0, K]\), as previously. Since the cash flows are due to coupons payments or principal repayments, the "ABS bond" price at time \(t\) is given by

\[
P_t = E_t \left[ \sum_{i \leq p, T_i \geq t} \exp(- \int_t^{T_i} r_u \, du) \{C_{T_i} \Delta_i O_K(T_i - 1) + \mathbb{1}(K \geq L(T_i - 1) + A_K(T_i - 1)) \cdot [A_K(T_i) - A_K(T_i - 1)] \} \right. \\
+ \left. \exp(- \int_t^{T_p} r_u \, du) O_K(T_p) \right],
\]

where \(C_{T_i}\) is the fixed or floating coupon associated with that tranche \(^{17}\), and \(\Delta_i\) is the coverage between \(T_{i-1}\) and \(T_i\). The final date \(T_p\) is not larger than the final legal maturity \(T^*\) of the deal (the largest maturity among the underlying loans). It can be considered as an expected call date.

First, let us deal with the simple fixed coupon case.

**Assumption (FC):** For every date \(T_i\), the related coupon rate is the same constant \(C_0\). In other words, \(C_{T_i} = C_0\) for all index \(i\).

Let us consider the case of fixed coupon bonds (or tranches) that are not impacted by amortization/repayments during the whole life of the deal, except at maturity (assumption (A)). To evaluate the bond price, it is sufficient to calculate the expectations

\[
F_1(s, \bar{s}) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) O_K(\bar{s}) \right],
\]

for every times \((s, \bar{s}), t \leq \bar{s} \leq s \leq T^*\). Note that \(F_1(s, s)\) is exactly the so-called previous equation (4). With similar arguments as previously, we get easily the following result.

**Theorem 3** Under the assumptions (A), (E), (IR) and (FC), the cash bond price is given by

\[
P_t = \sum_{i \leq p, T_i \geq t} C_0 \Delta_i F_1(T_i, T_i - 1) + E_1(T_p),
\]

\(^{16}\)As a consequence, we do not cover explicitly some path-dependent features, for instance credit triggers that would change the order of priority among tranche repayments. To be specific, the single correlation parameter between interest rate moves and expected losses is clearly not sufficient to model conveniently such features. This will provide avenues for further extensions of the current framework.

\(^{17}\)Most of the time, its value is known at the fixing date just before, i.e. at \(T_{i-1}\).
where $E_1$ had been defined in theorem 1 and, for every couple $(s, \bar{s})$, $t \leq \bar{s} \leq s$,

$$F_1(s, \bar{s}) = B(t, s) \exp \left( \int_{t}^{\bar{s}} \nu_{u, \bar{s}, Q_u} \, du \right) \cdot \text{Put} \left( EL(t, \bar{s}), K \exp \left( - \int_{t}^{\bar{s}} \nu_{u, \bar{s}, Q_u} \, du \right), \sigma(u, \bar{s}), \bar{s} - t \right).$$

To deal with the most senior tranche, we have to take into account the amortization process too. To get simple formulas, we assume in the next theorem that the likelihood of the events \{${E}L(s, s) + {A}(s, s) > 1$\} is negligible (w.r.t. the bond price itself). Thus, by mimicking theorem 2, we get easily:

**Theorem 4** Under the assumptions (A), (E), (IR), (AM) and (FC), the cash bond price of the whole portfolio (i.e. the tranche $[0, 100\%]$) is given by

$$P_T = \sum_{i, T_i \geq t, i \leq p} \{C_0 \Delta_i {E}_3^*(T_i, T_{i-1}) + {A}_1^*(T_i, T_i) - {A}_1^*(T_i, T_{i-1})\} + E_1(T_p),$$

where, for every couple $(s, \bar{s})$, $t \leq \bar{s} \leq s$,

$$E_3^*(s, \bar{s}) = B(t, s) \left[ 1 - A(t, \bar{s}) \exp \left( \int_{t}^{\bar{s}} \tau(u, \bar{s})\bar{\sigma}(u, s) \, du \right) \right. \left. - {E}L(t, \bar{s}) \exp \left( \rho \int_{t}^{\bar{s}} \sigma(u, \bar{s})\bar{\sigma}(u, s) \, du \right) \right], \quad (11)$$

and

$$A_1^*(s, \bar{s}) = B(t, s)A(t, \bar{s}) \exp \left( \int_{t}^{\bar{s}} \tau(u, \bar{s})\bar{\sigma}(u, s) \, du \right). \quad (12)$$

Let us now deal with the trickier case of floating rate bonds. To fix the ideas, let us assume that the coupon rate is Libor $^{18}$.

**Assumption (FlC):** At every date $T_i$, the paid coupon rate corresponds to the Libor rate that had been fixed at the previous coupon date, and is related to the same periodicity. In other words, $C_{T_i} = L(T_{i-1}, T_i)$.

Now, we have now to evaluate expressions like

$$F_2(s, \bar{s}) = E_t \left[ \exp \left( - \int_{t}^{s} r_u \, du \right) L(\bar{s}, s) O_K(\bar{s}) \right],$$

where, as usual, $\bar{s} \leq s$ and $\Delta$ is the coverage between $\bar{s}$ and $s$. In the appendix A, we prove that:

$^{18}$Adding a constant margin to this rate would be straightforward by applying theorems 3 and 4.
Theorem 5 Under the assumptions (A), (E), (IR) and (FlC), the cash bond price is given by

\[ P_t = \sum_{i \leq p, T_i \geq t} \Delta_i F_2(T_i, T_{i-1}) + E_1(T_p), \]

where, for every couple \((s, \bar{s})\), \(t \leq \bar{s} \leq s\),

\[ F_2(s, \bar{s}) = F_1(\bar{s}, \bar{s}) - F_1(s, \bar{s}). \]

Moreover, \(E_1\) and \(F_1\) have been defined in theorems 1 and 3 respectively.

Moreover, by the same reasoning, we deal easily with the last tranche.

Theorem 6 Under the assumptions (A), (E), (IR), (AM) and (FlC), the cash bond price of the whole portfolio (i.e. the tranche \([0, 100\%]\)) is given by

\[ P_t = \sum_{i, T_i \geq t, i \leq p} \{ \Delta_i F_2^*(T_i, T_{i-1}) + A_1^*(T_i, T_i) - A_1^*(T_i, T_{i-1}) \} + E_1(T_p), \]

where, for every couple \((s, \bar{s})\), \(t \leq \bar{s} \leq s\), we have

\[ F_2^*(s, \bar{s}) = E_3^*(\bar{s}, \bar{s}) - E_3^*(s, \bar{s}). \]

Remind that \(A_1^*\) and \(E_3^*\) have been defined in theorem 4.

We have got closed-form formulas for pricing cash MBS/ABS tranches. Note that the formulas above are relevant for pricing all the tranches under the so-called assumption (A). It is possible to remove the latter assumption and to get semi-analytical formulas: see in the appendix B.

5 Illustration: pricing of a real structured product

For the sake of illustration, we consider now a real recent trade in the light of our valuation model, under the assumption (A). It is a synthetic European RMBS with total issued notional around 2 billion euros. The maturity of the structure will be assumed 5 years, even if it is a call date\(^{19}\). The underlying loan portfolio has been tranched into six slices, with detachment points \(1\%, 3\%, 5\%, 7\%, 10\%\) and \(100\%\)\(^{20}\). A very wide super senior tranche is typical of such structures. As usual, default losses will be reimbursed from the top and default losses recorded from the bottom.

For the moment, assume we want to price and risk manage some of these tranches at inception. Thus, no loss has yet been recorded, and no amortization

\(^{19}\text{In other words, we do not try to evaluate the price of the embedded call option.}\)

\(^{20}\text{Actually, the true detachment points are slightly different.}\)
has occurred in the underlying pool. In practice, model parameters should be calibrated to some observed tranche prices of the current structure or of similar structures. Since the liquidity of such deals is very limited, it is usual to invoke some trader inputs, that would reflect his/her expectations in the light of historical data or recent trends typically. In our framework, we would like to guess which parameters are the most crucial ones in terms of calibration. To this goal, we calculate the present value impacts of some parameter changes. To simplify, we have assumed that all the volatilities we consider are constant in time. We have led a sensitivity analysis of tranche prices (in terms of par spreads) with respect to some input parameters: volatility of the Expected Loss, value of the default rate \( \theta \), of the correlation \( \rho \). Figures 1, 2 and 4 summarize the results. The reference set of parameters is the following: \( \theta = 0.4\% \), \( b = 0.5\% \), \( \rho = 30\% \), \( \sigma_0 = 85\% \), \( \bar{\sigma}_0 = 1\% \), \( \tau = 25\% \).

The most junior part of the capital structure beneficiates from high Expected Loss volatilities (figure 1). Indeed, in a fully deterministic framework, such first thin tranches would be entirely fed with our default rate assumption. Thus, more uncertainty concerning realized losses is good news for the owners of such tranches (all other things being equal). Obviously, it is the opposite for the owners of the most senior tranches. Mezzanine tranche profiles are humped, meaning they behave rather like senior tranches for low expected loss volatilities, and rather like equity ones for higher volatilities.

As expected, par spreads are monotonically increasing functions of default rates (figure 2). Its relation is almost linear, at least for a wide range of realistic default rates, and even for the most junior tranches. The latter effect is due to the trade-off between risky duration and expected loss: with higher default rates, the tranche expected loss is capped when its risky level will decrease, so increasing its par spread. For all the tranches (not only the most senior tranche under the assumption (A)), par spreads decrease when prepayment rates increase: figure 3. Indeed, the spot expected loss curves depend on this coefficient. Quicker the repayment process, smaller the expected losses in the whole portfolio. This phenomenon can be observed with all the tranches. In the case of the most senior tranche, its risky level is reduced significantly by increasing prepayment rates, cancelling almost of the latter expected loss effect. That is why, strangely, the super senior tranche is the less sensitive tranche w.r.t varying prepayment rates.

Moreover, except for the most senior tranche, the effect of the correlation between amortization/interest rates and default losses is very weak (figure 4). Indeed, under our assumption (A), these tranches are not hit by the amortization process. Thus, the latter correlation has an influence through the not risky discount factors only. Clearly, this induces a lot smaller effect on prices than a reduction of tranche notionals. At the opposite, the most senior tranche par spread is an increasing function of the correlation significantly: high correlations increase the likelihood of higher tranche reduction through the joint effect of
Figure 1: Par spreads of RMBS tranches as a function of Expected Loss volatility
Figure 2: Par spreads of RMBS tranches as a function of the default rate $\theta$
Figure 3: Par spreads of RMBS tranches as a function of the prepayment rate
Figure 4: Par spreads of RMBS tranches as a function of correlation $\rho$
default losses and quicker amortization. The investor requires a higher premium to be covered against the latter risk.

Now, assume that some losses have already been recorded: see figure 5. Here, the most junior tranches have already been fully or partially fed, but, since losses can be recovered 22, the processus can be reversed. Thus, even when realized losses are higher than 20%, the model spread of the tranche [0,1%] is not zero. Actually, it is around 2% of the remaining notional, that will be zero most of the time! This value can be interpreted as the probability to recover some part of the most junior tranche before maturity. Note the humped shapes of the par spreads of most tranches: for a given tranche, the associated par spreads reach their maxima when the realized losses hit that tranche. After, as explained above, par spreads decrease but do not reach zero. Note that the super senior tranche does not show such a profile, because only unrealistic large realized losses could possibly illustrate this phenomenon for this tranche.

6 Conclusion

Our goal was to build a model for pricing and for leading some relative value analysis of standard MBS/ABS products and of structures on these underlyings (CDO of ABS particularly). Concerning the latter products, and more generally structures that convey amortization and default risks simultaneously, the related markets are not as liquid as for interest rate or equity derivatives, most of the time. Thus, it would not be reasonable to propose models with a lot of parameters. There would induce a high risk of "overfitting", and some price could be paid in terms of risk management (poor deltas). Our specification appears to be a good compromise. We have taken into account the strong correlation between the dynamics of interest rates and the basket loss process, and even more between the dynamics of interest rates and the prepayment process in a realistic way. We are convinced that the assumption of independence between credit events and interest rates, so usual in the credit area, was not realistic here and we have proposed a tractable alternative. Closed-form formulas are highly valuable: they provide benchmarks without having to build huge IT infrastructure (for retrieving loan information, simulating numerous random factors and revaluating portfolios thousands of times). Finally, we have tried to keep a balance between the likelihood of the hypothesis, the number of the underlying random factors and the calibration issues.

The current framework can be extended towards several directions: alternative specifications in terms of the underlying processes or in terms of spot EL and A curves, addition of more random factors and correlation levels, comparison between several default/prepayment intensities assumptions, inclusion of some triggers or excess spread mechanisms etc. Particularly, an avenue for further research would be to replace the randomness of the amortized amount

22 This point is a particularity of the ABS sector w.r.t. corporate-based structures. And here, this point is integrated in the model through the diffusion specification.
Figure 5: Par spreads of RMBS tranches as a function of the current loss
A by the randomness of the prepayment rates \( b \) and of the "natural" amortization rate \( \xi_{t,T} \). Then, we would switch towards a three factor model instead of our current two factor model. But it would become more difficult to manage lognormal specifications, as it is well-known.

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References


A Proofs of the pricing formulas under the assumption (A)

A.1 Proof of theorem 1.

Under the $s$-forward neutral probability $Q_s$, the discount factor $(B(t, s))_{t, t \leq s}$ process is the numeraire and we get, for all the tranches except the most senior one,

$$E_1(s) = B(t, s)E_{t, Q_s}\left[(K - EL(s, s))^+\right],$$

and

$$E_2(s, \bar{s}) = B(t, s)E_{t, Q_s}\left[\{EL(s, s) \leq K\}EL(\bar{s}, \bar{s})\right].$$

A technical issue is coming from the fact we change one probability into another one every time. Formally, the processes $(EL(t, T))$ have not the same laws under all these probabilities. Actually, only the drifts are changing. It is possible to state explicitly all these drifts. Remind that the drift of $(EL(\cdot, T))$ under the risk neutral measure $Q$ is zero. By classical arguments (Brigo and Mercurio (2001)), we get: Under the $Q_s$ Forward measure, the drift of the process $EL(t, T)$ is given by

$$-\rho EL(t, T)\sigma(t, T) (\sigma_Q(t) - \bar{\sigma}(t, s)),$$

where $\sigma_Q$ is the volatility of the usual numeraire. Since this usual numeraire is the money market account, it has no volatility. Thus, $\sigma_Q(t) = 0$ and, for every $t, T$, the latter drift is

$$EL(t, T)\nu_{t, T, Q_s} = \rho EL(t, T)\sigma(t, T)\bar{\sigma}(t, s).$$

Therefore, under any Forward neutral probability $Q_s$, the Expected Loss processes are still lognormal:

$$EL(dt, T) = EL(t, T)(\nu_{t, T, Q_s}dt + \sigma(t, T)dW_t).$$

To evaluate the expectation (13), we can invoke the usual Black-Scholes formula:

$$E_1(s) = B(t, s)E_{t, Q_s}\left[(K - EL(s, s))^+\right]$$

$$= B(t, s)E_{t, Q_s}\left[K - EL(t, s) \exp\left(\int_t^s \nu_{u, s, Q_s} du - \int_t^s \sigma^2(u, s) du/2 + \int_t^s \sigma(u, s)dW_u\right)\right]^+$$

$$= B(t, s) \exp\left(\int_t^s \nu_{u, s, Q_s} du\right)\cdot E_{t, Q_s}\left[K_s^* - EL(t, s) \exp\left(-\int_t^s \sigma^2(u, s) ds/2 + \int_t^s \sigma(u, s)dW_u\right)\right]^+.$$

So, we get the formula for $E_1(s)$. 

28
For dealing with $E_2(s, \bar{s})$, choose now the numeraire $EL(\cdot, \bar{s})$. Under the probability $Q^E_{\bar{s}}$ that is induced by this new numeraire,

$$E_2(s, \bar{s}) = B(t, s) E_{t, Q^E_{\bar{s}}}[EL(\bar{s}, \bar{s}), E_{t, Q^E_{\bar{s}}} \{EL(s, s) \leq K\}],$$
or equivalently

$$E_2(s, \bar{s}) = B(t, s) EL(t, \bar{s}) \exp \left( \int_t^{\bar{s}} \nu_{u, \bar{s}, Q^E_{\bar{s}}} \, du \right) E_{t, Q^E_{\bar{s}}}[1 \{EL(s, s) \leq K\}].$$

But, under the probability $Q^E_{\bar{s}}$, the Expected Loss processes $EL(\cdot, s)$ follows the diffusion equation:

$$EL(dt, s) = EL(t, s) (\sigma(t, s)\sigma(t, \bar{s})dt + \sigma(t, s)dW_t).$$

Thus, we get an explicit expression for $E_2(s, \bar{s})$. □

A.2 Proof of theorem 2.

Under (A), we have

$$E^*_1(s) = B(t, s) E_{t, Q^s}[1 - EL(s, s) - A(s, s)],$$

and

$$E^*_2(s, \bar{s}) = B(t, s) E_{t, Q^s}[EL(\bar{s}, \bar{s})].$$

By our change of measure, the processes $(EL(t, T))$ and $(A(t, T))$ are no more martingales under the new measures. Concerning the Expected Loss process, we had found the $Q_s$-drift change (see equation (14)). This implies

$$E_{t, Q^s}[EL(\bar{s}, \bar{s})] = \exp \left( \int_t^{\bar{s}} \nu_{u, \bar{s}, Q^s} \, du \right) = \exp \left( \int_t^{\bar{s}} \sigma(u, \bar{s})\bar{\sigma}(u, s) \, du \right).$$

Similarly, we can deal with the "amortization" process $A(\cdot, s)$ as with the Expected Loss process. Thus, we get

$$E_{t, Q^s} [A(s, s)] = A(t, s) \exp \left( \int_t^s \tau(u, s)\bar{\sigma}(u, s) \, du \right),$$

so the result. □

A.3 Proof of theorem 5.

Under the $Q_s$ Forward measure, the drift of the process $EL(t, \bar{s})$ is given by $
u_{t, \bar{s}, Q_s}$, with our previous notations. By some standard conditional expectation
arguments, we get
\[
F_2(s, \bar{s}) = E_t \left[ \exp \left( -\int_t^\bar{s} r_u \, du \right) B(\bar{s}, s) L(\bar{s}, s) \Delta(\bar{s}, s) O_K(\bar{s}) \right]
\]
\[
= E_t \left[ \exp \left( -\int_t^\bar{s} r_u \, du \right) B(\bar{s}, s) \left\{ \frac{1}{B(\bar{s}, s)} - 1 \right\} O_K(\bar{s}) \right]
\]
\[
= E_t \left[ \exp \left( -\int_t^\bar{s} r_u \, du \right) \left\{ 1 - B(\bar{s}, s) \right\} O_K(\bar{s}) \right]
\]
\[
= F_{1,1}(\bar{s}, \bar{s}) - E_t \left[ \exp \left( -\int_t^\bar{s} r_u \, du \right) E_{\bar{s}} \left[ \exp \left( -\int_t^\bar{s} r_u \, du \right) O_K(\bar{s}) \right] \right]
\]
\[
= F_{1,1}(\bar{s}, \bar{s}) - F_{1,1}(s, \bar{s}). \square
\]

\section*{B Semi-analytical formulas without the assumption (A)}

In this appendix, we extend our formulas to deal with the general case, i.e. without assuming (A). Now, the amortization process can be related to any tranche, possibly the most senior one. Closed-form formula are no more available, but we can rely on semi-analytical formulas instead. Broadly speaking, the method is simple: conditionally on the value of the amortization process at some time horizon, the "base case" formulas apply, by shifting the relevant strikes. Then, integration w.r.t. the law of the expected amortized amount provides the result.

Let us consider first the previous synthetic structure and the evaluation of risky levels and default legs of all tranches (including the most senior one). Recall that the fair spread of the tranche \([0, K]\) is given by the relation
\[
s_{1,j} \{ RL_{t,K_j} - RL_{t,K_{j-1}} \} = DL_{t,K_j} - DL_{t,K_{j-1}}
\]
where risky levels are defined by
\[
RL_{t,K} = E_t \left[ \int_t^{T^*} \exp \left( -\int_t^s r_u \, du \right) O_K(s) \, ds \right],
\]
and default legs by
\[
DL_{t,K} = E_t \left[ \int_t^{T^*} \exp \left( -\int_t^s r_u \, du \right) 1(L(s) + A_K(s) \leq K)L(ds) \right].
\]

Thus, we need to evaluate quantities like
\[
E_1(s) = E_t \left[ \exp \left( -\int_t^s r_u \, du \right) (K - EL(s, s) - [A(s, s) - 1 + K]^+)^+ \right],
\]
and
\[
E_2(s, \bar{s}) = E_t \left[ \exp \left( -\int_t^s r_u \, du \right) 1\{EL(s, s) + [A(s, s) - 1 + K]^+ \leq K\} EL(\bar{s}, \bar{s}) \right],
\]
for every couple \((s, \bar{s})\), \(t \leq \bar{s} \leq s \leq T^*\). Actually, we will calculate first the quantity

\[
\mathcal{F}_1(s, \bar{s}) = E_t \left[ \exp \left( -\int_t^s r_u \, du \right) \left( K - EL(\bar{s}, \bar{s}) - [A(\bar{s}, \bar{s}) - 1 + K]^+ \right) \right],
\]

when \(\bar{s} \leq s\). Indeed, note that \(\mathcal{E}_1(s) = \mathcal{F}_1(s, s)\). Moreover, \(\mathcal{F}_1\) is the same as the so-called term \(F_1\) that had been calculated in section 4 under the assumption (A), and we will need \(\mathcal{F}_1\) for pricing coupon-bearing securities hereafter.

To fix the ideas, at time \(t\), the event \(A(\bar{s}, \bar{s}) = a\) will be identical to \(\int_t^s \tau(u, \bar{s}) \, dW_u = w(a)\), for some value \(w(a)\) that will depend on the spot curve \(A(t, \cdot)\). Clearly,

\[
\mathcal{F}_1(s, \bar{s}) = B(t, s) E_{t, Q}\left[ (K - EL(\bar{s}, \bar{s}) - [A(\bar{s}, \bar{s}) - 1 + K]^+) \right]
= B(t, s) E_{t, Q}\left[ E_{t, Q_s}\left[ (K - EL(\bar{s}, \bar{s}) - [a - 1 + K]^+) \big| A(\bar{s}, \bar{s}) = a\right] \right],
\]

and the conditional expectation can be evaluated easily. Here, the conditioning event is

\[
a = A(t, \bar{s}) \exp \left( \int_t^{\bar{s}} \sigma(u, s) \tau(u, \bar{s}) \, du - \frac{1}{2} \int_t^{\bar{s}} \tau^2(u, \bar{s}) \, du + \int_t^{\bar{s}} \tau(u, \bar{s}) \, dW_u \right),
\]

or equivalently

\[
\int_t^{\bar{s}} \tau(u, \bar{s}) \, dW_u = w(a).
\]

But we can decompose

\[
\int_t^{\bar{s}} \sigma(u, \bar{s}) \, dW_u = \xi_1 \int_t^{\bar{s}} \tau(u, \bar{s}) \, d\tilde{W}_u + \varepsilon,
\]

where

\[
\xi_1 = \frac{\rho \int_t^{\bar{s}} \sigma(u, \bar{s}) \tau(u, \bar{s}) \, du}{\int_t^{\bar{s}} \tau^2(u, \bar{s}) \, du}, \quad (15)
\]

and \(\varepsilon \sim \mathcal{N}(0, \mu_1^2)\), by setting

\[
\mu_1^2 = \int_t^{\bar{s}} \sigma^2(u, \bar{s}) \, du - \xi_1^2 \int_t^{\bar{s}} \tau^2(u, \bar{s}) \, du. \quad (16)
\]

Implicitly, note that the variance of \(\varepsilon\) depends on all the underlying volatility functions and arguments. Moreover, \(\mu_1^2\) is really positive, due to the Cauchy-Schwartz inequality. Thus, under \(Q_s\) and conditionally on \(A(\bar{s}, \bar{s}) = a\), the random variable \(EL(\bar{s}, \bar{s})\) can be written

\[
EL(\bar{s}, \bar{s}) = EL(t, \bar{s}) \exp \left( \int_t^{\bar{s}} \rho \sigma(u, \bar{s}) \bar{\sigma}(u, s) \, du - \frac{1}{2} \int_t^{\bar{s}} \sigma^2(u, \bar{s}) \, du + \xi_1 w(a) + \varepsilon \right),
\]

\[31\]
and

\[ E_{Q_s}\left( \left[ K - EL(s, \bar{s}) - [a - 1 + K]^+ \right] \right) \]

\[ = \text{Put} \left( \frac{EL(t, \bar{s}) \exp \left( \int_t^\top \rho \sigma(u, \bar{s}) \sigma(u, s) \ du - \frac{1}{2} \int_t^\top \sigma^2(u, \bar{s}) \ du + \xi_1 w(a) + \frac{1}{2} \mu_1^2 \right)}{\left( \int_t^\top \tau^2(u, \bar{s}) \ du \right)^{1/2}} \right) 1(K \geq [a - 1 + K]^+). \]

It is sufficient to integrate the latter formula w.r.t. the r.v. \( \int_t^\top \tau(u, \bar{s}) \ d\bar{W}_u \) to get the result:

**Theorem 7** Under (E), (IR) and (AM),

\[ F_1(s, \bar{s}) = B(t, s) \int \text{Put} \left( EL_u, K - [a_1(w) - 1 + K]^+, \mu_1, \bar{s} - t \right) \phi \left( \frac{w}{\left( \int_t^\top \tau^2(u, \bar{s}) \ du \right)^{1/2}} \right) 1(K \geq [a_1(w) - 1 + K]^+) dw, \tag{17} \]

where

\[ EL_u := EL(t, \bar{s}) \exp \left( \int_t^\top \rho \sigma(u, \bar{s}) \sigma(u, s) \ du - \frac{1}{2} \int_t^\top \sigma^2(u, \bar{s}) \ du + \xi_1 w + \frac{1}{2} \mu_1^2 \right), \]

where \( \xi_1 \) (resp. \( \mu_1 \)) are given by (15) (resp. (16)) and

\[ a_1(w) := A(t, \bar{s}) \exp \left( \int_t^\top \sigma(u, \bar{s}) \tau(u, s) \ du - \frac{1}{2} \int_t^\top \tau^2(u, \bar{s}) \ du + w \right). \]

Similarly and by leading the same change of numeraires as in theorem 1, we get

\[ E_2(s, \bar{s}) = B(t, s) EL(t, \bar{s}) \exp \left( \int_t^\top \nu_{u, \bar{s}, Q_s} \ du \right). E_{t, Q_s} \left[ 1\{EL(s, s) + [A(s, s) - 1 + K]^+] \leq K \} \right] \]

\[ = B(t, s) EL(t, \bar{s}) \exp \left( \int_t^\top \nu_{u, \bar{s}, Q_s} \ du \right) \cdot E_{t, Q_s} \left[ E_{t, Q_s} \left[ 1\{EL(s, s) + [a_2(w) - 1 + K]^+] \leq K \} \int_t^\top \tau(u, s) \ dW_u = w \right] \right] \]

where

\[ a_2(w) = A(t, s) \exp \left( \int_t^\top \rho \sigma(u, \bar{s}) \sigma(u, s) \ du - \frac{1}{2} \int_t^\top \tau^2(u, \bar{s}) \ du + w \right). \tag{18} \]

Note that the latter function is slightly different from the previous one \( a_1(w) \). Indeed, under \( Q_s^\top \), the instantaneous drift of \( A(t, s) \) is now proportional to \( \rho \sigma(t, \bar{s}) \tau(t, s) \). We deduce

\[ E_{t, Q_s} \left[ 1\{EL(s, s) + [a_2(w) - 1 + K]^+] \leq K \} | A(s, s) = a_2(w) \} \right] \]

\[ = 1(K \geq [a_2(w) - 1 + K]^+). \Phi \left( [\ln(K - [a_2(w) - 1 + K]^]) \right) \]

\[ - \ln EL(t, s) - \int_t^\top \sigma(u, s) \sigma(u, \bar{s}) \ du + \frac{1}{2} \int_t^\top \sigma^2(u, s) \ du - \xi_2 w \}/\mu_2, \]

32
where
\[ \xi_2 = \frac{\int_t^s \sigma(u,s)\tau(u,s)\,du}{\int_t^s \tau^2(u,s)\,du} \]  
(19)
and
\[ \mu_2^2 = \int_t^s \sigma^2(u,s)\,du - \xi_2^2 \int_t^s \tau^2(u,s)\,du. \]  
(20)

Therefore,

**Theorem 8** Under (E), (IR) and (AM), we have

\[ \mathcal{E}_2(s,\bar{s}) = B(t,s)EL(t,\bar{s}) \exp \left( \int_t^{\bar{s}} \nu_{u,\bar{s},Q_s} \,du \right) \left( \int \Phi \left( \frac{1}{\mu_2} \left\{ \ln(K - [a_2(u) - 1 + K]^+) - \ln EL(t,s) - \int_t^s \sigma(u,s)\sigma(u,\bar{s})\,du \right\} + \frac{1}{2} \int_t^s \sigma^2(u,s)\,du - \xi_2^2 \right) \cdot \phi \left( \frac{w}{(\int_t^s \tau^2(u,s)\,du)^{1/2}} \right) \frac{1(K \geq [a_2(u) - 1 + K]^+)dw}{(\int_t^s \tau^2(u,s)\,du)^{1/2}}, \right) \]

where \(a_2, \xi_2\) and \(\mu_2\) are defined by (18), (19) and (20) respectively.

**Corollary 1** Let us consider a base tranche \([0,K]\), \(K \in [0,1]\), of a synthetic ABS structure. Under the assumptions of theorems 7 and 8, its risky level is

\[ RL_{t,K} = \int_t^{T^*} \mathcal{F}_1(s,s)\,ds, \]

and its default leg is

\[ DL_{t,K} \simeq \sum_{i=1}^{p} [\mathcal{E}_2(T_i,T_i) - \mathcal{E}_2(T_i,T_{i-1})]. \]

To extend fully the results of the previous sections, it remains to tackle the case of cash structures. The next to last missing building block is

\[ \mathcal{F}_2(s,\bar{s}) = E_t \left[ \exp \left( - \int_t^s r_u \,du \right) L(\bar{s},s)O_K(\bar{s}) \right], \]

This term is necessary to calculate cash ABS structure prices. But note that, invoking the same arguments as in the proof of theorem 5, we have

\[ \mathcal{F}_2(s,\bar{s}) = B(t,\bar{s})E_{t,Q_s} [O_K(\bar{s})] - B(t,s)E_{t,Q_s} [O_K(s)] = \mathcal{F}_1(s,\bar{s}) - \mathcal{F}_1(s,\bar{s}). \]

The last missing building blocks are

\[ \mathcal{A}_1(s,\bar{s}) = E_t \left[ \exp \left( - \int_t^s r_u \,du \right) \mathbf{1}\{EL(\bar{s},\bar{s}) + [A(\bar{s},\bar{s}) - 1 + K]^+ \leq K\} A_K(s,s) \right], \]
Under (E), (IR) and (AM), we have

\[ \mathcal{A}_2(s, \bar{s}) = E_t \left[ \exp \left( - \int_t^s r_u \, du \right) 1 \{ EL(s, \bar{s}) + [A(\bar{s}, s) - 1 + K]^+ \leq K \} \mathcal{A}_K(s, \bar{s}) \right], \]

for every couples \((s, \bar{s})\), \(t \leq \bar{s} \leq s \leq T^*\). After some tedious calculations, it can be proved that:

**Theorem 9** Under (E), (IR) and (AM), we have

\[ \mathcal{A}_1(s, \bar{s}) = B(t, s) \int \Phi \left( \frac{1}{\mu_3} \left[ \ln(K - [a_3(w) - 1 + K]^+) - \ln EL(t, \bar{s}) - \int_t^s \rho \sigma(u, \bar{s}) \sigma(u, s) \, du \right. \right. \]
\[ + \frac{1}{2} \int_t^s \sigma^2(u, \bar{s}) \, du - \xi_3 w - \xi_4 \bar{w} \Big) \cdot \phi_{\rho}( \frac{w}{\sigma_{\bar{s}}}, \frac{\bar{w}}{\sigma_s} ) \]
\[ \left. \cdot [a_4(\bar{w}) - 1 + K]^+ 1(K \geq [a_3(w) - 1 + K]^+) \right] \, dw \, d\bar{w}, \]

where \(\phi_{\rho}\) denotes the density of a bivariate random vector of standard Gaussian r.v. with correlation parameter \(\rho\), and where we have set

\[ \xi_3 := \int_t^s \sigma(.)(\bar{s}) \tau(.)(\bar{s}) \cdot \int_t^s \tau(\bar{s})(.)(\bar{s}) - \int_t^s \sigma(\bar{s})(.) \tau(\bar{s})(.) \cdot \int_t^s \tau(\bar{s})(.) \tau(\bar{s})(.) \bigg| \],
\[ \xi_4 := \int_t^s \sigma(.)(\bar{s}) \tau(.)(\bar{s}) \cdot \int_t^s \tau(\bar{s})(.)(\bar{s}) - \int_t^s \sigma(\bar{s})(.) \tau(\bar{s})(.) \cdot \int_t^s \tau(\bar{s})(.) \tau(\bar{s})(.) \bigg| \],
\[ \mu_3^2 := \int_t^s \sigma^2(\bar{s})(.) \cdot \xi_3^2 \int_t^s \tau^2(\bar{s})(.) - \xi_4^2 \int_t^s \tau^2(\bar{s})(.) - 2\xi_3 \xi_4 \int_t^s \tau(\bar{s})(.) \tau(\bar{s})(.) , \]
\[ a_3(w) := A(t, \bar{s}) \exp \left( \int_t^s \tau(u, \bar{s}) \sigma(u, s) \, du - \frac{1}{2} \int_t^s \tau^2(u, \bar{s}) \, du + w \right), \]
\[ a_4(\bar{w}) := A(t, s) \exp \left( \int_t^s \tau(u, \bar{s}) \sigma(u, s) \, du - \frac{1}{2} \int_t^s \tau^2(u, \bar{s}) \, du + \bar{w} \right), \]
\[ \sigma^2 := \int_t^s \tau^2(u, s) \, du, \quad \rho := \left( \int_t^s \tau^2(\bar{s})(.) \int_t^s \tau^2(\bar{s})(.) \right)^{1/2} \].

Note that the previous term \(\mathcal{A}_1\) involves a two-dimensional integration. At the opposite, the term \(\mathcal{A}_2(s, \bar{s})\) is simpler and similar to \(\mathcal{E}_2\). We get easily
Theorem 10 Under (E), (IR) and (AM), we have

\[ A_2(s, \bar{s}) = \int B(t, s) \Phi \left( \frac{1}{\mu} \{ \ln(K - |a_5(w) - 1 + K|) - \ln{\mathrm{EL}(t, \bar{s})} - \int_{t}^{\bar{s}} \rho \sigma(u, \bar{s}) \sigma(u, s) \, du \right)
\]

\[ + \frac{1}{2} \int_{t}^{\bar{s}} \sigma^2(u, \bar{s}) \, du - \xi_5 w \}
\]

\[ \cdot \{ a_5(w) - 1 + K \}^+ \frac{1}{\sigma_{\bar{s}}} \left( K \geq |a_5(w) - 1 + K| \right) \} \, dw, \]

where

\[ \xi_5 = \frac{\int_{t}^{\bar{s}} \sigma(\cdot, \bar{s}) \tau(\cdot, \bar{s})}{\int_{t}^{\bar{s}} \tau^2(\cdot, \bar{s})}, \quad \mu_5^2 = \int_{t}^{\bar{s}} \sigma^2(\cdot, \bar{s}) - \xi_5^2 \int_{t}^{\bar{s}} \tau^2(\cdot, \bar{s}), \]

and

\[ a_5(w) = A(t, \bar{s}) \exp \left( \int_{t}^{\bar{s}} \bar{\sigma}(\cdot, s) \tau(\cdot, \bar{s}) - \frac{1}{2} \int_{t}^{\bar{s}} \tau^2(\cdot, \bar{s}) + w \right). \]

Thus, the previous theorems allow us to evaluate cash structures, as described in section 4.

Corollary 2 Under the assumptions (E), (IR), (AM) and (FC) and with our previous notations, the cash bond price of section 4 is

\[ P_t = \sum_{i, T_i \geq t} \{ C_0 \Delta_i F_1(T_i, T_{i-1}) + A_1(T_i, T_{i-1}) - A_2(T_i, T_{i-1}) + \mathcal{E}_1(T_p) \}. \]

Under the assumptions (E), (IR), (AM) and (FC), the related cash bond price is

\[ P_t = \sum_{i, T_i \geq t} \{ \Delta_i F_2(T_i, T_{i-1}) + A_1(T_i, T_{i-1}) - A_2(T_i, T_{i-1}) + \mathcal{E}_1(T_p) \}. \]