



The limits of granularity adjustments



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ABSTRACT

We provide a rigorous proof of granularity adjustment (GA) formulas to evaluate loss distributions and risk measures (value-at-risk) in the case of heterogeneous portfolios, multiple systematic factors and random recoveries. As a significant improvement with respect to the literature, we detail all the technical conditions of validity and provide an upper bound of the remainder term for finite portfolio sizes. Moreover, we deal explicitly with the case of general loss distributions, possibly with masses. For some simple portfolio models, we prove empirically that the granularity adjustments do not always improve the infinitely granular first-order approximations. This stresses the importance of checking some conditions of regularity before relying on such techniques. Smoothing the underlying loss distributions through random recoveries or exposures improves the GA performances in general.

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1. Introduction

Value-at-risk (VaR) remains the cornerstone of banking regulatory capital calculations, despite its well-known drawbacks (see the critics in Danielsson et al., 2001, for instance). Its use has spread throughout the fields of risk measurement and portfolio management: trading limits calculations, risk-return analysis, etc. Precise estimations of VaRs require computationally intensive processes, even for moderate-size portfolios, and particularly for high levels. Unfortunately, not every user is able to launch such a full evaluation process, nor is ready to wait several hours (or even days) every time a VaR is required. Therefore, there has been a need for approximate methods. These methods should be able to calculate quickly and efficiently VaRs, Expected Shortfalls, or other risk measures, for large portfolios and possibly complex instruments.

For a long time, it has been observed that the computation of VaRs is dramatically simplified under the assumption that bank portfolios are perfectly fine-grained (or infinitely granular, equivalently). In such situations, diversification fully eliminates idiosyncratic risk, so that the portfolio loss depends on systematic risk only. Since real-world portfolios are not perfectly fine-grained, there remains a residual of undiversified idiosyncratic risk. The impact of undiversified idiosyncratic risk on VaRs can be approximated analytically through the so-called “granularity

adjustments” (GA) techniques. In mathematical terms, they are based on some asymptotic expansions of the portfolio loss distributions, when the number of exposures is “large”.

The Basel 2 proposals for credit risk measurement has fueled research in this area since the beginning of this century: see Wilde (2001a), Gordy (2003). In particular, GAs allow closed-form calculations of approximated VaRs or expected shortfalls for some well-known industry models of credit portfolios. For instance, Wilde (2001b) provided GA formulas for CreditRisk+ (CSFP, 1997). Emmer and Tasche (2005) developed the same analysis for CreditMetrics (Gupton et al., 1997). These results have been reworked and/or amended in several papers, notably Gordy and Lütkebohmert (2012) in the case of CreditMetrics. In the increasing GA literature, seminal papers include Gouriéroux et al. (2000), Gordy (2003), Martin and Wilde (2002), etc. More recently, Antonov et al. (2007) applied such techniques to price CDOs. Gordy and Juneja (2010) showed that such techniques apply even with portfolio of options, for which nested simulations are required. Gagliardini and Gouriéroux (2011) provided a framework for asset pricing with factor models. Gordy and Marrone (2012) developed the GA theory under a mark-to-market credit risk perspective.

To the best of our knowledge, the specific orders of magnitude of GA approximations have never been studied in the literature. In other words, no paper has evaluated the error terms of such approximations. Moreover, the underlying conditions of regularity are often imprecise or too strong. Our goal will be to revisit the

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theory, by pushing the GA mathematics forward. Contrary to the current practice, we will not assume *a priori* that “the asymptotics work”, i.e. that we can rely on the infinitely granular assumption. At the opposite, we will revisit the theory by stating clear-cut granularity expansion results by keeping fixed the portfolio size n . Therefore, we do not state asymptotic results, but rather exact results for finite portfolio sizes. In Section 2, we establish two slightly different GA expansions. Under some sufficient conditions of regularity, we evaluate the errors induced by these approximations for general heterogenous credit portfolios. These results can be applied to any heterogenous and multi-factor credit risk model. In passing, we detail GAs of several multi-factor models, notably for a two-factor extension of the CreditMetrics-type model of Amraoui and Hitiér (2008). This model is the current market benchmark to price synthetic CDOs with random recoveries (see Appendix C.3).

Finally, we measure the quality of GA approximations by simulation, for some simple credit risk models. These specifications are a bit unusual but not unrealistic. They will play the role of “Toy models” because they allow closed-form calculations of GA terms. Theoretically and empirically, we show that the performances of granularity adjustments may be explained in the light of the regularity conditions we have exhibited previously. Particularly, they depend on the tail behavior of the characteristic functions of the underlying loss distributions: see Section 3. Proofs are gathered in the appendix.

2. A general formula for granularity adjustments

2.1. The framework

To fix the notations, consider a n -size credit portfolio. The exposure E_j of any counter party j will be assumed constant in time. Typically, such a quantity is positive because exposures are comparable to replacement costs of some risky positions in a portfolio.¹ We will be interested in the credit risk associated with this portfolio, between today and a given time horizon T . The associated “loss rate” of a given name j will be modeled as a random variable $Z_j, j = 1, \dots, n$. If no default occurs before the time horizon T , $Z_j = 0$. Otherwise, $Z_j > 0$ is the stochastic loss rate associated with j 's default. Then, the normalized portfolio loss (between today and the given time horizon T) is the random quantity

$$L_n := \sum_{j=1}^n A_{jn} Z_j, \quad (1)$$

where $A_{jn} := E_j / \sum_{j=1}^n E_j$ denotes the j -th share of exposure. Therefore, A_{jn} is the j -th percentage of the total risky exposure in the portfolio, and $\sum_{j=1}^n A_{jn} = 1$.

Actually, in a classical default risk mode, Z_j can be seen as the product of an indicator function $D_j \in \{0, 1\}$ and a loss-given-default (LGD) percentage of the face value of loan j . Here, $D_j = 1$ means “default of j before T ”. LGDs (also one minus recovery rates) are random percentages of the exposure shares A_{jn} , without modeling the exact time when these cash-flows are repaid. Equivalently, it means all repayments occur at the same time horizon T . Note that $1 - Z_j$ can be interpreted as the T -market value of j 's exposure, for a nominal of one, as usually under a market risk point-of-view. We assume that every variable Z_j is bounded from above by a constant \bar{Z}_j . This level will be one most of the time, but we do not exclude the possibility of negative recovery rates (i.e. losses larger than 100%).

We work under the static factor model framework, that is usual in credit risk portfolio modeling (see Koyluoglu and Hickman, 1998, Gordy, 2000, Gregory and Laurent, 2005, etc.): conditionally on a given random vector $\mathbf{X} \in \mathbb{R}^m$, the random variables $Z_j, j = 1, 2, \dots$, are mutually independent. In other words, the dependence between the individual losses is generated by some “systematic” random factors only, gathered into a vector \mathbf{X} .

The GA methodology works well when the number of underlying positions n is “large”, but the total portfolio notional is kept as a constant. In the literature, a portfolio is called “infinitely granular” when its size goes to the infinity ($n \rightarrow \infty$) and when every individual exposure share is infinitesimal with respect to the total portfolio size: $\lim_{n \rightarrow \infty} \sup_{j=1, \dots, n} |A_{jn}| = 0$. It is well-known that, under this infinitely granular assumption, the law of L_n is similar to the law of $E[L_n | \mathbf{X}]$. Since the latter law is a lot simpler than the former, most of the time, it is tempting to approximate the cdf or the quantiles of L_n by those of $E[L_n | \mathbf{X}]$. This provides a first-order approximation of risk measures like VaRs, that we call “EVaR”.

Instead of dealing with more or less ad hoc arguments, we will develop a mathematically rigorous Fourier analysis of the random loss L_n . First, we will calculate the relation between the characteristic functions of L_n and $E[L_n | \mathbf{X}]$, by some Taylor expansions. Second, some inversions of Fourier transforms will provide their corresponding relations in terms of cumulative distribution functions. Finally, we will deduce the link between their quantiles, and therefore their granularity adjustment terms.²

2.2. The classical proof of granularity adjustments

Let us recall the usual way of deriving a granularity adjustment. The standard method relies on the sensitivity of Value-at-Risk w.r.t. individual exposures, by invoking the results of Gouriéroux et al. (2000). To be short, they consider a continuum of losses between the true random loss L_n and its infinitely granular approximation:

$$L_n(\varepsilon) = E[L_n | \mathbf{X}] + \varepsilon \cdot (L_n - E[L_n | \mathbf{X}]),$$

where $\varepsilon \in [0, 1]$. For a given level $\alpha \in (0, 1)$, denote by $VaR_\alpha(L)$ the Value-at-Risk associated with an arbitrary random loss L . The case $\varepsilon = 1$ corresponds to the true portfolio loss, and the case $\varepsilon = 0$ corresponds to the loss of the infinitely granular portfolio. A limited expansion between $\varepsilon = 0$ and $\varepsilon = 1$ links together $VaR_\alpha(L_n)$ (the VaR of L_n) and its approximation $VaR_\alpha(E[L_n | \mathbf{X}])$ (the VaR of $E[L_n | \mathbf{X}]$ that we called “EVaR”):

$$VaR_\alpha(L_n) = VaR_\alpha(L_n(1)) = VaR_\alpha(E[L_n | \mathbf{X}]) + \frac{\partial VaR_\alpha(L_n(\varepsilon))}{\partial \varepsilon} \Big|_{\varepsilon=0} + \frac{1}{2} \frac{\partial^2 VaR_\alpha(L_n(\varepsilon))}{\partial \varepsilon^2} \Big|_{\varepsilon=0} + \dots$$

To lighten notations, we will denote $VaR_\alpha(E[L_n | \mathbf{X}])$ by $EVaR_{n,\alpha}$, and $E[L_n | \mathbf{X}]$ by $\mu_n(\mathbf{X})$. By simple calculations, it can be proved that

$$\frac{\partial VaR_\alpha(L_n(\varepsilon))}{\partial \varepsilon} \Big|_{\varepsilon=0} = E[L_n - \mu_n(\mathbf{X}) | \mu_n(\mathbf{X}) = EVaR_{n,\alpha}] = 0.$$

This property is always true. It does not depend on the fact that \mathbf{X} may be of dimension one and/or the function μ_n may be monotonic. Indeed, the event $\{\mu_n(\mathbf{X}) = EVaR_{n,\alpha}\}$ can be rewritten $\{\mathbf{X} \in \mathcal{A}\}$ for some measurable subset \mathcal{A} in \mathbb{R}^m . Since $E[L_n | \mathbf{X} \in \mathcal{A}]$ is an average of expected losses $E[L_n | \mathbf{X} = \mathbf{x}]$ when $\mathbf{x} \in \mathcal{A}$, then

$$E[L_n | \mu_n(\mathbf{X}) = EVaR_{n,\alpha}] = E[L_n | \mathbf{X} \in \mathcal{A}] = EVaR_{n,\alpha}.$$

² Note that Martin and Wilde (2002) have used the inversion of the Laplace transform to calculate derivatives of Value-at-Risk in a simplified framework, and they deduced the main “granularity adjustment” term. Nonetheless, they have not paid attention to the precision of this approximation.

¹ Nonetheless, nothing precludes the likelihood of negative or even random exposures in this article, if they are independent of the underlying default risks. In the latter case, simply replace the quantities A_{jn} below by their expectations.

Therefore, the granularity adjustment terms will come from the second order derivative of $\text{VaR}_\varepsilon(L_n(\varepsilon))$ w.r.t. ε . In the case of a univariate systematic factor \mathbf{X} , Gouriéroux et al. (2000) provide the corresponding formulas, but under some strong regularity conditions. In particular, they assume that the joint density of $(L_n, \mu_n(\mathbf{X}))$ exists w.r.t. the Lebesgue measure. Unfortunately, portfolio loss distributions often discrete, at least partly. Since empirical quantiles (and then VaR measures) can still be defined by generalized inverse functions, it is important to adapt the theoretical results in this case. This will be done below.

2.3. A refined proof of granularity adjustments

Now, we set some additional notations. Since we rely on some Fourier transform techniques, let χ_L and $\chi_{\mu(\mathbf{X})}$ be the characteristic functions of L_n and $\mu_n(\mathbf{X})$ respectively, i.e.

$$\chi_L(t) := E[\exp(itL_n)], \quad \text{and} \quad \chi_{\mu(\mathbf{X})}(t) := E[\exp(it\mu_n(\mathbf{X}))],$$

for any t . Since every random loss Z_j is upper bounded by a constant Z_j then, conditionally on the value of the systematic factor, the variance of the j -th loss exists and is defined by

$$V_j(\mathbf{X}) := E[Z_j^2 | \mathbf{X}] - E[Z_j | \mathbf{X}]^2.$$

Note that the laws of the individual losses Z_j may differ, for different indices j , as it is the case in practice. For any vector $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x}_{-(k)}$ denotes the $m - 1$ dimensional vector $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m)$.

Assumption A1. $\mu_n(\mathbf{X})$ has a density $f_{\mu(\mathbf{X})}$ w.r.t. the Lebesgue measure on \mathbb{R}^m .

Assumption A2. X_1 , the first component of \mathbf{X} , is an argument of $\mu_n(\mathbf{X})$ and $x_1 \mapsto \mu_n(x_1, \mathbf{x}_{-(1)})$ is strictly increasing and differentiable for every vector $\mathbf{x}_{-(1)}$. Moreover, we assume that the factor \mathbf{X} has a density $f_{\mathbf{X}}$ w.r.t. the Lebesgue measure on \mathbb{R}^m .

The latter technical conditions are satisfied easily in most of the factor models that have been considered in the literature. For convenience, A2 highlights the role of the first component of \mathbf{X} . Obviously, this index can be changed and the condition A2 modified.

More generally when $m \geq 2$, for any $v \in \mathbb{R}$ and $j = 1, \dots, n$, we can define

$$\kappa_j(v) := E[V_j(\mathbf{X}) | \mu_n(\mathbf{X}) = v] f_{\mu(\mathbf{X})}(v), \quad \text{under Assumption A1, and}$$

$$\kappa_j(v) := \int_{\{\mathbf{x} | \mu_n(\mathbf{x}) = v\}} V_j(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) \frac{d\mathbf{x}_{-(1)}}{\partial_{(1)} \mu_n(\mathbf{x})} \quad \text{under Assumption A2,}$$

where $\partial_{(k)} \mu_n$ denotes the partial derivative of μ_n w.r.t. its k -th argument, $k = 1, \dots, m$. These functions κ_j will provide the main terms of GAs hereafter.

Note that, in the case of a univariate factor model, i.e. $\mathbf{X} \in \mathbb{R}$, these formulas are respectively

$$\kappa_j(v) := V_j(\mu_n^{-1}(v)) f_{\mu(\mathbf{X})}(v), \quad \text{and} \quad \kappa_j(v) := \frac{V_j(\mu_n^{-1}(v)) f_{\mathbf{X}}(\mu_n^{-1}(v))}{\mu'_n(\mu_n^{-1}(v))}. \tag{2}$$

Even if multi-factor models are commonly used in practice, their GAs have been calculated very seldom in the literature, until now. Indeed, most of the time, the analytical difficulty is far higher than in the univariate case. Pykhtin (2004) has proposed to solve this problem by finding a comparable one-factor portfolio whose loss distribution is “similar to” the original multi-factor loss distribution. This technique has been extended and refined by Voropaev (2011). In the case of homogenous portfolios, Gagliardini and

Gouriéroux (2013) have provided exact GA under a simple stochastic volatility model. Note that Monte-Carlo acceleration techniques can provide valuable and general alternatives to GAs. Numerically, they may be particularly relevant in the multifactor case: see Glasserman et al. (2008), for instance.

In this paper, for the sake of illustration, a few functions κ_j and their corresponding GAs are provided in C, in the case of models that involve several systematic factors. In particular, in C.3, we have tackled a two-factor extension of the Amraoui and Hitier (2008) model, that constitutes the standard model in the market to price synthetic CDO tranches. This model is an extension of the Gaussian copula model of Li (2000) but with random recoveries.

Theorem 1. Under Assumption A1 or A2, and if the following conditions are fulfilled:

- (i) For every $j = 1, \dots, n$, the function $x \mapsto G_j(x) := \int_{-\infty}^x \kappa_j$ is bounded.
- (ii) Every function $\hat{G}_j : t \mapsto \int \exp(itu) \kappa_j(u) du$ is integrable, $j = 1, \dots, n$.
- (iii) There exist some constants $\varepsilon \in (0, 1)$ and $M = M_n > 0$ such that $\max_{j=1, \dots, n} M_n A_{jn} \bar{Z}_j \leq 1 - \varepsilon$.

Then, for any $x \in \mathbb{R}$,

$$\frac{1}{2} [P(L_n < x + 0) + P(L_n < x - 0)] = \frac{1}{2} [P(\mu_n(\mathbf{X}) < x + 0) + P(\mu_n(\mathbf{X}) < x - 0)] + T_{n,\infty}(x) + r_{n,M}(x),$$

where the granularity adjustment (of loss distributions) is

$$T_{n,\infty}(x) := \frac{1}{2} \sum_{j=1}^n A_{jn}^2 \kappa'_j(x),$$

and the remainder term $r_{n,M}(x)$ satisfies

$$\begin{aligned} |r_{n,M}(x)| &\leq \frac{1}{\pi} \left| \int_M^{+\infty} \text{Im}(\chi_L(t) \exp(-itx)/t) dt \right| \\ &+ \frac{1}{\pi} \left| \int_M^{+\infty} \text{Im}(\chi_{\mu(\mathbf{X})}(t) \exp(-itx)/t) dt \right| \\ &+ \frac{1}{2\pi} \sum_{j=1}^n A_{jn}^2 \left| \int_M^{+\infty} \text{Im}(\hat{G}_j(t) t \exp(-itx)) dt \right| \\ &+ \left[\frac{19}{18} + k_\varepsilon \right] \cdot \frac{|M|^3}{3\pi} \sum_{j=1}^n A_{jn}^3 \bar{Z}_j^3 \\ &+ \frac{M^4}{2} \left[\frac{14}{9} + k_\varepsilon \right]^2 \left\{ \sum_{j=1}^n A_{jn}^2 \bar{Z}_j^2 \right\}^2 \\ &\times \exp \left(\left[\frac{14}{9} + k_\varepsilon \right] M^2 \sum_{j=1}^n A_{jn}^2 \bar{Z}_j^2 \right), \end{aligned} \tag{3}$$

where

$$k_\varepsilon := \left[\frac{\ln(1/\varepsilon)}{(1-\varepsilon)^3} - \frac{1}{2(1-\varepsilon)} - \frac{1}{(1-\varepsilon)^2} \right].$$

See the proof in the appendix. The main objective of Theorem 1 is to evaluate whether GAs can improve the first order approximation (of the law of L_n by the law of $\mu_n(\mathbf{X})$). This will be the case when $T_{n,\infty}(x) \gg r_{n,M}(x)$ for some relevant values x (close to the targeted VaR, typically).

Note that the choice of a convenient value M depends on n , but our main result above is fundamentally with a fixed n value. Intuitively, the larger is the size n , the larger will be $M = M_n$, under the

constraint $\sup_j M_n A_{jn} \bar{Z}_j < 1 - \varepsilon$. This constraint precludes some situations where a small subset of exposures concentrates a significant part of the whole portfolio exposure. In other words, when the number of names n tends to the infinity, A_{jn} should tend to zero sufficiently quickly, so that the remaining terms in (3) are negligible w.r.t. the granularity adjustment $T_{n,\infty}(x)$ itself.

A similar constraint appeared in the seminal paper of Gordy (2003): to prove the first-order approximation of L_n by $\mu_n(\mathbf{X})$ almost surely, his Assumption (A-2) says that $E_n / \sum_{k=1}^n E_k = O(n^{-1/2-\zeta})$ for some $\zeta > 0$. Note that this implies $A_{jn} = O(j^{-1/2-\zeta})$ but not necessarily $A_{jn} = O(n^{-1/2-\zeta})$. In our paper, such conditions have to be revisited because, compared to Gordy (2003), we are dealing with higher order terms on one side (more demanding) but we prove only convergence in law on the other side (less demanding). Therefore, following Theorem 1, our required conditions on (A_{jn}) and (M_n) are written in terms of sums, like $M_n^p \sum_{j=1}^n A_{jn}^q$ for some couples of integers (p, q) (see (3)). Heuristically, we can see that our analysis will be globally more demanding than Condition (A-2) in Gordy (2003) (except in some pathological situations). Indeed, due to Theorem 1, we need to satisfy

$$M_n^4 \left\{ \sum_{j=1}^n A_{jn}^2 \right\}^2 = o \left(\sum_{j=1}^n A_{jn}^2 \kappa'_j(x) \right)$$

for some x . If the risk of every individual follows the same law (homogeneity), then κ_j does not depend on j and the latter inequality implies that $M_n^4 \sum_{j=1}^n A_{jn}^2 = o(1)$. Then, $\sum_{j=1}^n A_{jn}^2$ has to tend to zero because $M_n \rightarrow \infty$. But Gordy's condition (A-2) implies only $\sum_{j=1}^n A_{jn}^2$ converges.

Even if our theoretical results cover arbitrary individual exposure shares, consider the typical case of a balanced portfolio below: $A_{jn} = 1/n, j = 1, \dots, n$. Then, $T_{n,\infty}$ and GAs are of order n^{-1} , and we would like to check that $r_{n,M}(x) = o(n^{-1})$. A necessary condition, that will be discussed in Section 3, is $M_n^4 = o(n)$.

Note that the third-order VaR adjustments (see Eq. (11) in Voropaev, 2011, e.g.) involve the third moment of the individual losses Z_i . Interestingly, such quantities appear in Theorem 1, but indirectly, through their upper bounds Z_j^3 in Eq. (3). This is logical since our goal was only to upper bound the error terms of GA expansions.

It is well-known that the Fourier transform maps the space of rapidly decreasing functions onto itself (Yosida, 1980, Th. 1, p. 147). Therefore, when the law of losses follows such a behavior, in particular when it exhibits thin tails, it is likely the associated GA will provide “good” results. This intuition will be illustrated empirically in Section 3.

2.4. Granularity adjustments and Value-at-Risk approximations

Now, let us link the Value-at-Risk of the true loss L_n and of its “infinitely granular” approximation $\mu_n(\mathbf{X})$. For any non-decreasing function F and any $u \in [0, 1]$, let $F^{-1}(u) := \inf\{x \mid F(x) \geq u\}$ be a quantile function that is associated with F . This is the standard generalized inverse function, as defined in the literature. The cdf of any random variable Z will be denoted by F_Z .

To simplify, we suppose in this subsection that:

Assumption B. The laws of L_n and $\mu_n(\mathbf{X})$ are continuous.

Then, Theorem 1 implies that, under some technical assumptions and for every x ,

$$F_{L_n}(x) = F_{\mu_n(\mathbf{X})}(x) + T_{n,\infty}(x) + r_{n,M}(x).$$

We hope that the last term $r_{n,M}(x)$ can be seen as negligible, when M is well chosen and when the underlying distributions satisfy some regularity conditions.

Now, for a given $\alpha \in (0, 1)$, the value-at-risk of L_n at level α is defined as the generalized inverse of F_{L_n} , i.e.

$$VaR_\alpha(L_n) = F_{L_n}^{-1}(\alpha).$$

Most of the time, there exists a single possible value-at-risk for a given level α , except when F_{L_n} takes the value α on a non-empty interval $I_\alpha : F_{L_n}(x) = \alpha$ for every $x \in I_\alpha$. In this case, every number in the interior of I_α is a VaR_α candidate. Note that this situation is very unlikely in practice when different exposures per name are considered. In every case, under B, we satisfy the nice property

$$F_{L_n}(VaR_\alpha(L_n)) = \alpha, \quad \text{and} \quad F_{\mu_n(\mathbf{X})}(EVar_{n,\alpha}) = \alpha.$$

Theorem 2. Under Assumption B and the assumptions of Theorem 1, if $f_{\mu_n(\mathbf{X})}$, the density of $\mu_n(\mathbf{X})$, exists and is not zero in the neighborhood of $EVar_{n,\alpha}$ then

$$VaR_\alpha(L_n) = EVar_{n,\alpha} - \frac{T_{n,\infty}(VaR_\alpha(L_n)) + r_{n,M}(VaR_\alpha(L_n))}{f_{\mu_n(\mathbf{X})}(v_\alpha)},$$

for some real number M , chosen as in Theorem 2, and some number v_α such that

$$|v_\alpha - EVar_{n,\alpha}| \leq |EVar_{n,\alpha} - VaR_\alpha(L_n)|.$$

Proof. It is a simple consequence of a limited expansion. Indeed,

$$\begin{aligned} F_{\mu_n(\mathbf{X})}(EVar_{n,\alpha}) &= \alpha = F_{L_n}(VaR_\alpha(L_n)) \\ &= F_{\mu_n(\mathbf{X})}(VaR_\alpha(L_n)) + T_{n,\infty}(VaR_\alpha(L_n)) + r_{n,M}(VaR_\alpha(L_n)), \end{aligned}$$

and then

$$f_{\mu_n(\mathbf{X})}(v_\alpha) \cdot (EVar_{n,\alpha} - VaR_\alpha(L_n)) = T_{n,\infty}(VaR_\alpha(L_n)) + r_{n,M}(VaR_\alpha(L_n)),$$

for such a number v_α . \square

The size of the error term $r_{n,M}(VaR_\alpha(L_n))$ is related to the choice of M , to the tails of the characteristic functions of losses and to the behavior of the exposure shares $A_{jn}, j = 1, \dots, n$ for large n . Invoking Eq. (2) and assuming the remainder term $r_{n,M}$ is negligible w.r.t. $T_{n,\infty}$, we deduce the usual granularity adjustment of VaR easily. Some GA approximations of value-at-risk (or any other risk measure) values will be identified by the symbol \simeq hereafter.

Corollary 3. Under the assumptions of Theorem 2

$$VaR_\alpha(L_n) \simeq EVar_\alpha - \frac{1}{2f_{\mu_n(\mathbf{X})}(EVar_\alpha)} \sum_{j=1}^n A_{jn}^2 \kappa'_j(EVar_\alpha). \quad (4)$$

If the systematic factor $\mathbf{X} := X$ is univariate, if A1 and A2 are fulfilled, then

$$\kappa'_j(v) := \frac{d}{dv} \left[\frac{V_j(\mu_n^{-1}(v)) f_X(\mu_n^{-1}(v))}{\mu'_n(\mu_n^{-1}(v))} \right] = \frac{d}{dv} [V_j(\mu_n^{-1}(v)) f_{\mu_n(\mathbf{X})}(v)]. \quad (5)$$

Equivalently, we have

$$VaR_\alpha(L_n) \simeq EVar_\alpha - \frac{1}{2} \sum_{j=1}^n A_{jn}^2 \left[\frac{V'_j(VaR_{X,\alpha})}{\mu'_n(VaR_{X,\alpha})} + V_j(VaR_{X,\alpha}) \frac{f'_{L_n|X}(EVar_\alpha)}{f_{\mu_n(\mathbf{X})}(EVar_\alpha)} \right] \quad (6)$$

or even

$$VaR_\alpha(L_n) \simeq EVar_\alpha - \frac{1}{2f_X(VaR_{X,\alpha})} \sum_{j=1}^n A_{jn}^2 \frac{d}{dt} \left[\frac{V_j(t) f_X(t)}{\mu'_n(t)} \right]_{t=VaR_{X,\alpha}}. \quad (7)$$

Proof. Eq. (4) is just a consequence of Theorem 2. Due to the monotonicity of μ_n , note that $P(E[L_n | X] \leq t) = P(X \leq \mu_n^{-1}(t))$, and then

$$f_{\mu(X)}(t) = \frac{f_X(\mu_n^{-1}(t))}{\mu_n'(\mu_n^{-1}(t))}.$$

This proves Eq. (5). Since $\mu_n^{-1}(\text{VaR}_{E[L_n|X], \alpha}) = \text{VaR}_{X, \alpha}$, we deduce Eq. (6). We get Eq. (7) by setting $v = \mu_n(t)$ and writing derivatives w.r.t. t instead of v . \square

Therefore, we recover the granularity adjustment formulas, as they appeared in Wilde (2001a,b), Gordy (2003) or Martin and Wilde (2002), among others. Nonetheless, these formulas consider that $\text{E}VaR_x$ and $\text{VaR}_x(L_n)$ are sufficiently close so that we can use one of the other as arguments of κ_j and/or $f_{\mu(X)}$, in the granularity adjustment formula. Actually, in practice, this choice could matter, even if it should be negligible asymptotically. Moreover, the calculation that of $\text{E}VaR_x$ and $\text{VaR}_x(X)$ is far simpler than the evaluation of $\text{VaR}_x(L_n)$. The latter quantity is often estimated by simulation (i.e. with a statistical noise), when the former ones are obtained in closed form most of the time, i.e. with a very high level of accuracy. An analysis of the biases and the uncertainties induced by the use of simulation-based VaRs can be found in Gordy and Juneja (2010).

Therefore, we propose a new GA: set

$$\bar{\text{VaR}}_{n, \alpha} := \frac{1}{2}(\text{VaR}_x(L_n) + \text{E}VaR_{n, \alpha}). \tag{8}$$

Corollary 4. Under the assumptions of Theorem 2,

$$\text{VaR}_x(L_n) \simeq \text{E}VaR_x - \frac{1}{2f_{\mu(X)}(\bar{\text{VaR}}_{n, \alpha})} \sum_{j=1}^n A_{jn}^2 \kappa_j'(\text{VaR}_x(L_n)). \tag{9}$$

When the systematic factor $\mathbf{X} := X$ is univariate, and when A1 and A2 are fulfilled, we have

$$\begin{aligned} \text{VaR}_x(L_n) \simeq & \text{E}VaR_x - \frac{1}{2f_X(\mu_n^{-1}(\bar{\text{VaR}}_{n, \alpha}))} \cdot \frac{\mu_n'(\mu_n^{-1}(\bar{\text{VaR}}_{n, \alpha}))}{\mu_n'(\mu_n^{-1}(\text{VaR}_x(L_n)))} \\ & \times \sum_{j=1}^n A_{jn}^2 \frac{d}{dt} \left[\frac{V_j(t) f_X(t)}{\mu_n'(t)} \right]_{|t=\mu_n^{-1}(\text{VaR}_x(L_n))}. \end{aligned} \tag{10}$$

The latter formulas are slightly more complex than the traditional GA formulas in Corollary 3. Nonetheless, for theoretical reasons (see the proof of Corollary 3), we can hope that they could provide better results than the previous GA formulas, in practice. We will discuss this point in the empirical Section 3.

Actually, it should be noticed that the previous GA formulas belong to a more general class of granularity adjustments: for a given θ in $[0, 1]$, set

$$\bar{\text{VaR}}_{n, \alpha}(\theta) := \theta \text{VaR}_x(L_n) + (1 - \theta) \text{E}VaR_{n, \alpha}.$$

Therefore, there exists a continuum of GA formulas

$$\text{VaR}_x(L_n) \simeq \text{E}VaR_x - \frac{1}{2f_{\mu(X)}(\bar{\text{VaR}}_{n, \alpha}(\theta))} \sum_{j=1}^n A_{jn}^2 \kappa_j'(\bar{\text{VaR}}_{n, \alpha}(\theta)), \quad (\theta, \theta') \in [0, 1]^2. \tag{11}$$

In Corollaries 3 and 4, we have just chosen $(\theta, \theta') = (0, 0)$ and $(\theta, \theta') = (1/2, 1)$ respectively.

2.5. Granularity adjustments and Value-at-Risk approximations with discrete loss distributions

Now, we would like to remove Assumption B, to be able to deal with discontinuous L_n loss distributions. This case appears

naturally when exposures and recoveries are assumed to be fixed (a rough but standard assumption, unfortunately). On the other side and following the GA literature, we still assume that the distribution of $E[L_n | \mathbf{X}]$ is continuous, and that it owns a density w.r.t. the Lebesgue measure. Actually, most of the previous results still apply with slight modifications under this new framework.

To apply Theorem 1, we define the normalized cdf \tilde{F} by

$$\tilde{F}(x) := \frac{1}{2}[F(x - 0) + F(x + 0)],$$

for every x . Theorem 1 says that, under some conditions of regularity, for every x ,

$$\tilde{F}_{L_n}(x) = \tilde{F}_{\mu_n(X)}(x) + T_{n, \infty}(x) + r_{n, M}(x).$$

Now, for a given $\alpha \in (0, 1)$, the value-at-risk of L_n at level α is a corresponding quantile of F_{L_n} . To be consistent with Theorem 1, we decide that the VaRs are always defined as quantiles of \tilde{F}_{L_n} :

$$\text{VaR}_x(L_n) := \tilde{F}_{L_n}^{-1}(\alpha), \quad \alpha \in (0, 1).$$

On one side, let us assume that the levels α we consider are "reached" by the corresponding cdfs. In other words, α lies in the range of these cdfs:

$$\text{Assumption B}^*. \tilde{F}_{L_n}(\text{VaR}_x(L_n)) = \alpha.$$

This assumption may be strong in practice, if α chosen independently of the range of L_n 's values. Since we have assumed the law of $\mu_n(\mathbf{X})$ is continuous, note that we always have

$$\tilde{F}_{\mu_n(X)}(\text{E}VaR_{n, \alpha}) = F_{\mu_n(X)}(\text{E}VaR_{n, \alpha}) = \alpha.$$

By the same arguments as above, we get easily:

Proposition 5. If the distribution of L_n is general (possibly with discrete masses), Theorem 2, Corollary 3 and 4 still apply, replacing Assumption B with Assumption B*.

On the other side, if α does not belong to the image set of \tilde{F}_{L_n} then $\tilde{F}_{L_n}(\text{VaR}_x(L_n))$ is different from α , except when α lies in the middle of $[\tilde{F}_{L_n}(\text{VaR}_x(L_n) - 0), \tilde{F}_{L_n}(\text{VaR}_x(L_n) + 0)]$, due to our definition (8). This difference can be significant, and of the order of magnitude of granularity adjustments themselves.

For instance, consider the extreme case of perfect dependence between the r.v.s.' Z_i and a scalar systematic factor X , i.e. $Z_i = \mathbf{1}(X \leq a)$ for some constant a . Then, if all exposure shares are constant and equal to $1/n$, then L_n takes the two values 0 and 1 only. Thus, $\tilde{F}_{L_n} \in \{0, 1/2, 1\}$. In this case, the quantiles that can be invoked to apply Theorem 2 are related to $\alpha \in \{0, 1/2, 1\}$ only. The same reasoning applies when L_n takes the values $\{\ell_1, \dots, \ell_p\}$ only, and $\tilde{F}_{L_n}(\ell_j) = q_j, j = 1, \dots, p$. Then, with our conventions and strictly speaking, Proposition 5 can be invoked with the levels $\alpha \in \{q_1, \dots, q_p\}$ only.

To go one step beyond, assume that we define the loss quantiles by our usual rule $\text{VaR}_x(L_n) = \inf\{t \mid \tilde{F}_{L_n}(t) \geq \alpha\}$, and that $\tilde{F}_{L_n}(\text{VaR}_x(L_n)) = \alpha^* \neq \alpha$. Thus, since $\tilde{F}_{\mu_n(X)}(\text{E}VaR_{n, \alpha}) = \alpha$, we have $\tilde{F}_{\mu_n(X)}(\text{E}VaR_{n, \alpha}) = \tilde{F}_{\mu_n(X)}(\text{VaR}_x(L_n)) + T_{n, \infty}(\text{VaR}_x(L_n)) + r(\text{VaR}_x(L_n)) + \alpha - \alpha^*$,

and, under the conditions of regularity of Theorem 2, we get

$$\text{VaR}_x(L_n) = \text{E}VaR_{n, \alpha} - \frac{T_{n, \infty}(\text{VaR}_x(L_n)) + r(\text{VaR}_x(L_n)) + \alpha - \alpha^*}{f_{\mu(X)}(v_x)},$$

for some number v_x such that $|v_x - \text{VaR}_x(L_n)| \leq |\text{E}VaR_{n, \alpha} - \text{VaR}_x(L_n)|$. Unfortunately, the last additional term can be significant, i.e. not negligible w.r.t. the GA adjustment. For instance, it is at most $1/(2f_{\mu(X)}(v_x))$ in our example.

Therefore, in general, it will be more difficult to apply GA theoretical results when the underlying loss distributions are discrete, a rather common situation. That is why we advise to work under Assumption B*, i.e. to apply the granularity adjustments only at some quantile levels α such that $\tilde{F}_{L_n}(VaR_\alpha(L_n)) = \alpha$. In practice, it may be difficult to insure such a condition, unfortunately. Indeed, VaR-levels are often imposed by managers or regulators, whatever the loss distribution. A convenient solution should be to introduce random recovery rates systematically and then to get smooth loss distributions. Another way would be to introduce random exposures, as in Gordy and Marrone (2012). Such exposures may be driven by random credit spreads or interest rates, but any other market factor can be considered. They may be introduced by keeping the previous A_{jn} unchanged and deterministic, but by multiplying Z_j by a new random variable e_j , $E[e_j] = 1$, $j = 1, \dots, n$. The additional random variables (e_1, \dots, e_n) could be correlated too.

2.6. Granularity adjustments and Expected Shortfall approximations

It is tempting to get an equivalent of Corollary 3 but for expected shortfalls, in the sense of Acerbi and Tasche (2002): for any $\alpha \in (0, 1)$, the α -level expected shortfall of a loss function L is defined by

$$ES_\alpha(L) := \frac{1}{1-\alpha} E[L \cdot \mathbf{1}(L \geq VaR_\alpha(L))] - \frac{VaR_\alpha(L)}{1-\alpha} (\alpha - P(L < VaR_\alpha(L))),$$

when the value-at-risk is defined by $VaR_\alpha(L) = \inf\{t \mid F(t) \geq \alpha\}$. With the latter definition, we get a coherent measure of risk, that satisfies the noteworthy relation:

$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L) du.$$

Then, as noticed in Martin and Tasche (2007) or Gordy (2004), the granularity adjustment of an expected shortfall may be just deduced from the integration of granularity adjustments for a continuum of value-at-risks.

Here, under the assumptions of Corollary 3 and by integrating Eq. (4), we get

$$ES_\alpha(L_n) \simeq ES_\alpha(E[L_n \mid \mathbf{X}]) - \frac{1}{2(1-\alpha)} \sum_{j=1}^n A_{jn}^2 \int_\alpha^1 \frac{1}{f_{\mu(\mathbf{X})}(VaR_{\mu_n(\mathbf{X}),u})} \times \kappa_j'(VaR_{\mu_n(\mathbf{X}),u}) du.$$

Unfortunately, getting an upper bound of the remainder term seems to be significantly more tedious than for $VaR_\alpha(L_n)$, in particular because of the denominator in the latter equation.

An fruitful idea could be to note that, at least for continuous loss distributions,

$$ES_\alpha(L) = \frac{(-1)}{1-\alpha} \int_{VaR_\alpha(L)}^\infty u(1-F_L)(du) = VaR_\alpha(L) + \frac{1}{1-\alpha} \int_{VaR_\alpha(L)}^\infty (1-F_L)(u) du.$$

The remainder term of ES-granularity adjustments is thus related to the rest $\int_{VaR_\alpha(L)}^\infty r_{n,\infty}(x) dx$, where $r_{n,\infty}$ is detailed in the proof of Theorem 1. With the notations of the appendix, this implies to upper bound

$$\int_{VaR_\alpha(L)}^\infty \int_0^\infty Im\{E[\exp(it\mu_n(\mathbf{X}))\tilde{r}_\chi(\mathbf{X},t)] \exp(-itx)/t\} dt,$$

for an explicit but very complicated function $\tilde{r}_\chi(\mathbf{X},t)$. Clearly, the result will depend strongly on the behavior of the function

$$t \mapsto E[\exp(it\mu_n(\mathbf{X}))\tilde{r}_\chi(\mathbf{X},t)].$$

Due to the strength and the number of the required conditions, we stop trying to find an explicit GA upper bound for expected shortfalls here. We keep this task for further developments.

3. Empirical performances of granularity adjustments

Note that, in the literature, there are not so many empirical analysis of granularity adjustments. Historically, Gordy (2003) was the first one. His framework was a simplified CreditRisk + type model. Since the tails of the loss distributions generated by this model behave regularly and are thin, the performances of GA approximations appeared very convincing: see Table 1. Similar findings have been obtained with CreditMetrics-type, KMV Portfolio Manager-type models, or even the Basel 2 credit risk model: see Emmer and Tasche (2005), Gordy and Marrone (2012), for instance. This is due to the formal similarities between these models, in terms of loss distributions, as explained in Koyluoglu and Hickman (1998).

From the previous theoretical results, it appears clearly that the quality of granularity adjustments depends on the regularity of the underlying distributions. In other words, it is likely such adjustments do a good job for some models, and may do a less satisfying job for others, in the sense that GAs may not provide improvements w.r.t. the crude approximation $L_n \stackrel{law}{\approx} E[L_n \mid \mathbf{X}]$. In this section, we introduce different credit risk models. For convenience, the portfolios will be homogenous: $A_{jn} = 1/n$ for all $j = 1, \dots, n$. We compare their (true or numerically approximated) VaRs with their first-order approximations in the framework of infinitely granular portfolios, denoted by “EVAR”. Moreover, we measure to what extent granularity adjustments induce refined VaR measurements. The two GAs, as given by Corollaries 3 and 4, will be denoted by GA1 and GA2 respectively. The exact formulas of the GAs we consider in this section are detailed in B.

Note that our theoretical results allow to state when GAs will work, but not when some problems of accuracy will occur necessarily. In other words, we have stated sufficient conditions of regularity, but not necessary conditions. Nonetheless, we will check empirically the empirical relevance of the conditions of regularity induced by Theorem 1.

To be specific and with the previous notations, our granularity term $T_{n,\infty}(VaR_\alpha(L_n))$ is of order $1/n$ in the case of balanced portfolios. It is easy to check that the error term $r_{n,M}(x)$ is $o(T_{n,\infty}(x))$ (for every x) only if $M^4/n = o(1)$, due to the last term in the r.h.s. of (3). One task will be to check whether the other terms on the r.h.s. of this inequality are not too large compared to $T_{n,\infty}$. For every model, we will focus on the term $I_M(x) := \int_M^\infty Im(\chi_{\mu(x)}(t) \exp(-itx)/t) dt$, where x will be (close to) a value-at-risk of the loss distribution. Most of the time, it can be evaluated relatively easily, contrary to $\int_M^\infty Im(\chi_{L_n}(t) \exp(-itx)/t) dt$. Moreover, set $J_M(x) := \max_{j=1}^n \int_M^{+\infty} Im(\tilde{G}_j(t) t \exp(-itx)) dt$. It is necessary that this term tends to zero when $n \rightarrow \infty$ so that $r_{n,M}(x) = o(T_{n,\infty}(x))$.

Table 1
Direct and approximated VaR estimates. Portfolio size = 600 names. True Var estimated by simulation (300,000 draws). These numbers come from Gordy (2003), Table 3.

(in %)	99.0%	99.5%	99.9%
True VaR	4.577	5.522	7.872
Asymptotic VaR	4.220	5.109	7.260
GA	0.357	0.435	0.627
Approximated VaR	4.578	5.544	7.886
Error	0.001	0.022	0.014

3.1. Granularity adjustments in a continuous loss framework: Toy model 1

As a benchmark, we choose the most simple credit model as possible. Its main advantage relies in its tractability and by the fact that (true) VaRs can be calculated analytically. In this model, the random loss amount that is associated with the name i is still measured by the r.v. Z_i and the vector $(Z_i)_{i=1,\dots,n}$ is Gaussian. Actually, we assume a standard underlying one-factor model:

$$Z_i = a_i + \sigma_i \left(\rho_i X + \sqrt{1 - \rho_i^2} X_i^* \right), \tag{12}$$

where X and the $X_i^*, i = 1, \dots, n$, are (jointly) independent standard Gaussian variables. The parameters a_i, σ_i and ρ_i are positive constants, with $\rho_i \in [0, 1]$.

These assumptions are rather unusual in the credit area, particularly due to the likelihood of negative losses, and due to the unboundedness of the potential loss amounts. Even if some mark-to-market models can generate positive and/or negative credit losses, the standard framework avoid them, by considering only bonds or loans for instance. Nonetheless, we can say that, in this example, we are interested in market risk only.

Under this specification, the laws of L_n and $E[L_n | X]$ are Gaussian and their characteristic functions decrease towards zero at an exponential rate. Particularly, in the case of an homogenous pool of names, $I_M(x) = o(n^{-1})$ by choosing $M = n^a, a < 1/3$. Moreover, it is easy to check that the conditions (i) and (ii) are satisfied, because every \tilde{G}_j behaves like $t \mapsto \exp(-Ct^2)$ for some constants $C > 0$. Then $J_M(x) = o(n^{-1})$, obviously. Therefore, the remainder

term $r_{n,M}$ in Theorem 1 will be sufficiently low for a convenient choice of M , and we should get nice GAs.

This is what we observe, indeed. Tables 2 and 3 provide exact and approximated VaR calculations. As expected, the granularity approximations do a perfect job, by approximating the true VaR levels with a very high level of accuracy. Moreover, there are virtually no differences between our two GA formulas GA1 and GA2.

Interestingly, if we replace all the previous standard Gaussian variables by exponential random variables, with parameter one, the picture is less clear-cut: see Table 4. Taking into account the statistical uncertainty around the estimated value-at-risks, we observe that the GAs improve the EVaR calculations when the correlation parameter ρ_i is less than 0.5. For higher correlation levels, systematic risk dominates idiosyncratic risks, and the corresponding granularity adjustments become small. As a consequence, EVaRs estimate the underlying VaR levels very efficiently. Therefore, the remaining statistical uncertainty around our estimated VaRs prevents concluding anything definitive concerning GAs when correlations are high. Note that, with this specification, the function $t \mapsto \text{Im}(\tilde{G}_j(t)t \exp(-itx))$ is not integrable and our remaining term $r_{n,M}(x)$ cannot be negligible w.r.t. its corresponding GA. Therefore, even if our sufficient conditions are not fulfilled, this model provides satisfying GA results.

3.2. Granularity adjustments in an homogenous discrete one-factor Merton model: Toy model 2

Let us consider now discrete loss functions. For the time being, we restrict ourselves on the standard one-factor Merton-style model, as introduced in the seminal paper of Vasicek (2002), for which

Table 2

Performances of the “Toy model” 1 (closed-form calculations of VaR), as a function of the portfolio size n and the level $\alpha : a_i = 0.2, \sigma_i = 0.1, \rho_i = 0.25$. Results in %.

n	α	VaR	(VaR-EVaR)/VaR	(VaR-EVaR - GA1)/VaR	(VaR-EVaR; - GA2)/VaR
50	0.9	0.2365	1.899	-0.1331	-0.1664
500	0.9	0.2325	0.2052	-0.0015	-0.0019
5000	0.9	0.2321	0.0207	-1.55e-005	-1.938e-005
50,000	0.9	0.232	0.0021	-1.55e-007	-1.941e-007
50	0.99	0.2663	3.061	-0.2146	-0.2295
500	0.99	0.259	0.3343	-0.0025	-0.0026
5000	0.99	0.2582	0.0337	-2.53e-005	-2.688e-005
50,000	0.99	0.2582	0.0034	-2.53e-007	-2.69e-007

Table 3

Performances of the “Toy model” 1 (closed-form calculations of VaR), as a function of the correlations ρ_i and the level $\alpha : n = 5000, a_i = 0.2, \sigma_i = 0.1$. Results in %.

ρ_i	α	VaR	(VaR-EVaR)/VaR	(VaR-EVaR; - GA1)/VaR	(VaR-EVaR; - GA2)/VaR
0.05	0.9	0.2067	0.1213	-0.002374	-0.002968
0.15	0.9	0.2193	0.038	-8.236e-005	-0.0001029
0.25	0.9	0.2321	0.0206	-1.551e-005	-1.938e-005
0.35	0.9	0.2449	0.0131	-4.696e-006	-5.869e-006
0.45	0.9	0.2577	0.0088	-1.735e-006	-2.168e-006
0.55	0.9	0.2705	0.0060	-6.925e-007	-8.656e-007
0.65	0.9	0.2833	0.0040	-2.746e-007	-3.433e-007
0.75	0.9	0.2961	0.0025	-9.817e-008	-1.227e-007
0.85	0.9	0.3089	0.0013	-2.601e-008	-3.251e-008
0.95	0.9	0.3217	0.0004	-2.208e-009	-2.76e-009
0.05	0.99	0.2121	0.2146	-0.0042	-0.0044
0.15	0.99	0.235	0.0643	-0.0001395	-0.0001483
0.25	0.99	0.2582	0.0337	-2.53e-005	-2.688e-005
0.35	0.99	0.2815	0.0207	-7.416e-006	-7.881e-006
0.45	0.99	0.3047	0.0135	-2.663e-006	-2.83e-006
0.55	0.99	0.328	0.0089	-1.037e-006	-1.102e-006
0.65	0.99	0.3512	0.0058	-4.021e-007	-4.273e-007
0.75	0.99	0.3745	0.0036	-1.409e-007	-1.497e-007
0.85	0.99	0.3977	0.0019	-3.667e-008	-3.896e-008
0.95	0.99	0.421	0.0005	-3.063e-009	-3.255e-009

Table 4
Performances of the “Toy model” 1 when the random variables are exponential (1), as a function of the dependence parameter ρ_i . The portfolio size is $n = 1000$ and the level is $\alpha = 0.9$. $a_i = -1$, $\sigma_i = 1$ for all i . Calculation of VaR by simulations: 2,000,000 draws. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). Results in %.

ρ_i	VaR	EVaR	VaR-EVaR/VaR	VaR-EVaR – GA1/VaR	VaR-EVaR – GA2/VaR	$\hat{\sigma}(VaR)/VaR$
0.05	0.124	0.1139	8.179	0.1365	0.1772	0.072
0.15	0.3368	0.3341	0.824	-0.1433	-0.142	0.073
0.25	0.546	0.5439	0.3829	0.0395	0.03986	0.072
0.35	0.7442	0.7427	0.2091	0.04063	0.04076	0.078
0.45	0.9307	0.9292	0.1589	0.06372	0.06379	0.056
0.55	1.102	1.102	0.02413	-0.03341	-0.03341	0.098
0.65	1.256	1.257	-0.06917	-0.1045	-0.1046	0.104
0.75	1.387	1.388	-0.0951	-0.1162	-0.1162	0.094
0.85	1.486	1.484	0.1603	0.1493	0.1493	0.089
0.95	1.498	1.5	-0.1366	-0.14	-0.140	0.072

$$Z_i = \mathbf{1}(\sqrt{\rho_i}X + \sqrt{1 - \rho_i}X_i \leq b_i), \quad i = 1, \dots, n. \tag{13}$$

Here, the individual losses can take only two values in this model: zero or one. X and $X_i, i = 1, \dots, n$, are mutually independent Gaussian random variables. The default thresholds b_i are related to default probabilities by the usual relation $b_i = \Phi^{-1}(PD_i)$. The coefficient ρ_i is i -th asset correlation (between its asset value and the systematic risk X). Multi-factor versions of this model have been implemented in CreditMetrics ou Moody’s-KMV, for instance. For the sake of analytical tractability, we consider homogenous portfolios, for which

$$A_i = n^{-1}, \quad \rho_i = \rho, \quad PD_i = PD, \quad \forall i = 1, \dots, n.$$

In this case, we can apply our theoretical results and calculate the granularity adjustment

$$\frac{T_{n,\infty}(VaR_{E[L_n|X],\alpha})}{f_{\mu(X)}(VaR_{E[L_n|X],\alpha})} = \frac{(-1)}{2n} \left(2VaR_{E[L_n|X],\alpha} - 1 + \left[\sqrt{\frac{1-\rho}{\rho}} \Phi^{-1}(\alpha) - \Phi^{-1}(VaR_{E[L_n|X],\alpha}) \right] \times \frac{VaR_{E[L_n|X],\alpha}(1 - VaR_{E[L_n|X],\alpha})}{\phi \circ \Phi^{-1}(VaR_{E[L_n|X],\alpha})} \right).$$

This formula appeared already in the literature, notably [Emmer and Tasche \(2005\)](#), [Gagliardini and Gouriéroux \(2013\)](#) more recently.

For this model, GAs work pretty well, as seen in [Table 5](#). We could get such an intuition through an analysis of $\chi_{\mu(X)}$. Indeed, after a change of variable, we see that

$$\chi_{\mu(X)}(t) \propto \int_0^1 \exp(ity) \exp\left(-\frac{1}{2} \left[\frac{\Phi^{-1}(y)}{\sqrt{\rho}} - \frac{b(\rho+1)}{\sqrt{1-\rho}\sqrt{\rho}} \right]^2\right) dy.$$

Therefore, by doing q successive integrations by parts, it can be shown easily that $\chi_{\mu(X)}(t) = O(t^{-q})$ and $I_M(x) = O(M^{-q-1})$ as soon as $q < 1/\rho$. Therefore, recalling [Theorem 1](#), if $M^{-q-1} + M^2n^{-2} = o(n^{-1})$, the corresponding GA should work. The latter constraint is satisfied when $q > 3$, i.e. when $\rho < 1/3$. This is exactly what we observe in [Table 5](#): when $\rho < 1/3$, GAs improve significantly the EVaR infinitely granular approximations. Otherwise, when ρ is larger, we cannot state that this is still true (nor the opposite), due to the statistical uncertainty around VaR estimates.

By considering non standard structural-type credit portfolio models, we are leaving the previous “ideal world”, and GAs may be jeopardized. For instance, consider a second version of the “Toy model 2”, defined by

$$Z_i = \mathbf{1}(X_i \leq a_i + b_i X), \tag{14}$$

where $X_i, i = 1, \dots, n$ are n independent random variables uniform on $[0, 1]$. The systematic r.v. X will be independent of the X_i variables, and will follow various distributions:

1. a uniform distribution on $(0, 1)$;
2. a Bates distribution, that is the average of m uniform on $(0, 1)$ and independent random variables, $m \geq 2$;
3. a Beta distribution on $[0, 1]$, defined through its two positive parameters p and q , as usual.

Table 5
Performances of the “Toy model” 2 with Gaussian r.v.s. (calculation of VaR by simulations): 1000 names, 1,000,000 draws. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

(ρ, PD)	α	VaR	EVaR	VaR-EVaR/VaR	VaR-EVaR – GA1/VaR	$\hat{\sigma}(VaR)/VaR$
(0.1,0.1)	0.9	0.179	0.1778	0.6571	0.0892	0.0745
(0.1,0.3)	0.9	0.452	0.450	0.4356	0.1720	0.0494
(0.2,0.1)	0.9	0.215	0.2142	0.3870	0.0694	0.0973
(0.2,0.3)	0.9	0.522	0.5217	0.0530	-0.1016	0.0596
(0.3,0.1)	0.9	0.245	0.2442	0.3161	0.1024	0.1358
(0.3,0.3)	0.9	0.585	0.5840	0.1672	0.0598	0.0506
(0.4,0.1)	0.9	0.272	0.2716	0.1608	0.0059	0.1324
(0.4,0.3)	0.9	0.645	0.6441	0.1427	0.0631	0.0726
(0.5,0.1)	0.9	0.298	0.2978	0.0784	-0.0371	0.1664
(0.5,0.3)	0.9	0.705	0.7054	-0.0538	-0.1147	0.0614
(0.6,0.1)	0.9	0.324	0.3239	0.0216	-0.0653	0.1678
(0.6,0.3)	0.9	0.771	0.7705	0.0676	0.0209	0.0664
(0.7,0.1)	0.9	0.352	0.3512	0.2375	0.1732	0.2017
(0.7,0.3)	0.9	0.842	0.8414	0.0725	0.0373	0.0552
(0.8,0.1)	0.9	0.379	0.3811	-0.5601	-0.6058	0.2718
(0.8,0.3)	0.9	0.918	0.9178	0.0204	-0.0048	0.0452
(0.9,0.1)	0.9	0.417	0.4176	-0.1504	-0.1781	0.3123
(0.9,0.3)	0.9	0.986	0.9856	0.0400	0.0254	0.0135

Table 6

Performances of the “Toy model” 2 when X is uniform on $(0, 1)$ (calculation of VaR by simulations): 1000 names, 100,000 draws. $\hat{\sigma}(\text{VaR})$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

(a_i, b_i)	α	VaR	VaR–EVaR/VaR	VaR–EVaR – GA1/VaR	VaR–EVaR – GA2/VaR	$\hat{\sigma}(\text{VaR})/\text{VaR}$
(0.2,0.2)	0.9	0.381	0.2625	0.294	0.2937	0.111
(0.2,0.2)	0.99	0.412	3.398	3.423	3.419	0.0894
(0.2,0.4)	0.9	0.56	0.0001	–0.0107	–0.0107	0.0644
(0.2, 0.4)	0.99	0.605	1.488	1.472	1.47	0.0484
(0.4, 0.2)	0.9	0.581	0.1721	0.1583	0.1582	0.0085
(0.4, 0.2)	0.99	0.612	2.288	2.272	2.269	0.0711
(0.4, 0.4)	0.9	0.761	0.1314	0.0972	0.0971	0.0436
(0.4, 0.4)	0.99	0.803	0.8717	0.8349	0.834	0.0391

In the first case, the characteristic function of $E[L_n | X]$ is proportional to the function $t \mapsto \sin(\sigma t/2)/(\sigma t)$, where $\sigma = \sum_{i=1}^n A_i b_i$. Simple calculations show that $I_M(x)$ behaves like a sum of quantities $\sin(C.M)/M^2$, for some constants C . Therefore, in the case of homogenous portfolios, the upper bound obtained in Theorem 1 will be $O(1/M^2 + M^4/n^2)$ that cannot be $o(1/n)$, whatever the choice of M . Moreover, $t \mapsto t\hat{G}_j(t)$ behaves like a sum of $t \mapsto \exp(iCt)$ for some real constants C . Then, this function is not integrable in this case and $J_M(x)$ will not be $o(1)$. Then, there is a risk that GAs do not improve the VaR estimates in the case of a uniform X r.v. This is exactly what we check in Table 6: granularity adjustments look like small noises. They do not improve (but do not deteriorate!) VaR calculations, even after taking into account the statistical uncertainty around the empirical VaR estimates.

Since the assumption of a uniform systematic random variable is not particularly realistic, we have led the same experiment when X follows a Bates distribution with m degrees of freedom. The case $m = 1$ corresponds to the uniform law. Through an integration by parts argument, we check easily that the characteristic function of $\mu_n(X)$ is $O(t^{-m})$. Thus, by another integration by parts argument, we can show that $I_M(x)$ behaves as a sum of quantities $\sin(C(x).M)/M^{m+1}$, for some constants $C(x)$ that depend on x . Therefore, a sufficient condition for the relevance of GAs is given by $m > 3$ (set $M = n^a$ with $a \in (1/(m + 1); 1/4)$). Note that the asymptotic behavior of $t \mapsto t\hat{G}_j(t)$ is the same as a sum of functions $t \mapsto \sin(bt)^r/t^{m-1}$ for some integers $r \geq 1$. We deduce $J_M(x) = O(M^{-m+1})$, that tends to zero when $m > 1$. Actually, GAs work pretty well even when $m \geq 3$: see Table 7. Otherwise, when $m = 2$, GAs do not improve EVaRs, exactly as in Table 6. Note that the strength

of the GA improvements does not increase with m necessarily, when $m \geq 3$. At some stage, apparently, we can even observe a relative loss of performance. For instance, when $(a_i, b_i) = (0.2, 0.2)$ for every name, the best job is obtained with $m = 6$, but not with $m = 9$ or $m = 12$ that provide worse results than $m = 3$.

To complete this experiment, assume now that X follows a beta $B(p, q)$ distribution, $p, q > 0$. Then, the characteristic function of $\mu_n(X)$ is a confluent hypergeometric function $M(p, p + q, iCt)$ for some constant C : see Johnson and Kotz (1969, Equation (43); 1970, chap. 24). When t is large and positive, following Morse and Feshbach (1953), we have

$$M(p, p + q, it) \sim \frac{\Gamma(p + q)}{\Gamma(p)} |t|^{-q} \exp(-iq\pi/2 + i |t|) + \frac{\Gamma(p + q)}{\Gamma(q)} |t|^{-p} \exp(ip\pi/2).$$

By an integration by parts, the asymptotic behavior of $I_M(x)$ is $O(M^{-(q+1)}/(1-x) + M^{-(p+1)})$. Therefore, for a given level x , our previous upper bounds should tend to zero quicker than $M^{-\min(q,p)-1}$. Then, it is likely that GAs work when $\min(p, q) > 3$ (set $M = n^a$ with $a \in (1/(\min(p, q) + 1); 1/4)$). Moreover, simple calculations show that $\hat{G}_j(t)$ can be written as a sum of functions $t \mapsto \exp(Cit)/t^{\min(p,q)}$, C being a constant. Then, $J_M(x)$ tends to zero when $\min(p, q) > 1$.

In the latter case, the results of our simulations are provided in Table 8. Clearly, some cases are significantly less convincing than those of the Tables 2 and 3. As expected, depending on the parameters values p and q , the granularity adjustment can improve (or not) the approximation of the true VaR level. Apparently, granular-

Table 7

Performances of the “Toy model” 2 when X follows a Bates distribution with m degrees of freedom (calculation of VaR by simulations): 1000 names, 100,000 draws, $\alpha = 90\%$. $\hat{\sigma}(\text{VaR})$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

m	(a_i, b_i)	VaR	VaR–EVaR/VaR	VaR–EVaR – GA1/VaR	VaR–EVaR – GA2/VaR	$\hat{\sigma}(\text{VaR})/\text{VaR}$
2	(0.2,0.2)	0.358	0.7608	0.8012	0.7992	0.1351
2	(0.2,0.4)	0.512	0.2826	0.2806	0.2803	0.1077
2	(0.4,0.2)	0.558	0.4881	0.4782	0.478	0.0558
2	(0.4,0.4)	0.712	0.2032	0.1737	0.1737	0.0758
3	(0.2,0.2)	0.348	1.216	0.1076	0.0981	0.0439
3	(0.2,0.4)	0.489	0.2986	–0.1531	–0.1535	0.1027
3	(0.4,0.2)	0.548	0.7719	–0.0411	–0.0387	0.0523
3	(0.4,0.4)	0.69	0.3566	0.0525	0.0540	0.0675
6	(0.2,0.2)	0.336	1.617	–0.0573	–0.1159	0.0508
6	(0.2,0.4)	0.464	0.6179	–0.0750	–0.0849	0.0604
6	(0.4,0.2)	0.536	1.014	–0.209	–0.232	0.0786
6	(0.4,0.4)	0.664	0.4318	–0.0343	–0.0372	0.0557
9	(0.2,0.2)	0.331	1.856	–0.2436	–0.2802	0.0947
9	(0.2,0.4)	0.454	0.9446	0.0701	0.0537	0.0830
9	(0.4,0.2)	0.532	1.343	–0.1847	–0.1671	0.0108
9	(0.4,0.4)	0.654	0.6557	0.0675	0.0634	0.0766
12	(0.2,0.2)	0.328	1.987	–0.4698	–0.457	0.109
12	(0.2,0.4)	0.447	0.9025	–0.1269	–0.1446	0.0970
12	(0.4,0.2)	0.529	1.421	–0.3642	–0.2895	0.0607
12	(0.4,0.4)	0.647	0.6235	–0.0684	–0.0727	0.0252

Table 8
Performances of the “Toy model” 2 when X is beta (calculation of VaR by simulations): 1000 names, 100,000 draws. $a_i = b_i = 0.4$. $\hat{\sigma}(\text{VaR})$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

(p, q)	α	VaR	VaR-EVaR/VaR	VaR-EVaR – GA1/VaR	VaR-EVaR – GA2/VaR	$\hat{\sigma}(\text{VaR})/\text{VaR}$
(1,1)	0.9	0.76	0.0003	-0.0338	-0.0338	0.0679
(1,3)	0.9	0.615	0.1088	-0.1173	-0.1172	0.1084
(1,5)	0.9	0.55	0.4339	0.0683	0.0717	0.0575
(1,7)	0.9	0.515	0.5583	0.0503	0.0601	0.0993
(3,1)	0.9	0.787	0.1022	0.1212	0.1208	0.0518
(3,3)	0.9	0.703	0.2357	0.0039	0.0014	0.0807
(3,5)	0.9	0.64	0.239	-0.0779	-0.0788	0.1042
(3,7)	0.9	0.599	0.4959	0.0936	0.0942	0.0595
(5,1)	0.9	0.794	0.2949	0.3642	0.3632	0.0089
(5,3)	0.9	0.734	0.2528	-0.0116	-0.0182	0.0724
(5,5)	0.9	0.681	0.2039	-0.1249	-0.1273	0.0747
(5,7)	0.9	0.642	0.3444	-0.0501	-0.0521	0.0549
(1,1)	0.99	0.803	0.8722	0.8354	0.8345	0.0601
(1,3)	0.99	0.716	0.3047	-0.0562	-0.0541	0.0721
(1,5)	0.99	0.643	0.3504	-0.1211	-0.1132	0.1089
(1,7)	0.99	0.596	0.5373	-0.0647	-0.0449	0.1621
(3,1)	0.99	0.811	1.521	1.534	1.529	0.0389
(3,3)	0.99	0.761	0.4285	-0.1088	-0.1116	0.0415
(3,5)	0.99	0.709	0.4981	-0.0550	-0.0404	0.0445
(3,7)	0.99	0.667	0.671	0.0428	0.0716	0.1262
(5,1)	0.99	0.814	1.819	1.881	1.874	0.0489
(5,3)	0.99	0.778	0.8132	0.1006	0.0800	0.0867
(5,5)	0.99	0.737	0.7322	0.0823	0.1113	0.0858
(5,7)	0.99	0.7	0.6975	0.0088	0.0462	0.0927

ity adjustments improve significantly the infinitely granular approximation when $\min(p, q) \geq 3$. When $p = 1$ or $q = 1$, it is not always the case, particularly when $(p, q) = (1, 1)$. Interestingly, when one of the parameters p or q goes up, when the other is staying one, the performances of the obtained GAs are improved, in line with our analysis of $r_{n,M}$. Note there is no symmetry of our results w.r.t. the parameters p and q : when $(p, q) = (1, 5)$, GAs do a pretty well job, but not in the case $(p, q) = (5, 1)$.

3.3. Granularity adjustments in a realistic credit portfolio model: Toy model 3

Some people could argue that the identified difficulties above could arise due to the discreteness of the underlying loss distributions. That is why we test now a third specification, an extension of our Toy model 2. It will be more realistic than previously, by introducing individual random severities S_i (also called “losses given default”, or “LGD”), for every $i = 1, \dots, n$. The new loss model is

$$Z_i = S_i \mathbf{1}(X_i \leq a_i + b_i X), \quad S_i = c_i T_i + d_i X, \quad (15)$$

where all the r.v. X_i and $T_i, i = 1, \dots, n$, are mutually independent and follow a uniform distribution on $(0, 1)$. As previously, the random factor X will follow a Bates distribution and/or a Beta distribution, and will be independent of the other variables. To take into account the well-known dependency between recovery rates and systematic risk (see Altman et al., 2005, e.g.), we have set an explicit link between the systematic default risk factor X and the individual severities S_i , through the coefficients d_i : when X goes up, the systematic credit risk goes up, and so the severity rates. All the coefficients a_i, b_i, c_i and d_i belong to $(0, 1)$. For convenience, we have calculated GA1 only (usual granularity adjustments) in this model. Note that there are a lot of different ways of introducing recovery risk in such credit risk models: see Pykhtin (2003), Chen and Joslin (2012), Amraoui et al. (2012), among others. Our specification has been chosen for the sake of simplicity only.

With these new specifications, it is difficult to check the order of magnitude of the remainder term $r_{n,M}(x)$, or even of its components $I_M(x)$ and $J_M(x)$. Indeed, $\mu_n(X)$ is now a quadratic function of X , and finding explicit evaluations of our previous Fourier transforms

becomes an impossible task. Therefore, we will inspect the results only qualitatively, to check the robustness of GA techniques w.r.t. to relatively natural model modifications. Therefore, we will compare the Toy model 3 results with those obtained with the previous Toy model 2, notably in the case of independence between default and recovery risks (when $d_i = 0$ for every i).

When the law of X is Bates, the results are shown in Table 9. Globally, they confirm those obtained in Table 7: the higher m , the better are the GA results, and GAs improve first-order approximations when $m \geq 3$. Nonetheless, when $m = 2$, it is difficult to conclude anything because GA and EVaR approximations are very close. When there is no systematic recovery risk ($d_i = 0$), the additional amount of smoothness induced by idiosyncratic recoveries improves the relative GA performances, what is good news for practitioners. At the opposite, when there is a significant level of dependency between default and recovery risks, the gain obtained with GAs is reduced slightly, even if it remains noteworthy.

This is confirmed globally when X follows a Beta distribution. Table 10 corresponds to the case of random but purely idiosyncratic recoveries ($d_i = 0$, for every i). In the latter case, the results seem to be slightly better than those obtained in Table 8, i.e. without smoothing individual losses: GAs improve EVaR approximations when $\min(p, q) \geq 2$. After the introduction of systematic recovery risk, EVARs evaluate VaRs very efficiently and we cannot conclude anything for sure concerning GA improvements because of the statistical uncertainty around VaRs.³ The corresponding empirical results can be provided under request.

To conclude this empirical section, it would be nice to understand/predict when GAs provide good results or not. Apparently, this does not seem to be a problem of discreteness “per se”: there are some doubts around the continuous Toy model 1 with exponential random variables, and the discrete Toy model 2 provides convenient results for some particular specifications and some parameter values. Nonetheless, smoothing the underlying losses

³ “Estimation of the GA by simulation is difficult enough, because simulation noise tends to swamp the small gap between VaR and asymptotic VaR.” (Gordy and Marrone, 2012)

Table 9

Performances of the “Toy model” 3 when X follows a Bates distribution with m degrees of freedom (calculation of VaR by simulations): 1000 names, 100,000 draws, $\alpha = 90\%$, $(a_i, b_i) = (0.4, 0.4)$. $\hat{\sigma}(\text{VaR})$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

m	(c_i, d_i)	VaR	VaR-EVaR/VaR	VaR-EVaR – GA1/VaR	$\hat{\sigma}(\text{VaR})/\text{VaR}$
2	(0.4,0)	0.1426	0.4786	0.4725	0.0632
2	(0.4,0.2)	0.2528	0.1841	0.1795	0.1256
2	(0.4,0.4)	0.3630	0.0838	0.0849	0.1377
3	(0.4,0)	0.1382	0.5799	0.0602	0.0560
3	(0.4,0.2)	0.2369	0.3125	0.1217	0.0972
3	(0.4,0.4)	0.3355	0.2396	0.1079	0.1179
4	(0.4,0)	0.1359	0.1359	0.0336	0.0498
4	(0.4,0.2)	0.2289	0.2070	-0.0246	0.0939
4	(0.4,0.4)	0.3220	0.2138	0.0537	0.1117
6	(0.4,0)	0.1333	0.8972	0.0256	0.0399
6	(0.4,0.2)	0.2192	0.3712	0.0712	0.0693
6	(0.4,0.4)	0.3053	0.2745	0.0665	0.0899
8	(0.4,0)	0.1317	0.9808	-0.0456	0.0368
8	(0.4,0.2)	0.2137	0.3440	-0.0136	0.0727
8	(0.4,0.4)	0.2960	0.4362	0.1882	0.0916
10	(0.4,0)	0.1308	1.0932	-0.0689	0.0327
10	(0.4,0.2)	0.2100	0.3899	-0.0184	0.0631
10	(0.4,0.4)	0.2900	0.3051	0.0210	0.0769
12	(0.4,0)	0.1302	1.1484	-0.1363	0.0355
12	(0.4,0.2)	0.2074	0.4460	-0.0079	0.0574
12	(0.4,0.4)	0.2858	0.3851	0.0690	0.0735

Table 10

Performances of the “Toy model” 3 when X is beta (calculation of VaR by simulations): 1000 names, 100,000 draws. $a_i = b_i = 0.4$ and $\alpha = 90\%$. $\hat{\sigma}(\text{VaR})$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

(c_i, d_i)	(p, q)	VaR	VaR-EVaR/VaR	VaR-EVaR – GA1/VaR	$\hat{\sigma}(\text{VaR})/\text{VaR}$
(0.2,0)	(1,1)	0.0760	0.0972	0.0849	0.0509
(0.2,0)	(1,2)	0.0675	0.1907	-0.0705	0.0816
(0.2,0)	(1,3)	0.0616	0.2730	-0.1046	0.0899
(0.2,0)	(1,5)	0.0551	0.5276	-0.0700	0.0835
(0.2,0)	(1,7)	0.0516	0.8675	0.0491	0.0735
(0.2,0)	(2,1)	0.0781	0.2579	0.3163	0.0335
(0.2,0)	(2,2)	0.0723	0.3475	0.04554	0.0537
(0.2,0)	(2,3)	0.0674	0.3683	-0.0252	0.0623
(0.2,0)	(2,5)	0.0608	0.4662	-0.0867	0.0634
(0.2,0)	(2,7)	0.0565	0.8470	0.1351	0.0704
(0.2,0)	(3,1)	0.0789	0.4956	0.6215	0.0261
(0.2,0)	(3,2)	0.0745	0.2745	-0.0802	0.0431
(0.2,0)	(3,3)	0.0703	0.3327	-0.1016	0.0564
(0.2,0)	(3,5)	0.0641	0.5544	-0.0069	0.0612
(0.2,0)	(3,7)	0.0599	0.6558	-0.0360	0.0650
(0.2,0)	(5,1)	0.0798	0.8648	1.1233	0.0225
(0.2,0)	(5,2)	0.0767	0.5210	0.0551	0.0284
(0.2,0)	(5,3)	0.0736	0.5502	0.0203	0.0397
(0.2,0)	(5,5)	0.0683	0.6440	0.0277	0.0459
(0.2,0)	(5,7)	0.0644	0.6931	-0.0179	0.0537
(0.2,0)	(7,1)	0.0802	1.1040	1.4942	0.0204
(0.2,0)	(7,2)	0.0777	0.6953	0.1158	0.0262
(0.2,0)	(7,3)	0.0752	0.6514	0.0192	0.0332
(0.2,0)	(7,5)	0.0708	0.6817	-0.0057	0.0393
(0.2,0)	(7,7)	0.0672	0.7331	-0.0241	0.0455

and adding more idiosyncratic noise cannot worsen the picture (Toy model 3).

Martin and Tasche (2007) have warned against the existence of local maxima, particularly of $\mu_n(\mathbf{X})$, that can deteriorate GAs. After a careful inspection of the empirical loss distributions given by our Toy models, it does not seem to be the case here. Since we consider homogenous portfolios and a very large number of random draws, the empirical loss distributions look very smooth. Moreover, since $\mu_n(X)$ is a linear function of X in our examples, its law is very smooth and does not show local maxima.

Actually, it seems to us that GA techniques suffer when the underlying loss distributions exhibit tails that do not decrease to zero sufficiently rapidly. The tail behavior of these distributions can be measured through the regularity and the decay rate of their associated characteristic functions, as proposed in Theorem 1.

Indeed, it is known that the Fourier transform maps the Schwartz space (the space of infinitely differentiable functions decaying rapidly at infinity) into itself. For a given model, a preliminary analysis in terms of Fourier transforms often provides useful indications concerning the relevance of GAs, as we have seen with Toy model 2 (Gaussian) for instance.

4. Conclusions

In this article, we have proved a general formula for granularity adjustments. Our framework encompasses multi-factor models, random recoveries and discrete loss distributions. Through some Fourier transform techniques, we have exhibited an upper bound of the error terms of such approximations, and introduced several GA-based formulas for VaR calculations. We have tested the

performances of granularity expansions in the case of several simple credit portfolio specifications.

For rapidly decreasing loss distributions, for instance Gaussian or Poisson-like loss distributions, GAs improve EVar approximations very significantly. Otherwise, it is possible to exhibit some rather simple models that provide less clear-cut conclusions. There are situations, that can often be guessed from our theoretical results, where GAs do not seem to improve EVaRs, even after taking account the statistical uncertainty around VaR estimates. Nonetheless, we have not found any example where GAs deteriorate EVaRs clearly.

We have stressed the benefit from introducing recovery risks and/or random exposures in credit risk models, before applying granularity approximations. We advise to check the regularity of the underlying loss distribution of the model, before using “blindly” such approximations. We have provided some sufficient conditions to be insured these techniques can be used in safety. Nonetheless, our empirical results are fragile and other experiments are surely necessary. In particular, it would be good to evaluate the true VaRs by simulation very precisely, but this implies considering several millions of random draws at least. Then, the evaluation of the VaR statistical uncertainties by bootstrap would become a computationally intensive experiment.

Note that the same GA analysis could be led with Expected Shortfalls, even if obtaining the equivalent of [Theorem 2](#) would be significantly more painful. But there are some hopes to get better empirical results in the most demanding models. Indeed, as noticed in [Martin and Tasche \(2007\)](#), GAs for Expected Shortfalls seem to behave better than for VaRs, especially when there are local modes in the distribution tails (even if this was not the case with the toy models we have considered, apparently).

Finally, note that other expansion techniques exist in the literature. For instance, [Voropaev \(2011\)](#) approximated VaR or Expected Shortfall levels recently through infinite expansions based on Hermite polynomial. There, the successive terms decrease like $\rho^p/p!$, $p = 1, 2, \dots$, for some correlation level ρ , the portfolio size being fixed. Alternatively, usual GA successive terms decrease as n^{-r} for “large” portfolio sizes n and successive powers $r > 0$. Therefore, the nature of these asymptotic expansions are very different and need further investigation.

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Appendix A. Proof of [Theorem 1](#)

Denote by $p_j(\mathbf{X})$ the default probability of j before T , conditionally on the factor \mathbf{X} . Therefore, by setting

$$\psi_j(\mathbf{X}, t) := p_j(\mathbf{X}) (E[\exp(itA_{jn}Z_j) | D_j = 1, \mathbf{X}] - 1) = E[\exp(itA_{jn}Z_j) | \mathbf{X}] - 1,$$

and due to the conditional independence assumption, we can write

$$\begin{aligned} \chi_L(t) &= E[E[\exp(itL_n) | \mathbf{X}]] = E\left[\prod_{j=1}^n E[\exp(itA_{jn}Z_j) | \mathbf{X}]\right] \\ &= E\left[\exp\left(\sum_{j=1}^n \ln\{1 + \psi_j(\mathbf{X}, t)\}\right)\right] \\ &= E\left[\exp\left(\sum_{j=1}^n \psi_j(\mathbf{X}, t) - \psi_j^2(\mathbf{X}, t)/2 + r_{\chi j}(\mathbf{X}, t)\right)\right] \end{aligned} \tag{A.1}$$

where

$$r_{\chi j}(\mathbf{X}, t) := \psi_j^3(\mathbf{X}, t) \int_0^1 \frac{u^2}{(1 + u\psi_j(\mathbf{X}, t))} du.$$

The latter relation is obtained by integrating the relation

$$\frac{1}{1 + \psi_j u} = 1 - \psi_j u + \frac{\psi_j^2 u^2}{1 + \psi_j u}$$

between 0 and 1. A Taylor expansion with an integral form remainder provides

$$\begin{aligned} \psi_j(\mathbf{X}, t) &= E[\exp(itA_{jn}Z_j) | \mathbf{X}] - 1 \\ &= itA_{jn}E[Z_j | \mathbf{X}] - \frac{t^2}{2}A_{jn}^2E[Z_j^2 | \mathbf{X}] + r_{\psi j}(\mathbf{X}, t), \end{aligned} \tag{A.2}$$

where

$$r_{\psi j}(\mathbf{X}, t) := -\frac{it^3A_{jn}^3}{2} \int_0^1 E[\exp(itvA_{jn}Z_j)Z_j^2 | \mathbf{X}](1 - v)^2 dv.$$

By putting [\(A.2\)](#) into [\(A.1\)](#), we get

$$\begin{aligned} \chi_L(t) &= E\left[\exp\left(\sum_{j=1}^n \left\{itA_{jn}E[Z_j | \mathbf{X}] - \frac{t^2}{2}A_{jn}^2E[Z_j^2 | \mathbf{X}]\right\}\right.\right. \\ &\quad \left.\left.+ \frac{t^2}{2}A_{jn}^2E[Z_j^2 | \mathbf{X}] + \bar{r}_{\chi j}(\mathbf{X}, t)\right)\right], \end{aligned}$$

where $\bar{r}_{\chi j}(\mathbf{X}, t)$ can be specified explicitly:

$$\begin{aligned} \bar{r}_{\chi j}(\mathbf{X}, t) &= r_{\psi j}(\mathbf{X}, t) - \frac{1}{2}\left\{\frac{t^2}{2}A_{jn}^2E[Z_j^2 | \mathbf{X}] - r_{\psi j}(\mathbf{X}, t)\right\}^2 + itA_{jn}E[Z_j \\ &\quad | \mathbf{X}]\left\{\frac{t^2}{2}A_{jn}^2E[Z_j^2 | \mathbf{X}] - r_{\psi j}(\mathbf{X}, t)\right\} + r_{\chi j}(\mathbf{X}, t). \end{aligned}$$

By another Taylor expansion, we get finally

$$\begin{aligned} \chi_L(t) &= E\left[\exp\left(it\mu_n(\mathbf{X}) - \frac{t^2}{2}\sum_{j=1}^n A_{jn}^2V_j(\mathbf{X}) + \sum_{j=1}^n \bar{r}_{\chi j}(\mathbf{X}, t)\right)\right] \\ &= E\left[\exp(it\mu_n(\mathbf{X})) \cdot \left\{1 - \frac{t^2}{2}\sum_{j=1}^n A_{jn}^2V_j(\mathbf{X}) + \bar{r}_{\chi}(\mathbf{X}, t)\right\}\right], \end{aligned}$$

where

$$\begin{aligned} \bar{r}_{\chi}(\mathbf{X}, t) &:= \sum_{j=1}^n \bar{r}_{\chi j}(\mathbf{X}, t) + \xi(\mathbf{X}, t)^2 \int_0^1 (1 - u) \exp(\xi(\mathbf{X}, t)u) du, \\ \xi(\mathbf{X}, t) &:= -\frac{t^2}{2}\sum_{j=1}^n A_{jn}^2V_j(\mathbf{X}) + \sum_{j=1}^n \bar{r}_{\chi j}(\mathbf{X}, t). \end{aligned}$$

We deduce

$$\begin{aligned} \chi_L(t) &= \chi_{\mu(\mathbf{X})}(t) - \frac{t^2}{2}\sum_{j=1}^n A_{jn}^2E[\exp(it\mu_n(\mathbf{X}))V_j(\mathbf{X})] \\ &\quad + E[\exp(it\mu_n(\mathbf{X}))\bar{r}_{\chi}(\mathbf{X}, t)]. \end{aligned} \tag{A.3}$$

A careful inspection of the proof of [Theorem 4.4.3.](#) in [Kawata \(1972\)](#) provides:

Lemma 6. *Let F be a bounded non decreasing function on \mathbb{R} . Let χ_F be the associated characteristic function, i.e. $\chi_F(x) := \int \exp(itx) dF(t)$. Then, for every real number x ,*

$$\frac{1}{2}[F(x + 0) + F(x - 0)] = \frac{F(+\infty)}{2} - \frac{1}{\pi} \int_0^{+\infty} \text{Im}\left(\chi_F(t) \frac{\exp(-itx)}{t}\right) dt.$$

This is the version of the [Zolotarev \(1957\)](#) formula, but adapted to possibly discontinuous distributions. Note that the consider the

usual average between $F(x + 0)$ and $F(x - 0)$, and not any arbitrary weighted average. This is necessary to invoke powerful tools of the usual Fourier theory, in particular Fourier's single integral Theorem (Th. 4.3.1. in Kawata, 1972). By applying the latter formula to the cdfs of the loss distributions L_n and $\mu_n(\mathbf{X})$, we get particularly

$$\frac{1}{2}[P(L_n < x + 0) + P(L_n < x - 0)] = \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} \text{Im} \left(\frac{\chi_L(t)}{t \exp(itx)} \right) dt,$$

$$\begin{aligned} & \frac{1}{2}[P(\mu_n(\mathbf{X}) < x + 0) + P(\mu_n(\mathbf{X}) < x - 0)] \\ &= \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} \text{Im} \left(\frac{\chi_{\mu(\mathbf{X})}(t)}{t \exp(itx)} \right) dt, \end{aligned}$$

for every x . Set

$$T_{n,M}(x) := \frac{1}{2\pi} \sum_{j=1}^n A_{jn}^2 \int_0^M \text{Im} \{ E[\exp(it\mu_n(\mathbf{X}))V_j(\mathbf{X})] t \exp(-itx) \} dt,$$

for any $M \in \mathbb{R} = \mathbb{R} \cup \{+\infty\}$. Clearly, the granularity adjustment will come from the term $T_{n,\infty}(x)$ (see the second term on the r.h.s. of Eq. (A.3)). Note that

$$\begin{aligned} E[\exp(it\mu_n(\mathbf{X}))V_j(\mathbf{X})] &= E[\exp(it\mu_n(\mathbf{X}))E[V_j(\mathbf{X}) | \mu_n(\mathbf{X})]] \\ &= \int \exp(itv)\kappa_j(v) dv \text{ under A1, and} \end{aligned}$$

$$\begin{aligned} E[\exp(it\mu_n(\mathbf{X}))V_j(\mathbf{X})] &= \int \exp(it\mu_n(\mathbf{x}))V_j(\mathbf{x})f_X(\mathbf{x}) d\mathbf{x} \\ &= \int \exp(itv)\kappa_j(v) dv \text{ under A2.} \end{aligned}$$

Note that $v \mapsto G_j(v) := \int_{-\infty}^v \kappa_j$ is a continuous and monotonic function. We have assumed it is bounded and non decreasing. Its associated characteristic function is $\widehat{G}_j(v) = \int \exp(itv)\kappa_j(t) dt$. Applying Lemma 6, we get

$$G_j(x) = \frac{G_j(+\infty)}{2} - \frac{1}{\pi} \int_0^{+\infty} \text{Im}(\widehat{G}_j(t) \exp(-itx)/t) dt.$$

Actually, such a function is derivable almost surely. Since $\widehat{G}_j \in L^1(\mathbb{R})$, we can apply the dominated convergence theorem and calculate derivatives inside the integrand (see Billingsley (1995), Theorem 16.8 for details). We get

$$G'_j(x) = \frac{1}{\pi} \int_0^{+\infty} \text{Im}(\widehat{G}_j(t) i \exp(-itx)) dt.$$

Since we have assumed the function $t \mapsto t\widehat{G}_j(t)$ belongs to L^1 , we get further

$$\begin{aligned} G''_j(x) &= \frac{1}{\pi} \int_0^{+\infty} \text{Im}(\widehat{G}_j(t) t \exp(-itx)) dt \\ &= \frac{1}{\pi} \int_0^{+\infty} \text{Im}(E[\exp(it\mu_n(\mathbf{X}))V_j(\mathbf{X})] t \exp(-itx)) dt. \end{aligned}$$

The previous term $T_{n,\infty}(x)$ can be rewritten

$$T_{n,\infty}(x) := \frac{1}{2} \sum_{j=1}^n A_{jn}^2 G''_j(x) = \frac{1}{2} \sum_{j=1}^n A_{jn}^2 \kappa'_j(x).$$

Therefore, we have got the interesting result:

$$\begin{aligned} & \frac{1}{2}[P(L_n < x + 0) + P(L_n < x - 0)] \\ &= \frac{1}{2}[P(\mu_n(\mathbf{X}) < x + 0) + P(\mu_n(\mathbf{X}) < x - 0)] \\ &+ \frac{1}{2\pi} \sum_{j=1}^n A_{jn}^2 \kappa'_j(x) - \frac{1}{\pi} \\ &\times \int_0^{+\infty} \text{Im} \{ E[\exp(it\mu_n(\mathbf{X}))\tilde{r}_\chi(\mathbf{X}, t)] \exp(-itx)/t \} dt. \end{aligned} \quad (\text{A.4})$$

It remains to deal with the remainder term $R_\infty(x)$, where we set for any $M \in \mathbb{R}$

$$R_M(x) := \frac{(-1)}{\pi} \int_0^M \text{Im} \{ E[\exp(it\mu_n(\mathbf{X}))\tilde{r}_\chi(\mathbf{X}, t)] \exp(-itx)/t \} dt.$$

To evaluate $R_\infty(x)$, we cannot use the same method as for $T_{n,\infty}(x)$. Indeed, $E[\exp(it\mu_n(\mathbf{X}))\tilde{r}_\chi(\mathbf{X}, t)]$ is not the Fourier transform of a non decreasing function. Moreover, this integral cannot be calculated explicitly, unfortunately. Actually, to show that the latter remainder term $R_\infty(x)$ is negligible w.r.t. the ‘‘granularity-type’’ term $T_{n,\infty}(x)$, we come back to Eq. (A.3) and restrict the domain of integration to some interval $[0, M]$. To be specific, by integrating (A.3), we get

$$\begin{aligned} \frac{1}{\pi} \int_0^M \text{Im} \left(\chi_L(t) \frac{\exp(-itx)}{t} \right) dt &= \frac{1}{\pi} \int_0^M \text{Im} \left(\chi_{\mu(\mathbf{X})}(t) \frac{\exp(-itx)}{t} \right) dt \\ &- T_{n,M}(x) - R_M(x) \end{aligned} \quad (\text{A.5})$$

for every $M > 0$. But, as we have seen before,

$$\begin{aligned} \frac{1}{\pi} \int_0^M \text{Im} \left(\chi_L(t) \frac{\exp(-itx)}{t} \right) dt &= \frac{1}{2} - \frac{1}{2}[P(L_n < x + 0) + P(L_n < x - 0)] \\ &- R_{L,M}^*(x), \end{aligned} \quad (\text{A.6})$$

where

$$R_{L,M}^*(x) := \frac{1}{\pi} \int_M^{+\infty} \text{Im}(\chi_L(t) \exp(-itx)/t) dt.$$

Note that

$$|R_{L,M}^*(x)| \leq \frac{1}{\pi M} \int_M^{+\infty} |\chi_L|.$$

Similarly,

$$\begin{aligned} & \frac{1}{\pi} \int_0^M \text{Im}(\chi_{\mu(\mathbf{X})}(t) \exp(-itx)/t) dt \\ &= \frac{1}{2} - \frac{1}{2}[P(\mu_n(\mathbf{X}) < x + 0) + P(\mu_n(\mathbf{X}) < x - 0)] - R_{L|\mathbf{X},M}^*(x), \end{aligned}$$

where

$$R_{L|\mathbf{X},M}^*(x) := \frac{1}{\pi} \int_M^{+\infty} \text{Im}(\chi_{\mu(\mathbf{X})}(t) \exp(-itx)/t) dt$$

and

$$|R_{L|\mathbf{X},M}^*(x)| \leq \frac{1}{\pi M} \int_M^{+\infty} |\chi_{\mu(\mathbf{X})}|.$$

Moreover, the same reasoning as above proves that

$$\begin{aligned} T_{n,M}(x) &= T_{n,\infty}(x) - R_{T,M}^*(x), \\ R_{T,M}^*(x) &= \frac{1}{2\pi} \sum_{j=1}^n A_{jn}^2 \int_M^{+\infty} \text{Im}(\widehat{G}_j(t) t \exp(-itx)) dt. \end{aligned}$$

Note that

$$|R_{T,M}^*(x)| \leq \frac{1}{2\pi} \sum_{j=1}^n A_{jn}^2 \int_M^{+\infty} |t\widehat{G}_j(t)| dt.$$

To tackle $R_M(x)$, note that $\tilde{r}_\chi(\mathbf{X}, t)$ can be rewritten as a sum of terms that will be considered as negligible w.r.t. $T_{n,\infty}$. Indeed, for every $0 \leq t \leq M$ and every j, \mathbf{X} ,

$$|\psi_j(\mathbf{X}, t)| \leq tA_{jn} |E[Z_j | \mathbf{X}]| \leq tA_{jn}\bar{Z}_j,$$

by the mean value theorem. Actually, since we choose M such that $MA_{jn}\bar{Z}_j \leq 1 - \varepsilon$ for every j and n , we obtain

$$|r_{Z_j}(\mathbf{X}, t)| \leq |\psi_j^3(\mathbf{X}, t)| \int_0^1 \frac{u^2}{1 - (1 - \varepsilon)u} du := |\psi_j^3(\mathbf{X}, t)| \cdot k_\varepsilon.$$

Note that

$$|r_{\psi,j}(\mathbf{X}, t)| \leq \frac{t^3 A_{jn}^3 \bar{Z}_j E[Z_j^2 | \mathbf{X}]}{6} \leq \frac{t^3 A_{jn}^3 \bar{Z}_j^3}{6}.$$

Therefore,

$$\begin{aligned} |\bar{r}_{\chi,j}(\mathbf{X}, t)| &\leq |r_{\psi,j}(\mathbf{X}, t)| + \frac{1}{2} \left\{ \frac{t^2}{2} A_{jn}^2 \bar{Z}_j^2 + |r_{\psi,j}(\mathbf{X}, t)| \right\}^2 \\ &\quad + |t| A_{jn} \bar{Z}_j \left\{ \frac{t^2}{2} A_{jn}^2 \bar{Z}_j^2 + |r_{\psi,j}(\mathbf{X}, t)| \right\} + |r_{\chi,j}(\mathbf{X})| \\ &\leq \left[\frac{3}{2} + k_\varepsilon \right] \cdot (t A_{jn} \bar{Z}_j)^3. \end{aligned}$$

We deduce

$$|\bar{r}_\chi(\mathbf{X}, t)| \leq \left[\frac{19}{18} + k_\varepsilon \right] \cdot |t|^3 \sum_{j=1}^n A_{jn}^3 \bar{Z}_j^3 + \xi_{n,t}^2 \exp(\xi_{n,t}),$$

where

$$\xi_{n,t} := \frac{t^2}{2} \sum_{j=1}^n A_{jn}^2 V_j(\mathbf{X}) + \left[\frac{19}{18} + k_\varepsilon \right] \cdot |t|^3 \sum_{j=1}^n A_{jn}^3 \bar{Z}_j^3.$$

Therefore, we obtain

$$\begin{aligned} |R_M(x)| &\leq E \left[\frac{1}{\pi} \int_0^M |\bar{r}_\chi(\mathbf{X}, t)| / t dt \right] \\ &\leq \left[\frac{19}{18} + k_\varepsilon \right] \cdot \frac{|M|^3}{3\pi} \sum_{j=1}^n A_{jn}^3 \bar{Z}_j^3 + \frac{1}{\pi} \int_0^M E[\xi_{n,t}^2 \exp(\xi_{n,t})] \frac{dt}{t}. \end{aligned}$$

Since

$$0 \leq \xi_{n,t} \leq \frac{t^2}{2} \left[1 + \frac{19}{9} + 2k_\varepsilon \right] \sum_{j=1}^n A_{jn}^2 \bar{Z}_j^2 := t^2 \sigma^2,$$

then we have

$$\begin{aligned} \int_0^M E[\xi_{n,t}^2 \exp(\xi_{n,t})] \frac{dt}{t} &\leq \int_0^M \sigma^4 t^3 \exp(t^2 \sigma^2) dt \\ &\leq \exp(M^2 \sigma^2) \cdot [\exp(-M^2 \sigma^2) - 1 + M^2 \sigma^2] / 2 \\ &\leq \frac{M^4 \sigma^4}{2} \exp(M^2 \sigma^2) \\ &\leq \frac{M^4}{2} \left[\frac{14}{9} + k_\varepsilon \right]^2 \left\{ \sum_{j=1}^n A_{jn}^2 \bar{Z}_j^2 \right\}^2 \exp \left(\left[\frac{14}{9} + k_\varepsilon \right] \sum_{j=1}^n A_{jn}^2 \bar{Z}_j^2 M^2 \right), \end{aligned}$$

proving the given upper bound for $R_M(x)$. Finally, we deduce

$$\begin{aligned} \frac{1}{2} [P(L_n < x + 0) + P(L_n < x - 0)] &= \frac{1}{2} [P(\mu_n(\mathbf{X}) < x + 0) \\ &\quad + P(\mu_n(\mathbf{X}) < x - 0)] \\ &\quad + T_{n,\infty}(x) - R_{L,M}^*(x) \\ &\quad + R_M(x) + R_{L,\mathbf{X},M}^*(x) - R_{T,M}^*(x), \end{aligned}$$

proving the result. \square

Appendix B. Details of GA calculations for the models in Section 3

For a given level α and every particular model considered in Section 3, we detail the approximated VaRs:

$$E\text{VaR}_\alpha := E\text{VaR}_\alpha = E[L_n | X = \text{VaR}_\alpha(X)].$$

The associated granularity adjustments, as given in Corollaries 3 and 4, are

$$\text{VaR}_\alpha(L_n) \simeq E\text{VaR}_\alpha - \text{GA}_k, \quad k = 1, 2,$$

where

$$\text{GA}_1 := \frac{1}{2f_X(\text{VaR}_{\alpha,X})} \sum_{j=1}^n A_{jn}^2 \frac{d}{dt} \left[\frac{V_j(t)f_X(t)}{\mu'_n(t)} \right]_{|t=\text{VaR}_{\alpha,X}},$$

and, by setting $\bar{V}_{n,\alpha} = (\text{VaR}_\alpha(L_n) + E\text{VaR}_\alpha)/2$,

$$\begin{aligned} \text{GA}_2 &:= \frac{1}{2f_X(\mu_n^{-1}(\text{VaR}_{n,\alpha}))} \cdot \frac{\mu'_n(\mu_n^{-1}(\text{VaR}_{n,\alpha}))}{\mu'_n(\mu_n^{-1}(\text{VaR}_\alpha(L_n)))} \\ &\quad \times \sum_{j=1}^n A_{jn}^2 \frac{d}{dt} \left[\frac{V_j(t)f_X(t)}{\mu'_n(t)} \right]_{|t=\mu_n^{-1}(\text{VaR}_\alpha(L_n))}. \end{aligned}$$

B.1. Toy model 1

$$E\text{VaR}_\alpha = \left(\sum_{i=1}^n A_i a_i \right) + \left(\sum_{i=1}^n A_i \sigma_i \rho_i \right) \text{VaR}_{\alpha,X} := C_0 + C_1 \Phi^{-1}(\alpha).$$

Since the distribution of the losses L_n is Gaussian, we get a closed-form formula value-at-risk

$$\text{VaR}_\alpha(L_n) = C_0 + \sigma \Phi^{-1}(\alpha),$$

$$\sigma^2 = E[L_n^2] - E[L_n]^2 = \sum_{i=1}^n A_i^2 \sigma_i^2 + \sum_{i,j=1}^n A_i A_j \sigma_i \sigma_j \rho_i \rho_j.$$

$$\text{GA}_1 = \frac{(-1)}{2} \cdot \frac{\sum_{i=1}^n A_i^2 \sigma_i^2 (1 - \rho_i^2)}{\sum_{i=1}^n A_i \sigma_i \rho_i} \cdot \Phi^{-1}(\alpha).$$

By setting

$$\tilde{V}_\alpha := \frac{\text{VaR}_\alpha(L_n) - C_0}{C_1}, \quad \check{V}_\alpha := \frac{\bar{V}_{n,\alpha} - C_0}{C_1},$$

we get

$$\text{GA}_2 = \frac{(-1)}{2\phi(\tilde{V}_\alpha)} \cdot \frac{\sum_{i=1}^n A_i^2 \sigma_i^2 (1 - \rho_i^2)}{\sum_{i=1}^n A_i \sigma_i \rho_i} \cdot \tilde{V}_\alpha \phi(\tilde{V}_\alpha).$$

Replacing the Gaussian r.v.s by exponential laws with parameter one, we get

$$\text{GA}_1 = \frac{(-1)}{2} \cdot \frac{\sum_{i=1}^n A_i^2 \sigma_i^2 (1 - \rho_i^2)}{\sum_{i=1}^n A_i \sigma_i \rho_i}.$$

B.2. Toy model 2

Here, we estimate numerically the value-at-risk $\text{VaR}_\alpha(L_n)$ and $\text{VaR}_{\alpha,X}$ (in the case of a Bates law). Since X belongs to $(0, 1)$, the expectation of the losses (conditionally on X) is a linear function of X . We deduce

$$E\text{VaR}_\alpha = \left(\sum_{i=1}^n A_i b_i \right) \text{VaR}_{\alpha,X} + \left(\sum_{i=1}^n A_i a_i \right) := C \cdot \text{VaR}_{\alpha,X} + D. \quad (\text{B.1})$$

When X follows a Bates distribution with m degrees of freedom, its cdf and its density are given by

$$F_X(t) := \frac{1}{m!} \sum_{k=0}^{\lfloor mt \rfloor} (-1)^k C_m^k (mt - k)^m, \quad \text{and}$$

$$f_X(t) := \frac{m}{(m-1)!} \sum_{k=0}^{\lfloor mt \rfloor} (-1)^k C_m^k (mt - k)^{m-1}, \quad t \in (0, 1).$$

Then, we get

$$\begin{aligned} \text{GA}_1 &= \frac{1}{2C f_X(\text{VaR}_{\alpha,X})} \sum_{i=1}^n A_i^2 [b_i(1 - a_i - b_i t) f_X(t) - b_i(a_i + b_i t) f_X(t) \\ &\quad + (a_i + b_i)(1 - a_i - b_i t) f'_X(t)]_{|t=\text{VaR}_{\alpha,X}}. \end{aligned}$$

By setting

$$\tilde{V}_\alpha := \frac{\text{VaR}_\alpha(L_n) - D}{C}, \quad \check{V}_\alpha := \frac{\tilde{V}_{n,\alpha} - D}{C},$$

we obtain

$$GA_2 = \frac{1}{2Cf_X(\tilde{V}_\alpha)} \sum_{i=1}^n A_i^2 [b_i(1 - a_i - b_i t)f_X(t) - b_i(a_i + b_i t)f_X(t) + (a_i + b_i)(1 - a_i - b_i t)f'_X(t)]_{|t=\tilde{V}_\alpha}.$$

Moreover, when X follows a Beta law with parameters (p, q) ,

$$GA_1 = \frac{1}{2C} \sum_{i=1}^n A_i^2 \left[\left\{ \frac{p-1}{t} - \frac{q-1}{1-t} \right\} \cdot \{a_i(1 - a_i) + b_i(1 - 2a_i)t - b_i^2 t^2\} + \{b_i(1 - 2a_i) - 2b_i^2 t\} \right]_{|t=\text{VaR}_{X,\alpha}}, \quad \text{and}$$

$$GA_2 = \frac{1}{2C} \cdot \frac{\tilde{V}_\alpha^{p-1}(1 - \tilde{V}_\alpha)^{q-1}}{\tilde{V}_\alpha^{p-1}(1 - \tilde{V}_\alpha)^{q-1}} \sum_{i=1}^n A_i^2 \left[\left\{ \frac{p-1}{t} - \frac{q-1}{1-t} \right\} \cdot \{a_i(1 - a_i) + b_i(1 - 2a_i)t - b_i^2 t^2\} + \{b_i(1 - 2a_i) - 2b_i^2 t\} \right]_{|t=\tilde{V}_\alpha}.$$

B.3. Toy model 3

As previously, we estimate numerically the value-at-risks of $\text{VaR}_\alpha(L_n)$ and $\text{VaR}_{X,\alpha}$. Now, the expectation of the losses, conditionally on X , is a quadratic (monotonic) function of X : $\mu_n(x) = C_0 + C_1x + C_2x^2$, where

$$C_0 = \sum_{i=1}^n \frac{A_i a_i c_i}{2}, \quad C_1 = \sum_{i=1}^n A_i \left(a_i d_i + \frac{b_i c_i}{2} \right), \quad C_2 = \sum_{i=1}^n A_i b_i d_i.$$

Then, $\mu'_n(x) = C_1 + 2C_2x$. With our notations, we have

$$V_i(x) = (a_i + b_i x) \cdot \left(\frac{c_i^2}{3} + c_i d_i x + d_i^2 x^2 \right) - (a_i + b_i x) \cdot \left(\frac{c_i}{2} + d_i x \right)^2,$$

and we calculate easily its derivative. Whatever the law of X , we can write

$$GA_1 = \frac{1}{2f_X(\text{VaR}_{X,\alpha})} \sum_{i=1}^n A_i^2 \frac{d\zeta_i}{dt}(t)_{|t=\text{VaR}_{X,\alpha}},$$

with $\zeta_i(t) = V_i(t)f_X(t)/\mu'_n(t)$.

Therefore,

$$GA_1 = \frac{1}{2(C_1 + 2C_2 \text{VaR}_{X,\alpha})} \sum_{i=1}^n A_i^2 \left\{ \text{Var}'_j(t) + \frac{V_i(t)f'_X(t)}{f_X(t)} - 2 \frac{V_i(t)C_2}{C_1 + 2C_2 t} \right\}_{|t=\text{VaR}_{X,\alpha}}.$$

Appendix C. Details of GA calculations for some multi-factor models

Now, we provide some details on GA calculations when the studied models involve several systematic random factors, in other words when \mathbf{X} belongs to \mathbb{R}^m with $m \geq 2$. Here, the associated GA is given by Theorem 2:

$$GA := \frac{T_{n,\infty}(\text{VaR}_\alpha(L_n))}{f_{\mu(\mathbf{X})}(\text{VaR}_\alpha(L_n))} = \frac{1}{2f_{\mu(\mathbf{X})}(\text{VaR}_\alpha(L_n))} \sum_{j=1}^n A_{jn}^2 \kappa'_j(\text{VaR}_\alpha(L_n)).$$

C.1. Granularity adjustments in a multi-factor Gaussian loss framework

Let us generalize our "Toy model 1" by introducing the bivariate random $X = (X_1, X_2)'$ and the random loss

$$Z_i = a_i + \sigma_i(\rho_{1i}X_1 + \rho_{2i}X_2 + \sqrt{1 - \rho_{1i}^2 - \rho_{2i}^2}X_i^*), \quad (C.1)$$

where X_1, X_2 and the $X_i^*, i = 1, \dots, n$, are (jointly) independent standard Gaussian variables. The parameters a_i, σ_i, ρ_{1i} and ρ_{2i} are positive constants, with $\rho_{ki} \in [0, 1], k = 1, 2$. Typically, X_1 (resp. X_2) can be interpreted as a systematic country (resp. industry) risk, or the opposite.

Here, $V_i(\mathbf{x}) = \sigma_i^2(1 - \rho_{1i}^2 - \rho_{2i}^2) := v_i$ and

$$E[L_n | \mathbf{X}] = \sum_{i=1}^n A_i a_i + \sum_{i=1}^n A_i \sigma_i \rho_{1i} X_1 + \sum_{i=1}^n A_i \sigma_i \rho_{2i} X_2 := C_0 + C_1 X_1 + C_2 X_2.$$

Thus $\partial_{(1)} \mu_n(\mathbf{x}) = C_1$, and

$$\kappa_i(v) = \frac{v_i}{C_1} \int \phi\left(\frac{v - C_0 - C_2 t}{C_1}\right) \phi(t) dt = \frac{v_i}{\sqrt{C_1^2 + C_2^2}} \phi\left(\frac{v - C_0}{\sqrt{C_1^2 + C_2^2}}\right).$$

We deduce easily

$$GA_1 = \frac{(-1)}{2} \cdot \frac{\sum_{i=1}^n A_i^2 \sigma_i^2 (1 - \rho_{1i}^2 - \rho_{2i}^2)}{\sqrt{C_1^2 + C_2^2}} \cdot \Phi^{-1}(\alpha),$$

that extends the formula we got in B.1. Similar arguments allow explicit calculation when $\mathbf{X} \in \mathbb{R}^m, m \geq 3$.

C.2. Granularity adjustments in a multi-factor discrete loss model

Let us consider the simplest multi-factor extension of our Toy model 2 as possible:

$$Z_i = \mathbf{1}(X_i \leq a_i + b_i X_1 + c_i X_2), \quad (C.2)$$

where X_1, X_2 and $X_i, i = 1, \dots, n$ are $n + 2$ independent random variables uniform on $[0, 1]$. All the coefficients are nonnegative, and assume $0 < a_i + b_i + c_i < 1$ for any i .

In this case,

$$V_i(\mathbf{x}) = (a_i + b_i x_1 + c_i x_2)(1 - a_i - b_i x_1 - c_i x_2), \quad \text{and}$$

$$E[L_n | \mathbf{X}] = \sum_{i=1}^n A_i a_i + \sum_{i=1}^n A_i b_i X_1 + \sum_{i=1}^n A_i c_i X_2 := \gamma_0 + \gamma_1 X_1 + \gamma_2 X_2.$$

Let assume that $0 < \gamma_1 \leq \gamma_2$ (otherwise, switch the roles of X_1 and X_2). Thus $\partial_{(1)} \mu_n(\mathbf{x}) = \gamma_1$. Let

$$I(v, u) = \int_0^u \mathbf{1}(v - \gamma_0 - \gamma_2 t \in [0, \gamma_1]) \left(a_i + b_i \frac{v - \gamma_0 - \gamma_2 t}{\gamma_1} + c_i t \right) \times \left(1 - a_i - b_i \frac{v - \gamma_0 - \gamma_2 t}{\gamma_1} - c_i t \right) dt,$$

for any $u \in [0, 1]$. Due to the fixed support of the underlying distributions, it is necessary to distinguish three cases to evaluate κ_i :

- If $v \in [\gamma_0, \gamma_0 + \gamma_1]$, then $\kappa_i(v) = I(v, (v - \gamma_0)/\gamma_2)$;
- If $v \in [\gamma_0 + \gamma_1, \gamma_0 + \gamma_2]$, then $\kappa_i(v) = I(v, (v - \gamma_0)/\gamma_2) - I(v, (v - \gamma_0 - \gamma_1)/\gamma_2)$;
- If $v \in [\gamma_0 + \gamma_2, \gamma_0 + \gamma_1 + \gamma_2]$, then $\kappa_i(v) = I(v, 1) - I(v, (v - \gamma_0 - \gamma_1)/\gamma_2)$;
- $\kappa_i(v) = 0$ otherwise.

Elementary calculations provide

$$I(v, u) = \left(a_i + b_i \frac{v - \gamma_0}{\gamma_1} \right) \left(1 - a_i - b_i \frac{v - \gamma_0}{\gamma_1} \right) \frac{u}{\gamma_1} + \left[\left(a_i + b_i \frac{v - \gamma_0}{\gamma_1} \right) \left(\frac{\gamma_2 b_i}{\gamma_1} - c_i \right) + \left(1 - a_i - b_i \frac{v - \gamma_0}{\gamma_1} \right) \left(c_i - \frac{\gamma_2 b_i}{\gamma_1} \right) \right] \frac{u^2}{2\gamma_1} + \left(\frac{\gamma_2 b_i}{\gamma_1} - c_i \right) \left(c_i - \frac{\gamma_2 b_i}{\gamma_1} \right) \frac{u^3}{3\gamma_1}.$$

To calculate the corresponding GA, it is necessary to derive κ_i , or the partial derivatives of I equivalently, a rather simple task that is left for the reader. Therefore, it remains to evaluate the density of $\mu_n(\mathbf{X})$, a linear combination of the two independent r.v.s X_1 and X_2 . Simple calculations provide

$$f_{\mu(\mathbf{X})}(t) = \frac{1}{\gamma_1 \gamma_2} \{ \mathbf{1}(t \in [\gamma_0, \gamma_0 + \gamma_1]) \cdot (t - \gamma_0) + \mathbf{1}(t \in [\gamma_0 + \gamma_1, \gamma_0 + \gamma_2]) + \mathbf{1}(t \in [\gamma_0 + \gamma_2, \gamma_0 + \gamma_1 + \gamma_2]) \cdot (\gamma_0 + \gamma_1 + \gamma_2 - t) \}.$$

C.3. Granularity adjustments of a two-factor CreditMetrics-type model with stochastic recoveries

Now, let us consider a two-dimensional systematic factor $\mathbf{X} = (X, Y)$, a bivariate Gaussian vector with $E[X] = E[Y] = 0$, $E[X^2] = E[Y^2] = 1$ and $E[XY] = \theta$. Our model will be

$$Z_i = LGD_i \cdot \mathbf{1}(\sqrt{\rho_i}X + \sqrt{1 - \rho_i}X_i^* \leq \Phi^{-1}(p_i)),$$

where the idiosyncratic risks X_i^* are univariate independent standard Gaussian random variables, and p_i is the unconditional default probability of i . Moreover, we assume that the individual “loss-given-default” are random:

$$LGD_i = (1 - \bar{R}_i) \frac{\Phi\left(\frac{\Phi^{-1}(\bar{p}_i) - \sqrt{v}Y}{\sqrt{1-v}}\right)}{\Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}X}{\sqrt{1-\rho_i}}\right)},$$

where \bar{R}_i , v and \bar{p}_i are positive constants. This is a two-factor extension of the model of Amraoui and Hitier (2008). During the recent credit crisis, the latter model has become a standard to price CDO tranches under the Gaussian copula framework by introducing random recoveries. Taking this randomness into account was necessary to calibrate the usual CDO pricing models to market quotes. Clearly, this model is an extension of Li’s (2000) Gaussian copula model: see Amraoui et al. (2012) for a discussion. In particular, such a parametrization allows the equality between the individual expected losses as given by the model and by the corresponding CDS quotes respectively. This consistency is guaranteed by imposing that

$$E[Z_i] = (1 - \bar{R}_i)\bar{p}_i = (1 - R_0)p_i,$$

where R_0 is the constant recovery rate that is used by the market conventionally (40% usually).

Let us assume that $\bar{p}_i = \bar{p}$ for every $i = 1, \dots, n$. Note that it does not mean that every name in the basket has the same (unconditional) default probability. Therefore, even with this restriction, the basket cannot be considered as homogenous, because individual default probabilities and recovery rates are different across different obligors, and we can calibrate the remaining parameters to individual expected losses. Then, the conditional expected loss is

$$\mu_n(\mathbf{X}) = \left(\sum_{i=1}^n A_{in}(1 - \bar{R}_i) \right) \Phi\left(\frac{\Phi^{-1}(\bar{p}) - \sqrt{v}Y}{\sqrt{1-v}}\right).$$

Note that it depends only on Y , the second component of \mathbf{X} , what simplifies our calculations significantly. For convenience, let us set $\bar{\mu} := \sum_{i=1}^n A_{in}(1 - \bar{R}_i)$. By simple calculations, we get

$$E[V_i(\mathbf{X}) | \mu_n(\mathbf{X}) = \mu] = (1 - \bar{R}_i)^2 \left(\frac{\mu}{\bar{\mu}}\right)^2 \cdot [I_i(\mu) - 1], \quad \text{where}$$

$$I_i(\mu) := \int \frac{\phi\left(\frac{x - \theta y}{\sqrt{1 - \theta^2}}\right)}{\Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}x}{\sqrt{1 - \rho_i}}\right)} \frac{dx}{\sqrt{1 - \theta^2}}$$

for some $y = y(\mu)$ such that

$$\Phi\left(\frac{\Phi^{-1}(\bar{p}) - \sqrt{v}y}{\sqrt{1-v}}\right) = \frac{\mu}{\bar{\mu}}, \quad \text{or } y(\mu) = \left\{ \Phi^{-1}(\bar{p}) - \sqrt{1-v}\Phi^{-1}\left(\frac{\mu}{\bar{\mu}}\right) \right\} / \sqrt{v}. \tag{C.3}$$

The integral $I_i(\mu)$ above is convergent when $1 - \theta^2 < (1 - \rho_i)/\rho_i$, what is not a strong constraint with usual correlation levels ρ_i . Moreover, the density of $\mu_n(\mathbf{X})$ is given by

$$f_{\mu(\mathbf{X})}(t) = \frac{\sqrt{1-v}}{\sqrt{v}\bar{\mu}} \cdot \frac{1}{\phi \circ \Phi^{-1}(t/\bar{\mu})} \phi\left(\frac{\Phi^{-1}(\bar{p}) - \sqrt{1-v}\Phi^{-1}(t/\bar{\mu})}{\sqrt{v}}\right).$$

Therefore, invoking Eq. (4), the granularity adjustment is given here by

$$GA = \frac{1}{2f_{\mu(\mathbf{X})}(E\text{VaR}_x)} \sum_{i=1}^n A_{in}^2 \kappa_i'(E\text{VaR}_x),$$

$$\kappa_i(\mu) = E[V_i(\mathbf{X}) | \mu_n(\mathbf{X}) = \mu] f_{\mu(\mathbf{X})}(\mu).$$

With the notations of Eq. (C.3), we get easily

$$\begin{aligned} \frac{\kappa_i'}{\kappa_i}(\mu) &= \frac{2}{\mu} + \frac{I_i'(\mu)}{I_i(\mu) - 1} + \frac{\sqrt{1-v}\Phi^{-1}(\bar{p}) - (1-2v)\Phi^{-1}(\mu/\bar{\mu})}{v\mu\phi \circ \Phi^{-1}(\mu/\bar{\mu})}, \\ I_i'(\mu) &= \frac{\theta\sqrt{1-v}}{\bar{\mu}(1-\theta^2)\sqrt{v}} \int \frac{\left(\frac{\theta y(\mu) - x}{\sqrt{1-\theta^2}}\right) \phi\left(\frac{\theta y(\mu) - x}{\sqrt{1-\theta^2}}\right)}{\Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho_i}x}{\sqrt{1-\rho_i}}\right)} \cdot \frac{dx}{\phi\left(\frac{\Phi^{-1}(\bar{p}) - \sqrt{v}y(\mu)}{\sqrt{1-v}}\right)}. \end{aligned}$$

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