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Hedging default risks of CDOs in Markovian contagion models

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We describe a replicating strategy of CDO tranches based upon dynamic trading of the corresponding credit default swap index. The aggregate loss follows a homogeneous Markov chain associated with contagion effects. Default intensities depend upon the number of defaults and are calibrated onto an input loss surface. Numerical implementation can be carried out thanks to a recombining tree. We examine how input loss distributions drive the credit deltas. We find that the deltas of the equity tranche are lower than those computed in the standard base correlation framework. This is related to the dynamics of dependence between defaults.

Keywords: Actuarial science; Asset management; Defaultable securities; Correlation modelling

1. Introduction

When dealing with CDO tranches, the market approach to the derivation of credit default swap deltas consists of bumping the credit curves of the names and computing the ratios of changes in the present value of the CDO tranches and the hedging credit default swaps. This involves a pricing engine for CDO tranches, usually a mixture of copula and base correlation approaches, leading to 'market deltas'. The only rationale for this *modus operandi* is local hedging with respect to credit spread risks, provided that the trading books are marked-to-market with the same pricing engine. Even when dealing with small changes in credit spreads, there is no guarantee that this would lead to appropriate hedging strategies, especially to cover large spread widenings and possibly defaults. For instance, one can envisage changes in base correlation correlated with changes in credit spreads. A number of CDO hedging anomalies in the base correlation approach are reported by Morgan and Mortensen (2007). Moreover, the standard approach is not associated with a replicating theory, thus inducing the possibility of unexplained drifts and time decay effects in the present value of hedged portfolios (Petrelli *et al*. 2007).

Unfortunately, the trading desk cannot rely on a sound theory to determine replicating prices of CDO tranches. This is partly due to the dimensionality issue, and partly to the stacking of credit spread and default risks. Laurent (2006) considers the case of multivariate intensities in a conditionally independent framework and shows that, for large portfolios where default risks are well diversified, one can concentrate on the hedging of credit spread risks and control the hedging errors. In this approach, the key assumption is the absence of contagion effects, which implies that credit spreads of survival names do not jump at default times, or, equivalently, that defaults are not informative. Whether one should rely on this assumption should be considered with caution as discussed by Das *et al*. (2007). Anecdotal evidence such as the failures of Delphi, Enron, Parmalat and WorldCom shows mixed results.

In this paper, we take an alternative route, concentrating on default risks, credit spreads and dependence dynamics driven by the arrival of defaults. We calculate so-called 'credit deltas', which are the present value impacts of default event on a given CDO tranche, divided by the present value impact of the hedging instrument (here the underlying index) under the same scenario.† Contagion models were introduced to the credit field by

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†Let us stress that the computed exposure at default is not equal to the usual 'value on default' or iOmega. In our model, the arrival of a default is associated with a shift in credit spreads and in base correlations due to contagion effects, while the value on default is usually computed under the assumption of constant spreads and correlations.

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Davis and Lo (2001), Jarrow and Yu (2001) and further studied by Yu (2007). Schönbucher and Schubert (2001) show that copula models exhibit contagion effects and relate jumps of credit spreads at default times to the partial derivatives of the copula. This is also the framework used by Bielecki *et al.* (2007a) to address the hedging issue. A similar but somehow more tractable approach has been considered by Frey and Backhaus (2008), since the latter paper considers Markovian models of contagion. In a copula model, the contagion effects are computed from the dependence structure of default times, while in contagion models the intensity dynamics are the inputs from which the dependence structure of default times is derived. In both approaches, credit spread shifts occur only at default times. Thanks to this quite simplistic assumption, and provided that no simultaneous defaults occur, it can be shown that the CDO market is complete, i.e. CDO tranche cash-flows can be fully replicated by dynamically trading individual credit spread swaps or, in some cases, by trading the credit default swap index.

Frey and Backhaus (2007) considered the hedging of CDO tranches in a Markov chain credit risk model allowing for spread and contagion risk. In this framework, when the hedging instruments are credit default swaps with a given maturity, the market is incomplete. In order to derive dynamic hedging strategies, Frey and Backhaus (2007) use risk minimization techniques. In a multivariate Poisson model, Elouerkhaoui (2006) also addresses the hedging problem thanks to the risk minimization approach. As can be seen from the previous papers, practical implementation can be cumbersome, especially when dealing hedging ratios at different points in time and different states.

As far as applications are concerned, calibration of the credit dynamics to market inputs is critical. Calibrations of Markov chain models similar to ours have recently been considered by a number of authors, including van der Voort (2006), Schönbucher (2006), Arnsdorf and Halperin (2007), de Koch *et al.* (2007), Epple *et al.* (2007), Lopatin and Misirpashaev (2007), Herbertsson (2008a,b) and Cont and Minca (2008). The aim of the previous papers was to construct arbitrage-free, consistent with market inputs, Markovian models of aggregate losses, possibly in incomplete markets, without detailing the feasibility and implementation of replication strategies. Regarding the hedging issue, a nice feature of our specification is that the market inputs completely determine the credit dynamics, thanks to the forward Kolmogorov equations. This parallels the approach of Dupire (1994) in the equity derivatives context. Thanks to this feature and the completeness of the market, one can unambiguously derive dynamic hedging strategies of CDO tranches. This can be seen as a benchmark for the study of more sophisticated, model- or criteria-dependent, hedging strategies.

For the paper to be self-contained, we recall in section 2 the mathematics behind the perfect replicating strategy. The main tool there is a martingale representation theorem for multivariate point processes. In section 3, we restrict ourselves to the case of homogeneous portfolios with Markovian intensities, which results in a dramatic dimensionality reduction for the (risk-neutral) valuation of CDO tranches and also the hedging of such tranches. We find that the aggregate loss is associated with a pure birth process, which is now well documented in the credit literature. In line with several recent papers, section 4 provides the calibration procedures of such contagion models based on the marginal distributions of the number of defaults. Section 5 details the computation of replicating strategies of CDO tranches with respect to the credit default swap index, through a recombining tree on the aggregate loss. We analyse the dependency of the hedging strategy on the chosen recovery rate. We also discuss how hedging strategies are related to dependence assumptions in Gaussian copula and base correlation frameworks.

2. Theoretical framework

2.1. Default times

Throughout the paper, we will consider n obligors and a random vector of default times $(\tau_1, \ldots, \tau_n)$ defined on a probability space (Ω, A, P). We denote by $N_1(t) = 1_{\{\tau_1 \le t\}}, \ldots, N_n(t) = 1_{\{\tau_n \le t\}}$ the default indicator processes and by $H_{i,t} = \sigma(N_i(s), s \le t)$, $i = 1, \ldots, n$, $H_t = \vee_{i=1}^n H_{i,t}$. $(H_t)_{t \in \mathbb{R}^+}$ is the natural filtration associated with the default times.

We denote by $\tau^1, \ldots, \tau^n$ the ordered default times and assume that no simultaneous defaults can occur, i.e. $\tau^1 < \cdots < \tau^n$, P-a.s. This assumption is important with respect to the completeness of the market. As shown below, it allows us to dynamically hedge basket default swaps and CDOs with n credit default swaps.†

Moreover, we assume that there exist some (P, H_t) intensities for the counting processes $N_i(t)$, $i = 1, \ldots, n$, i.e. there exist some (non-negative) H_t-predictable processes $\alpha_1^P, \ldots, \alpha_n^P$, such that $t \to N_i(t) - \int_0^t \alpha_i^P(s)\mathrm{d}s$ are (P, H_t) martingales.

2.2. Market assumptions

For the sake of simplicity, let us assume for a while that instantaneous digital default swaps are traded on the names. An instantaneous digital credit default swap on name i traded at t provides a payoff equal to $\mathrm{d}N_i(t) - \alpha_i(t)\mathrm{d}t$ at $t + \mathrm{d}t$. $\mathrm{d}N_i(t)$ is the payment on the default leg and $\alpha_i(t)\mathrm{d}t$ is the (short-term) premium on the default swap. Note that considering such instantaneous digital default swaps rather than actually traded credit default swaps is not a limitation for our purposes. This can rather be seen as a convenient choice of basis from a

†In the general case where multiple defaults could occur, we have to consider possibly 2^n states, and we would require non-standard credit default swaps with default payments conditionally on all sets of multiple defaults to hedge CDO tranches.

theoretical point of view. Of course, we will compute credit deltas with respect to traded credit default swaps in the applications below.†

Since we deal with the filtration generated by default times, the credit default swap premiums are deterministic between two default events. Therefore, we restrict ourselves to a market where only default risks occur and credit spreads themselves are driven by the occurrence of defaults. In our simple setting, there is no specific credit spread risk. This corresponds to the framework of Bielecki *et al.* (2007a,b).

For simplicity, we further assume that (continuously compounded) default-free interest rates are constant and equal to r. Given an initial investment V_0 and H_t-predictable processes $\delta_1(\bullet), \dots, \delta_n(\bullet)$ associated with a self-financed trading strategy in instantaneous digital credit default swaps, we attain at time T the payoff $V_0 e^{rT} + \sum_{i=1}^{n} \int_0^T \delta_i(s) e^{r(T-s)} (dN_i(s) - \alpha_i(s) ds)$. By definition, $\delta_i(s)$ is the nominal amount of instantaneous digital credit default swap on name i held at time s. This induces a net cash-flow of $\delta_i(s) \times (dN_i(s) - \alpha_i(s)ds)$ at time $s + ds$, which has to be invested in the default-free savings account up to time T.

2.3. Hedging and martingale representation theorem

From the absence of arbitrage opportunities, $\alpha_1, \dots, \alpha_n$ are non-negative H_t-predictable processes. From the same reason, $\{\alpha_i(t) > 0\} =^{P-=\text{a.s.}} \{\alpha_i^P(t) > 0\}$. Under mild regularity assumptions, there exists a probability Q equivalent to P such that the instantaneous credit default swap premiums $\alpha_1, \dots, \alpha_n$ are the (Q, H_t) intensities associated with the default times (Brémaud 1981, chapter VI).‡ Therefore, from now on, the premiums will be denoted $\alpha_1^Q, \dots, \alpha_n^Q$ and we will work under the probability Q.

Let us consider some H_T-measurable Q-integrable payoff M. Since M depends upon the default indicators of the names up to time T, this encompasses the cases of CDO tranches and basket default swaps, provided that recovery rates are deterministic. Thanks to the integral representation theorem of point process martingales (Brémaud 1981, chapter III), there exists some H_t-predictable processes $\theta_1, \dots, \theta_n$ such that

$$M = E^Q[M] + \sum_{i=1}^{n} \int_0^T \theta_i(s) \Big(dN_i(s) - \alpha_i^Q(s) ds \Big).$$

As a consequence, we can replicate M with the initial investment $E^Q[M e^{-rT}]$ and the trading strategy based on instantaneous digital credit default swaps defined by $\delta_i(s) = \theta_i(s) e^{-r(T-s)}$ for $0 \le s \le T$ and $i = 1, \dots, n$. Let us remark that the replication price at time t is provided by $V_t = E^Q[M e^{-r(T-t)} | H_t]$.§

While the use of the representation theorem guarantees that, in our framework, any basket default swap can be perfectly hedged with respect to default risks, it does not provide a practical way of constructing hedging strategies. As is the case with interest rate or equity derivatives, exhibiting hedging strategies involves certain Markovian assumptions (see subsection 2.3 and section 4).

3. Homogeneous Markovian contagion models

3.1. Intensity specification

In the contagion approach, one starts from a specification of the risk-neutral pre-default intensities $\alpha_1^Q, \dots, \alpha_n^Q$.¶ In the framework of the previous section, the risk-neutral default intensities depended upon the complete history of defaults. More simplistically, it is often assumed that they depend only upon the current credit status, i.e. the default indicators, thus $\alpha_i^Q(t)$, $i \in \{1, \dots, n\}$, is a deterministic function of $N_1(t), \dots, N_n(t)$. In this paper, we will remain in this Markovian framework, i.e. the pre-default intensities will take the form $\alpha_i^Q(t, N_1(t), \dots, N_n(t))$.⊥ Popular examples are the models of Kusuoka (1999), Jarrow and Yu (2001) and Yu (2007), where the intensities are affine functions of the default indicators. The connection between contagion models and Markov chains is described in the book of Lando (2004) and was further discussed by Herbertsson (2008a).

Another practical issue is related to name heterogeneity. Modelling all possible interactions amongst names leads to a huge number of contagion parameters and high-dimensional problems, thus to numerical issues. For this practical purpose, we will further restrict ourselves to models where all the names share the same risk-neutral intensity.‖ This can be viewed as a reasonable assumption for CDO tranches on large indices, although this is obviously an issue with equity tranches for which idiosyncratic risk is an important feature. Since pre-default

†Note that the instantaneous credit default swaps are not exposed to spread risk but only to default risk.

‡We remark that the assumption of no simultaneous defaults also holds for Q.

§Note that $M = E^Q[M|H_t] + \sum_{i=1}^{n} \int_t^T \theta_i(s)(dN_i(s) - \alpha_i^Q(s)ds)$. As a consequence, we readily obtain $M = V_t e^{r(T-t)} + \sum_{i=1}^{n} \int_t^T \theta_i(s)(dN_i(s) - \alpha_i^Q(s)ds)$, which provides the time t replication price of M. We also remark that, for a small time interval dt, $V_{t+dt} \approx V_t(1 + r\,dt) + \sum_{i=1}^{n} \delta_i(t)(dN_i(t) - \alpha_i^Q(t)\,dt)$, which is consistent with market practice and regular rebalancing of the replicating portfolio. An investor who wants to be compensated at time t against the price fluctuations of M during a short period dt has to invest V_t in the risk-free asset and take positions $\delta_1, \dots, \delta_n$ in the n instantaneous digital credit default swaps. Recall that there is no initial charge to enter into a credit default swap position.

¶After the default of name i, the intensity is equal to zero: $\alpha_i^Q(t) = 0$ on $\{t \ge \tau_i\}$.

⊥This Markovian assumption may be questionable, since the contagion effect of a default event may vanish as time goes by. The Hawkes process, which was used in the credit field by Giesecke and Goldberg (2006) and Errais *et al.* (2007), provides such an example of a more complex time dependence. Other specifications with the same aim are discussed by Lopatin and Misirpashaev (2007).

‖This means that the pre-default intensities have the same functional dependence to the default indicators.

risk-neutral default intensities $\alpha_1^Q, \ldots, \alpha_n^Q$ are equal, we will further denote these individual pre-default intensities by $\alpha_\bullet^Q$.

For further tractability, we will further rely on a strong name homogeneity assumption, that individual pre-default intensities only depend upon the number of defaults. Let us denote by $N(t) = \sum_v^n N_i(t)$ the number of defaults at time t within the pool of assets. Pre-default intensities thus take the form $\alpha_\bullet^Q(t, N(t))$.† This is related to mean-field approaches (Frey and Backhaus 2007). As for parametric specifications, we can envisage additive effects, i.e. the pre-default name intensities take the form $\alpha_\bullet^Q(t) = \alpha + \beta N(t)$ for some constants α, β as mentioned by Frey and Backhaus (2008), corresponding to the 'linear counterparty risk model',‡ or multiplicative effects in the spirit of Davis and Lo (2001), i.e. the pre-default intensities take the form $\alpha_\bullet^Q(t) = \alpha \times \beta^{N(t)}$. Of course, we could envisage a non-parametric model. Below, we provide a calibration procedure for such unconstrained intensities on market inputs.

For simplicity, we will further assume a constant recovery rate equal to R and a constant exposure among the underlying names. The aggregate fractional loss at time t is given by $L(t) = (1 - R)(N(t)/n)$. As a consequence of the no simultaneous defaults assumption, the intensity of $L(t)$ or of $N(t)$ is simply the sum of the individual default intensities and is itself only a function of the number of defaults process. Let us denote by $\lambda(t, N(t))$ the risk-neutral loss intensity. This is related to the individual pre-default risk-intensities by

$$\lambda(t, N(t)) = (n - N(t)) \times \alpha_\bullet^Q(t, N(t)).$$

We are thus typically in a bottom-up approach, where one starts with the specification of name intensities and thus derives the dynamics of the aggregate loss.

3.2. Risk-neutral pricing

We remark that, in a Markovian homogeneous contagion model, the process $N(t)$ is a Markov chain (under the risk-neutral probability Q) and, more precisely, a pure birth process, according to Karlin and Taylor (1975) terminology,§ since only single defaults can occur.¶ The generator of the chain, $\Lambda(t)$, is quite simple:

$$\Lambda(t) = \begin{pmatrix} -\lambda(t,0) & \lambda(t,0) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda(t,1) & \lambda(t,1) & 0 & & & 0 \\ 0 & & \bullet & \bullet & & & 0 \\ 0 & & & & \bullet & & 0 \\ 0 & & & & & \bullet & 0 \\ 0 & & & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Such a simple model of the number of default dynamics was considered by Schönbucher (2006) where it is called the 'one-step representation of the loss distribution'. Our paper can be seen as a bottom-up view of the previous model, where the risk-neutral prices can actually be viewed as replicating prices. As an example of this approach, let us consider the replication price of a European payoff with payment date T, such as a 'zero-coupon tranchelet', paying $1_{\{N(T)=k\}}$ at time T for some $k \in \{0, 1, \ldots, n\}$. Let us denote by $V(t, N(t)) = e^{-r(T-t)} Q(N(T) = k | N(t))$ the time t replication price and by $V(t, \bullet)$ the price vector whose components are $V(t, 0), V(t, 1), \ldots, V(t, n)$ for $0 \leq t \leq T$. We can thus relate the price vector $V(t, \bullet)$ to the terminal payoff using the transition matrix $Q(t, T)$ between dates t and T:

$$V(t, \bullet) = e^{-r(T-t)} Q(t, T) V(T, \bullet),$$

where $V(T, N(T)) = \delta_k(N(T))$. The transition matrix solves for the Kolmogorov backward and forward equations $\partial Q(t, T)/\partial t = -\Lambda(t) Q(t, T)$ and $\partial Q(t, T)/\partial T = Q(t, T)\Lambda(T)$. In the time homogeneous case, i.e. when the generator is a constant $\Lambda(t) = \Lambda$, the transition matrix can be written in exponential form, $Q(t, T) = \exp((T - t)\Lambda)$.⊥

These ideas have been put into practice by van der Voort (2006), Arnsdorf and Halperin (2007), de Koch *et al.* (2007), Epple *et al.* (2007), Lopatin and

†We remark that, on $\{\tau_i > t\}$, $N(t) = \sum_{j \neq i} N_j(t)$, so that the pre-default intensity of name i actually only depends on the credit status of the other names.

‡Ding *et al.* (2006) consider the case where the intensity of the loss process is linear in the number of defaults. The loss distribution is then negative binomial.

§According to Feller's terminology, we should speak of a pure death process. Since we later refer to Karlin and Taylor (1975), we prefer their terminology.

¶Regarding the assumption of no simultaneous defaults, we also refer to Putyatin *et al.* (2005), Brigo *et al.* (2007) and Walker (2007b). Allowing for multiple defaults could actually ease the calibration onto senior CDO tranche quotes.

⊥Since $e^{-rt} \times V(t, N(t))$ is a (Q, H_t) martingale and using Ito–Doeblin's formula, it can be seen that V solves for the backward Kolmogorov equations:

$$\frac{\partial V(t, N(t))}{\partial t} + \lambda(t, N(t)) \times (V(t, N(t) + 1) - V(t, N(t))) = rV(t, N(t)).$$

Misirpashaev (2007), Herbertsson (2008a) and Herbertsson and Rootzén (2008). These papers focus on the pricing of credit derivatives, while our concern here is the feasibility and implementation of replicating strategies.

3.3. Computation of credit deltas

We recall that the credit delta with respect to name i is the amount of hedging instruments (the index here, but possibly an ith credit default swap) that should be bought to be protected against a sudden default of name i. A nice feature of homogeneous contagion models is that the credit deltas are the same for all (non-defaulted) names, which results in a dramatic dimensionality reduction.

Let us consider a European† type payoff and denote its replication price at time t by $V(t,\bullet)$. In order to compute the credit deltas, let us remark that, by Ito's lemma,

$$\begin{aligned}\mathrm{d}V(t,N(t)) &= \frac{\partial V(t,N(t-))}{\partial t}\mathrm{d}t + (V(t,N(t)) - V(t,N(t-)))\mathrm{d}N(t)\\ &= \frac{\partial V(t,N(t-))}{\partial t}\mathrm{d}t + (V(t,N(t-)+1)\\ &\quad - V(t,N(t-)))\mathrm{d}N(t).\end{aligned}$$

The second term on the right-hand side of the latter expression, $V(t,N(t-)+1) - V(t,N(t-))$, is associated with the jump in the price process when a default occurs in the credit portfolio, i.e. $\mathrm{d}N(t)=1$. Due to the fact that $\mathrm{d}N(t) = \sum_{i=1}^n \mathrm{d}N_i(t)$ and, since $\mathrm{e}^{-r(T-t)}V(t,N(t))$ is a Q-martingale, it can be seen using Itô's lemma that V solves for the backward Kolmogorov equations:

$$\frac{\partial V(t,k)}{\partial t} + \lambda(t,k)\times(V(t,k+1) - V(t,k)) = rV(t,k),\quad k=0,\dots,n-1,$$

where the terminal conditions are given by the payoff function at time T. We end up with

$$\begin{aligned}\mathrm{d}V(t,N(t)) &= rV(t,N(t-))\mathrm{d}t\\ &\quad + \sum_{i=1}^n (V(t,N(t-)+1) - V(t,N(t-)))\\ &\quad \times(\mathrm{d}N_i(t) - \alpha_\bullet^Q(t,N(t-))(1-N_i(t))\mathrm{d}t).\end{aligned}$$

As a consequence, the credit deltas with respect to the individual instantaneous default swaps are equal to

$$\delta_i(t) = V(t,N(t-)+1) - V(t,N(t-)),$$

for $0 \le t \le T$ and $i=1,\dots,n$.

Let us denote by

$$V_I(t,k) = \mathrm{e}^{-r(T-t)}E^Q\left[1 - \frac{N(T)}{n}\middle| N(t)=k\right]$$

the time t price of the equally weighted portfolio involving defaultable discount bonds, and set

$$\delta_I(t,N(t-)) = \frac{V(t,N(t-)+1) - V(t,N(t-))}{V_I(t,N(t-)+1) - V_I(t,N(t-))}.$$

It can readily be seen that

$$\begin{aligned}\mathrm{d}V(t,N(t)) &= r\times(V(t,N(t-)) - \delta_I(t,N(t-))V_I(t,N(t-)))\mathrm{d}t\\ &\quad + \delta_I(t,N(t-))\mathrm{d}V_I(t,N(t)).\end{aligned}$$

As a consequence, we can perfectly hedge a European-type payoff, say a zero-coupon CDO tranche, using only the index portfolio and the risk-free asset.‡ The hedge ratio, with respect to the index portfolio, is actually equal to

$$\delta_I(t,N(t-)) = \frac{V(t,N(t-)+1) - V(t,N(t-))}{V_I(t,N(t-)+1) - V_I(t,N(t-))}.$$

The previous hedging strategy is feasible provided that $V_I(t,N(t-)+1) \neq V_I(t,N(t-))$. The usual case corresponds to some positive dependence, thus $\alpha_\bullet^Q(t,0) \le \alpha_\bullet^Q(t,1) \le \cdots \le \alpha_\bullet^Q(t,n-1)$. Therefore, $V_I(t,N(t-)+1) < V_I(t,N(t-))$.§ The decrease in the index portfolio value is the consequence of a direct default effect (one name defaults) and an indirect effect related to a positive shift in the credit spreads associated with the non-defaulted names.

The idea of building a hedging strategy based on the change in value at default times was introduced by Arvanitis and Laurent (1999). The rigorous construction of a dynamic hedging strategy in a univariate case can be found in Blanchet-Scalliet and Jeanblanc (2004). Our result can be seen as a natural extension to the multivariate case, provided that we deal with Markovian homogeneous models: we simply need to deal with the number of defaults $N(t)$ and the index portfolio $V_I(t,N(t))$ instead of a single default indicator $N_i(t)$ and the corresponding defaultable discount bond price.

Although this is not needed in the computation of dynamic hedging strategies, we can actually build a bridge between the above Markov chain approach for the aggregate loss and well-known models involving credit migration (see sppendix A).

4. Calibration of loss intensities

Another nice feature of the homogeneous Markovian contagion model is that the loss dynamics or, equivalently, the default intensities can be determined from market inputs such as CDO tranche premiums. Since the risk-neutral dynamics are unambiguously derived from market inputs, so will be dynamic hedging strategies of CDO tranches. This greatly facilitates empirical studies,

†At this stage, for notational simplicity, we assume that there are no intermediate payments. This corresponds, for instance, to the case of zero-coupon CDO tranches with up-front premiums. The more general case is considered in section 4.

‡As above, in order to ease the exposition, we neglect at this stage actual payoff features such as premium payments, amortization schemes, etc. This is detailed in section 4.

§In the case where $\alpha_\bullet^Q(t,0) = \alpha_\bullet^Q(t,1) = \cdots = \alpha_\bullet^Q(t,n-1)$, there are no contagion effects and default dates are independent. We still have $V_I(t,N(t-)+1) < V_I(t,N(t-))$ since $V_I(t,N(t-))$ is linear in the number of surviving names.

since the replicating figures do not depend upon unobserved and difficult to calibrate parameters.

The construction of the implied Markov chain for the aggregate loss parallels that of Dupire (1994), who constructed a local volatility model from call option prices. Derman and Kani (1994) and Rubinstein (1994) used similar ideas to build up implied trees. Laurent and Leisen (2000) showed how an implied Markov chain can be derived from a discrete set of option prices. In these approaches, the calibration of the implied dynamics on market inputs involves forward Kolmogorov equations. For a complete set of CDO tranche premiums or, equivalently, for a complete set of a number of default distributions, Schönbucher (2006) provided the construction of the loss intensities. For the paper to be self-contained, we detail and comment on this in appendix B. Lopatin and Misirpashaev (2007) and Cont and Minca (2008) also detail the similarities between Dupire's approach and the building of the one-step Markov chain of Schönbucher (2006).

In practical applications, we can only rely on a discrete set of loss distributions corresponding to liquid CDO tranche maturities. In the examples below, we will calibrate the loss intensities given a single calibration date T. For simplicity, we will be given the number of default probabilities $p(T,k)$, $k=0,1,\ldots,n$.† Here, and in the sequel, we assume that the loss intensities are time homogeneous: the intensities do not depend on time but only on the number of realized defaults. We further denote by $\lambda_k=\lambda(t,k)$, for $0\le t\le T$, the loss intensity for $k=0,1,\ldots,n-1$.‡ The computation of the loss intensities λ_k from the number of default probabilities is quite similar to Epple *et al.* (2007). For the paper to be self-contained, it is detailed in appendix C.

An alternative calibrating approach can be found in Herbertsson (2008a) or in Arnsdorf and Halperin (2007). In Herbertsson (2008a), the name intensities $\alpha_{\bullet}^{Q}(t,N(t))$ are time homogeneous, piecewise linear in the number of defaults (the node points are given by standard detachment points) and they are fitted to spread quotes by a least-square numerical procedure. Arnsdorf and Halperin (2007) propose a piecewise constant parameterization of name intensities (which are referred to as 'contagion factors') in time. When intensities are also piecewise linear in the number of defaults, they use a 'multi-dimensional solver' to calibrate onto the observed tranche prices.§ In the same vein, Frey and Backhaus (2007, 2008) introduce a parametric form for the function $\lambda(t,k)$, a variant of the 'convex counterparty risk model', and fit the parameters to tranche spreads. Lopatin and Misirpashaev (2007) express the loss intensity $\lambda(t,k)$ as a polynomial function of an auxiliary variable involving the number of defaults.

5. Computation of credit deltas through a recombining tree

5.1. Building up a tree

We now address the computation of CDO tranche deltas with respect to the credit default swap index of the same maturity. As for the hedging instrument, the premium is set at the inception of the deal and remains fixed, corresponding to market convention. We do not take into account roll dates every six months and trade the same index series up to maturity. Switching from one hedging instrument to another could be dealt with very easily in our framework and is closer to market practice, but we thought that using the same underlying across the tree would simplify the exposition.¶

The (fractional) loss at time t is given by $L(t)=(1-R)(N(t)/n)$. Let us consider a tranche with attachment point a and detachment point b, $0\le a\le b\le 1$. Up to some minor adjustment for the premium leg (see below), the credit default swap index is a $[0,1]$ tranche. We denote by $O(N(t))$ the outstanding nominal on a tranche. This is equal to $b-a$ if $L(t)<a$, to $b-L(t)$ if $a\le L(t)<b$ and to 0 if $L(t)\ge b$.

Let us recall that, for a European-type payoff, the price vector fulfils $V(t,\bullet)=\mathrm{e}^{-r(t'-t)}Q(t,t')V(t',\bullet)$ for $0\le t\le t'\le T$. The transition matrix can be expressed as $Q(t,t')=\exp(\Lambda(t'-t))$, where Λ is the generator matrix associated with the number of defaults. Note that, in the time homogeneous framework discussed in the previous section, the generator matrix does not depend on time.

For practical implementation, we will be given a set of node dates $t_0=0,\ldots,t_i,\ldots,t_{n_s}=T$. For simplicity, we will further consider a constant time step $\Delta=t_1-t_0=\cdots=t_i-t_{i-1}=\cdots$; this assumption can easily be relaxed. The most simple discrete-time approximation one can think of is $Q(t_i,t_{i+1})\simeq Id+\Lambda(t_i)\times(t_{i+1}-t_i)$, which leads to $Q(N(t_{i+1})=k+1|N(t_i)=k)\simeq\lambda_k\Delta$ and $Q(N(t_{i+1})=k|N(t_i)=k)\simeq 1-\lambda_k\Delta$. For large λ_k, the transition probabilities can become negative. Thus, we will rather use $Q(N(t_{i+1})=k+1|N(t_i)=k)\simeq 1-\mathrm{e}^{-\lambda_k\Delta}$ and $Q(N(t_{i+1})=k|N(t_i)=k)\simeq\mathrm{e}^{-\lambda_k\Delta}$.

Under the previous approximation the number of defaults process can be described via a recombining tree as in van der Voort (2006). One could clearly envisage using continuous Markov chain techniques,⊥ but the tree implementation is quite intuitive from a financial point of

†Clearly, this involves more information that one could directly access through the quotes of liquid CDO tranches, especially with respect to small and large numbers of defaults. As for the computation of the number of default probabilities from quoted CDO tranche premiums, we refer to Krekel and Partenheimer (2006), Galiani *et al.* (2006), Meyer-Dautrich and Wagner (2007), Parcell and Wood (2007), Walker (2007a) and Torresetti *et al.* (2007). Practical issues related to the calibration inputs are also discussed by van der Voort (2006).
‡Therefore, the pre-default name intensity is such that $\alpha_{\bullet}^{Q}(t,N(t))=\lambda_{N(t)}/(n-N(t))$. Let us recall that $\lambda(t,n)=0$.
§In both approaches, there are as many unknown parameters as available market quotes.
¶Actually, the credit deltas at inception are the same whatever the choice.
⊥For such approaches, we refer to Moler and Van Loan (2003) and Herbertsson (2008a) regarding the numerical issues.

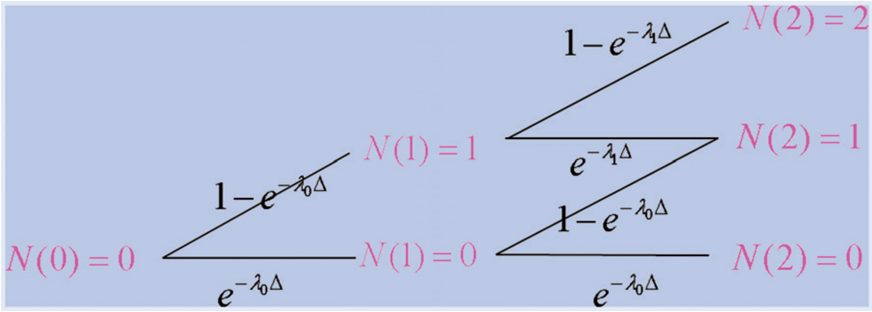

Figure 1. Number of defaults tree.

view as it corresponds to the implied binomial tree of Derman and Kani (1994). Convergence of the discrete-time Markov chain to its continuous limit is a rather standard issue and will not be detailed here (figure 1).

5.2. Computation of hedge ratios for CDO tranches

Let us denote by $d(i,k)$ the value at time t_i when $N(t_i)=k$ of the default payment leg of the CDO tranche.† The default payment at time t_{i+1} is equal to $O(N(t_i))-O(N(t_{i+1}))$. Thus, $d(i,k)$ is given by the following recurrence equation:

$$d(i,k)=\mathrm{e}^{-r\Delta}((1-\mathrm{e}^{-\lambda_k\Delta})\times(d(i+1,k+1)+O(k)-O(k+1))+\mathrm{e}^{-\lambda_k\Delta}d(i+1,k)).$$

Let us now deal with a (unitary) premium leg. We denote the regular premium payment dates by $T_1,\dots,T_p$ and, for simplicity, we assume that $\{T_1,\dots,T_p\}\subset\{t_0,\dots,t_{n_s}\}$. Let us consider some date t_{i+1} and set l such that $T_l<t_{i+1}\le T_{l+1}$. Whatever t_{i+1}, there is an accrued premium payment of $(O(N(t_i))-O(N(t_{i+1})))\times(t_{i+1}-T_l)$. If $t_{i+1}=T_{l+1}$, i.e. t_{i+1} is a regular premium payment date, there is an extra premium cash-flow at time t_{i+1} of $O(N(T_{l+1}))\times(T_{l+1}-T_l)$. Thus, if t_{i+1} is a regular premium payment date, the total premium payment is equal to $O(N(t_i))\times(T_{l+1}-T_l)$.

Let us denote by $r(i,k)$ the value at time t_i when $N(t_i)=k$ of the unitary premium leg.‡ If $t_{i+1}\in\{T_1,\dots,T_p\}$, $r(i,k)$ is given by

$$r(i,k)=\mathrm{e}^{-r\Delta}(O(k)\times(T_{l+1}-T_l)+(1-\mathrm{e}^{-\lambda_k\Delta})\times r(i+1,k+1)+\mathrm{e}^{-\lambda_k\Delta}r(i+1,k)).$$

If $t_{i+1}\notin\{T_1,\dots,T_p\}$, then

$$r(i,k)=\mathrm{e}^{-r\Delta}((1-\mathrm{e}^{-\lambda_k\Delta})\times(r(i+1,k+1)+(O(k)-O(k+1))\times(t_{i+1}-T_l))+\mathrm{e}^{-\lambda_k\Delta}r(i+1,k)).$$

The CDO tranche premium is equal to $s=d(0,0)/r(0,0)$. The value of the CDO tranche (buy protection case) at time t_i when $N(t_i)=k$ is given by $V_{\mathrm{CDO}}(i,k)=d(i,k)-sr(i,k)$. The equity tranche needs to be dealt with slightly differently since its spread is set to $s=500$ bp. However, the value of the CDO equity tranche is still given by $d(i,k)-sr(i,k)$.

As for the credit default swap index, we will denote by $r_{IS}(i,k)$ and $d_{IS}(i,k)$ the values of the premium and default legs. We define the credit default swap index spread at time t_i when $N(t_i)=k$ by $s_{IS}(i,k)\times r_{IS}(i,k)=d_{IS}(i,k)$.§ The up-front premium of the credit default swap index, bought at inception, at node (i,k) is given by $V_{IS}(i,k)=d_{IS}(i,k)-s_{IS}(0,0)\times r_{IS}(i,k)$. The default leg of the credit default swap index is computed as a standard default leg of a $[0,100\%]$ CDO tranche. Thus, in the recursion equation giving $d_{IS}(i,k)$ we write the outstanding nominal for k defaults as $O(k)=1-[k(1-R)/n]$, where R is the recovery rate and n the number of names. According to standard market rules, the premium leg of the credit default swap index needs a slight adaptation since the premium payments are based only upon the number of non-defaulted names and do not take into account recovery rates. As a consequence, the outstanding nominal to be used in the recursion equations providing $r_{IS}(i,k)$ is such that $O(k)=1-(k/n)$.

As usual in binomial trees, $\delta(i,k)$ is the ratio of the difference of the option value (at time t_{i+1}) in the upper state ($k+1$ defaults) and lower state (k defaults) and the corresponding difference for the underlying asset. In our case, both the CDO tranche and the credit default swap index are 'dividend-baring'. For instance, when the number of defaults switches from k to $k+1$, the default leg of the CDO tranche is associated with a default payment of $O(k)-O(k+1)$. Similarly, given the above discussion, when the number of defaults switches from k to $k+1$, the premium leg of the CDO tranche is associated with an accrued premium payment of $-s\times 1_{t_{i+1}\notin\{T_1,\dots,T_p\}}(O(k)-O(k+1))\times(t_{i+1}-T_l)$.¶

†We consider the value of the default leg immediately after t_i. Thus, we do not consider a possible default payment at t_i in the calculation of $d(i,k)$.
‡As for the default leg, we consider the value of the premium leg immediately after t_i. Thus, we do not take into account a possible premium payment at t_i in the calculation of $r(i,k)$ either.
§This is an approximation of the index spread since, according to market rules, the first premium payment is reduced.
¶If $t_{i+1}\in\{T_1,\dots,T_p\}$, the premium payment is the same whether the number of defaults is equal to k or $k+1$. Therefore, it does not appear in the computation of the credit delta.

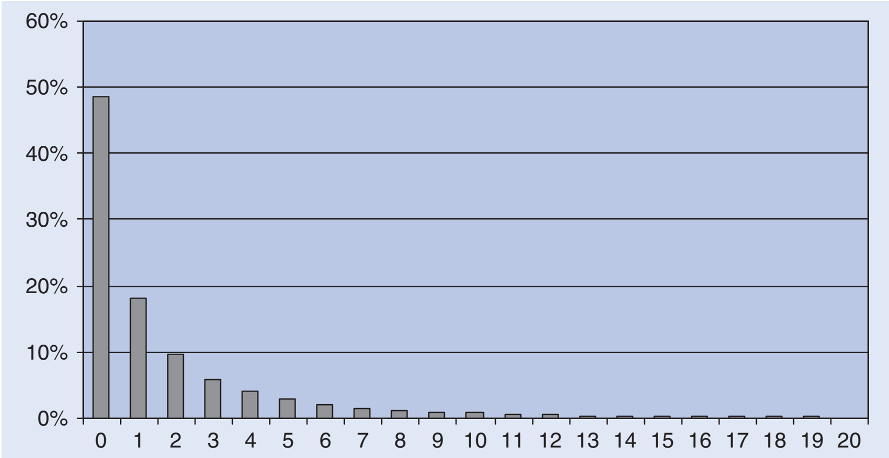

Figure 2. Distribution of the number of defaults. Number of defaults on the x axis.

Thus, when a default occurs the change in value of the CDO tranche is the outcome of a capital gain of $V_{\mathrm{CDO}}(i+1,k+1)-V_{\mathrm{CDO}}(i+1,k)$ and of a cash-flow of $(O(k)-O(k+1))\times(1-s\times 1_{t_{i+1}\notin\{T_1,\ldots,T_p\}}\times(t_{i+1}-T_l))$.

The credit delta of the CDO tranche at node (i,k) with respect to the credit default swap index is thus given by

$$\delta(i,k)=\frac{\begin{array}{l}V_{\mathrm{CDO}}(i+1,k+1)-V_{\mathrm{CDO}}(i+1,k)\\ \quad+(O(k)-O(k+1))\times(1-s\times 1_{t_{i+1}\notin\{T_1,\ldots,T_p\}}\\ \quad\times(t_{i+1}-T_l))\end{array}}{\begin{array}{l}V_{IS}(i+1,k+1)-V_{IS}(i+1,k)\\ \quad+[(1-R)/n]-(1/n)\times s_{IS}(0,0)\times 1_{t_{i+1}\notin\{T_1,\ldots,T_p\}}\\ \quad\times(t_{i+1}-T_l)\end{array}}.$$

We remark that using the previous credit deltas leads to a perfect replication of a CDO tranche within the tree, which is feasible since the approximating discrete market is complete.

In the next section, we compute CDO tranche credit deltas with respect to the credit default swap index in two steps. We first calibrate loss intensities from a one-factor Gaussian copula loss distribution. This allows us to examine how the recovery rate assumption and the correlation between defaults impact credit deltas. We then calibrate loss intensities from a loss distribution associated with a market base correlation structure and we compare our 'default risk' deltas with 'credit spread' deltas computed on the basis of a bump in the credit default swap index spread. In particular, we investigate spread deltas computed from the standard market approach and spread deltas recently obtained by Arnsdorf and Halperin (2007) and Eckner (2007).

5.3. *Model calibrated on a loss distribution associated with a Gaussian copula*

In this numerical illustration, the loss intensities λ_k are computed from a loss distribution generated from a one-factor Gaussian copula. The correlation parameter is equal to $\rho=30\%$,† the credit spreads are all equal to 20 basis points per annum, the recovery rate is such that $R=40\%$ and the maturity is $T=5$ years. The number of names is $n=125$. Figure 2 shows the distribution of the number of defaults.

Loss intensities λ_k are calibrated as previously discussed up to $k=49$ defaults. Under the Gaussian copula assumption, the default probabilities $p(5,k)$ are insignificant for $k>49$.‡ To avoid numerical difficulties, we computed the corresponding λ_k by linear extrapolation.§

As can be seen from figure 3, loss intensities change almost linearly with respect to the number of defaults. Such linear behavior of loss intensities was also found by Lopatin and Misirpashaev (2007). Our results can also be

†ρ is the correlation between default events in a one-factor homogeneous Gaussian copula model where the time t conditional default probability (the probability that a name defaults before t given the latent factor V) is defined by

$$\tilde{p}_t=\Phi\left(\frac{-\sqrt{\rho}\,V+\Phi^{-1}(\bar{p}_t)}{\sqrt{1-\rho}}\right),$$

where Φ is the cumulative standard Gaussian density and $\bar{p}_t$ is the time t marginal default probability. In former versions of the paper, ρ was associated with a conditional default probability defined by

$$\tilde{p}_t=\Phi\left(\frac{-\rho\,V+\Phi^{-1}(\bar{p}_t)}{\sqrt{1-\rho^2}}\right).$$

‡$\sum_{k\geq 50}p(5,k)\simeq 3\times 10^{-4}$, $p(5,50)\simeq 3.2\times 10^{-5}$ and $p(5,125)\simeq 4\times 10^{-12}$.

§We checked that various choices of loss intensities for a large number of defaults had no effect on the computation of the deltas. We stress that this applies for the Gaussian copula case since the loss distribution has thin tails. For the market case example, we proceeded differently.

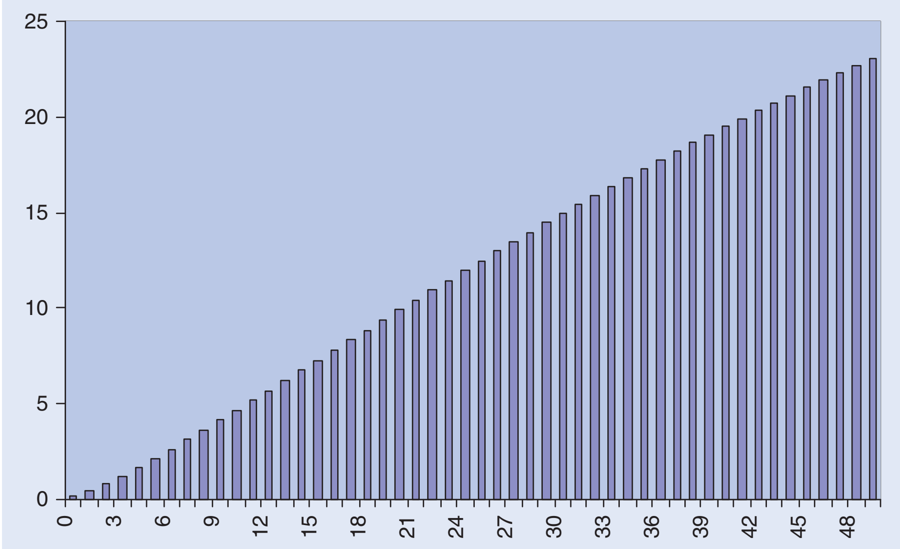

Figure 3. Loss intensities λ_k, $k = 0, \ldots, 49$.

related to the analysis of Ding *et al*. (2006), who deal with a dynamic model where the loss intensity is actually linear in the number of defaults.

Table 1 shows the dynamics of the credit default swap index spreads $s_{IS}(i,k)$ along the nodes of the tree. The continuously compounded default free rate is $r = 3\%$ and the time step is $\Delta = 1/365$. It can be seen that default arrivals are associated with rather large jumps of credit spreads. For instance, if a (first) default occurs after a quarter, the credit default swap index spread jumps from 18 to 70 bps. An extra default by this time leads to an index spread of 148 bps.

The credit deltas with respect to the credit default swap index $\delta(i,k)$ were computed for the $[0,3\%]$ and $[3,6\%]$ CDO tranches (see tables 2 and 3). As for the equity tranche, it can be seen that the credit deltas are positive and decrease up to zero. This is not surprising given that a buy protection equity tranche involves a short put position over the aggregate loss with a 3% strike. This is associated with positive deltas, negative gammas and thus decreasing deltas. When the number of defaults is above six, the equity tranche is exhausted and the deltas obviously are equal to zero.

At inception, the credit delta of the equity tranche is equal to 54%, whilst it is only equal to 25% for the $[3,6\%]$ tranche, which is deeper out of the money (see table 3). Moreover, the $[3,6\%]$ CDO tranche involves a call spread position over the aggregate loss. As a consequence, the credit deltas are positive and first increase (positive gamma effect) and then decrease (negative gamma) up to zero as soon as the tranche is fully amortized.

Given the recovery rate assumption of 40%, the outstanding nominal of the $[3,6\%]$ tranche is equal to 3% for six defaults and to 2.64% for seven defaults. One might thus be led to believe that, at the sixth default, the $[3,6\%]$ tranche should behave almost like an equity tranche. However, as can be seen from table 3, the credit delta of the equity tranche is much lower: around 1% instead of 50%. This is due to dramatic shifts in credit spreads when moving from the no-defaults to the six defaults state (see table 1). In the latter case, the expected loss on the tranche is much larger, which is consistent with smaller deltas given the call spread payoff.

Table 1. Dynamics of credit default swap index spread $s_{IS}(i,k)$ in basis points per annum.

Nb Defaults	Weeks 0	14	56	84
0	20	18	14	13
1	0	70	54	46
2	0	148	112	93
3	0	243	182	150
4	0	350	261	215
5	0	466	347	285
6	0	589	437	359
7	0	719	531	436
8	0	856	630	516
9	0	997	732	598
10	0	1142	839	683

Table 2. Deltas of the $[0,3\%]$ equity tranche with respect to the credit default swap index.

Nb Defaults	OutStanding Nominal	Weeks 0	14	56	84
0	3.00%	0.538	0.591	0.755	0.859
1	2.52%	0	0.238	0.381	0.508
2	2.04%	0	0.074	0.137	0.212
3	1.56%	0	0.026	0.044	0.070
4	1.08%	0	0.011	0.017	0.024
5	0.60%	0	0.005	0.007	0.009
6	0.12%	0	0.001	0.001	0.001
7	0.00%	0	0	0	0

Table 3. Deltas of the [3, 6%] tranche with respect to the credit default swap index.

Nb Defaults	OutStanding Nominal	Weeks 0	14	56	84
0	3.00%	0.255	0.254	0.219	0.171
1	3.00%	0	0.280	0.349	0.357
2	3.00%	0	0.167	0.294	0.389
3	3.00%	0	0.068	0.158	0.265
4	3.00%	0	0.026	0.065	0.128
5	3.00%	0	0.014	0.027	0.053
6	3.00%	0	0.010	0.016	0.025
7	2.64%	0	0.008	0.011	0.015
8	2.16%	0	0.006	0.008	0.010
9	1.68%	0	0.004	0.005	0.007
10	1.20%	0	0.003	0.003	0.004
11	0.72%	0	0.002	0.002	0.002
12	0.24%	0	0.001	0.001	0.001
13	0.00%	0	0	0	0

5.4. *Sensitivity of hedging strategies to the recovery rate assumption*

The previous deltas were computed under the assumption that the recovery rate was equal to 40%, which is a standard but somewhat arbitrary assumption. We further investigate the dependence of the dynamic hedging strategy with respect to the choice of recovery rate. For our robustness study to be meaningful, we will modify the recovery rates but keep the loss surface (or, equivalently, the CDO tranche premiums) unchanged. This implies a change in the number of defaults distribution. The procedure is detailed in appendix D.

Table 4 shows the credit deltas at the initial date for various CDO tranches under different recovery assumptions. Fortunately, the recovery rate assumption has a small effect on the computed credit deltas.

5.5. *Dependence of hedging strategies upon the correlation parameter*

Let us recall that the recombining tree is calibrated on a loss distribution over a given time horizon. The shape of the loss distribution depends critically upon the correlation parameter, which, up to now, has been set at $\rho = 30\%$. Decreasing the dependence between default events leads to a thinner right-tail of the loss distribution and smaller contagion effects. We detail here the effects of varying the correlation parameter on the hedging strategies.

For simplicity, we firstly focus the analysis on the equity tranche and shift the correlation parameter from 30% to 10%. It can be seen from tables 2 and 5 that the credit deltas are much higher in the latter case. After 14 weeks, prior to the first default, the credit delta is equal to 59% for a 30% correlation and to 96% when the correlation parameter is equal to 10%.

Table 4. Deltas at inception for different recovery rates.

Tranches	Recovery Rates 10%	20%	30%	40%	50%
[0–3%]	0.554	0.547	0.542	0.538	0.528
[3–6%]	0.251	0.254	0.254	0.255	0.257
[6–9%]	0.129	0.130	0.130	0.131	0.131

Table 5. Deltas of the [0, 3%] equity tranche with respect to the credit default swap index, $\rho = 10\%$ (credit deltas can be above one in the no-default case. This is due to the amortization scheme of the premium leg. We detail in the next section the impact of the premium leg on credit deltas).

Nb Defaults	OutStanding Nominal	Weeks 0	14	42	84
0	3.00%	0.931	0.960	1.009	1.058
1	2.52%	0	0.694	0.785	0.910
2	2.04%	0	0.394	0.485	0.645
3	1.56%	0	0.179	0.233	0.352
4	1.08%	0	0.072	0.092	0.145
5	0.60%	0	0.027	0.032	0.046
6	0.12%	0	0.004	0.005	0.007
7	0.00%	0	0	0	0

To further investigate how changes in correlation levels alter credit deltas, we computed the market value of the default leg of the equity tranche at a 14 week horizon as a function of the number of defaults under different correlation assumptions (see figure 4). The market value of the default leg, on the y axis, is computed as the sum of expected discounted cash-flows posterior to this 14 week horizon date and the accumulated defaults cash-flows paid before.† We also plotted the accumulated losses, which represent the intrinsic value of the equity tranche default leg. Unsurprisingly, we recognize some typical concave patterns associated with a short put option payoff.

As can be seen from figure 4, prior to the first default, the value of the default leg of the equity tranche decreases as the correlation parameter increases from 0 to 40%. However, after the first default, the ordering of default leg values is reversed. This can easily be understood, since larger correlations are associated with larger jumps in credit spreads at default arrivals and thus larger changes in the expected discounted cash-flows associated with the default leg of the equity tranche.‡

†For simplicity, we neglected the compounding effects over this short period.
‡We remark that the larger the correlation, the larger the change in market value of the default leg of the equity tranche at the arrival of the first default. Indeed, in a high correlation framework, this default means a relatively greater likelihood of default for the surviving names. This is not inconsistent with the previous results showing a decrease in credit deltas when the correlation parameter increases. The credit delta is the ratio of the change in value in the equity tranche and of the change in value in the credit default swap index. For a larger correlation parameter, the change in value in the credit default swap index is also larger due to magnified contagion effects.

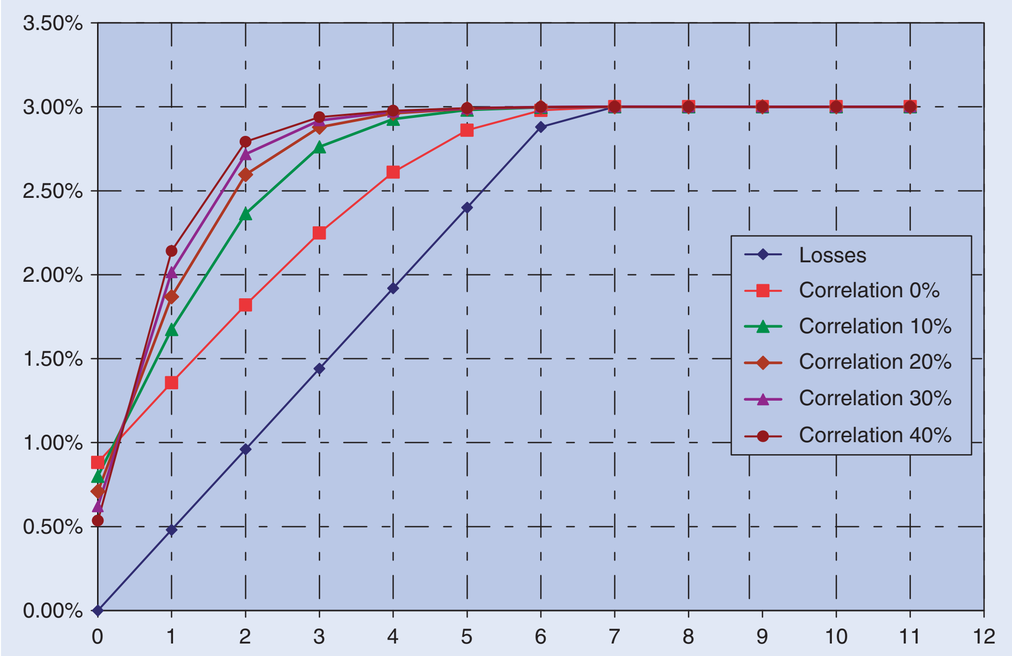

Figure 4. Market value of equity default leg under different correlation assumptions. Number of defaults on the x axis.

Therefore, varying the correlation parameter is associated with two opposing mechanisms.

- The first is related to a typical negative vanna effect.† Increasing correlation lowers loss 'volatility' and leads to smaller expected losses on the equity tranche. In a standard option pricing framework, this should lead to an increase in the credit delta of the short put position on the loss.
- This is superseded by the shifts due to contagion effects. Increasing correlation is associated with large contagion effects and thus larger jumps in credit spreads at the arrival of defaults. This, in turn, leads to a larger jump in the market value of the credit index default swap. Let us recall that the default leg of the equity tranche exhibits a concave payoff and thus a negative gamma. As a consequence, the credit delta, i.e. the ratio between the change in value of the option and the change in value of the underlying, decreases.

5.6. *Taking into account a base correlation structure*

Up to now, the probabilities of the number of defaults were computed using a Gaussian copula and a single correlation parameter. In this example, we use a steep upward-sloping base correlation curve for the iTraxx, typical of June 2007, as input to derive the distribution of the probabilities of the number of defaults (see table 6).

Table 6. Base correlations with respect to attachment points.

3%	6%	9%	12%	22%
18%	28%	36%	42%	58%

The maturity is still equal to 5 years, the recovery rate to 40% and the credit spreads to 20 bps. The default-free rate is now 4%.

Rather than spline interpolation of base correlations, we used a parametric model of the 5 year loss distribution to fit the market quotes and compute the probabilities of the number of defaults. This produces arbitrage-free and smooth distributions that ease the calculation of the loss intensities.‡ Figure 5 shows the number of defaults distribution. This is rather different from the 30% flat correlation Gaussian copula case, both for small and large losses. For instance, the probability of no defaults dropped from 48.7 to 19.5%, while the probability of a single default rose from 18.2 to 36.5%. Let us stress that these figures are for illustrative purposes. The market does not provide direct information on first losses and thus the shape of the left tail of the loss distribution is a controversial issue. As for the right-tail, we have $\sum_{k\geq 50} p(5,k) \simeq 1.4\times 10^{-3}$ and $p(5,50)\simeq 3.3\times 10^{-6}$, $p(5,125)\simeq 1.38\times 10^{-3}$. The cumulative probabilities of large numbers of defaults are larger compared with the Gaussian copula case. The probability of the names defaulting altogether is also quite large, corresponding to some kind of Armageddon risk. Once again, these figures need to be considered with caution, corresponding to high senior and super-senior tranche premiums and disputable

†We recall that, in option pricing, the vanna is the sensitivity of the delta to a unit change in volatility.
‡We also computed the number of defaults distribution using entropic calibration. Although we could still compute loss intensities, the pattern with respect to the number of defaults was not monotonic. Depending on market inputs, direct calibration onto CDO tranche quotes can lead to shaky figures.

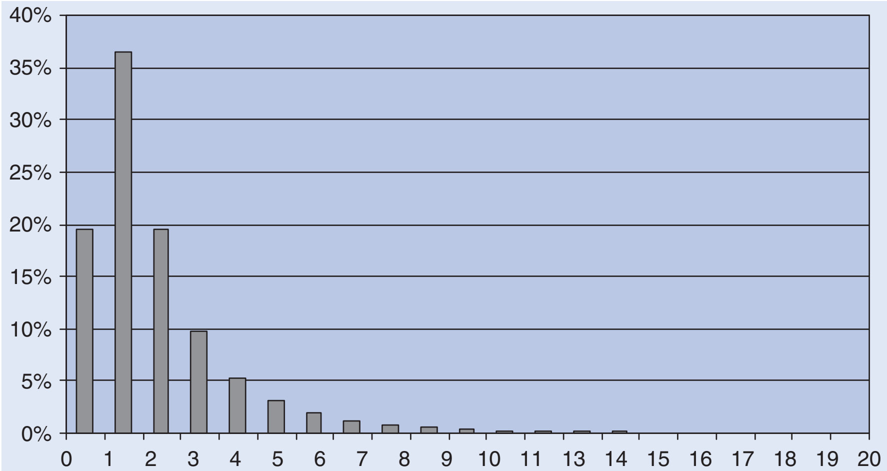

Figure 5. Distribution of the number of defaults. Number of defaults on the x axis.

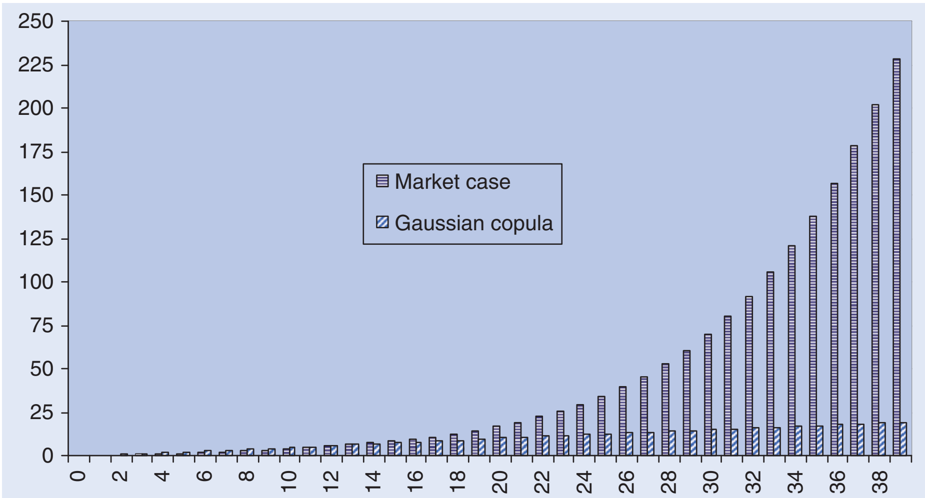

Figure 6. Loss intensities for the Gaussian copula and market case examples. Number of defaults on the x axis.

assumptions concerning the probability of all names defaulting.

Figure 6 shows the loss intensities calibrated onto market inputs compared with the loss intensities based on Gaussian copula inputs up to 39 defaults.† As can be seen, the loss intensity increases much quicker with the number of defaults compared with the Gaussian copula approach. The average relative change in the loss intensities is 19%, whereas it is 16% when computed under the Gaussian copula assumption. Unsurprisingly, a steep base correlation curve is associated with fatter upper tails of the loss distribution and magnified contagion effects.

Table 7 shows the dynamics of the credit default swap index spreads $s_{IS}(i,k)$ along the nodes of the tree. As for

Table 7. Dynamics of credit default swap index spread $s_{IS}(i,k)$ in basis points per annum.

Nb Defaults	Weeks 0	14	56	84
0	20	19	17	16
1	0	31	23	20
2	0	95	57	43
3	0	269	150	98
4	0	592	361	228
5	0	1022	723	490
6	0	1466	1193	905
7	0	1870	1680	1420
8	0	2243	2126	1945
9	0	2623	2534	2423
10	0	3035	2939	2859

†Contrary to the Gaussian copula example, we computed the complete set of loss intensities using the procedure described in appendix C.

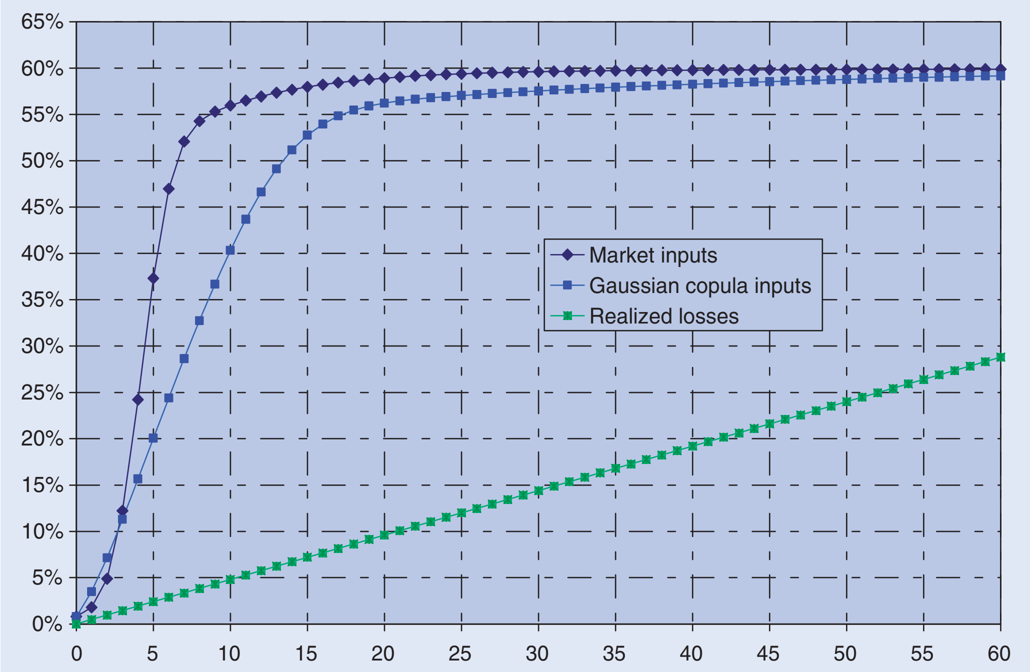

Figure 7. Expected losses on the credit portfolio after 14 weeks over a five year horizon (y axis) with respect to the number of defaults (x axis) using market and Gaussian copula inputs.

tree implementation, the time step is still $\Delta = 1/365$. We remark that, up to 12 defaults, loss intensities calibrated from market inputs are on the whole smaller than in the Gaussian copula case. The contagion effect is smaller in the 30% correlation Gaussian copula in low default states and greater for high default states. Unsurprisingly, market quotes lead to smaller index spreads of up to two defaults at 14 weeks (see tables 1 and 7). This is also consistent with figure 7, where the conditional expected losses in the two approaches cross each other at the third default. However, as mentioned above, this detailed pattern has to be considered with caution, since it involves the probabilities of 0, 1 and 2 defaults which are not directly observed in the market. After two defaults, credit spreads become larger when calibrated from market inputs.

From figure 7 we can investigate the credit spread dynamics when using market inputs. We plotted the conditional (with respect to the number of defaults) expected loss $E[L(T)|N(t)]$ for $T = 5$ years and $t = 14$ weeks for the previous market inputs and for the 30% flat correlation Gaussian copula case. The conditional expected loss is expressed as a percentage of the nominal of the portfolio.† We also plotted the accumulated losses on the portfolio. The expected losses are greater than the accumulated losses due to positive contagion effects. There are dramatic differences between the Gaussian copula and the market inputs examples. In the Gaussian copula case, the expected loss is almost linear with respect to the number of defaults in a wide range (say up to 15 defaults). The pattern is quite different when using market inputs with huge nonlinear effects. This shows large

Table 8. Delta of the [0, 3%] equity tranche with respect to the credit default swap index.

Nb Defaults	OutStanding Nominal	Weeks 0	14	56	84
0	3.00%	0.645	0.731	0.953	1.038
1	2.52%	0	0.329	0.584	0.777
2	2.04%	0	0.091	0.197	0.351
3	1.56%	0	0.023	0.045	0.090
4	1.08%	0	0.008	0.011	0.018
5	0.60%	0	0.004	0.003	0.004
6	0.12%	0	0.001	0.001	0.001
7	0.00%	0	0	0	0

contagion effects after a few defaults, as can also be seen from table 7 and figure 6. This rather explosive behavior was also observed by Herbertsson (2008b, tables 3 and 4) and by Cont and Minca (2008, figures 1 and 3). Lopatin and Misirpashaev (2007) also found that contagion effects are magnified when using market data compared with Gaussian copula inputs.

Table 8 shows the dynamic deltas associated with the equity tranche. We note that the credit deltas drop quite quickly to zero with the occurrence of defaults. This is not surprising given the surge in credit spreads and dependencies after the first default (see figure 7): after only a few defaults the equity tranche is virtually exhausted.

It is noteworthy that the credit deltas $\delta(i,k)$ can be decomposed into a default leg delta $\delta_d(i,k)$ and a premium leg delta $\delta_r(i,k)$ as $\delta(i,k) = \delta_d(i,k) - s\delta_r(i,k)$,

†Thus, given a recovery rate of 40%, the maximum expected loss is 60%.

Table 9. Deltas of the default leg of the [0, 3%] equity tranche with respect to the credit default swap index ($\delta_d(i,k)$).

Nb Defaults	OutStanding Nominal	Weeks 0	14	56	84
0	3.00%	0.541	0.617	0.823	0.910
1	2.52%	0	0.279	0.510	0.690
2	2.04%	0	0.072	0.166	0.304
3	1.56%	0	0.016	0.034	0.072
4	1.08%	0	0.004	0.006	0.012
5	0.60%	0	0.002	0.002	0.002
6	0.12%	0	0.001	0.000	0.000
7	0.00%	0	0	0	0

Table 10. Deltas of the premium leg of the [0, 3%] equity tranche with respect to the credit default swap index ($s\delta_r(i,k)$).

Nb Defaults	OutStanding Nominal	Weeks 0	14	56	84
0	3.00%	−0.104	−0.113	−0.113−0	−0.128
1	2.52%	0	−0.050	−0.074	−0.087
2	2.04%	0	−0.018	−0.031	−0.047
3	1.56%	0	−0.007	−0.011	−0.018
4	1.08%	0	−0.004	−0.004	−0.006
5	0.60%	0	−0.002	−0.002	−0.002
6	0.12%	0	−0.001	0.000	0.000
7	0.00%	0	0	0	0

with

$$\delta_d(i,k) = \frac{d(i+1,k+1) - d(i+1,k) + O(k) - O(k+1)}{V_{IS}(i+1,k+1) - V_{IS}(i+1,k) + [(1-R)/n] - (1/n) \times s_{IS}(0,0) \times 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} \times (t_{i+1} - T_l)}$$

and

$$\delta_r(i,k) = \frac{r(i+1,k+1) - r(i+1,k) + (O(k) - O(k+1))1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} \times (t_{i+1} - T_l)}{V_{IS}(i+1,k+1) - V_{IS}(i+1,k) + [(1-R)/n] - (1/n) \times s_{IS}(0,0) \times 1_{t_{i+1} \notin \{T_1,\ldots,T_p\}} \times (t_{i+1} - T_l)}.$$

Tables 9 and 10 detail the credit deltas associated with the default and premium legs of the equity tranche. As can be seen from table 8, credit deltas for the equity tranche may be slightly above one when no default has occurred. Table 10 shows that this is due to the amortization scheme of the premium leg, which is associated with significant negative deltas. Let us recall that premium payments are based on the outstanding nominal. The arrival of defaults thus reduces the commitment to pay. Furthermore, the increase in credit spreads due to contagion effects involves a decrease in the expected outstanding nominal. When considering the default leg only, we are led to credit deltas that actually remain within the standard 0–100% range. The default leg delta of the equity tranche with respect to the credit default swap index is initially equal to 54.1%. Let us also remark that credit deltas of the default leg gradually increase with time, which is consistent with a decrease in time value.

Table 11. Market delta spreads and model deltas (a default event) at inception.

	[0–3%]	[3–6%]	[6–9%]	[9–12%]	[12–22%]
Market deltas	27	4.5	1.25	0.6	0.25
Model deltas	21.5	4.63	1.63	0.9	0.6

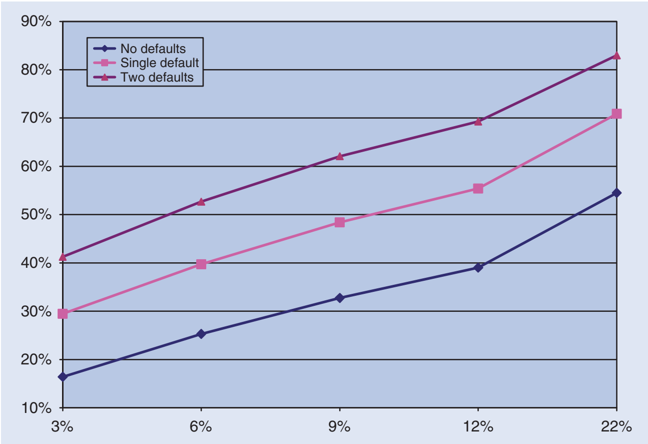

Figure 8. Dynamics of the base correlation curve with respect to the number of defaults. Detachment points on the x axis. Base correlations on the y axis.

5.7. *Comparison with standard market practice*

We further examine the credit deltas of the different tranches at inception. These are compared with the deltas as computed by market participants under the previous base correlation structure assumption (see table 11). These market deltas are calculated by bumping the credit curves by one basis point and computing the changes in present value of the tranches and of the credit default swap index. Once the credit curves are bumped, the moneyness varies, but the market practice is to keep the base correlations constant when recalculating the CDO tranches. This corresponds to the so-called 'sticky strike' rule. The delta is the ratio of the change in present value of the tranche to the change in present value of the credit default swap index divided by the tranche's nominal. For example, a credit delta of an equity tranche previously equal to one would now lead to a figure of 33.33.

First, we can see that the outlines are roughly the same, which is noticeable since the two approaches are completely different. We can then remark that the model deltas are smaller for the equity tranche compared with the market deltas, while they are larger for the other tranches.

These discrepancies can be understood from the dynamics of the dependence between defaults embedded in the Markovian contagion model. Figure 8 shows the base correlation curves at a 14 week horizon, when the number of defaults is equal to zero, one or two. We can see that the arrival of the first defaults is associated with

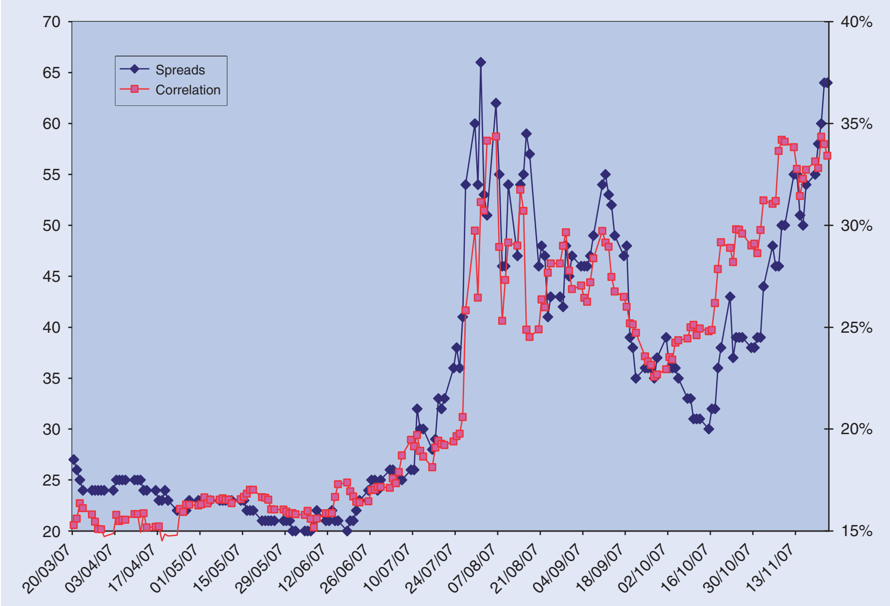

Figure 9. Credit spreads on the five year iTraxx index (series 7) in bps on the left axis. Implied correlation on the equity tranche on the right axis.

parallel shifts in the base correlation curves. This increase in dependence counterbalances the increase of credit spreads and expected losses on the equity tranche and lowers the credit delta. The model deltas can be thought of as the 'sticky implied tree' model deltas of Derman (1999). These are suitable in a regime of fear corresponding to systematic credit shifts.

The summer 2007 credit crisis provides evidence that implied correlations tend to increase with credit spreads and thus with expected losses. Figure 9 shows the dynamics of the five year iTraxx credit spread and of the implied correlation of the equity tranche. Over this period the correlation between the two series was 91%. This clearly favors the contagion model and once again suggests a flaw in the 'sticky strike' market practice.

We also thought it insightful to compare our model deltas and the results provided by Arnsdorf and Halperin (2007, figure 7) (see table 12).

The market conditions are slightly different since the computations were performed in March 2007, thus the maturity is slightly less than five years. The market deltas are quoted deltas provided by major trading firms. We can see that these are quite close to the previous market deltas since the computation methodology involving the Gaussian copula and the base correlation is quite standard. The model deltas (corresponding to 'model B' of Arnsdorf and Halperin 2007) have a different meaning from ours: theirs are related to credit spread deltas rather than default risk deltas and are not related to a dynamic replicating strategy. However, it is noteworthy that the model deltas of Arnsdorf and Halperin (2007) are quite similar to ours, and thus rather far away from market deltas. Although this is not a formal proof, it appears from figure 4 that (systemic) gammas are rather small prior to the first default. If we could view a shock on the credit spreads as a small shock on the expected loss while

Table 12. Market and model deltas as in Arnsdorf and Halperin (2007).

	[0–3%]	[3–6%]	[6–9%]	[9–12%]	[12–22%]
Market deltas	26.5	4.5	1.25	0.65	0.25
Model deltas	21.9	4.81	1.64	0.79	0.38

Table 13. Market deltas, 'intensity' model credit deltas in Eckner (2007) and contagion model deltas.

Tranches	[0–3%]	[3–7%]	[7–10%]	[10–15%]	[15–30%]
Market deltas	18.5	5.5	1.5	0.8	0.4
AJD deltas	21.7	6.0	1.1	0.4	0.1
Contagion model deltas	17.9	6.3	2.5	1.3	0.8

a default event induces a larger shock (but not so large given the risk diversification at the index level) on the expected loss, the similarity between the different model deltas is not so surprising. As above, model deltas are smaller for the equity tranche and larger for the other tranches when compared with market deltas.

We also compare our model deltas with credit deltas obtained by Eckner (2007, table 5) within an affine jump-diffusion intensity model where the model parameters were calibrated on CDX NA IG5 quotes of December 2005 (see table 13). In the latter framework, credit deltas are computed from sensitivities of CDO tranche and index prices with respect to a uniform and relative shift of individual intensities. We compute our contagion model deltas from loss intensities calibrated on the same data set.

Even though the approaches are completely different, once again the outlines are similar. Let us remark that the equity tranche deltas computed by Eckner are larger according to some 'sticky delta' rule.

6. Conclusions

The lack of internally consistent methods to hedge CDO tranches has paved the way for a variety of local hedging approaches that do not guarantee the full replication of tranche payoffs. This may not appear to be a practical issue when trade margins are high and holding periods short. However, we believe that there might be a growing concern from investment banks concerning the long-term credit risk management of trading books as the market matures.

A homogeneous Markovian contagion model can be implemented as a recombining binomial tree and thus provides a strikingly easy way to compute dynamic replicating strategies of CDO tranches. While such models have recently been considered for the pricing of exotic basket credit derivatives, our main concern here is to provide a rigorous framework for the hedging issue.

We do not aim at providing a definitive answer to the thorny issue of hedging CDO tranches. For this purpose, we would also need to tackle name heterogeneity, possible non-Markovian effects in the dynamics of credit spreads, non-deterministic intensities between two default dates, the occurrence of multiple defaults, stochastic recovery rates, etc. A fully comprehensive approach to the hedging of CDO tranches is likely to be quite cumbersome both on economic and numerical grounds.

However, from a practical perspective, we believe that our approach might be useful to assess the default exposure of CDO tranches by quantifying the credit contagion effects in a reasonable way. We also found some noticeable similarities between credit spread deltas as computed under the standard base correlation methodology and the default risk deltas as computed from our recombining tree. A closer look at the discrepancies between the two approaches suggests some inconsistency in the market approach as far as the dynamics of the correlation is involved. Taking into account such dynamic effects lowers the credit deltas of the equity tranche and therefore increases the credit deltas of the senior tranches. From a risk management perspective, understanding how credit deltas are related to base correlation curves requires a coupling of standard vanna analysis and the study of contagion and dynamic dependence effects.

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Appendix A: Dynamics of defaultable discount bonds and credit spreads

Let us derive the dynamics of a (digital) defaultable discount bond associated with name $i \in \{1, \dots, n\}$ and maturity T. The corresponding payoff at time T is equal to $1_{\{\tau_i > T\}} = 1 - N_i(T)$. Let us now consider a portfolio of the previously defined defaultable bonds with holdings equal to $1/n$ for all names. The portfolio payoff is equal to $V_I(T, N(T)) = 1 - [N(T)/n]$. The replication price at time t given that $N(t) = k$ of such a portfolio is equal to

$$V_I(t,k) = \mathrm{e}^{-r(T-t)} E^Q\left[1 - \frac{N(T)}{n} \,\middle|\, N(t) = k\right].$$

Since the names are exchangeable, the $n-k$ non-defaulted names have the same price, which is thus $V_I(t,k)/(n-k)$. Thus the price time t of the defaultable discount bond, $B_i(t,T)$, is given by

$$\begin{aligned} B_i(t,T) &= (1 - N_i(t)) \times \frac{V_I(t, N(t))}{n - N(t)}, \quad V_I(t, \bullet) \\ &= \mathrm{e}^{-r(T-t)} Q(t,T) V_I(T, \bullet), \end{aligned}$$

where the pre-default intensity of τ_i is equal to $\alpha^Q_\bullet(t, N(t)) = \lambda(t, N(t))/[n - N(t)]$. When $N(t) = n$, $\alpha^Q_\bullet(t, N(t)) = 0$ and $B_i(t,T) = 0$. We remark that the defaultable discount bond price follows a Markov chain with $n+1$ states $\{N(t) = 0, N_i(t) = 0\}, \dots,$

$\{N(t) = n-1, N_i(t) = 0\}$ and $\{N_i(t) = 1\}$. The generator matrix, $\Lambda(t)$, is equal to

$$\begin{pmatrix} -\lambda(t,0) & ((n-1)/n)\lambda(t,0) & 0 & 0 & 0 & 0 & \lambda(t,0)/n \\ 0 & -\lambda(t,1) & ((n-2)/(n-1))\lambda(t,1) & 0 & & & \lambda(t,1)/(n-1) \\ 0 & 0 & \bullet & \bullet & & & \bullet \\ 0 & & & & \bullet & & \bullet \\ 0 & & & & & & \bullet \\ 0 & & & & & -\lambda(t,n-1) & \lambda(t,n-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the dynamics of the defaultable bond prices can be viewed as a special case of that studied by Jarrow *et al.* (1997), although the economic interpretation of the states slightly differs.

Appendix B: Calibration equations for a complete set of a number of default probabilities

While the pricing and thus the hedging involves a backward procedure, calibration is associated with forward Kolmogorov differential equations. We show here a non-parametric fitting procedure of a possibly non-time-homogeneous pure birth process onto a complete set of marginal distributions of a number of defaults. This is quite similar to that described by Schönbucher (2006), although the purpose is different since the aim of the previous paper was to construct arbitrage-free, consistent with a complete loss surface, Markovian model of aggregate losses, possibly in incomplete markets, without detailing the feasibility and implementation of replication strategies.

We further denote the marginal number of default probabilities by $p(t,k) = Q(N(t) = k)$ for $0 \le t \le T$, $k = 0, 1, \dots, n$.

In the case of a pure birth process, the forward Kolmogorov equations can be written as

$$\frac{\mathrm{d}p(t,k)}{\mathrm{d}t} = \lambda(t,k-1)\,p(t,k-1) - \lambda(t,k)\,p(t,k), \quad \text{for} \quad k = 1, \dots, n, \quad \frac{\mathrm{d}p(t,0)}{\mathrm{d}t} = -\lambda(t,0)\,p(t,0).$$

Since the space state is finite, there are no regularity issues and these equations admit a unique solution (see below for practical implementation). We refer to Karlin and Taylor (1975) for more details concerning the forward equations in the case of a pure birth process. These forward equations can be used to compute the loss intensity dynamics $t \in [0,T] \to \lambda(t, N(t))$, due to

$$\lambda(t,0) = -\frac{1}{p(t,0)}\frac{\mathrm{d}p(t,0)}{\mathrm{d}t}, \quad \lambda(t,k) = \frac{1}{p(t,k)}\left[\lambda(t,k-1)\,p(t,k-1) - \frac{\mathrm{d}p(t,k)}{\mathrm{d}t}\right],$$

for $k = 1, \dots, n$, $0 \le t \le T$.

We can also write

$$\lambda(t,k) = -\frac{1}{p(t,k)}\frac{\mathrm{d}\sum_{m=0}^{k} p(t,m)}{\mathrm{d}t} = -\frac{1}{Q(N(t)=k)}\frac{\mathrm{d}Q(N(t) \le k)}{\mathrm{d}t}.$$

Finally, the name intensities are provided by

$$\alpha_{\bullet}^{Q}(t, N(t)) = \frac{\lambda(t, N(t))}{n - N(t)}.$$

This shows that, under the assumption of no simultaneous defaults, we can fully recover the loss intensities from the marginal distributions of the number of defaults. However, despite its simplicity, the previous approach (the inference of $\lambda(t,k)$ from the default probabilities $p(t,m)$) involves theoretical and practical issues.

With respect to the theoretical issues, we deal with the assumption of no simultaneous defaults. We show below that, under standard no-arbitrage requirements, (pseudo)-loss intensities might still be computed but they may fail to reconstruct the input number of defaults distributions. Whatever the model, the marginal number of defaults probabilities must fulfil $0 \le p(t,m) \le 1$, $\forall\,(t,m) \in [0,T] \times \{0, 1, \dots, n-1\}$, $\sum_{m=0}^{n} p(t,m) = 1$, $\forall t \in [0,T]$, and since $N(t)$ is non-decreasing, $\sum_{m=0}^{k} p(t,m) \ge \sum_{m=0}^{k} p(t',m)$, $\forall k \in \{0, 1, \dots, n\}$, $\forall t, t' \in [0,T]$ and $t \le t'$. This implies that $\lambda(t,k)$, as computed from the above equation, are non-negative. Moreover, since

$$\sum_{m=0}^{n} p(t,m) = 1, \qquad \frac{\mathrm{d}\sum_{m=0}^{n} p(t,m)}{\mathrm{d}t} = 0,$$

$\lambda(t,n) = 0$, i.e. $\{N(t) = n\}$ is absorbing. In other words, standard no-arbitrage constraints on the probabilities of the number of defaults guarantee the existence of non-negative (pseudo)-loss intensities with the required boundary conditions. However, we conclude that this (pseudo)-loss intensity may fail to reconstruct the input number of defaults distributions. The no simultaneous defaults assumption implies particularly that $\mathrm{d}p(t,m)/\mathrm{d}t = 0$ for $t = 0$ and $m > 1$. If this constraint is not fulfilled by market inputs, we will not be able to reconstruct the input $p(t,m)$ from the (pseudo)-loss intensities.

On practical grounds, the computation of $p(t,m)$ usually involves an arbitrary smoothing procedure and hazardous extrapolations for small time horizons.

For these reasons, we believe that it is more appropriate and reasonable to calibrate the Markov chain of aggregate losses on a discrete set of meaningful market inputs corresponding to liquid maturities.

Appendix C: Calibration of time-homogeneous loss intensities

Solving for the forward equations provides $p(T,0) = e^{-\lambda_0 T}$ and $p(T,k) = \lambda_{k-1} \int_0^T e^{-\lambda_k (T-s)} p \times (s, k-1) ds$ for $1 \le k \le n-1$ (see Karlin and Taylor 1975 for more details). The previous equations can be used to determine $\lambda_0, \ldots, \lambda_{n-1}$ iteratively, even if our calibration inputs are the default probabilities at the single date T.

Assume for the moment that the intensities $\lambda_0, \ldots, \lambda_{n-1}$ are known, positive and distinct.† To solve the forward equations, we assume that the default probabilities can be written as $p(t,k) = \sum_{i=0}^{k} a_{k,i} e^{-\lambda_i t}$ for $0 \le t \le T$ and $k = 0, \ldots, n-1$.‡ Set $a_{0,0} = 1$, and the recurrence equations $a_{k,i} = [\lambda_{k-1}/(\lambda_k - \lambda_i)] a_{k-1,i}$ for $i = 0, 1, \ldots, k-1$, $k = 1, \ldots, n-1$ and $a_{k,k} = -\sum_{i=0}^{k-1} a_{k,i}$. Then we can readily check that, if satisfied, these equations provide solutions of the forward PDE. Since it is well known that these solutions are unique, this means we have obtained explicitly the solution of the forward PDE, knowing the intensities $(\lambda_k)_{k=1,\ldots,n}$.

Therefore, using $p(0,k) = 0$ and $\lambda_0 = -\ln(p(T,0))/T$, we can compute iteratively $\lambda_1, \ldots, \lambda_{n-1}$ by solving the univariate nonlinear implicit equations $p(T,k) = \sum_{i=0}^{k} a_{k,i} e^{-\lambda_i T}$, or, equivalently,

$$\sum_{i=0}^{k-1} a_{k-1,i} e^{-\lambda_i T} \times \left(\frac{1 - e^{-(\lambda_k - \lambda_i)T}}{\lambda_k - \lambda_i} \right) = \frac{p(T,k)}{\lambda_{k-1}}, \quad k = 1, \ldots, n-1.$$

It can readily be seen that, for any $k \in \{0, \ldots, n-1\}$, $p(T,k)$ is a decreasing function of λ_k, taking value $\lambda_{k-1} \int_0^T p(s, k-1) \, ds$ for $\lambda_k = 0$ and with a limit equal to zero as λ_k tends to infinity. In other words, the previous λ_k equations have a unique solution provided that

$$p(T,k) < \lambda_{k-1} \times \left(\sum_{i=0}^{k-1} a_{k-1,i} \times \left(\frac{1 - e^{-\lambda_i T}}{\lambda_i} \right) \right), \quad \text{for } k = 1, \ldots, n-1.$$

Note that, in practice, all the intensities λ_k will be different (almost surely). Thus, starting from the T-default probabilities only, we have found the explicit solutions of the forward equations and the intensities $(\lambda_k)_{k=1,\ldots,n}$ that would be consistent with these probabilities.

Appendix D: Tree computations for different recovery rates

Given a recovery rate R, the (fractional) loss at time t on the credit portfolio is such that $L(t) = (1-R)[N(t)/n]$. The mapping

$$(t, \tilde{k}) \in [0,T] \times [0,1] \to EL(t, \tilde{k}) = E^Q \left[\min(\tilde{k}, L(t)) \right] = \sum_{m=1}^{n} \min\left(\tilde{k}, \frac{m(1-R)}{n} \right) p(t,m)$$

is known as the 'loss surface'. We can compute the probabilities of the number of defaults from $EL(t, \tilde{k})$. It can quickly be checked that the probabilities of the number of defaults are given by

$$p(t,k) = \frac{n}{R-1} \times \begin{pmatrix} EL\left(t, \frac{(k-1)\times(1-R)}{n}\right) \\ -2EL\left(t, \frac{k\times(1-R)}{n}\right) \\ +EL\left(t, \frac{(k+1)\times(1-R)}{n}\right) \end{pmatrix},$$

for $k = 1, \ldots, n-1$, and

$$p(t,n) = \frac{n}{1-R} \times \left(EL(t, 1-R) - EL\left(t, \frac{n-1}{n} \times (1-R)\right) \right).$$

Finally, $p(t,0)$ is obtained from $\sum_{k=0}^{n} p(t,k) = 1$. This provides the dependence of the probabilities of the number of defaults with respect to the recovery rate R.

†Due to the last assumption, the described calibration approach is not highly regarded by numerical analysts (see Moler and Van Loan (2003) for a discussion). However, it is well suited in our case studies.

‡Since $\lambda_n = 0$, $p(t,n)$ takes a slightly different form. Its detailed expression is useless here since we only need to deal with $p(t,0), \ldots p(t, n-1)$ to calibrate $\lambda_0, \ldots, \lambda_{n-1}$. Let us also remark that $p(t,n)$ can equally be recovered from $p(t,n) = \lambda_{n-1} \int_0^t p(s, n-1) \, ds$ or from $\sum_{k=0}^{n} p(t,k) = 1$.