The limits of Granularity Adjustments

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Abstract

We provide a rigorous proof of granularity adjustment (GA) formulas to evaluate loss distributions and risk measures (value-at-risk) in the case of heterogenous portfolios, multiple systemic factors and random recoveries. As a significant improvement with respect to the literature, we detail all the technical conditions of validity and provide an upper bound of the remainder term at a finite distance. Moreover, we deal explicitly with the case of general loss distributions, possibly with masses. For some simple portfolio models, we prove empirically that the granularity adjustments do not always improve the infinitely granular first-order approximations. This stresses the importance of checking our conditions of regularity before relying of such techniques. And smoothing the underlying loss distributions through random recoveries or exposures improves the GA performances in general.

Keywords: Credit portfolio model; Granularity adjustment; Value-at-risk; Fourier Transform
JEL Classification: G32; G17

1. Introduction

Value-at-risk (VaR) remains the corner stone of banking regulatory capital calculations, despite its well-known drawbacks (see the critics in Danielsson et al., 2001, for instance). Its use has spread throughout the fields of risk measurement and portfolio management: trading limits calculations, risk-return analysis, etc. Precise estimations of VaRs require computationally intensive processes, even for moderate-size portfolios, and particularly for high levels. Unfortunately, every user is not able to launch such a full evaluation process, or is not ready to wait several hours (or even days) every time a VaR is required. Therefore, there has been a need for approximated methods. These methods should be able to calculate quickly and efficiently VaRs, Expected Shortfalls, or other risk measures, for large portfolios and possibly complex instruments.

For a long time, it has been observed that the computation of VaRs is dramatically simplified under the assumption that bank portfolios are perfectly fine-grained (or infinitely granular, equivalently). In such situations, diversification fully eliminates idiosyncratic risk, so that the portfolio loss depends on systemic risk only. Since real-world portfolios are not perfectly fine-grained, it remains a residual of undiversified idiosyncratic risk in portfolios. The impact of undiversified idiosyncratic risk on VaRs can be approximated analytically through the so-called "granularity adjustments" (GA) techniques. In mathematical terms, they are based on asymptotic expansions of portfolio loss distributions, when the number of exposures is "large".

The Basel 2 proposals for credit risk measurement has fueled research in this area since the beginning of this century: see Wilde (2001a), Gordy (2003). In particular, GAs allow closed-form calculations of approximated VaRs or expected shortfalls for some well-known industry models of credit portfolios. For

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instance, Wilde (2001b) provided GA formulas for CreditRisk+ (CSFP, 1997). Emmer and Tasche (2005) developed the same analysis for CreditMetrics (Gupton et al., 1997). These results have been reworked and/or amended in several papers, notably Gordy and Lütkebohmert (2012) in the case of CreditMetrics. In the increasing GA literature, seminal papers include Gouriéroux, Laurent and Scaillet (2000), Gordy (2003), Martin and Wilde (2002), etc. More recently, Antonov et al. (2007) applied such techniques to price CDOs. Gordy and Juneja (2010) showed that such techniques apply even with portfolio of options, for which nested simulations are required. Gagliardini and Gouriéroux (2011) provided a framework for asset pricing with factor models. Gordy and Marrone (2012) developed the GA theory under a mark-to-market credit risk perspective.

To the best of our knowledge, the specific orders of magnitude of GA approximations have never been studied in the literature. In other words, no paper has evaluated the error terms of such approximations. Moreover, the underlying conditions of regularity are often imprecise or too strong. Our goal will be to revisit the theory, by pushing the GA mathematics forward. Contrary to the current practice, we will not assume *a priori* that "the asymptotics work", i.e. that we can rely on the infinitely granular assumption. At the opposite, we will revisit the theory by stating clear-cut granularity expansion results by keeping fixed the portfolio size n. Therefore, we do not state asymptotic results, but rather exact results at a finite distance. In Section 2, we establish two slightly different GA expansions. Under some sufficient conditions of regularity, we evaluate the errors induced by these approximations for general heterogenous credit portfolios.

Finally, we measure the quality of GA approximations by simulation, for some simple credit risk models. These specifications are a bit unusual but not unrealistic. They will play the role of "toy models" because they allow closed-form calculations of GA terms. Theoretically and empirically, we show that the performances of granularity adjustments may be explained in the light of the regularity conditions we have exhibited previously. Particularly, they depend on the tail behavior of the characteristic functions of the underlying loss distributions: see Section 3. Proofs are gathered in the appendix.

2. A general formula for granularity adjustments

2.1. The framework

To fix the notations, consider a n-size credit portfolio. The exposure E_j of any counterparty j will be assumed constant in time. Such a quantity is positive because exposures are comparable to replacement costs of some risky positions in a portfolio. Nonetheless, nothing precludes negative exposures in this article. We will be interested in the credit risk associated to this portfolio, between today and a given time horizon T. The associated "loss rate" of a given name j will be modeled as a random variable Z_j, $j = 1, \ldots, n$. If no default occurs before the time horizon T, $Z_j = 0$. Otherwise, $Z_j > 0$ is the stochastic loss rate associated to j's default. Then, the normalized portfolio loss (between today and the given time horizon T) is the random quantity

$$L_n := \sum_{j=1}^{n} A_{jn} Z_j, \tag{1}$$

where $A_{jn} := E_j / \sum_{j=1}^{n} E_j$ denotes the j-th share of exposure. Therefore, A_{jn} is the j-th percentage of the total risky exposure in the portfolio, and $\sum_{j=1}^{n} A_{jn} = 1$.

Actually, in a classical default risk mode, Z_j can be seen as the product of an indicator function $D_j \in \{0, 1\}$ and a loss-given-default (LGD) percentage of the face value of loan j. Here, $D_j = 1$ means "default of j before T". LGDs (also one minus recovery rates) are random percentages of the exposure shares A_{jn}, without modeling the exact time when these cash-flows are repaid. Equivalently, it means all repayments occur at the same time horizon T. Note that $1 - Z_j$ can be interpreted as the T-market value of j's exposure, for a nominal of one, as usually under a market risk point-of-view. We assume that every variable Z_j is bounded from above by a constant $\bar{Z}_j$. This level will be one most of the time, but we do not exclude the possibility of negative recovery rates (i.e. losses larger than 100%).

We work under the static factor model framework, that is usual in credit risk portfolio modeling (see Koyluoglu and Hickman, 1998, Gordy, 2000, Gregory and Laurent, 2005, etc): conditionally on a given

random vector $\mathbf{X} \in \mathbb{R}^m$, the random variables Z_j, $j = 1, 2, \ldots$, are mutually independent. In other words, the dependence between the individual losses is generated by some "systemic" random factors only, gathered into a vector $\mathbf{X}$.

The GA methodology works well when the number of underlyings n is "large", but the total portfolio notional is kept as a constant. In the literature, a portfolio is called "infinitely granular" when its size goes to the infinity ($n \to \infty$) and when every individual exposure share is infinitesimal with respect to the total portfolio size: $\lim_{n\to\infty} \sup_{j=1,\ldots,n} |A_{jn}| = 0$. It is well-known that, under this infinitely granular assumption, the law of L_n is similar to the law of $E[L_n|\mathbf{X}]$. Since the latter law is a lot more simpler than the former, most of the time, it is tempting to approximate the cdf or the quantiles of L_n by those of $E[L_n|\mathbf{X}]$. This provides a first-order approximation of risk measures like VaRs, that we call "$EVaR$".

Instead of dealing with more or less ad-hoc arguments, we will lead a mathematically rigorous Fourier analysis of the random loss L_n. First, we will calculate the relation between the characteristic functions of L_n and $E[L_n|\mathbf{X}]$, by some Taylor expansions. Second, some inversions of Fourier transforms will provide their corresponding relations in terms of cumulative distribution functions. Finally, we will deduce the link between their quantiles, and therefore their granularity adjustment terms [2].

2.2. The classical proof of granularity adjustments

Let us recall the usual way of deriving a granularity adjustment. The standard method relies on the sensitivity of Value-at-Risk w.r.t. individual exposures, by invoking the results of Gouriéroux, Laurent and Scaillet (2000). To be short, they consider a continuum of losses between the true random loss L_n and its infinitely granular approximation:

$$L_n(\varepsilon) = E[L_n|\mathbf{X}] + \varepsilon.\left(L_n - E[L_n|\mathbf{X}]\right),$$

where $\varepsilon \in [0, 1]$. For a given level $\alpha \in (0, 1)$, denote by $VaR_\alpha(\varepsilon)$ the Value-at-Risk associated with the loss $L_n(\varepsilon)$. The case $\varepsilon = 1$ corresponds to the true portfolio loss, and the case $\varepsilon = 0$ corresponds to the loss of the infinitely granular portfolio. A limited expansion between $\varepsilon = 0$ and $\varepsilon = 1$ links together $VaR_{L_n,\alpha}$ (the VaR of L_n) and its approximation $VaR_{E[L_n|\mathbf{X}],\alpha}$ (the VaR of $E[L_n|\mathbf{X}]$ or $EVaR$):

$$VaR_{L_n,\alpha} = VaR_\alpha(1) = VaR_{E[L_n|\mathbf{X}],\alpha} + \frac{\partial VaR_\alpha(\varepsilon)}{\partial\varepsilon}_{|\varepsilon=0} + \frac{1}{2}\frac{\partial^2 VaR_\alpha(\varepsilon)}{\partial^2\varepsilon}_{|\varepsilon=0} + \cdots$$

By simple calculations, it can be proved that

$$\frac{\partial VaR_\alpha(\varepsilon)}{\partial\varepsilon}_{|\varepsilon=0} = E\left[L_n - E[L_n|\mathbf{X}] \mid E[L_n|\mathbf{X}] = VaR_{E[L_n|\mathbf{X}],\alpha}\right] = 0.$$

This property is always true. It does not depend on the fact that $\mathbf{X}$ may be of dimension one and/or the function $x \mapsto E[L_n|X = x]$ may be monotonic. Indeed, the event $\{E[L_n|\mathbf{X}] = VaR_{E[L_n|\mathbf{X}],\alpha}\}$ can be rewritten $\{\mathbf{X} \in \mathcal{A}\}$ for some measurable subset $\mathcal{A}$ in $\mathbb{R}^m$. Since $E[L_n|\mathbf{X} \in \mathcal{A}]$ is an average of expected losses $E[L_n|\mathbf{X} = \mathbf{x}]$ when $\mathbf{x} \in \mathcal{A}$, then

$$E\left[L_n|E[L_n|\mathbf{X}] = VaR_{E[L_n|\mathbf{X}],\alpha}\right] = E[L_n \mid \mathbf{X} \in \mathcal{A}] = VaR_{E[L_n|\mathbf{X}],\alpha}.$$

Therefore, the granularity adjustment terms will come from the second order derivative of $VaR_\alpha(\varepsilon)$ w.r.t. ε. In the case of a univariate systemic factor X, Gouriéroux, Laurent and Scaillet (2000) provide the corresponding formulas, but under some strong regularity conditions. In particular, they assume that the joint density of $(L_n, E[L_n|\mathbf{X}])$ exists w.r.t. the Lebesgue measure. Unfortunately, portfolio loss distributions often discrete, at least partly. Since empirical quantiles (and then VaR measures) can still be defined by generalized inverse functions, it is important to adapt the theoretical results in this case. This will be done below.

[2] Note that Martin and Wilde (2002) have used the inversion of the Laplace transform technique to calculate derivatives of Value-at-Risk in a simplified framework, and they deduced the main "granularity adjustment" term. Nonetheless, they have not paid attention to the precision of this approximation.

2.3. A refined proof of granularity adjustments

Now, we set some additional notations. Since we rely on Fourier transform techniques, let χ_{L_n} and $\chi_{E[L_n|\mathbf{X}]}$ be the characteristic functions of L_n and $E[L_n|\mathbf{X}]$ respectively, i.e.

$$\chi_{L_n}(t) := E\left[\exp\left(itL_n\right)\right], \text{ and } \chi_{E[L_n|\mathbf{X}]}(t) := E\left[\exp\left(itE[L_n|\mathbf{X}]\right)\right],$$

for any t. To lighten the notations, such functions will be denoted χ_L and $\chi_{E[L|\mathbf{X}]}$ simply. Moreover, let $\mu_n(\mathbf{x}) := E[L_n|\mathbf{X} = \mathbf{x}]$ be the expected portfolio loss, conditionally on the value of the factor $\mathbf{X}$.

Since every random loss Z_j is upper bounded by a constant $\bar{Z}_j$ then, conditionally on the value of the "systemic" factor, the variance of j-th losses exists and is defined by

$$Var_j(\mathbf{X}) := E[Z_j^2|\mathbf{X}] - E[Z_j|\mathbf{X}]^2.$$

Note that the laws of the individual losses Z_j may differ, for different indices j, what is the case in practice. For any vector $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{x}_{-(k)}$ denotes the $m-1$ dimensional vector $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m)$.

Assumption A. X_1, the first component of $\mathbf{X}$, is an argument of $\mu_n(\mathbf{X})$ and $x_1 \mapsto \mu_n(x_1, \mathbf{x}_{-(1)})$ is strictly increasing and differentiable for every vector $\mathbf{x}_{-(1)}$. Moreover, we assume that the factor $\mathbf{X}$ has a density $f_{\mathbf{X}}$ w.r.t. the Lebesgue measure on $\mathbb{R}^m$.

The latter technical condition is satisfied easily in most of the factor models that have been considered in the literature. For convenience, this condition highlights the role of the first component of $\mathbf{X}$. Obviously, this index can be changed and the condition A modified.

In the case of a univariate factor model, i.e. $\mathbf{X} \in \mathbb{R}$, let

$$\kappa_j(v) := \frac{Var_j(\mu_n^{-1}(v)) f_X(\mu_n^{-1}(v))}{\mu_n'(\mu_n^{-1}(v))}. \tag{2}$$

This function will provide the key element of the granularity adjustments we state hereafter.

More generally when $m \geq 2$, for any $v \in \mathbb{R}$ and $j = 1, \ldots, n$, define

$$\kappa_j(v) := \int_{\{\mathbf{x}|\mu_n(\mathbf{x})=v\}} Var_j(\mathbf{x}) f_X(\mathbf{x}) \frac{d\mathbf{x}_{-(1)}}{\partial_{(1)}\mu_n(\mathbf{x})},$$

where $\partial_{(k)}\mu_n$ denotes the partial derivative of μ_n w.r.t. its k-th argument, $k = 1, \ldots, m$. These functions κ_j will provide the main term of GA in the multi-factor case. Even if multi-factor models are commonly used in practice, their GAs have not been calculated in the literature until now (to the best of our knowledge). Therefore, for the sake of illustration, a few functions κ_j and their corresponding GAs are provided in Appendix C, in the case of models that involve several systemic factors.

Theorem 1. *If Assumption A and if the following conditions are fulfilled:*

(i) For every $j = 1, \ldots, n$, the function $x \mapsto G_j(x) := \int_{-\infty}^{x} \kappa_j$ is bounded.

(ii) Every function $\hat{G}_j : t \mapsto \int \exp(itu)\kappa_j(u)\, du$ is integrable, $j = 1, \ldots, n$.

(iii) There exist some constants $\varepsilon \in (0,1)$ and $M = M_n > 0$ such that $\max_{j=1,\ldots,n} M_n A_{jn} \bar{Z}_j \leq 1 - \varepsilon$.

Then, for any $x \in \mathbb{R}$,

$$\frac{1}{2}\left[P(L_n < x + 0) + P(L_n < x - 0)\right] = \frac{1}{2}\left[P(E[L_n|\mathbf{X}] < x + 0) + P(E[L_n|\mathbf{X}] < x - 0)\right] + \quad T_{n,\infty}(x) + r_{n,M}(x),$$

where the granularity adjustment is

$$T_{n,\infty}(x) := \frac{1}{2}\sum_{j=1}^{n} A_{jn}^2 \kappa_j'(x),$$

and the remainder term $r_{n,M}(x)$ satisfies

$$|r_{n,M}(x)| \leq \frac{1}{\pi}|\int_M^{+\infty} Im\left(\chi_L(t)\exp(-itx)/t\right)\,dt| + \frac{1}{\pi}|\int_M^{+\infty} Im\left(\chi_{E[L|\mathbf{X}]}(t)\exp(-itx)/t\right)\,dt|$$
$$+ \quad \frac{1}{2\pi}\sum_{j=1}^n A_{jn}^2|\int_M^{+\infty} Im\left(\hat{G}_j(t)t\exp(-itx)\right)\,dt| + [\frac{19}{18}+k_\varepsilon].\frac{|M|^3}{3\pi}\sum_{j=1}^n A_{jn}^3\bar{Z}_j^3$$
$$+ \quad \frac{M^4}{2}[\frac{14}{9}+k_\varepsilon]^2\left\{\sum_{j=1}^n A_{jn}^2\bar{Z}_j^2\right\}^2\exp([\frac{14}{9}+k_\varepsilon]M^2\sum_{j=1}^n A_{jn}^2\bar{Z}_j^2), \tag{3}$$

where

$$k_\varepsilon := \left[\frac{\ln(1/\varepsilon)}{(1-\varepsilon)^3} - \frac{1}{2(1-\varepsilon)} - \frac{1}{(1-\varepsilon)^2}\right].$$

See the proof in the appendix. The main objective of Theorem 1 is to evaluate whether GAs can improve the first order approximation (of the law of L_n by the law of $E[L_n|\mathbf{X}]$). This will be the case when $T_{n,\infty}(x) >> r_{n,M}(x)$ for some relevant values x (close to the targeted VaR, typically).

Note that the choice of a convenient value M depends on n, but our main result above is fundamentally "at a finite distance". Intuitively, the larger is the size n, the larger will be M_n, under the constraint $\sup_j M_n A_{jn}\bar{Z}_j < 1-\varepsilon$. This constraint precludes some situations where a small subset of exposures concentrates a significant part of the whole portfolio exposure. In other words, when the number of names n tends to the infinity, A_{jn} should tend to zero sufficiently quickly, so that the remaining terms in (3) are negligible w.r.t. the granularity adjustment $T_{n,\infty}(x)$ itself.

A similar constraint appeared in the seminal paper of Gordy (2003): to prove the first-order approximation of L_n by $E[L_n|\mathbf{X}]$ almost surely, his Assumption (A-2) says that $E_n/\sum_{k=1}^n E_k = O(n^{-1/2-\zeta})$ for some $\zeta > 0$. Note that this implies $A_{jn} = O(j^{-1/2-\zeta})$ but not necessarily $A_{jn} = O(n^{-1/2-\zeta})$. In our paper, such conditions have to be revisited because, compared to Gordy (2003), we are dealing with higher order terms on one side (more demanding) but we prove only convergence in law on the other side (less demanding). Therefore, following Theorem 1, our required conditions on (A_{jn}) and (M_n) are written in terms of sums, like $M_n^p\sum_{j=1}^n A_{jn}^q$ for some couples of integers (p,q) (see (3)). Heuristically, we can see that our analysis will be globally more demanding than Condition (A-2) in Gordy (2003) (except in some pathological situations). Indeed, due to Theorem 1, we need to satisfy

$$M_n^4\{\sum_{j=1}^n A_{jn}^2\}^2 << \sum_{j=1}^n A_{jn}^2\kappa_j'(x)$$

for some x. This is almost equivalent to $M_n^4\sum_{j=1}^n A_{jn}^2 << 1$. Then, $\sum_{j=1}^n A_{jn}^2$ has to tend to zero. But Gordy's condition (A-2) implies only $\sum_{j=1}^n A_{jn}^2$ converges.

Even if our theoretical results cover arbitrary individual exposure shares, we will focus on the typical case of a balanced portfolio below: $A_{jn} = 1/n$, $j = 1,\ldots,n$. In this case, $T_{n,\infty}$ and GAs are of order n^{-1}, and we would like to check that $r_{n,M}(x) = o(n^{-1})$. A necessary condition, that will be discussed in Section 3, is $M_n^4 = o(n)$.

2.4. Granularity adjustments and Value-at-Risk approximations

Now, let us link the Value-at-Risk of the true loss L_n and of its "infinitely granular" approximation $E[L_n|\mathbf{X}]$. For any non-decreasing function F and any $u \in [0,1]$, let $F^-(u) := \inf\{x|F(x) \geq u\}$ be a quantile function that is associated to F. This is the standard generalized inverse function, as defined in the literature. The cdf of any random variable Z will be denoted by F_Z.

To simplify, we suppose in this subsection that:

Assumption B. The laws of L_n and $E[L_n|\mathbf{X}]$ are continuous.

Then, Theorem 1 implies that, under some technical assumptions and for every x,

$$F_{L_n}(x) = F_{E[L_n|\mathbf{X}]}(x) + T_{n,\infty}(x) + r_{n,M}(x).$$

We hope that the last term $r_{n,M}(x)$ can be seen as negligible, when M is well chosen and when the underlying distributions satisfy some regularity conditions.

Now, for a given $\alpha \in (0,1)$, the value-at-risk of L_n at level α is defined as the generalized inverse of F_{L_n}, i.e.

$$VaR_{L_n,\alpha} = F^-_{L_n}(\alpha).$$

Most of the time, there exists a single possible value-at-risk for a given level α, except when F_{L_n} takes the value α on a non-empty interval I_α: $F_{L_n}(x) = \alpha$ for every $x \in I_\alpha$. In this case, every number in the interior of I_α is a VaR_α candidate. Note that this situation is very unlikely in practice when different exposures per name are considered. In every case, under B, we satisfy the nice property

$$F_{L_n}(VaR_{L_n,\alpha}) = \alpha, \quad \text{and} \quad F_{E[L_n|\mathbf{X}]}(VaR_{E[L_n|\mathbf{X}],\alpha}) = \alpha.$$

Theorem 2. *Under Assumption B and the assumptions of Theorem 1, if $f_{L_n|\mathbf{X}}$, the density of $E[L_n|\mathbf{X}]$, exists and is not zero in the neighborhood of $VaR_{E[L_n|\mathbf{X}],\alpha}$ then*

$$VaR_{L_n,\alpha} = VaR_{E[L_n|\mathbf{X}],\alpha} - \frac{T_{n,\infty}(VaR_{L_n,\alpha}) + r_{n,M}(VaR_{L_n,\alpha})}{f_{L_n|\mathbf{X}}(v_\alpha)},$$

for some real number M, chosen as in Theorem 2, and some number v_α such that

$$|v_\alpha - VaR_{E[L_n|\mathbf{X}],\alpha}| \leq |VaR_{E[L_n|\mathbf{X}],\alpha} - VaR_{L_n,\alpha}|.$$

Proof. It is a simple consequence of a limited expansion. Indeed,

$$\begin{aligned} F_{E[L_n|\mathbf{X}]}(VaR_{E[L_n|\mathbf{X}],\alpha}) = \alpha &= F_{L_n}(VaR_{L_n,\alpha}) \\ &= F_{E[L_n|\mathbf{X}]}(VaR_{L_n,\alpha}) + T_{n,\infty}(VaR_{L_n,\alpha}) + r_M(VaR_{L_n,\alpha}), \end{aligned}$$

and then

$$f_{L_n|\mathbf{X}}(v_\alpha).\left(VaR_{E[L_n|\mathbf{X}],\alpha} - VaR_{L_n,\alpha}\right) = T_{n,\infty}(VaR_{L_n,\alpha}) + r_{n,M}(VaR_{L_n,\alpha}),$$

for such a number v_α. □

The size of the error term $r_M(VaR_{L_n,\alpha})$ is related to the choice of M, to the tails of the characteristic functions of losses and to the behavior of the exposure shares A_{jn}, $j = 1, \ldots, n$ for large n. Invoking Equation (2) and assuming the remainder term $r_{n,M}$ is negligible w.r.t. $T_{n,\infty}$, we deduce the usual granularity adjustment easily.

Corollary 3. *Under the assumptions of Theorem 2,*

$$VaR_{L_n,\alpha} \simeq VaR_{E[L_n|X],\alpha} - \frac{1}{2f_{L_n|\mathbf{X}}(VaR_{E[L_n|X],\alpha})} \sum_{j=1}^{n} A_{jn}^2 \kappa_j'(VaR_{E[L_n|X],\alpha}), \tag{4}$$

If the systemic factor $\mathbf{X} := X$ is univariate and μ_n is invertible, then

$$\kappa_j'(v) := \frac{d}{dv}\left[\frac{Var_j(\mu_n^{-1}(v))f_X(\mu_n^{-1}(v))}{\mu_n'(\mu_n^{-1}(v))}\right] = \frac{d}{dv}\left[Var_j(\mu_n^{-1}(v))f_{L_n|X}(v)\right]. \tag{5}$$

Equivalently, we have

$$VaR_{L_n,\alpha} \simeq VaR_{E[L_n|X],\alpha} - \frac{1}{2}\sum_{j=1}^{n} A_{jn}^2 \left[\frac{Var_j'(VaR_{X,\alpha})}{\mu_n'(VaR_{X,\alpha})} + Var_j(VaR_{X,\alpha})\frac{f'_{L_n|X}(VaR_{E[L_n|X],\alpha})}{f_{L_n|X}(VaR_{E[L_n|X],\alpha})}\right] \tag{6}$$

or even

$$VaR_{L_n,\alpha} \simeq VaR_{E[L_n|X],\alpha} - \frac{1}{2f_X(VaR_{X,\alpha})}\sum_{j=1}^{n} A_{jn}^2 \frac{d}{dt}\left[\frac{Var_j(t)f_X(t)}{\mu_n'(t)}\right]_{|t=VaR_{X,\alpha}}. \tag{7}$$

PROOF. Equation (4) is just a consequence of Theorem 2. Due to the monotonicity of μ_n, note that $P(E[L_n|X] \leq t) = P(X \leq \mu_n^{-1}(t))$, and then

$$f_{L_n|X}(t) = \frac{f_X(\mu_n^{-1}(t))}{\mu_n'(\mu_n^{-1}(t))}.$$

This proves Equation (5). Since $\mu_n^{-1}(VaR_{E[L_n|X],\alpha}) = VaR_{X,\alpha}$, we deduce Equation (6). We get Equation (7) by setting $v = \mu_n(t)$ and writing derivatives w.r.t. t instead of v. □

Therefore, we recover the granularity adjustment formulas, as they appeared in Wilde (2001a,b), Gordy (2003) or Martin and Wilde (2002), among others. Nonetheless, these formulas consider that $VaR_{E[L_n|X],\alpha}$ and $VaR_{L_n,\alpha}$ are sufficiently close so that we can use one of the other as arguments of κ_j and/or $f_{L_n|\mathbf{X}}$, in the granularity adjustment formula. Actually, in practice, this choice could matter, even if it should be negligible asymptotically. This is why we propose a new GA: set

$$\bar{V}ar_{n,\alpha} := \frac{1}{2}(VaR_{L_n,\alpha} + VaR_{E[L_n|\mathbf{X}],\alpha}). \tag{8}$$

Corollary 4. *Under the assumptions of Theorem 2,*

$$VaR_{L_n,\alpha} \simeq VaR_{E[L_n|X],\alpha} - \frac{1}{2f_{L_n|\mathbf{X}}(\bar{V}aR_{n,\alpha})} \sum_{j=1}^{n} A_{jn}^2 \kappa_j'(VaR_{L_n,\alpha}). \tag{9}$$

When the "systemic" factor $\mathbf{X} := X$ is univariate and μ_n is invertible, we have

$$\begin{aligned} VaR_{L_n,\alpha} &\simeq VaR_{E[L_n|X],\alpha} \\ &- \frac{1}{2f_X(\mu_n^{-1}(\bar{V}aR_{n,\alpha}))} \cdot \frac{\mu_n'(\mu_n^{-1}(\bar{V}aR_{n,\alpha}))}{\mu_n'(\mu_n^{-1}(VaR_{L_n,\alpha}))} \sum_{j=1}^{n} A_{jn}^2 \frac{d}{dt}\left[\frac{Var_j(t) f_X(t)}{\mu_n'(t)}\right]_{|t=\mu_n^{-1}(VaR_{L_n,\alpha})}. \end{aligned} \tag{10}$$

The latter formulas are slightly more complex than the traditional GA formulas in Corollary 3. Nonetheless, for theoretical reasons (see the proof of Corollary 3), there are some hopes their relevance is slightly higher. We will discuss this point in the empirical Section 3.

Actually, it should be noticed that the previous GA formulas belong to a more general class of granularity adjustments: for a given θ in $[0,1]$, set

$$\bar{V}ar_{n,\alpha}(\theta) := \theta VaR_{L_n,\alpha} + (1-\theta)VaR_{E[L_n|\mathbf{X}],\alpha}.$$

Therefore, there exists a continuum of GA formulas

$$VaR_{L_n,\alpha} \simeq VaR_{E[L_n|X],\alpha} - \frac{1}{2f_{L_n|X}(\bar{V}aR_{n,\alpha}(\theta))} \sum_{j=1}^{n} A_{jn}^2 \kappa_j'(\bar{V}aR_{n,\alpha}(\theta')), \quad (\theta,\theta') \in [0,1]^2. \tag{11}$$

In Corollaries 3 and 4, we have just chosen $(\theta,\theta') = (0,1)$ and $(\theta,\theta') = (1/2,0)$ respectively.

2.5. Granularity adjustments and Value-at-Risk approximations with discrete loss distributions

Now, we would like to remove Assumption B, to be able to deal with discontinuous L_n loss distributions. This case appears naturally when exposures and recoveries are assumed to be fixed (a rough but standard assumption, unfortunately). On the other side and following the GA literature, we still assume that the distribution of $E[L_n|\mathbf{X}]$ is continuous, and that it owns a density w.r.t. the Lebesgue measure. Actually, most of the previous results still apply with slight modifications under this new framework.

To apply Theorem 1, we define the normalized cdf $\tilde{F}$ by

$$\tilde{F}(x) := \frac{1}{2}[F(x-0) + F(x+0)],$$

for every x. Theorem 1 says that, under some conditions of regularity, for every x,

$$\tilde{F}_{L_n}(x) = \tilde{F}_{E[L_n|\mathbf{X}]}(x) + T_{n,\infty}(x) + r_{n,M}(x).$$

Now, for a given $\alpha \in (0,1)$, the value-at-risk of L_n at level α is a corresponding quantile of F_{L_n}. To be consistent with Theorem 1, we decide that the VaRs are always defined as quantiles of $\tilde{F}_{L_n}$:

$$VaR_{L_n,\alpha} := \tilde{F}^-_{L_n}(\alpha), \quad \alpha \in (0,1).$$

On one side, let us assume that the levels α we consider are "reached" by the corresponding cdfs. In other words, α lies in the range of these cdfs:

Assumption B.* $\tilde{F}_{L_n}(VaR_{L_n,\alpha}) = \alpha$.

This assumption may be strong in practice, if α chosen independently of the range of L_n's values. Since we have assumed the law of $E[L_n|\mathbf{X}]$ is continuous, note that we always have

$$\tilde{F}_{E[L_n|\mathbf{X}]}(VaR_{E[L_n|\mathbf{X}],\alpha}) = F_{E[L_n|\mathbf{X}]}(VaR_{E[L_n|\mathbf{X}],\alpha}) = \alpha.$$

By the same arguments as above, we get easily:

Proposition 5. *If the distribution of L_n is general (possibly with discrete masses), Theorem 2, Corollary 3 and 4 still apply, replacing Assumption B with Assumption B*.*

On the other side, if α does not belong to the image set of $\tilde{F}_{L_n}$ then $\tilde{F}_{L_n}(VaR_{L_n,\alpha})$ is different from α, except when α lies in the middle of $[\tilde{F}_n(VaR_{L_n,\alpha} - 0), \tilde{F}_n(VaR_{L_n,\alpha} + 0)]$, due to our definition (8). This difference can be significant, and of the order of magnitude of granularity adjustments themselves.

For instance, consider the extreme case of perfect dependence between the r.v.s' Z_i and a scalar systemic factor X, i.e. $Z_i = \mathbf{1}(X \leq a)$ for some constant a. Then, if all exposure shares are constant and equal to $1/n$, then L_n takes the two values 0 and 1 only. Thus, $\tilde{F}_{L_n} \in \{0, 1/2, 1\}$. In this case, the quantiles that can be invoked to apply Theorem 2 are related to $\alpha \in \{0, 1/2, 1\}$ only. The same reasoning applies when L_n takes the values $\{\ell_1, \ldots, \ell_p\}$ only, and $\tilde{F}_{L_n}(\ell_j) = q_j$, $j = 1, \ldots, p$. Then, with our conventions and strictly speaking, Proposition 5 apply with the levels $\alpha \in \{q_1, \ldots, q_p\}$ only.

To go one step beyond, assume that we define the loss quantiles by our usual rule $VaR_{L_n,\alpha} = \inf\{t|\tilde{F}_{L_n}(t) \geq \alpha\}$, and that $\tilde{F}_{L_n}(VaR_{L_n,\alpha}) = \alpha^* \neq \alpha$. Thus, since $\tilde{F}_{E[L_n|\mathbf{X}]}(VaR_{E[L_n|\mathbf{X}],\alpha}) = \alpha$, we have

$$\tilde{F}_{E[L_n|\mathbf{X}]}(VaR_{E[L_n|\mathbf{X}],\alpha}) = \tilde{F}_{E[L_n|\mathbf{X}]}(VaR_{L_n,\alpha}) + T_{n,\infty}(VaR_{L_n,\alpha}) + r(VaR_{L_n,\alpha}) + \alpha - \alpha^*,$$

and, under the conditions of regularity of Theorem 2, we get

$$VaR_{L_n,\alpha} = VaR_{E[L_n|\mathbf{X}],\alpha} - \frac{T_{n,\infty}(VaR_{L_n,\alpha}) + r(VaR_{L_n,\alpha}) + \alpha - \alpha^*}{f_{L_n|\mathbf{X}}(v_\alpha)},$$

for some number v_α such that $|v_\alpha - VaR_{L_n,\alpha}| \leq |VaR_{E[L_n|\mathbf{X}],\alpha} - VaR_{L_n,\alpha}|$. Unfortunately, the last additional term can be significant, i.e. not negligible w.r.t. the GA adjustment. For instance, it is at most $1/(2f_{L_n|\mathbf{X}}(v_\alpha))$ in our example.

Therefore, in general, il will be more difficult to apply GA theoretical results when the underlying loss distributions are discrete, a rather common situation. That is why we advise to work under Assumption B*, i.e. to apply the granularity adjustments only at some quantile levels α such that $\tilde{F}_{L_n}(VaR_{L_n,\alpha}) = \alpha$. In practice, it may be difficult to insure such a condition, unfortunately. Indeed, VaR-levels are often imposed by managers or regulators, whatever the loss distribution. A convenient solution should be to introduce random recovery rates systematically and then to get smooth loss distributions. Another way would be to introduce random exposures, as in Gordy and Marrone (2012). Such exposures may be driven by random credit spreads or interest rates, but any other market factor can be considered. They may be introduced by keeping the previous A_{jn} unchanged and deterministic, but by multiplying Z_j by a new (possibly correlated) random variable e_j, $E[e_j] = 1$, $j = 1, \ldots, n$.

2.6. Granularity adjustments and Expected Shortfall approximations

It is tempting to get an equivalent of Corollary 3 but for expected shortfalls, in the sense of Acerbi and Tasche (2002): for any $\alpha \in (0,1)$, the α-level expected shortfall of a loss function L is defined by

$$ES_\alpha(L) := \frac{1}{1-\alpha} E[L.\mathbf{1}(L \geq VaR_\alpha(L))] - \frac{VaR_\alpha(L)}{1-\alpha} \left(\alpha - P(L < VaR_\alpha(L))\right),$$

when the value-at-risk is defined by $VaR_\alpha(L) = \inf\{t | F(t) \geq \alpha\}$. With the latter definition, we get a coherent measure of risk, that satisfies the noteworthy relation:

$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L)\, du.$$

Then, as noticed in Martin and Tasche (2007) or Gordy (2004), the granularity adjustment of an expected shortfall may be just deduced from the integration of granularity adjustments for a continuum of value-at-risks.

Here, under the assumptions of Corollary 3 and by integrating Equation (4), we get

$$ES_\alpha(L_n) \simeq ES_\alpha(E[L_n|\mathbf{X}]) - \frac{1}{2(1-\alpha)} \sum_{j=1}^n A_{jn}^2 \int_\alpha^1 \frac{1}{f_{L_n|\mathbf{X}}(VaR_{E[L_n|\mathbf{X}],u})} \kappa_j'(VaR_{E[L_n|\mathbf{X}],u})\, du.$$

Unfortunately, getting an upper bound of the remainder term seems to be significantly more tedious than for $VaR_\alpha(L_n)$, in particular because of the denominator in the latter equation.

An fruitful idea could be to note that, at least for continuous loss distributions,

$$ES_\alpha(L) = \frac{(-1)}{1-\alpha} \int_{VaR_\alpha(L)}^\infty u\,(1-F_L)(du) = VaR_\alpha(L) + \frac{1}{1-\alpha} \int_{VaR_\alpha(L)}^\infty (1-F_L)(u)\, du.$$

The remainder term of ES-granularity adjustments is thus related to the rest $\int_{VaR_\alpha(L)}^\infty r_{n,\infty}(x)\, dx$, where $r_{n,\infty}$ is detailed in the proof of Theorem 1. With the notations of the appendix, this implies to upper bound

$$\int_{VaR_\alpha(L)}^\infty \int_0^\infty Im\left\{E\left[\exp\left(itE[L_n|\mathbf{X}]\right)\tilde{r}_\chi(\mathbf{X},t)\right]\exp(-itx)/t\right\}\, dt,$$

for an explicit but very complicated function $\tilde{r}_\chi(\mathbf{X},t)$. Clearly, the result will depend strongly on the behavior of the function

$$t \mapsto E\left[\exp\left(itE[L_n|\mathbf{X}]\right)\tilde{r}_\chi(\mathbf{X},t)\right].$$

Due to the strength and the number of the required conditions, we stop work on finding an explicit GA upper bound for expected shortfalls. We keep this task for further developments.

3. Empirical performances of Granularity Adjustments

Note that, in the literature, there are not so many empirical analysis of granularity adjustments. Historically, Gordy (2003) was the first one. His framework was a simplified CreditRisk+ type model. Since the tails of the loss distributions generated by this model behave regularly and are thin, the performances of GA approximations appeared very convincing: see Table C.1. Similar findings have been obtained with CreditMetrics-type, KMV Portfolio Manager-type models, or even the Basel 2 credit risk model: see Emmer and Tasche (2005), Gordy and Marrone (2012), for instance. This is due to the formal similarities between these models, in terms of loss distributions, as explained in Koyluoglu and Hickman (1998).

From the previous theoretical results, it appears clearly that the quality of granularity adjustments depends on the regularity of the underlying distributions. In other words, it is likely such adjustments do a good job for some models, and may do a bad job for others, in the sense that GAs do not provide

improvements w.r.t. the crude approximation $L_n \overset{law}{=} E[L_n|\mathbf{X}]$. In this section, we introduce different credit risk models. For convenience, the portfolios will be homogenous: $A_{jn} = 1/n$ for all $j = 1, \ldots, n$. We compare their (true or numerically approximated) VaRs with their first-order approximations in the framework of infinitely granular portfolios. Such approximations are we denoted by "$EVaR$". Moreover, we measure to what extent granularity adjustments induce refined VaR measurements. The two GAs, as given by Corollaries 3 and 4, will be denoted by GA1 and GA2 respectively. The exact formulas of the GAs we consider in this section are detailed in Appendix B.

Note that our theoretical results allow to state when GAs will work, but not when some problems of accuracy will occur necessarily. In other words, we have stated sufficient conditions of regularity, but not necessary conditions. We will check empirically the empirical relevance of the conditions of regularity induced by Theorem 1.

To be specific and with the previous notations, our granularity term $T_{n,\infty}(VaR_{L_n,\alpha})$ is of order $1/n$ in the case of balanced portfolios. It is easy to check that the error term $r_{n,M}(x)$ is $o(T_{n,\infty}(x))$ (for every x) only if $M_n^4/n = o(1)$, due to the last term in the r.h.s. of (3). One task will be to check whether the other terms on the r.h.s. of this inequality are not too large compared to $T_{n,\infty}$. For every model, we will focus on the term $I_M(x) := \int_M^\infty Im(\chi_{E[L_n|X]}(t)\exp(-itx)/t)\,dt$, where x will be (close to) a value-at-risk of the loss distribution. Most of the time, it can be evaluated relatively easily, contrary to $\int_M^\infty Im(\chi_{L_n}(t)\exp(-itx)/t)\,dt$. Moreover, set $J_M(x) := \max_{j=1}^n \int_M^{+\infty} Im\left(\hat{G}_j(t)t\exp(-itx)\right)\,dt$. It is necessary that this term tends to zero when $n \to \infty$ so that $r_{n,M}(x) << T_{n,\infty}(x)$.

3.1. Granularity adjustments in a Gaussian loss framework

As a benchmark, we choose the most simple credit model as possible. Its main advantage relies in its tractability and by the fact (true) VaRs can be calculated analytically. In this model, the random loss amount that is associated with the name i is still measured by the r.v. Z_i and the vector $(Z_i)_{i=1,\ldots,n}$ is Gaussian. Actually, we assume a standard underlying one-factor model:

$$Z_i = a_i + \sigma_i(\rho_i X + \sqrt{1-\rho_i^2}X_i^*), \tag{12}$$

where X and the X_i^*, $i = 1, \ldots, n$, are (jointly) independent standard Gaussian variables. The parameters a_i, σ_i and ρ_i are positive constants, with $\rho_i \in [0, 1]$.

These assumptions are clearly unrealistic, particularly due to the likelihood of negative losses, and due to the unboundedness of the potential loss amounts. Nonetheless, under this specification, the laws of L_n and $E[L_n|X]$ are Gaussian and their characteristic functions decrease towards zero at an exponential rate. Particularly, in the case of an homogenous pool of names, $I_M(x) = o(n^{-1})$ by choosing $M = n^a$, $a < 1/3$. Moreover, it is easy to check that the conditions (i) and (ii) are satisfied, because every $\hat{G}_j$ behaves like $t \mapsto \exp(-Cst^2t^2)$ for some constants Cst. Then $J_M(x) = o(n^{-1})$, obviously. Therefore, the remainder term $r_{n,M}$ in Theorem 1 will be sufficiently low for a convenient choice of M, and we should get nice GAs.

This is what we observe, indeed. Tables C.2 and C.3 provide exact and approximated VaR calculations. As expected, granularity approximations do a perfect job, by approximating the true VaR levels with a very high level of accuracy. Moreover, there are virtually no differences between our two GA formulas GA1 and GA2.

3.2. Granularity adjustments in a simple discrete loss framework

Unfortunately, these nice performances of granularity adjustments can be put in jeopardy relatively easily. Indeed, by considering non standard structural-type credit portfolio models, the picture tends to change and we are leaving the previous "ideal world". For instance, consider the second "Toy model" defined by

$$Z_i = \mathbf{1}(X_i \leq a_i + b_i X), \tag{13}$$

where X_i, $i = 1, \ldots, n$ are n independent random variables uniform on $[0, 1]$. Here, the individual losses can take only two values in this model: zero or one. The systemic r.v. X will be independent of the X_i variables, and will follow various distributions:

1. a uniform distribution on $(0,1)$;
2. a Bates distribution, that is the average of m uniform on $(0,1)$ and independent random variables, $m \geq 2$;
3. a Beta distribution on $[0,1]$, defined through its two positive parameters p and q, as usual.

In the first case, the characteristic function of $E[L_n|X]$ is proportional to the function $t \mapsto \sin(\sigma t/2)/(\sigma t)$, where $\sigma = \sum_{i=1}^{n} A_i b_i$. Simple calculations show that $I_M(x)$ behaves like a sum of quantities $\sin(Cst.M)/M^2$, for some constants Cst. Therefore, in the case of homogenous portfolios, the upper bound obtained in Theorem 1 will be $O(1/M^2 + M^4/n^2)$ that cannot be $o(1/n)$, whatever the choice of M_n. Moreover, $t \mapsto t\hat{G}_j(t)$ behaves like a sum of $t \mapsto \exp(i.Cst.t)$ for some real constants Cst. Then, this function is not integrable in this case and $J_M(x)$ will not be $o(1)$. Then, it is unlikely that GAs improve the VaR estimates in the case of a uniform X r.v. This is exactly what we check in Table C.4: granularity adjustments look like small noises. They do not improve VaR calculations, even after taking into account the statistical uncertainty around the empirical VaR estimates.

Since the assumption of a uniform systemic random variable is not particularly realistic, we have led the same experiment when X follows a Bates distribution with m degrees of freedom. The case $m = 1$ corresponds to the uniform law. Through an integration by parts argument, we check easily that the characteristic function of $E[L_n|X]$ is $O(t^{-m})$. Thus, by another integration by parts argument, we can show that $I_M(x)$ behaves as a sum of quantities $\sin(Cst(x).M)/M^{m+1}$, for some constants $Cst(x)$ that depend on x. Therefore, a sufficient condition for the relevance of GAs is given by $m > 3$ (set $M_n = n^a$ with $a \in (1/(m+1); 1/4)$). Note that the asymptotic behavior of $t \mapsto t\hat{G}_j(t)$ is the same as a sum of functions $t \mapsto \sin(bt)^r/t^{m-1}$ for some integers $r \geq 1$. We deduce $J_M(x) = O(M^{-m+1})$, that tends to zero when $m > 1$. Actually, GAs work pretty well even when $m \geq 3$: see Table C.5. Note that the strength of the GA improvements does not increase with m when $m \geq 3$ necessarily. At some stage, apparently, we can even observe a relative loss of performance. For instance, when $(a_i, b_i) = (0.2, 0.2)$ for every name, the best job is obtained with $m = 6$, but not with $m = 9$ or $m = 12$ that provide worse results than $m = 3$.

To complete this experiment, assume now that X follows a beta $B(p,q)$ distribution, $p, q > 0$. Then, the characteristic function of $E[L_n|X]$ is a confluent hypergeometric function $M(p, p+q, iCt)$ for some constant C: see Johnson and Kotz (1969, Equation (43) ; 1970, chap. 24). When t is large and positive, following Morse and Feshback (1953), we have

$$M(p, p+q, it) \sim \frac{\Gamma(p+q)}{\Gamma(p)}|t|^{-q}\exp(-iq\pi/2 + i|t|) + \frac{\Gamma(p+q)}{\Gamma(q)}|t|^{-p}\exp(ip\pi/2).$$

By an integration by parts, the asymptotic behavior of $I_M(x)$ is $O(M^{-(q+1)}/(1-x) + M^{-(p+1)})$. Therefore, for a given level x, our previous upper bounds should tend to zero quicker than $M^{-\min(q,p)-1}$. Then, it is likely that GAs work when $\min(p,q) > 3$ (set $M_n = n^a$ with $a \in (1/(\min(p,q)+1); 1/4)$). Moreover, simple calculations show that $\hat{G}_j(t)$ can be written as a sum of functions $t \mapsto \exp(Cst.it)/t^{\min(p,q)}$. Then, $J_M(x)$ tends to zero when $\min(p,q) > 1$.

In the latter case, the results of our simulations are provided in Table C.6. Clearly, they are significantly less convincing than those of Tables C.2 and C.3. As expected, depending on the parameters values p and q, the granularity adjustment can improve or deteriorate the approximation of the true VaR level. Apparently, granularity adjustments improve significantly the infinitely granular approximation when $\min(p,q) \geq 3$. When $p = 1$ or $q = 1$, they work worse than the standard first order approximation $EVaR_\alpha$. Interestingly, when one of the parameters p or q goes up, when the other is staying one, the performances of the obtained GAs are improved. Note there is no symmetry of our results w.r.t. the parameters p and q: when $(p,q) = (1,5)$, GAs do a pretty well job, but not in the case $(p,q) = (5,1)$.

3.3. Granularity adjustments in a realistic credit portfolio model

To challenge our previous results, we have tested a third specification. It will be more realistic than previously, by introducing individual random severities S_i (also called "losses given default", or "LGD"), for every $i = 1, \ldots, n$. The new loss model is

$$Z_i = S_i.\mathbf{1}(X_i \leq a_i + b_i X), \quad S_i = c_i T_i + d_i X, \tag{14}$$

where all the r.v. X_i and T_i, $i = 1, \ldots, n$, are mutually independent and follow a uniform distribution on $(0, 1)$. As previously, the random factor X will follow a Bates distribution and/or a Beta distribution, and will be independent of the other variables. To take into account the well-known dependency between recovery rates and systemic risk (see Altman et al. 2005, e.g.), we have set an explicit link between the systemic default risk factor X and the individual severities S_i, through the coefficients d_i: when X goes up, the systemic credit risk goes up, and so the severity rates. All the coefficients a_i, b_i, c_i and d_i belong to $(0, 1)$. For convenience, we have calculated GA1 only (usual granularity adjustments) in this model. Note that there are a lot of different ways of introducing recovery risk in such credit risk models: see Pykhtin (2003), Chen and Joslin (2012), Amraoui et al. (2012), among others. Our specification has been chosen for the sake of simplicity only.

With these new specifications, it is difficult to check the order of magnitude of the remainder term $r_{n,M}(x)$, or even of its components $I_M(x)$ and $J_M(x)$. Indeed, $E[L_n|X]$ is now a quadratic function of X, and finding explicit evaluations of our previous Fourier transforms becomes an impossible task. Therefore, we will inspect the results only qualitatively, to check the robustness of GA techniques w.r.t. to relatively natural model modifications. Therefore, we will compare the Toy model 3 results with those obtained with the previous Toy model 2, or even in the case of independence between default and recovery risks (when $d_i = 0$ for every i).

When the law of X is Bates, the results are shown in Table C.7. Globally, they confirm those obtained in Table C.5: the higher m, the better are the GA results, and GAs improve first-order approximations when $m \geq 3$. Nonetheless, when there is no systemic recovery risk ($d_i = 0$), the additional amount of smoothness induced by idiosyncratic recoveries improves GA performances, what is good news for practitioners. At the opposite, when there is dependency between default and recovery risks, these performances suffer slightly, even if they remain acceptable. Nonetheless, it is difficult to draw very clear-cut conclusions, due to the statistical uncertainty around our empirical VaR estimates.

This is confirmed when X follows a Beta distribution: see Tables C.8, C.9 and C.10. The first table corresponds to the case of random but purely idiosyncratic recoveries ($d_i = 0$, for every i). In the latter case, the results seem to be slightly better than those obtained in Table C.6, i.e. with smoothing individual losses: GAs improve $EVaR_\alpha$ approximations when $\min(p, q) \geq 2$. The picture is a bit changed after the introduction of systemic recovery risk. This is mainly due to much better $EVaR_\alpha$ approximations. Then, taking account the statistical VaR uncertainty, it remains almost no space for GA improvements. Therefore, most of the time, we are unable to say whether GAs improve the first-order approximations in the latter case. In practice, this means: "all other things being equal", the higher is the systemic risk, the less relevant are GAs.

"Estimation of the GA by simulation is difficult enough, because simulation noise tends to swamp the small gap between VaR and asymptotic VaR." (Gordy and Marrone, 2012)

4. Conclusion

In this article, we have proved a general formula for granularity adjustments. Our framework encompasses multi-factor models, random recoveries and discrete loss distributions. Through some Fourier transform techniques, we have exhibited an upper bound of the error terms of such approximations, and introduced several GA-based formulas for VaR calculations. We have tested the performances of granularity expansions in the case of several simple credit portfolio specifications. It appears that, outside Gaussian or Poisson-like loss distributions, it is relatively easy to observe disappointing approximated VaR levels. We have stressed the benefit from introducing recovery risks and/or random exposures in credit risk models, before applying granularity approximations. We advise to check the regularity of the underlying loss distribution of the model, before using "blindly" such approximations. We have provided some sufficient conditions to be insured these techniques can be used in safety. Nonetheless, our results are fragile and other empirical experiments are surely necessary. A comparison with other expansion techniques, like those introduced recently by Voropaev (2011), is another avenue for further research. Finally, the same analysis could be led with Expected Shortfalls, even if obtaining the equivalent of Theorem 2 would be significantly more painful. But there are some hopes to get better empirical results in the most

demanding models. Indeed, as noticed in Martin and Tasche (2007), GAs for Expected Shortfalls seem to behave better than for VaRs when there are local modes in the distribution tails.

Appendix A. Proof of Theorem 1.

Denote by $p_j(\mathbf{X})$ the default probability of j before T, conditionally on the factor $\mathbf{X}$. Therefore, by setting

$$\psi_j(\mathbf{X},t) := p_j(\mathbf{X})\left(E[\exp(itA_{jn}Z_j)|D_j=1,\mathbf{X}]-1\right) = E[\exp(itA_{jn}Z_j)|\mathbf{X}]-1,$$

and due to the conditional independence assumption, we can write

$$\begin{aligned}\chi_L(t) = E\left[E[\exp(itL_n)|\mathbf{X}]\right] &= E\left[\prod_{j=1}^{n} E[\exp(itA_{jn}Z_j)|\mathbf{X}]\right] \\ &= E\left[\exp\left(\sum_{j=1}^{n}\ln\left\{1+\psi_j(\mathbf{X},t)\right\}\right)\right] \\ &= E\left[\exp\left(\sum_{j=1}^{n}\psi_j(\mathbf{X},t)-\psi_j^2(\mathbf{X},t)/2+r_{\chi,j}(\mathbf{X},t)\right)\right] \qquad (A.1)\end{aligned}$$

where

$$r_{\chi,j}(\mathbf{X},t) := \psi_j^3(\mathbf{X},t)\int_0^1 \frac{u^2}{(1+u\psi_j(\mathbf{X},t))}\,du.$$

The latter relation is obtained by integrating the relation

$$\frac{1}{1+\psi_j u} = 1-\psi_j u+\frac{\psi_j^2 u^2}{1+\psi_j u}$$

between 0 and 1. A Taylor expansion with an integral form remainder provides

$$\psi_j(\mathbf{X},t) = E[\exp(itA_{jn}Z_j)|\mathbf{X}]-1 = itA_{jn}E[Z_j|\mathbf{X}]-\frac{t^2}{2}A_{jn}^2E[Z_j^2|\mathbf{X}]+r_{\psi,j}(\mathbf{X},t), \qquad (A.2)$$

where

$$r_{\psi,j}(\mathbf{X},t) := -\frac{it^3A_{jn}^3}{2}\int_0^1 E\left[\exp(itvA_{jn}Z_j)Z_j^3|\mathbf{X}\right](1-v)^2\,dv.$$

By putting (A.2) into (A.1), we get

$$\chi_L(t) = E\left[\exp\left(\sum_{j=1}^{n}\{itA_{jn}E[Z_j|\mathbf{X}]-\frac{t^2}{2}A_{jn}^2E[Z_j^2|\mathbf{X}]\}+\frac{t^2}{2}A_{jn}^2E[Z_j|\mathbf{X}]^2+\bar{r}_{\chi,j}(\mathbf{X},t)\right)\right],$$

where $\bar{r}_{\chi,j}(\mathbf{X},t)$ can be specified explicitly:

$$\begin{aligned}\bar{r}_{\chi,j}(\mathbf{X},t) &= r_{\psi,j}(\mathbf{X},t)-\frac{1}{2}\{\frac{t^2}{2}A_{jn}^2E[Z_j^2|\mathbf{X}]-r_{\psi,j}(\mathbf{X},t)\}^2 \\ &+ itA_{jn}E[Z_j|\mathbf{X}]\{\frac{t^2}{2}A_{jn}^2E[Z_j^2|\mathbf{X}]-r_{\psi,j}(\mathbf{X},t)\}+r_{\chi,j}(\mathbf{X},t).\end{aligned}$$

By another Taylor expansion, we get finally

$$\begin{aligned}\chi_L(t) &= E\left[\exp\left(itE[L_n|\mathbf{X}]-\frac{t^2}{2}\sum_{j=1}^{n}A_{jn}^2Var_j(\mathbf{X})+\sum_{j=1}^{n}\bar{r}_{\chi,j}(\mathbf{X},t)\right)\right] \\ &= E\left[\exp\left(itE[L_n|\mathbf{X}]\right)\cdot\left\{1-\frac{t^2}{2}\sum_{j=1}^{n}A_{jn}^2Var_j(\mathbf{X})+\tilde{r}_\chi(\mathbf{X},t)\right\}\right],\end{aligned}$$

where

$$\tilde{r}_\chi(\mathbf{X},t) := \sum_{j=1}^{n} \bar{r}_{\chi,j}(\mathbf{X},t) + \xi(\mathbf{X},t)^2 \int_0^1 (1-u)\exp(\xi(\mathbf{X},t)u)\,du,$$

$$\xi(\mathbf{X},t) := -\frac{t^2}{2}\sum_{j=1}^{n} A_{jn}^2 Var_j(\mathbf{X}) + \sum_{j=1}^{n} \bar{r}_{\chi,j}(\mathbf{X},t).$$

We deduce

$$\begin{aligned}\chi_L(t) = \chi_{E[L|\mathbf{X}]}(t) - \frac{t^2}{2}\sum_{j=1}^{n} A_{jn}^2 E\left[\exp\left(itE[L_n|\mathbf{X}]\right) Var_j(\mathbf{X})\right] \\ + \quad E\left[\exp\left(itE[L_n|\mathbf{X}]\right)\tilde{r}_\chi(\mathbf{X},t)\right].\end{aligned} \tag{A.3}$$

A careful inspection of the proof of Theorem 4.4.3. in Kawata (1972) provides:

Lemma 6. *Let F be a bounded non decreasing function on $\mathbb{R}$. Let χ_F be the associated characteristic function, i.e. $\chi_F(x) := \int \exp(itx)\,dF(t)$. Then, for every real number x,*

$$\frac{1}{2}\left[F(x+0)+F(x-0)\right] = \frac{F(+\infty)}{2} - \frac{1}{\pi}\int_0^{+\infty} Im\left(\chi_F(t)\frac{\exp(-itx)}{t}\right)\,dt.$$

This is the version of the Zolotarev (1957) formula, but adapted to possibly discontinuous distributions. By applying this formula to the cdfs of the loss distributions L_n and $E[L_n|\mathbf{X}]$, we get particularly

$$\frac{1}{2}\left[P(L_n < x+0)+P(L_n < x-0)\right] = \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} Im\left(\frac{\chi_L(t)}{t\exp(itx)}\right)\,dt,$$

$$\frac{1}{2}\left[P(E[L_n|\mathbf{X}] < x+0)+P(E[L_n|\mathbf{X}] < x-0)\right] = \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} Im\left(\frac{\chi_{E[L|\mathbf{X}]}(t)}{t\exp(itx)}\right)\,dt,$$

for every x. Set

$$T_{n,M}(x) := \frac{1}{2\pi}\sum_{j=1}^{n} A_{jn}^2 \int_0^M Im\left\{E\left[\exp\left(itE[L_n|\mathbf{X}]\right) Var_j(\mathbf{X})\right] t\exp(-itx)\right\}\,dt,$$

for any $M \in \bar{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$. Clearly, the granularity adjustment will come from the term $T_{n,\infty}(x)$ (see the second term on the r.h.s. of Equation (A.3)). Note that

$$E\left[\exp\left(itE[L_n|\mathbf{X}]\right) Var_j(\mathbf{X})\right] = \int \exp\left(it\mu_n(\mathbf{x})\right) Var_j(\mathbf{x}) f_X(\mathbf{x})\,d\mathbf{x} = \int \exp(itv)\kappa_j(v)\,dv,$$

where, with obvious notations,

$$\kappa_j(v) := \int_{\{\mathbf{x}|\mu_n(\mathbf{x})=v\}} Var_j(\mathbf{x}) f_X(\mathbf{x})\,\frac{d\mathbf{x}_{-(1)}}{\partial_{(1)}\mu_n(\mathbf{x})}.$$

Note that $v \mapsto G_j(v) := \int_{-\infty}^{v}\kappa_j$ is a continuous and monotonic function. We have assumed it is bounded and non decreasing. Its associated characteristic function is $\hat{G}_j(v) = \int \exp(itv)\kappa_j(t)\,dt$. Applying Lemma 6, we get

$$G_j(x) = \frac{G_j(+\infty)}{2} - \frac{1}{\pi}\int_0^{+\infty} Im\left(\hat{G}_j(t)\exp(-itx)/t\right)\,dt.$$

Actually, such a function is derivable almost surely. Since $\hat{G}_j \in L^1(\mathbb{R})$, we can apply the dominated convergence theorem and calculate derivates inside the integrand (see Billingsley (1995), Theorem 16.8 for details). We get

$$G_j'(x) = \frac{1}{\pi}\int_0^{+\infty} Im\left(\hat{G}_j(t) i\exp(-itx)\right)\,dt.$$

Since we have assumed the function $t \mapsto t\hat{G}_j(t)$ belongs to L^1, we get further

$$\begin{aligned} G_j''(x) &= \frac{1}{\pi}\int_0^{+\infty} Im\left(\hat{G}_j(t)t\exp(-itx)\right)\,dt \\ &= \frac{1}{\pi}\int_0^{+\infty} Im\left(E\left[\exp\left(itE[L_n|\mathbf{X}]\right)Var_j(X)\right]t\exp(-itx)\right)\,dt. \end{aligned}$$

The previous term $T_{n,\infty}(x)$ can be rewritten

$$T_{n,\infty}(x) := \frac{1}{2}\sum_{j=1}^{n} A_{jn}^2 G_j''(x) = \frac{1}{2}\sum_{j=1}^{n} A_{jn}^2 \kappa_j'(x).$$

Therefore, we have got the interesting result:

$$\begin{aligned} &\frac{1}{2}\left[P(L_n < x+0) + P(L_n < x-0)\right] \\ &= \frac{1}{2}\left[P(E[L_n|\mathbf{X}] < x+0) + P(E[L_n|\mathbf{X}] < x-0)\right] + \frac{1}{2\pi}\sum_{j=1}^{n} A_{jn}^2\kappa_j'(x) \\ &- \frac{1}{\pi}\int_0^{+\infty} Im\left\{E\left[\exp\left(itE[L_n|\mathbf{X}]\right)\tilde{r}_\chi(\mathbf{X},t)\right]\exp(-itx)/t\right\}\,dt. \end{aligned} \tag{A.4}$$

It remains to deal with the remainder term $R_\infty(x)$, where we set for any $M \in \bar{\mathbb{R}}$

$$R_M(x) := \frac{(-1)}{\pi}\int_0^{M} Im\left\{E\left[\exp\left(itE[L_n|\mathbf{X}]\right)\tilde{r}_\chi(\mathbf{X},t)\right]\exp(-itx)/t\right\}\,dt.$$

To evaluate $R_\infty(x)$, we cannot use the same method as for $T_{n,\infty}(x)$. Indeed, $E\left[\exp\left(itE[L_n|\mathbf{X}]\right)\tilde{r}_\chi(\mathbf{X},t)\right]$ is not the Fourier transform of a non decreasing function. Moreover, this integral cannot be calculated explicitly, unfortunately. Actually, to show that the latter remainder term $R_\infty(x)$ is negligible w.r.t. the "granularity-type" term $T_{n,\infty}(x)$, we come back to Equation (A.3) and restrict the domain of integration to some interval $[0, M]$. To be specific, by integrating (A.3), we get

$$\frac{1}{\pi}\int_0^{M} Im\left(\chi_L(t)\frac{\exp(-itx)}{t}\right)\,dt = \frac{1}{\pi}\int_0^{M} Im\left(\chi_{E[L|\mathbf{X}]}(t)\frac{\exp(-itx)}{t}\right)\,dt - T_{n,M}(x) - R_M(x) \tag{A.5}$$

for every $M > 0$. But, as we have seen before,

$$\frac{1}{\pi}\int_0^{M} Im\left(\chi_L(t)\frac{\exp(-itx)}{t}\right)\,dt = \frac{1}{2} - \frac{1}{2}\left[P(L_n < x+0) + P(L_n < x-0)\right] - R_{L,M}^*(x), \tag{A.6}$$

where

$$R_{L,M}^*(x) := \frac{1}{\pi}\int_M^{+\infty} Im\left(\chi_L(t)\exp(-itx)/t\right)\,dt.$$

Note that

$$|R_{L,M}^*(x)| \leq \frac{1}{\pi M}\int_M^{+\infty} |\chi_L|.$$

Similarly,

$$\begin{aligned} \frac{1}{\pi}\int_0^{M} Im\left(\chi_{E[L|\mathbf{X}]}(t)\exp(-itx)/t\right)\,dt &= \frac{1}{2} - \frac{1}{2}\left[P(E[L_n|\mathbf{X}] < x+0)\right. \\ &+ \left. P(E[L_n|\mathbf{X}] < x-0)\right] - R_{L|\mathbf{X},M}^*(x), \end{aligned}$$

where

$$R_{L|\mathbf{X},M}^*(x) := \frac{1}{\pi}\int_M^{+\infty} Im\left(\chi_{E[L|\mathbf{X}]}(t)\exp(-itx)/t\right)\,dt$$

and

$$|R^*_{L|\mathbf{X},M}(x)| \leq \frac{1}{\pi M}\int_M^{+\infty} |\chi_{E[L|\mathbf{X}]}|.$$

Moreover, the same reasoning as above proves that

$$T_{n,M}(x) = T_{n,\infty}(x) - R^*_{T,M}(x),$$

$$R^*_{T,M}(x) = \frac{1}{2\pi}\sum_{j=1}^n A_{jn}^2 \int_M^{+\infty} Im\left(\hat{G}_j(t)t\exp(-itx)\right)\, dt.$$

Note that

$$|R^*_{T,M}(x)| \leq \frac{1}{2\pi}\sum_{j=1}^n A_{jn}^2 \int_M^{+\infty} |t\hat{G}_j(t)|\, dt.$$

To tackle $R_M(x)$, note that $\tilde{r}_\chi(\mathbf{X},t)$ can be rewritten as a sum of terms that will be considered as negligible w.r.t. $T_{n,\infty}$. Indeed, for every $0 \leq t \leq M$ and every $j, \mathbf{X}$,

$$|\psi_j(\mathbf{X},t)| \leq tA_{jn}|E[Z_j|\mathbf{X}]| \leq tA_{jn}\bar{Z}_j,$$

by the mean value theorem. Actually, since we choose M such that $MA_{jn}\bar{Z}_j \leq 1-\varepsilon$ for every j and n, we obtain

$$|r_{\chi,j}(\mathbf{X},t)| \leq |\psi_j^3(\mathbf{X},t)| \int_0^1 \frac{u^2}{1-(1-\varepsilon)u}\, du := |\psi_j^3(\mathbf{X},t)|.k_\varepsilon.$$

Note that

$$|r_{\psi,j}(\mathbf{X},t)| \leq \frac{t^3A_{jn}^3}{6}\bar{Z}_j E[Z_j^2|\mathbf{X}] \leq \frac{t^3A_{jn}^3\bar{Z}_j^3}{6}.$$

Therefore,

$$\begin{aligned}|\bar{r}_{\chi,j}(\mathbf{X},t)| &\leq |r_{\psi,j}(\mathbf{X},t)| + \frac{1}{2}\{\frac{t^2}{2}A_{jn}^2\bar{Z}_j^2 + |r_{\psi,j}(\mathbf{X},t)|\}^2 \\ &+ \quad |t|A_{jn}\bar{Z}_j\{\frac{t^2}{2}A_{jn}^2\bar{Z}_j^2 + |r_{\psi,j}(\mathbf{X},t)|\} + |r_{\chi,j}(\mathbf{X})| \leq [\frac{3}{2}+k_\varepsilon].(tA_{jn}\bar{Z}_j)^3.\end{aligned}$$

We deduce

$$|\tilde{r}_\chi(\mathbf{X},t)| \leq [\frac{19}{18}+k_\varepsilon].|t|^3\sum_{j=1}^n A_{jn}^3\bar{Z}_j^3 + \xi_{n,t}^2\exp(\xi_{n,t}),$$

where

$$\xi_{n,t} := \frac{t^2}{2}\sum_{j=1}^n A_{jn}^2 Var_j(\mathbf{X}) + [\frac{19}{18}+k_\varepsilon].|t|^3\sum_{j=1}^n A_{jn}^3\bar{Z}_j^3.$$

Therefore, we obtain

$$\begin{aligned}|R_M(x)| &\leq E\left[\frac{1}{\pi}\int_0^M |\tilde{r}_\chi(\mathbf{X},t)|/t\, dt\right] \\ &\leq \quad [\frac{19}{18}+k_\varepsilon].\frac{|M|^3}{3\pi}\sum_{j=1}^n A_{jn}^3\bar{Z}_j^3 + \frac{1}{\pi}\int_0^M E[\xi_{n,t}^2\exp(\xi_{n,t})]\,\frac{dt}{t}.\end{aligned}$$

Since

$$0 \leq \xi_{n,t} \leq \frac{t^2}{2}[1+\frac{19}{9}+2k_\varepsilon]\sum_{j=1}^n A_{jn}^2\bar{Z}_j^2 := t^2\sigma^2,$$

then we have

$$\begin{aligned}\int_0^M E[\xi_{n,t}^2 \exp(\xi_{n,t})]\,\frac{dt}{t} &\le \int_0^M \sigma^4 t^3 \exp\left(t^2\sigma^2\right)\,dt\\ &\le \exp\left(M^2\sigma^2\right).[\exp(-M^2\sigma^2)-1+M^2\sigma^2]/2\\ &\le \frac{M^4\sigma^4}{2}\exp\left(M^2\sigma^2\right)\\ &\le \frac{M^4}{2}[\frac{14}{9}+k_\varepsilon]^2\left\{\sum_{j=1}^n A_{jn}^2\bar{Z}_j^2\right\}^2\exp([\frac{14}{9}+k_\varepsilon]\sum_{j=1}^n A_{jn}^2\bar{Z}_j^2M^2),\end{aligned}$$

proving the given upper bound for $R_M(x)$. Finally, we deduce

$$\begin{aligned}\frac{1}{2}\left[P(L_n<x+0)+P(L_n<x-0)\right] &= \frac{1}{2}\left[P(E[L_n|\mathbf{X}]<x+0)+P(E[L_n|\mathbf{X}]<x-0)\right]\\ &+ \quad T_{n,\infty}(x)-R_{L,M}^*(x)+R_M(x)+R_{L|\mathbf{X},M}^*(x)-R_{T,M}^*(x),\end{aligned}$$

proving the result. □

Appendix B. Details of GA calculations for the models in Section 3

For a given level α and every particular model considered in Section 3, we detail the approximated VaRs:

$$EVaR_\alpha := VaR_{E[L_n|X],\alpha} = E[L_n|X=VaR_\alpha(X)].$$

The associated granularity adjustments, as given in Corollaries 3 and 4, are

$$VaR_{L_n,\alpha} \simeq VaR_{E[L_n|X],\alpha} - GA_k,\ \ k=1,2,$$

where

$$GA_1 := \frac{1}{2f_X(VaR_{X,\alpha})}\sum_{j=1}^n A_{jn}^2\frac{d}{dt}\left[\frac{Var_j(t)f_X(t)}{\mu_n'(t)}\right]_{|t=VaR_{X,\alpha}},$$

and, by setting $\bar{V}_{n,\alpha}=\left(VaR_{L_n,\alpha}+EVaR_\alpha\right)/2$,

$$GA_2 := \frac{1}{2f_X(\mu_n^{-1}(\bar{V}aR_{n,\alpha}))}\cdot\frac{\mu_n'(\mu_n^{-1}(\bar{V}aR_{n,\alpha}))}{\mu_n'(\mu_n^{-1}(VaR_{L_n,\alpha}))}\sum_{j=1}^n A_{jn}^2\frac{d}{dt}\left[\frac{Var_j(t)f_X(t)}{\mu_n'(t)}\right]_{|t=\mu_n^{-1}(VaR_{L_n,\alpha})}.$$

Appendix B.1. Toy Model 1

$$EVaR_\alpha = \left(\sum_{i=1}^n A_i a_i\right)+\left(\sum_{i=1}^n A_i\sigma_i\rho_i\right)VaR_{X,\alpha} := C_0 + C_1\Phi^{-1}(\alpha).$$

Since the distribution of the losses L_n is Gaussian, we get a closed-form formula value-at-risk

$$VaR_{L_n,\alpha} = C_0+\sigma\Phi^{-1}(\alpha),$$

$$\sigma^2 = E[L_n^2]-E[L_n]^2 = \sum_{i=1}^n A_i^2\sigma_i^2+\sum_{i,j=1}^n A_iA_j\sigma_i\sigma_j\rho_i\rho_j.$$

$$GA_1 = \frac{(-1)}{2}\cdot\frac{\sum_{i=1}^n A_i^2\sigma_i^2(1-\rho_i^2)}{\sum_{i=1}^n A_i\sigma_i\rho_i}\cdot\Phi^{-1}(\alpha).$$

By setting

$$\tilde{V}_\alpha := \frac{VaR_{L_n,\alpha}-C_0}{C_1},\ \ \check{V}_\alpha := \frac{\bar{V}_{n,\alpha}-C_0}{C_1},$$

we get

$$GA_2 = \frac{(-1)}{2\phi(\check{V}_\alpha)}\cdot\frac{\sum_{i=1}^n A_i^2\sigma_i^2(1-\rho_i^2)}{\sum_{i=1}^n A_i\sigma_i\rho_i}\cdot\tilde{V}_\alpha\phi(\tilde{V}_\alpha).$$

Appendix B.2. Toy Model 2

Here, we estimate numerically the value-at-risk $VaR_{L_n,\alpha}$ and $VaR_{X,\alpha}$ (in the case of a Bates law). Since X belongs to $(0,1)$, the expectation of the losses (conditionally on X) is a linear function of X. We deduce

$$EVaR_\alpha = \left(\sum_{i=1}^n A_i b_i\right) VaR_{X,\alpha} + \left(\sum_{i=1}^n A_i a_i\right) := C.VaR_{X,\alpha} + D. \tag{B.1}$$

When X follows a Bates distribution with m degrees of freedom, its cdf and its density are given by

$$F_X(t) := \frac{1}{m!}\sum_{k=0}^{\lfloor mt\rfloor}(-1)^k C_m^k (mt-k)^m, \text{ and}$$

$$f_X(t) := \frac{m}{(m-1)!}\sum_{k=0}^{\lfloor mt\rfloor}(-1)^k C_m^k (mt-k)^{m-1}, \ t\in(0,1).$$

Then, we get

$$\begin{aligned} GA_1 = \frac{1}{2Cf_{\mathbf{X}}(VaR_{X,\alpha})}\sum_{i=1}^n A_i^2 \left[b_i(1-a_i-b_it)f_X(t)\right. \\ - \quad \left. b_i(a_i+b_it)f_X(t) + (a_i+b_i)(1-a_i-b_it)f_X'(t)\}\right]_{|t=VaR_{X,\alpha}}. \end{aligned}$$

By setting

$$\tilde{V}_\alpha := \frac{VaR_{L_n,\alpha}-D}{C}, \ \check{V}_\alpha := \frac{\bar{V}_{n,\alpha}-D}{C},$$

we obtain

$$\begin{aligned} GA_2 = \frac{1}{2Cf_{\mathbf{X}}(\check{V}_\alpha)}\sum_{i=1}^n A_i^2 \left[b_i(1-a_i-b_it)f_X(t)\right. \\ - \quad \left. b_i(a_i+b_it)f_X(t) + (a_i+b_i)(1-a_i-b_it)f_X'(t)\}\right]_{|t=\check{V}_\alpha}. \end{aligned}$$

Moreover, when X follows a Beta law with parameters (p,q),

$$\begin{aligned} GA_1 = \frac{1}{2C}\sum_{i=1}^n A_i^2 \left[\{\frac{p-1}{t}-\frac{q-1}{1-t}\}.\{a_i(1-a_i)+b_i(1-2a_i)t-b_i^2t^2\}\right. \\ + \quad \left.\{b_i(1-2a_i)-2b_i^2t\}\right]_{|t=VaR_{X,\alpha}}, \text{ and} \end{aligned}$$

$$\begin{aligned} GA_2 = \frac{1}{2C}\cdot\frac{\tilde{V}_\alpha^{p-1}(1-\tilde{V}_\alpha)^{q-1}}{\check{V}_\alpha^{p-1}(1-\check{V}_\alpha)^{q-1}}\sum_{i=1}^n A_i^2 \left[\{\frac{p-1}{t}-\frac{q-1}{1-t}\}.\{a_i(1-a_i)\right. \\ + \quad \left. b_i(1-2a_i)t-b_i^2t^2\}+\{b_i(1-2a_i)-2b_i^2t\}\right]_{|t=\tilde{V}_\alpha}. \end{aligned}$$

Appendix B.3. Toy Model 3

As previously, we estimate numerically the value-at-risks of $VaR_{L_n,\alpha}$ and $VaR_{X,\alpha}$. Now, the expectation of the losses, conditionally on X, is a quadratic (monotonic) function of X: $\mu_n(x) = C_0+C_1x+C_2x^2$, where

$$C_0 = \sum_{i=1}^n \frac{A_ia_ic_i}{2}, \ C_1 = \sum_{i=1}^n A_i(a_id_i+\frac{b_ic_i}{2}), \ C_2 = \sum_{i=1}^n A_ib_id_i.$$

Then, $\mu_n'(x) = C_1 + 2C_2 x$. With our notations, we have

$$Var_i(x) = (a_i + b_i x).(\frac{c_i^2}{3} + c_i d_i x + d_i^2 x^2) - (a_i + b_i x).(\frac{c_i}{2} + d_i x)^2,$$

and we calculate easily its derivative. Whatever the law of X, we can write

$$GA_1 = \frac{1}{2f_{\mathbf{X}}(VaR_{X,\alpha})} \sum_{i=1}^{n} A_i^2 \frac{d\zeta_i}{dt}(t)_{|t=VaR_{X,\alpha}},$$

with $\zeta_i(t) = Var_i(t) f_X(t)/\mu_n'(t)$. Therefore,

$$GA_1 = \frac{1}{2(C_1 + 2C_2 VaR_{X,\alpha})} \sum_{i=1}^{n} A_i^2 \left\{ Var_j'(t) + \frac{Var_i(t) f_X'(t)}{f_X(t)} - 2\frac{Var_i(t) C_2}{C_1 + 2C_2 t} \right\}_{|t=VaR_{X,\alpha}}.$$

Appendix C. Details of GA calculations for some multi-factor models

Now, we provide some details on GA calculations when the studied models involve several systemic random factors, in other words when $\mathbf{X}$ belongs to $\mathbb{R}^m$ with $m \geq 2$. Here, the associated GA is given by Theorem 2:

$$GA := \frac{T_{n,\infty}(VaR_{L_n,\alpha})}{f_{L_n|\mathbf{X}}(VaR_{L_n,\alpha})} = \frac{1}{2f_{L_n|\mathbf{X}}(VaR_{L_n,\alpha})} \sum_{j=1}^{n} A_{jn}^2 \kappa_j'(VaR_{L_n,\alpha}).$$

Appendix C.1. Granularity adjustments in a multi-factor Gaussian loss framework

Let us generalize our "Toy model 1" by introducing the bivariate random $X = (X_1, X_2)'$ and the random loss

$$Z_i = a_i + \sigma_i(\rho_{1i} X_1 + \rho_{2i} X_2 + \sqrt{1 - \rho_{1i}^2 - \rho_{2i}^2} X_i^*), \tag{C.1}$$

where X_1, X_2 and the X_i^*, $i = 1, \ldots, n$, are (jointly) independent standard Gaussian variables. The parameters a_i, σ_i, ρ_{1i} and ρ_{2i} are positive constants, with $\rho_{ki} \in [0, 1]$, $k = 1, 2$. Typically, X_1 (resp. X_2) can be interpreted as a systemic country (resp. industry) risk, or the opposite.

Here, $Var_i(\mathbf{x}) = \sigma_i^2(1 - \rho_{1i}^2 - \rho_{2i}^2) := \nu_i$ and

$$E[L_n|\mathbf{X}] = \sum_{i=1}^{n} A_i a_i + \sum_{i=1}^{n} A_i \sigma_i \rho_{1i} X_1 + \sum_{i=1}^{n} A_i \sigma_i \rho_{2i} X_2 := C_0 + C_1 X_1 + C_2 X_2.$$

Thus $\partial_{(1)}\mu_n(\mathbf{x}) = C_1$, and

$$\kappa_i(v) = \frac{\nu_i}{C_1} \int \phi\left(\frac{v - C_0 - C_2 t}{C_1}\right) \phi(t)\, dt = \frac{\nu_i}{\sqrt{C_1^2 + C_2^2}} \phi\left(\frac{v - C_0}{\sqrt{C_1^2 + C_2^2}}\right).$$

We deduce easily

$$GA_1 = \frac{(-1)}{2} \cdot \frac{\sum_{i=1}^{n} A_i^2 \sigma_i^2 (1 - \rho_{1i}^2 - \rho_{2i}^2)}{\sqrt{C_1^2 + C_2^2}} \cdot \Phi^{-1}(\alpha),$$

that extends the formula we got in Appendix B.1. Similar arguments allow explicit calculation when $\mathbf{X} \in \mathbb{R}^m$, $m \geq 3$.

Appendix C.2. Granularity adjustments in a multi-factor discrete loss model

Let us consider the simplest multi-factor extension of our Toy Model 2 as possible:

$$Z_i = \mathbf{1}(X_i \leq a_i + b_i X_1 + c_i X_2), \tag{C.2}$$

where X_1, X_2 and X_i, $i = 1, \ldots, n$ are $n+2$ independent random variables uniform on $[0,1]$. All the coefficients are nonnegative, and assume $0 < a_i + b_i + c_i < 1$ for any i.

In this case,

$$Var_i(\mathbf{x}) = (a_i + b_i x_1 + c_i x_2)(1 - a_i - b_i x_1 - c_i x_2), \text{ and}$$

$$E[L_n|\mathbf{X}] = \sum_{i=1}^{n} A_i a_i + \sum_{i=1}^{n} A_i b_i X_1 + \sum_{i=1}^{n} A_i c_i X_2 := \gamma_0 + \gamma_1 X_1 + \gamma_2 X_2.$$

Let assume that $0 < \gamma_1 \leq \gamma_2$ (otherwise, switch the roles of X_1 and X_2). Thus $\partial_{(1)}\mu_n(\mathbf{x}) = \gamma_1$. Let

$$I(v,u) = \int_0^u \mathbf{1}(v - \gamma_0 - \gamma_2 t \in [0, \gamma_1])(a_i + b_i \frac{v - \gamma_0 - \gamma_2 t}{\gamma_1} + c_i t)(1 - a_i - b_i \frac{v - \gamma_0 - \gamma_2 t}{\gamma_1} - c_i t)\, dt,$$

for any $u \in [0,1]$. Due to the fixed support of the underlying distributions, it is necessary to distinguish three cases to evaluate κ_i:

- If $v \in [\gamma_0, \gamma_0 + \gamma_1]$, then $\kappa_i(v) = I(v, (v - \gamma_0)/\gamma_2)$;
- If $v \in [\gamma_0 + \gamma_1, \gamma_0 + \gamma_2]$, then $\kappa_i(v) = I(v, (v - \gamma_0)/\gamma_2) - I(v, (v - \gamma_0 - \gamma_1)/\gamma_2)$;
- If $v \in [\gamma_0 + \gamma_2, \gamma_0 + \gamma_1 + \gamma_2]$, then $\kappa_i(v) = I(v, 1) - I(v, (v - \gamma_0 - \gamma_1)/\gamma_2)$;
- $\kappa_i(v) = 0$ otherwise.

Elementary calculations provide

$$\begin{aligned} I(v,u) &= (a_i + b_i \frac{v - \gamma_0}{\gamma_1})(1 - a_i - b_i \frac{v - \gamma_0}{\gamma_1}) \frac{u}{\gamma_1} \\ &+ \left[(a_i + b_i \frac{v - \gamma_0}{\gamma_1})(\frac{\gamma_2 b_i}{\gamma_1} - c_i) + (1 - a_i - b_i \frac{v - \gamma_0}{\gamma_1})(c_i - \frac{\gamma_2 b_i}{\gamma_1}) \right] \frac{u^2}{2\gamma_1} + (\frac{\gamma_2 b_i}{\gamma_1} - c_i)(c_i - \frac{\gamma_2 b_i}{\gamma_1}) \frac{u^3}{3\gamma_1}. \end{aligned}$$

To calculate the corresponding GA, it is necessary to derive κ_i, or the partial derivatives of I equivalently, a rather simple task that is left for the reader. Therefore, it remains to evaluate the density of $E[L_n|\mathbf{X}]$, a linear combination of the two independent r.v.s X_1 and X_2. Simple calculations provide

$$\begin{aligned} f_{L_n|\mathbf{X}}(t) = \frac{1}{\gamma_1 \gamma_2} &\{ \mathbf{1}(t \in [\gamma_0, \gamma_0 + \gamma_1]).(t - \gamma_0) \\ &+ \mathbf{1}(t \in [\gamma_0 + \gamma_1, \gamma_0 + \gamma_2]) + \mathbf{1}(t \in [\gamma_0 + \gamma_2, \gamma_0 + \gamma_1 + \gamma_2]).(\gamma_0 + \gamma_1 + \gamma_2 - t) \}. \end{aligned}$$

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(in %)	99.0%	99.5%	99.9%
True VaR	4.577	5.522	7.872
Asymptotic VaR	4.220	5.109	7.260
GA	0.357	0.435	0.627
Approximated VaR	4.578	5.544	7.886
Error	0.001	0.022	0.014

Table C.1: Direct and approximated VaR estimates. Portfolio size=600 names. True Var estimated by simulation (300000 draws). These numbers come from Gordy (2003), Table 3.

n	α	VaR	(VaR-EVaR)/VaR	(VaR-EVaR - GA1)/VaR	(VaR-EVaR - GA2)/VaR
50	0.9	0.2365	1.899	-0.1331	-0.1664
500	0.9	0.2325	0.2052	-0.0015	-0.0019
5000	0.9	0.2321	0.0207	-1.55e-005	-1.938e-005
50000	0.9	0.232	0.0021	-1.55e-007	-1.941e-007
50	0.99	0.2663	3.061	-0.2146	-0.2295
500	0.99	0.259	0.3343	-0.0025	-0.0026
5000	0.99	0.2582	0.0337	-2.53e-005	-2.688e-005
50000	0.99	0.2582	0.0034	-2.53e-007	-2.69e-007

Table C.2: Performances of the "toy model" 1 (closed-form calculations of VaR), as a function of the portfolio size n and the level α: $a_i = 0.2$, $\sigma_i = 0.1$, $\rho_i = 0.25$. Results in %.

ρ_i	α	VaR	(VaR-EVaR)/VaR	(VaR-EVaR - GA1)/VaR	(VaR-EVaR - GA2)/VaR
0.05	0.9	0.2067	0.1213	-0.002374	-0.002968
0.15	0.9	0.2193	0.038	-8.236e-005	-0.0001029
0.25	0.9	0.2321	0.0206	-1.551e-005	-1.938e-005
0.35	0.9	0.2449	0.0131	-4.696e-006	-5.869e-006
0.45	0.9	0.2577	0.0088	-1.735e-006	-2.168e-006
0.55	0.9	0.2705	0.0060	-6.925e-007	-8.656e-007
0.65	0.9	0.2833	0.0040	-2.746e-007	-3.433e-007
0.75	0.9	0.2961	0.0025	-9.817e-008	-1.227e-007
0.85	0.9	0.3089	0.0013	-2.601e-008	-3.251e-008
0.95	0.9	0.3217	0.0004	-2.208e-009	-2.76e-009
0.05	0.99	0.2121	0.2146	-0.0042	-0.0044
0.15	0.99	0.235	0.0643	-0.0001395	-0.0001483
0.25	0.99	0.2582	0.0337	-2.53e-005	-2.688e-005
0.35	0.99	0.2815	0.0207	-7.416e-006	-7.881e-006
0.45	0.99	0.3047	0.0135	-2.663e-006	-2.83e-006
0.55	0.99	0.328	0.0089	-1.037e-006	-1.102e-006
0.65	0.99	0.3512	0.0058	-4.021e-007	-4.273e-007
0.75	0.99	0.3745	0.0036	-1.409e-007	-1.497e-007
0.85	0.99	0.3977	0.0019	-3.667e-008	-3.896e-008
0.95	0.99	0.421	0.0005	-3.063e-009	-3.255e-009

Table C.3: Performances of the "toy model" 1 (closed-form calculations of VaR), as a function of the correlations ρ_i and the level α: $n = 5000$, $a_i = 0.2$, $\sigma_i = 0.1$. Results in %.

(a_i, b_i)	α	VaR	$\frac{VaR-EVaR}{VaR}$	$\frac{VaR-EVaR-GA1}{VaR}$	$\frac{VaR-EVaR-GA2}{VaR}$	$\frac{\hat{\sigma}(VaR)}{VaR}$
(0.2,0.2)	0.9	0.381	0.2625	0.294	0.2937	0.111
(0.2,0.2)	0.99	0.412	3.398	3.423	3.419	0.1223
(0.2,0.4)	0.9	0.56	0.0001	-0.0107	-0.0107	0.0744
(0.2, 0.4)	0.99	0.605	1.488	1.472	1.47	0.0440
(0.4, 0.2)	0.9	0.581	0.1721	0.1583	0.1582	0.0379
(0.4, 0.2)	0.99	0.612	2.288	2.272	2.269	0.0258
(0.4, 0.4)	0.9	0.761	0.1314	0.0972	0.0971	0.0664
(0.4, 0.4)	0.99	0.803	0.8717	0.8349	0.834	0.0546

Table C.4: Performances of the "toy model" 2 when X is uniform on $(0, 1)$ (calculation of VaR by simulations): 1000 names, 100000 draws. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

m	(a_i, b_i)	VaR	$\frac{VaR-EVaR}{VaR}$	$\frac{VaR-EVaR-GA1}{VaR}$	$\frac{VaR-EVaR-GA2}{VaR}$	$\frac{\hat{\sigma}(VaR)}{VaR}$
2	(0.2,0.2)	0.358	0.7608	0.8012	0.7992	0.1351
2	(0.2,0.4)	0.512	0.2826	0.2806	0.2803	0.1077
2	(0.4,0.2)	0.558	0.4881	0.4782	0.478	0.0558
2	(0.4,0.4)	0.712	0.2032	0.1737	0.1737	0.0758
3	(0.2,0.2)	0.348	1.216	0.1076	0.0981	0.0439
3	(0.2,0.4)	0.489	0.2986	-0.1531	-0.1535	0.1027
3	(0.4,0.2)	0.548	0.7719	-0.0411	-0.0387	0.0523
3	(0.4,0.4)	0.69	0.3566	0.0525	0.0540	0.0675
6	(0.2,0.2)	0.336	1.617	-0.0573	-0.1159	0.0508
6	(0.2,0.4)	0.464	0.6179	-0.0750	-0.0849	0.0604
6	(0.4,0.2)	0.536	1.014	-0.209	-0.232	0.0786
6	(0.4,0.4)	0.664	0.4318	-0.0343	-0.0372	0.0557
9	(0.2,0.2)	0.331	1.856	-0.2436	-0.2802	0.0947
9	(0.2,0.4)	0.454	0.9446	0.0701	0.0537	0.0830
9	(0.4,0.2)	0.532	1.343	-0.1847	-0.1671	0.0108
9	(0.4,0.4)	0.654	0.6557	0.0675	0.0634	0.0766
12	(0.2,0.2)	0.328	1.987	-0.4698	-0.457	0.109
12	(0.2,0.4)	0.447	0.9025	-0.1269	-0.1446	0.0970
12	(0.4,0.2)	0.529	1.421	-0.3642	-0.2895	0.0607
12	(0.4,0.4)	0.647	0.6235	-0.0684	-0.0727	0.0252

Table C.5: Performances of the "toy model" 2 when X follows a Bates distribution with m degrees of freedom (calculation of VaR by simulations): 1000 names, 100000 draws, $\alpha = 90\%$. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

(p,q)	α	VaR	$\frac{VaR-EVaR}{VaR}$	$\frac{VaR-EVaR-GA1}{VaR}$	$\frac{VaR-EVaR-GA2}{VaR}$	$\frac{\hat{\sigma}(VaR)}{VaR}$
(1, 1)	0.9	0.76	0.0003	-0.0338	-0.0338	0.0679
(1, 3)	0.9	0.615	0.1088	-0.1173	-0.1172	0.1084
(1, 5)	0.9	0.55	0.4339	0.0683	0.0717	0.0575
(1, 7)	0.9	0.515	0.5583	0.0503	0.0601	0.0993
(3, 1)	0.9	0.787	0.1022	0.1212	0.1208	0.0518
(3, 3)	0.9	0.703	0.2357	0.0039	0.0014	0.0807
(3, 5)	0.9	0.64	0.239	-0.0779	-0.0788	0.1042
(3, 7)	0.9	0.599	0.4959	0.0936	0.0942	0.0595
(5, 1)	0.9	0.794	0.2949	0.3642	0.3632	0.0089
(5, 3)	0.9	0.734	0.2528	-0.0116	-0.0182	0.0724
(5, 5)	0.9	0.681	0.2039	-0.1249	-0.1273	0.0747
(5,7)	0.9	0.642	0.3444	-0.0501	-0.0521	0.0549
(1, 1)	0.99	0.803	0.8722	0.8354	0.8345	0.0601
(1, 3)	0.99	0.716	0.3047	-0.0562	-0.0541	0.0721
(1, 5)	0.99	0.643	0.3504	-0.1211	-0.1132	0.1089
(1, 7)	0.99	0.596	0.5373	-0.0647	-0.0449	0.1621
(3, 1)	0.99	0.811	1.521	1.534	1.529	0.0389
(3, 3)	0.99	0.761	0.4285	-0.1088	-0.1116	0.0415
(3, 5)	0.99	0.709	0.4981	-0.0550	-0.0404	0.0445
(3, 7)	0.99	0.667	0.671	0.0428	0.0716	0.1262
(5, 1)	0.99	0.814	1.819	1.881	1.874	0.0489
(5, 3)	0.99	0.778	0.8132	0.1006	0.0800	0.0867
(5, 5)	0.99	0.737	0.7322	0.0823	0.1113	0.0858
(5, 7)	0.99	0.7	0.6975	0.0088	0.0462	0.0927

Table C.6: Performances of the "toy model" 2 when X is beta (calculation of VaR by simulations): 1000 names, 100000 draws. $a_i = b_i = 0.4$. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

m	(c_i, d_i)	VaR	$\frac{VaR-EVaR}{VaR}$	$\frac{VaR-EVaR-GA1}{VaR}$	$\frac{\hat{\sigma}(VaR)}{VaR}$
2	(0.4,0)	0.1426	0.4786	0.4725	0.0632
2	(0.4,0.2)	0.2528	0.1841	0.1795	0.1256
2	(0.4,0.4)	0.3630	0.0838	0.0849	0.1377
3	(0.4,0)	0.1382	0.5799	0.0602	0.0560
3	(0.4,0.2)	0.2369	0.3125	0.1217	0.0972
3	(0.4,0.4)	0.3355	0.2396	0.1079	0.1179
4	(0.4,0)	0.1359	0.7191	0.0336	0.0498
4	(0.4,0.2)	0.2289	0.2070	-0.0246	0.0939
4	(0.4,0.4)	0.3220	0.2138	0.0537	0.1117
6	(0.4,0)	0.1333	0.8972	0.0256	0.0399
6	(0.4,0.2)	0.2192	0.3712	0.0712	0.0693
6	(0.4,0.4)	0.3053	0.2745	0.0665	0.0899
8	(0.4,0)	0.1317	0.9808	-0.0456	0.0368
8	(0.4,0.2)	0.2137	0.3440	-0.0136	0.0727
8	(0.4,0.4)	0.2960	0.4362	0.1882	0.0916
10	(0.4,0)	0.1308	1.0932	-0.0689	0.0327
10	(0.4,0.2)	0.2100	0.3899	-0.0184	0.0631
10	(0.4,0.4)	0.2900	0.3051	0.0210	0.0769
12	(0.4,0)	0.1302	1.1484	-0.1363	0.0355
12	(0.4,0.2)	0.2074	0.4460	-0.0079	0.0574
12	(0.4,0.4)	0.2858	0.3851	0.0690	0.0735

Table C.7: Performances of the "toy model" 3 when X follows a Bates distribution with m degrees of freedom (calculation of VaR by simulations): 1000 names, 100000 draws, $\alpha = 90\%$, $(a_i, b_i) = (0.4, 0.4)$. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

(c_i, d_i)	(p, q)	VaR	$\frac{VaR-EVaR}{VaR}$	$\frac{VaR-EVaR-GA1}{VaR}$	$\frac{\hat{\sigma}(VaR)}{VaR}$
(0.2,0)	(1,1)	0.0760	0.0972	0.0849	0.0509
(0.2,0)	(1,2)	0.0675	0.1907	-0.0705	0.0816
(0.2,0)	(1,3)	0.0616	0.2730	-0.1046	0.0899
(0.2,0)	(1,5)	0.0551	0.5276	-0.0700	0.0835
(0.2,0)	(1,7)	0.0516	0.8675	0.0491	0.0735
(0.2,0)	(2,1)	0.0781	0.2579	0.3163	0.0335
(0.2,0)	(2,2)	0.0723	0.3475	0.04554	0.0537
(0.2,0)	(2,3)	0.0674	0.3683	-0.0252	0.0623
(0.2,0)	(2,5)	0.0608	0.4662	-0.0867	0.0634
(0.2,0)	(2,7)	0.0565	0.8470	0.1351	0.0704
(0.2,0)	(3,1)	0.0789	0.4956	0.6215	0.0261
(0.2,0)	(3,2)	0.0745	0.2745	-0.0802	0.0431
(0.2,0)	(3,3)	0.0703	0.3327	-0.1016	0.0564
(0.2,0)	(3,5)	0.0641	0.5544	-0.0069	0.0612
(0.2,0)	(3,7)	0.0599	0.6558	-0.0360	0.0650
(0.2,0)	(5,1)	0.0798	0.8648	1.1233	0.0225
(0.2,0)	(5,2)	0.0767	0.5210	0.0551	0.0284
(0.2,0)	(5,3)	0.0736	0.5502	0.0203	0.0397
(0.2,0)	(5,5)	0.0683	0.6440	0.0277	0.0459
(0.2,0)	(5,7)	0.0644	0.6931	-0.0179	0.0537
(0.2,0)	(7,1)	0.0802	1.1040	1.4942	0.0204
(0.2,0)	(7,2)	0.0777	0.6953	0.1158	0.0262
(0.2,0)	(7,3)	0.0752	0.6514	0.0192	0.0332
(0.2,0)	(7,5)	0.0708	0.6817	-0.0057	0.0393
(0.2,0)	(7,7)	0.0672	0.7331	-0.0241	0.0455

Table C.8: Performances of the "toy model" 3 when X is beta (calculation of VaR by simulations): 1000 names, 100000 draws. $a_i = b_i = 0.4$ and $\alpha = 90\%$. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

(c_i, d_i)	(p, q)	VaR	$\frac{VaR-EVaR}{VaR}$	$\frac{VaR-EVaR-GA1}{VaR}$	$\frac{\hat{\sigma}(VaR)}{VaR}$
(0.2,0.4)	(1,1)	0.3499	-0.1705	-0.1712	0.1282
(0.2,0.4)	(1,2)	0.2509	-0.1532	-0.1920	0.2485
(0.2,0.4)	(1,3)	0.1933	0.3126	0.2572	0.2758
(0.2,0.4)	(1,5)	0.1355	0.1939	0.1050	0.3071
(0.2,0.4)	(1,7)	0.1084	0.1662	0.0410	0.2709
(0.2,0.4)	(2,1)	0.3735	0.1491	0.1580	0.0619
(0.2,0.4)	(2,2)	0.3049	-0.2064	-0.2555	0.1559
(0.2,0.4)	(2,3)	0.2499	-0.0427	-0.1066	0.1948
(0.2,0.4)	(2,5)	0.1843	-0.1039	-0.1934	0.2228
(0.2,0.4)	(2,7)	0.1478	-0.0147	-0.1303	0.2248
(0.2,0.4)	(3,1)	0.3819	-0.0514	-0.0324	0.0477
(0.2,0.4)	(3,2)	0.3290	0.0697	0.0111	0.1152
(0.2,0.4)	(3,3)	0.2824	0.0191	-0.0534	0.1408
(0.2,0.4)	(3,5)	0.2161	-0.1361	-0.2306	0.1768
(0.2,0.4)	(3,7)	0.1771	0.0156	-0.1006	0.1879
(0.2,0.4)	(5,1)	0.3892	0.0331	0.0724	0.0303
(0.2,0.4)	(5,2)	0.3537	0.1532	0.0760	0.0622
(0.2,0.4)	(5,3)	0.3163	0.2578	0.1682	0.0981
(0.2,0.4)	(5,5)	0.2579	0.1562	0.0496	0.1338
(0.2,0.4)	(5,7)	0.2179	0.0595	-0.0643	0.1475
(0.2,0.4)	(7,1)	0.3923	0.0307	0.0904	0.0221
(0.2,0.4)	(7,2)	0.3658	0.1359	0.0404	0.0526
(0.2,0.4)	(7,3)	0.3356	0.1450	0.0383	0.0643
(0.2,0.4)	(7,5)	0.2844	0.2914	0.1721	0.1002
(0.2,0.4)	(7,7)	0.2458	0.2192	0.0858	0.1194

Table C.9: Performances of the "toy model" 3 when X is beta (calculation of VaR by simulations): 1000 names, 100000 draws. $a_i = b_i = 0.4$ and $\alpha = 90\%$. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.

(c_i, d_i)	(p, q)	VaR	$\frac{VaR-EVaR}{VaR}$	$\frac{VaR-EVaR-GA1}{VaR}$	$\frac{\hat{\sigma}(VaR)}{VaR}$
(0.2,0.8)	(1,1)	0.6225	-0.1630	-0.1616	0.1462
(0.2,0.8)	(1,2)	0.4348	0.0596	0.0304	0.2512
(0.2,0.8)	(1,3)	0.3255	-0.0698	-0.1092	0.3366
(0.2,0.8)	(1,5)	0.2160	0.0906	0.0332	0.3434
(0.2,0.8)	(1,7)	0.1663	0.3989	0.3230	0.3169
(0.2,0.8)	(2,1)	0.6696	0.0371	0.0466	0.0704
(0.2,0.8)	(2,2)	0.5371	0.0619	0.0215	0.1600
(0.2,0.8)	(2,3)	0.4339	-0.0859	-0.1369	0.2209
(0.2,0.8)	(2,5)	0.3068	-0.0127	-0.0794	0.2295
(0.2,0.8)	(2,7)	0.2393	0.0716	-0.0100	0.2611
(0.2,0.8)	(3,1)	0.6856	-0.0751	-0.0567	0.0472
(0.2,0.8)	(3,2)	0.5838	-0.0913	-0.1410	0.1118
(0.2,0.8)	(3,3)	0.4936	-0.3069	-0.3673	0.1637
(0.2,0.8)	(3,5)	0.3687	0.2097	0.1352	0.1841
(0.2,0.8)	(3,7)	0.2942	0.6223	0.5347	0.2019
(0.2,0.8)	(5,1)	0.6992	0.0128	0.0493	0.0303
(0.2,0.8)	(5,2)	0.6311	0.0896	0.0228	0.0668
(0.2,0.8)	(5,3)	0.5605	-0.0604	-0.1373	0.1148
(0.2,0.8)	(5,5)	0.4478	0.1658	0.0772	0.1437
(0.2,0.8)	(5,7)	0.3711	-0.2095	-0.3098	0.1660
(0.2,0.8)	(7,1)	0.7055	0.0147	0.0695	0.0233
(0.2,0.8)	(7,2)	0.6530	0.0860	0.0024	0.0563
(0.2,0.8)	(7,3)	0.5967	0.1357	0.0432	0.0794
(0.2,0.8)	(7,5)	0.4983	0.1021	0.0006	0.1081
(0.2,0.8)	(7,7)	0.4247	-0.0065	-0.1177	0.1232

Table C.10: Performances of the "toy model" 3 when X is beta (calculation of VaR by simulations): 1000 names, 100000 draws. $a_i = b_i = 0.4$ and $\alpha = 90\%$. $\hat{\sigma}(VaR)$ is the estimated stdev of the empirical VaR (Bootstrap, 500 replications). All ratios are in percentages.