Multi-factor Granularity Adjustments for Market and Counterparty Risks

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Abstract

Approximated analytical calculations of loss distributions and risk measures are often accurate with factor models when portfolios become more fine-grained. Such calculations can be improved by granularity adjustment (GA) techniques, when there remains a significant amount of undiversified idiosyncratic risk. We explain why it is so difficult to obtain analytic approximations of risk measures through granularity adjustment, when the underlying portfolio losses depend on several systematic factors. We propose several flexible families of models to manage the market and/or the counterparty risk of portfolios of financial assets. Explicit closed-form formulas based on granularity adjustments are provided, to approximate value-at-risks. We take into account random exposures, random recoveries and default risk simultaneously. Such models can be applied to portfolios of bonds, loans, stocks, or even derivatives. We prove the accuracy of such analytic approximations through simulations, when the vectors of systematic factors are gaussian or elliptical, more generally.

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1 Introduction

Risk measures, especially value-at-risk (VaR), provide the foundations of financial risk managements and regulation, in finance (Basel 3) as well as in insurance (Solvency 2). In particular, these measures are required to calculate minimum amounts of regulatory capital. Besides, most financial institutions calculate VaRs and expected shortfalls on a regular basis, for different internal purposes: risk monitoring, asset allocation, economic capital allocation, etc. Therefore, the ability of calculating such risk measures quickly and efficiently has been recognized as a strategic challenge, especially for the largest institutions.

Some "brute-force" solutions as Monte-Carlo VaR methods are too time-consuming and cannot be called "on the fly" in practice. Parametric VaR models are based on strong distributional assumptions and are not relevant for credit risk purposes. The accuracy of historical VaR techniques depends heavily on the choice of the reference historical dataset. Fortunately, approximated analytical calculations of VaRs are most often possible with factor models, by far the usual situation: when a portfolio becomes more fine-grained, i.e. when the largest individual exposures account for a negligible share of the total portfolio exposure, idiosyncratic risk is diversified away at the portfolio level. Therefore, the portfolio loss distribution is close to the distribution of its expected loss given the underlying factors: portfolio losses approximately depend only on systematic risk. Typically, the latter distribution is a lot simpler than the initial loss distribution and can be obtained analytically for a lot of market and/or credit portfolio models.

Unfortunately, the risk associated to real portfolios most often depends on a significant amount of undiversified idiosyncratic risk. Indeed, the majority of medium-sized or specialized institutions portfolios do not diversify away all their idiosyncratic risk. Actually, the previous approximation can be refined by granularity adjustment (GA) techniques. Potentially, they can be applied to any risk-factor model. They provide additional "idiosyncratic" terms in the asymptotic expansion of portfolio loss distributions and of their associated risk measures.

Most available GA formulas are related to one-factor models: historically, Wilde (2001a), Martin and Wilde (2002), and Gordy (2003) introduced the technique and applied it to the "Basel 2" model; Wilde (2001b) provided the formulas for a single-factor version of CreditRisk+; Emmer and Tasche (2005), refined by Gordy and Lütkebohmert (2013), made the same task for CreditMetrics, etc. Actually, GA techniques can be applied in models with multiple systematic factors (Gordy 2003). However, only very few papers have provided *explicit* GA formulas. Tasche (2006) pointed out the difficulty and detailed loss distributions in the case of two gaussian factors, but without analytical VaR approximations. Gagliardini and Gouriéroux (2013) proposed such a formula for a simple two-factor stochastic volatility model. Recently, Fermanian (2014) proposed other simple examples of multifactorial systematic variables models, particularly in the case of CDO pricing with random recoveries.

Note that Pykhtin (2004) proposed to solve the multi-factor problem by building a comparable one-factor portfolio whose loss distribution is close to the original multi-factor loss distribution. This intuition has been extended and refined by Voropaev (2011). In the same spirit, Garcia Cespedes *et al.* (2006) multiplied stand-alone capital charges by some multi-factor adjustments to reflect diversification effects. Nonetheless, such ideas, even valuable, do not contend with the technical difficulties in getting well-grounded asymptotic GA formulas that would result from several systematic factors.

Clearly, the majority of portfolio models depend by far on several systematic factors. For instance, CreditMetrics or Moody's KMV Portfolio Manager invoke dozens of industry/country systematic factors. The current standard way of pricing some structured credit products such as CDOs is to rely on at least two correlated systematic factors, to drive simultaneously default events and recovery levels. A lot of ABS products are priced and risk managed by assuming several global "market" factors (Libor rates, house price indices, GDP growth rates, etc.) induce the main trends in the market. Therefore, it is highly desirable to obtain GA formulas for a large range of useful and realistic models. Unfortunately, this is not so easy to exhibit closed-form GA formulas. We will explain and illustrate the successive obstacles that make this task difficult.

Moreover, almost all the GA literature has adopted an actuarial point of view and focus on credit risk only, when most models in risk management are "mark-to-market". This has been pointed out by Gordy and Marrone (2012). They have extended the GA methodology to random exposures, mainly rating-based. Nonetheless, their approach is limited to univariate systematic factors. Here, we will consider tractable multi-factor models where risks may be due to default events, recoveries, and other financial factors that drive exposures.

In this paper, we propose several families of models for which granularity adjustments are calculated, when the systematic variables are multivariate. Granularity adjustment formulas are recalled and discussed in Section 2. Section 3 deals with portfolios that are exposed to counterparty risks, a mix of default risks and random exposures. In Section 4, we reconsider the market risk of a portfolio of assets. At the end of every section, we evaluate the accuracy of our GAs by simulation.

2 Multi-factor Granularity adjustments

Consider a portfolio with n risky exposures. Every exposure depends on its own market and/or credit risk but all these risks are dependent, obviously. The key assumption is the mutual independence of the n underlying individual risky exposures, given a vector of systematic random factors $\mathbf{X} \in \mathbb{R}^m$. This vector $\mathbf{X}$ summarizes the market trends that will occur between now and our time horizon T. Typically, $\mathbf{X}$ reflects the realizations of future macro-economic hazards, financial variables or exogenous factors that influence systematic risk: natural catastrophes, pandemic, wars, etc.

Formally, the portfolio loss between today and our given time horizon T is $L_n = \sum_{i=1}^n A_{in} Z_i$, where A_{in} denotes the share of i-th value in the total portfolio value at $t = 0$. By construction, $\sum_{i=1}^n A_{in} = 1$. Moreover, the random variables Z_i, $i = 1, \ldots, n$ are mutually independent given $\mathbf{X}$. They measure the random loss associated to the i-th risky position between $t = 0$ and $t = T$, as a percentage of the current exposure. Actually, Z_i is a "loss fractions" that combines the riskiness of i-th exposure and of its loss-given-default (LGD). Note that the total current value of the portfolio is not specified and will be equal to one.

A portfolio is called infinitely granular (or fine-grained) when its size n is "large" and when the portion of every individual exposure i is negligible compared with the total size of the portfolio, i.e. $\lim_{n \to \infty} \sup_{i=1,\ldots,n} |A_{in}| = 0$. Under this hypothesis, the law of L_n is asymptotically the same as the law of $\mathbb{E}[L_n|\mathbf{X}]$, i.e. the expectation of the underlying losses given the vector of systematic random factors. Since the latter variable is much more manageable than the former, we can approximate the quantiles of L_n by those of $\mathbb{E}[L_n|\mathbf{X}]$. In other words, when the portfolio is infinitely granular, approximate $VaR_\alpha(L_n) =: VaR_{n,\alpha} = \inf\{x \,|\, P(L_n \leq x) \geq 1 - \alpha\}$, $\alpha \in (0, 1)$, by the VaR of the expected loss $\mu(\mathbf{X}) = \mathbb{E}[L_n|\mathbf{X}]$, the latter one being denoted by $EVaR_\alpha(L_n)$, or simpler $EVaR_{n,\alpha}$. Roughly, it can be proved that $VaR_{n,\alpha} \simeq EVaR_{n,\alpha}$ when n tends to infinity and the portfolio is infinitely granular. See Gordy (2003) for details.

Actually, the latter approximation can be refined, i.e. some amount of idiosyncratic risk can be put in an approximated formula. Assume that $\mu(\mathbf{X})$ has a continuous density f_μ with regard to the Lebesgue measure. Denote by $\mathbb{V}(Z_i|\mathbf{X}=\mathbf{x})$ the conditional variance of Z_i given $\mathbf{X}=\mathbf{x}$. For every $i=1,\dots,n$, define $\kappa_i(y)=\mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X})|\mu(\mathbf{X})=y\right]f_\mu(y)$ and $T_{n,\infty}(y)=\sum_{i=1}^n A_{in}^2\kappa_i'(y)/2$. Under certain technical conditions and when n tends to the infinity, the portfolio VaR can be approximated by the granularity adjustment formula

$$VaR_\alpha(L_n)\simeq VaRGA_{n,\alpha}:=EVaR_{n,\alpha}-\frac{T_{n,\infty}(EVaR_{n,\alpha})}{f_\mu(EVaR_{n,\alpha})}. \qquad (1)$$

See Gordy (2004) and Fermanian (2014). Thus far, most applications of GA formulas assume a univariate systematic factor $\mathbf{X}:=X\in\mathbb{R}$. Then, it is convenient to calculate κ_i because a single realization of X induces the event $\{\mu(X)=y\}$. Then, we get $\kappa_i(y)=\mathbb{V}(Z_i|X=\mu^{-1}(y))f_\mu(y)$, that can often be derived analytically.

When $\mathbf{X}$ is a vector, i.e. $m\geq 2$, things are significantly more difficult because the event $\{\mu(\mathbf{X})=y\}$ is related to a lot of $\mathbf{X}$ values in general. Typically, when the law of $\mathbf{X}$ is continuous, such values belong to some complicated manifolds that cannot be described easily. Therefore, the calculation of κ_i becomes unfeasible, what has discouraged most authors. Actually, "difficult" does not mean "impossible". Our goal will be to exhibit some flexible families of models for which such GA analytical formulas can be explicitly obtained.

In practical terms, we have to fufill several requirements to obtain closed-form GA formulas in a multi-factor setting:

(i) The conditional expected loss $\mu(\mathbf{x})$ has to admit a simple analytical expression and, most often, its density f_μ can be calculated. Since the expected loss of the portfolio given $\mathbf{X}$ is the sum of individual expected losses, working with families of distributions that are "stable by aggregation" is attractive.

(ii) The conditional variance $\mathbb{V}(Z_j|\mathbf{X} = \mathbf{x})$ also has to admit a rather simple expression.

(iii) We have to calculate $\mathbb{E}[\mathbb{V}(Z_i|\mathbf{X})|\mu(\mathbf{X}) = y]$ and its derivative analytically. Without any model restriction, this is unfeasible in general.

3 Granularity Adjustment formulas for counterparty risk

3.1 Model specifications

Since the last financial crisis, the risk management of counterparty risk has become a strategic topic for most financial institutions. This is the risk of losses due to the default of some counterparties. In general, its evaluation is sensitive because exposures are random. In theory, this task would necessitate multivariate dynamic models driving credit spread/rating risk and other random factors (equity, interest rates, FX, etc.). Therefore, it is tempting to rely on some approximated models, that aggregate default risks and risky exposures. This is the logic behind the models we introduce now.

As before, the future market values (exposures) at time T will be random and independent given the systematic factor $\mathbf{X} \in \mathbb{R}^m$. The individual default probability of obligor i given $\mathbf{X}$ is denoted by $p_i(\mathbf{X})$. In a depressing environment, $p_i(\mathbf{X})$ tends to become higher. Moreover, exposures and default events are mutually independent given $\mathbf{X}$. This is weaker than the common assumption of unconditional independence between both random quantities. In particular, some wrong-way risks can be taken into account, when exposures and default likelihoods both depend on $\mathbf{X}$.

Let us illustrate these ideas with a simple and intuitive framework.

Assume that the position i is related to a "bullet" bond. If the corresponding obligor has defaulted, then the associated loss is the (random) value of its loss-given-default (LGD). Recovery rates depend on the economic environment heavily. They tend to decrease during depressions (see Altman *et al.* 2005). When macro-factors are included in $\mathbf{X}$, this justifies the specification

$$Z_i|\mathbf{X} \sim \begin{cases} \mu_i(\mathbf{X}) & \text{with probability} \quad p_i(\mathbf{X}), \\ 0 & \text{else,} \end{cases} \tag{2}$$

for some deterministic function $\mu_i(\mathbf{X})$. Therefore, $\mu_i(\mathbf{X})$ is simply the LGD of bond i given $\mathbf{X}$, when measured as a percentage of the bond notional (or of the bond price today). Note that there is no idiosyncratic LGD risk in (2). We get easily $\mathbb{E}[Z_i|\mathbf{X} = \mathbf{x}] = p_i(\mathbf{x})\mu_i(\mathbf{x})$, $\mathbb{V}(Z_i|\mathbf{X} = \mathbf{x}) = p_i(\mathbf{x})\mu_i^2(\mathbf{x})(1 - p_i(\mathbf{x}))$.

For other securities, particularly derivatives, the laws of future valuations could be reasonably approximated by mixtures of gaussian random variables: for some deterministic functions $\mu_i(\cdot)$ and $\sigma_i(\cdot)$, set

$$Z_i|\mathbf{X} \sim \begin{cases} \max\left(\mathcal{N}(\mu_i(\mathbf{X}), \sigma_i^2(\mathbf{X})), 0\right) & \text{with probability} \quad p_i(\mathbf{X}), \\ 0 & \text{else.} \end{cases} \tag{3}$$

Due to the assumed independence between the random variables Z_i given $\mathbf{X}$, $i = 1, \ldots, n$, the model is fully specified.

Denoting by Φ (resp. ϕ) the cumulative distribution function (resp. density) of a standardized random gaussian variable, simple calculations provide:

Lemma 1 *Under (3),*

$$\mathbb{E}[Z_i|\mathbf{X} = \mathbf{x}] = p_i(\mathbf{x})\mu_i(\mathbf{x})\Phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right) + p_i(\mathbf{x})\sigma_i(\mathbf{x})\phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right), \text{ and}$$

$$\begin{aligned} \mathbb{V}(Z_i|\mathbf{X} = \mathbf{x}) &= p_i(\mathbf{x})\left[(\sigma_i(\mathbf{x})^2 + \mu_i(\mathbf{x})^2)\Phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right) + \mu_i(\mathbf{x})\sigma_i(\mathbf{x})\phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right)\right] \\ &- \ p_i(\mathbf{x})^2\left[\mu_i(\mathbf{x})\Phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right) + \sigma_i(\mathbf{x})\phi\left(\frac{\mu_i(\mathbf{x})}{\sigma_i(\mathbf{x})}\right)\right]^2. \end{aligned}$$

Note that both cases (bonds or other securities) could be encompassed in the same model specification (3). Indeed, by considering degenerated gaussian variables, for which $\sigma_i(\mathbf{x}) = 0$, we recover model (2). The model specification (3) can be invoked when bond LGDs depend on some idiosyncratic factors, for instance linked to the debt structure of the firm or to the management of collateral.

To manage the nonlinearities induced by Φ and ϕ, we need a simplifying assumption.

Assumption (A). The distribution of the loss fraction Z_i knowing $\mathbf{X}$ is given by (3) for all $i = 1, \ldots, n$. Moreover, for every i, $\sigma_i(\mathbf{x}) = a_i|\mu_i(\mathbf{x})|$ for some non-negative constant a_i and almost every $\mathbf{x} \in \mathbb{R}^m$.

Let us discuss the realism of the latter assumption.

- If the i-th exposure is related to a long stock or bond position, then $\mu_i(\mathbf{X})$ is the stock/bond level of loss at default given $\mathbf{X}$, but divided by the initial stock/bond value. It is comparable to a LGD. Assumption (A) is reasonable, because the level of uncertainty around LGDs is intuitively higher for high LGDs.

- If the position i is related to a derivative as a long call, the same arguments apply. The single annoying situation occurs when $\mu_i(\mathbf{X})$ is close to zero, in which case there will be small recorded losses, under (A). We could correct such situations by adding to $Z_i|\mathbf{X}$ a small fixed amount of losses in every case. Alternatively, when the derivative value can become positive or negative between $t = 0$ and T (as for a swap), $\mu_i(\mathbf{X})$ has an arbitrary sign and the specification (3) is able to manage such situations.

- It is not necessary to include LGD random variables in specification (3) explicitly. Indeed, it would appear as a multiplicative factor of $\mu_i(\mathbf{X})$ and $\sigma_i(\mathbf{X})$. Formally, this would not change the model specification. Under (A), the absolute value of $\mu_i(\mathbf{x})$ can be interpreted as the expected loss of the defaulted security i given $\mathbf{X} = \mathbf{x}$

and that i *is defaulted*, once multiplied by A_{in} and a scaling factor depending on a_i.

Therefore, for most financial products, there exists a deterministic function $b_i(\mathbf{x})$ exists s.t. $\mathbb{E}[Z_i|\mathbf{X}=\mathbf{x}] = b_i(\mathbf{x})p_i(\mathbf{x})\mu_i(\mathbf{x})$. For bonds and model (2), set $b_i = 1$ obviously and $a_i = 0$. Otherwise, $b_i(\mathbf{x}) = \Phi(s_i(\mathbf{x})/a_i) + a_i s_i(\mathbf{x})\phi(s_i(\mathbf{x})/a_i)$ in the case of model (3), where $s_i(\mathbf{x}) \in \{1, -1\}$ is the sign of $\mu_i(\mathbf{x})$. We neglect the case $\mu_i(\mathbf{X}) = 0$, whose probability is assumed to be zero. We deduce

$$\mu(\mathbf{x}) = \mathbb{E}[L_n|\mathbf{X}=\mathbf{x}] = \sum_{i=1}^{n} A_{in} b_i(\mathbf{x}) p_i(\mathbf{x}) \mu_i(\mathbf{x}), \text{ and} \tag{4}$$

$$\mathbb{V}(Z_i|\mathbf{X}=\mathbf{x}) = e_i(\mathbf{x})p_i(\mathbf{x})\mu_i(\mathbf{x})^2 - b_i(\mathbf{x})^2 p_i(\mathbf{x})^2 \mu_i(\mathbf{x})^2, \tag{5}$$

by setting

$$e_i(\mathbf{x}) := \left[(a_i^2+1)\Phi\left(\frac{s_i(\mathbf{x})}{a_i}\right) + a_i s_i(\mathbf{x})\phi\left(\frac{s_i(\mathbf{x})}{a_i}\right)\right].$$

To go on and beside (A), we have to make an additional assumption concerning the heterogeneity among individual positions and/or default probabilities.

3.2 Linkage between conditional probabilities and individual exposures

The simplest way of getting closed-form GA formulas under (A) is to assume

Assumption (B.1). For every $i = 1, \ldots, n$, $p_i(\mathbf{x})\mu_i(\mathbf{x}) = c_i x_1$ for some constant c_i and every $\mathbf{x} \in \mathbb{R}^m$. Moreover, the law of X_1 is continuous.

Without a lack of generality, we have particularized the first component of the systematic vector $\mathbf{X}$. The latter assumption is connecting conditional default probabilities and exposures for every individual position. It can be interpreted as the existence of a common driver X_1 for all "individual expected losses", given $\mathbf{X}$. as in the usual model for CDO pricing with random recoveries (see Amraoui et al., 2012), in which conditional default probabilities multiplied by conditional LGD are constrained to obtain tractable formulas and easy calibrations w.r.t. CDS quotes. More generally, a factor default model for $(p_1(\mathbf{x}), \ldots, p_n(x))$ is first stated and calibrated w.r.t. historical default rates or multi-name credit derivatives. Afterwards, under (B.1), every c_i is calibrated to the expected loss of the i-th position.

In the case of our model (3) and under Assumption (A) and (B.1), we get the simple expressions

$$\mathbb{E}[L_n|\mathbf{X}] = \left(\sum_{i=1}^{n} A_{in} b_i(\mathbf{X}) c_i\right) X_1 := \beta(\mathbf{X}).X_1, \text{ and} \tag{6}$$

$$\mathbb{V}(Z_i|\mathbf{X}) = c_i^2 X_1^2 \left[\frac{e_i(\mathbf{X})}{p_i(\mathbf{X})} - b_i^2(\mathbf{X})\right]. \tag{7}$$

Since the distribution of X_1 is continuous, the event $\mu_i(\mathbf{X}) = 0$ is of measure zero. Moreover, the sign of $\mu_i(\mathbf{x})$ is entirely determined by the signs of X_1 and c_i. Then, $\beta(\mathbf{X})$, $e_i(\mathbf{X})$ and $b_i(\mathbf{X})$ take only two values: for every $i = 1, \ldots, n$ and a.e., there are constants s.t. $\beta(\mathbf{X}) = \beta_1 \mathbf{1}(X_1 > 0) + \beta_2 \mathbf{1}(X_1 < 0)$, $e_i(\mathbf{X}) = e_{i,1}\mathbf{1}(X_1 > 0) + e_{i,2}\mathbf{1}(X_1 < 0)$, and $b_i(\mathbf{X}) = b_{i,1}\mathbf{1}(X_1 > 0) + b_{i,2}\mathbf{1}(X_1 < 0)$. We will denote $\beta(X_1)$, $e_i(X_1)$ and $b_i(X_1)$ from now on.

Set $\mathbb{E}[1/p_i(\mathbf{X})|X_1 = x_1] := \zeta_i(x_1)$, whose functional form is deduced from the chosen factor model of joint default probabilities. Deduce

$$\mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) \,|\, \mu(\mathbf{X}) = y\right] = \mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) \,|\, X_1 = \frac{y}{\beta_1}\right] = \frac{c_i^2 y^2}{\beta_1^2}\left[e_{i,1}\zeta_i\left(\frac{y}{\beta_1}\right) - b_{i,1}^2\right], \tag{8}$$

when $X_1 > 0$. Similarly, if $X_1 < 0$, then

$$\mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) \,|\, \mu(\mathbf{X}) = y\right] = \frac{c_i^2 y^2}{\beta_2^2}\left[e_{i,2}\zeta_i\left(\frac{y}{\beta_2}\right) - b_{i,2}^2\right]. \tag{9}$$

When $X_1 > 0$, $\mu_i(\mathbf{X})$ and $\beta(X_1)$ have a constant sign almost surely. However, in every case, $EVaR_{n,\alpha}$ is proportional to the α-quantile of X_1, as usual in the literature: $EVaR_{n,\alpha} = F_{X_1}^{-1}(\alpha).\left(\beta_1 \mathbf{1}(X_1 > 0) + \beta_2 \mathbf{1}(X_1 > 0)\right)$, where F_{X_1} denotes the cdf of X_1.

Moreover, if f_{X_1} (the density of X_1 w.r.t. the Lebesgue measure) exists, then the density of $E[L_n|\mathbf{X}]$ is

$$f_\mu(y) = \frac{\mathbf{1}(y<0)}{\beta_1} f_{X_1}\left(\frac{y}{\beta_1}\right) + \frac{\mathbf{1}(y \geq 0)}{\beta_2} f_{X_1}\left(\frac{y}{\beta_2}\right). \tag{10}$$

We get GA formulas under (B.1) because the functions κ_i can be obtained by (8), (9) and (10).

Typically, in a lot of credit portfolio models, the systematic factor $\mathbf{X}$ is Gaussian whose components can be chosen as independent after a re-parametrization. Thus, it is easy to evaluate the law of $\mathbf{X}$ conditional to X_1 in such a gaussian situation, and sometimes to calculate the function ζ_i analytically.

Nonetheless, Assumption (B.1) may appear as not very realistic. Indeed, most of the time, the (joint) law of the default events is specified independently of the law of exposures. The latter ones can often be seen as "exogenously" specified. Then, it is difficult to consider that an upward impact on $p_i(\mathbf{X})$ will be perfectly counter-balanced by a downward shift of $\mu_i(\mathbf{X})$, when X_1 is kept constant. The following specification is an attempt to solve this lack of realism.

3.3 Linkage of conditional probabilities/exposures among positions

Here, let us come back one step backwards, by still working under (A) but by leaving (B.1) out. Now, we will assume a certain amount of similarity among the individual default probabilities and among the individual random losses. This will provide an alternative family of counterparty risk models.

Assumption (B.2). For every $i = 1, \ldots, n$ and $\mathbf{x} \in \mathbb{R}^m$, $p_i(\mathbf{x}) = \pi_i p(\mathbf{x})$ and $\mu_i(\mathbf{x}) = \nu_i + \omega_i q(\mathbf{x})$, for some given functions p and q and some constants π_i, ν_i and ω_i. Moreover, $\mu_i(\mathbf{X})$ has a constant sign for almost every $\mathbf{X}$-realization.

Note that Assumption (B.2) is equivalent to imposing a two (systematic) factor model, driven by $(p(\mathbf{X}), q(\mathbf{X}))$. Indeed, the joint law of $(Z_1, \ldots, Z_n)$ given $\mathbf{X}$ is the same given

$(p(\mathbf{X}), q(\mathbf{X}))$. This assumption can be weakened (see below).

Under (B.2), we are able to manage the case of long/short positions, credit derivatives, bonds, etc., in the same framework, playing with the constants ν_i and ω_i and their signs. For instance, if $q(\mathbf{x})$ is high during stressed periods in the credit market, a protection buyer (resp. seller) Credit Default Swap position will be typically associated with $\omega_i > 0$ (resp. $\omega_i < 0$). Concerning default probabilities given $\mathbf{X}$, it makes sense to assume they are driven by an aggregated factor $p(\cdot)$. This reflects the likelihood of future states of the credit cycle. The coefficients π_i can be seen as rating-based scaling factors.

Assumption (B.2) is rather natural and realistic for homogenous portfolio. In this case, the way the systematic factors $\mathbf{X}$ drive the individual losses is similar across all the names.

Keeping in mind (4) and (5), (B.2) implies

$$\mu(\mathbf{x}) = \mathbb{E}[L_n|\mathbf{X}=\mathbf{x}] = \sum_{i=1}^{n} A_{in} b_i(\mathbf{x}) \pi_i p(\mathbf{x})[\nu_i + \omega_i q(\mathbf{x})] := p(\mathbf{x})[A(\mathbf{x}) + B(\mathbf{x}) q(\mathbf{x})],$$

$$A(\mathbf{x}) := \sum_{i=1}^{n} A_{in} b_i(\mathbf{x}) \pi_i \nu_i,\ B(\mathbf{x}) := \sum_{i=1}^{n} A_{in} b_i(\mathbf{x}) \pi_i \omega_i, \text{ and}$$

$$\mathbb{V}(Z_i|\mathbf{X}=\mathbf{x}) = e_i(\mathbf{x}) \pi_i p(\mathbf{x})[\nu_i + \omega_i q(\mathbf{x})]^2 - b_i(\mathbf{x})^2 \pi_i^2 p(\mathbf{x})^2 [\nu_i + \omega_i q(\mathbf{x})]^2.$$

For convenience and to simplify calculations, we have assumed in this subsection that $\mu_i(\mathbf{X}) = \nu_i + \omega_i q(\mathbf{X})$ (and then $b_i(\mathbf{X})$) has a constant sign almost everywhere. This is natural for a lot of securities. In the case of derivatives, this constraint can be seen as a lack of generality, but extended formulas can be written nonetheless. They are left to the reader.

Then, $A := A(\mathbf{x})$, $B := B(\mathbf{x})$ are constants, and denote by g the joint density of

$(p(\mathbf{X}), q(\mathbf{X}))$. By definition, $EVaR_{n,\alpha}$ is the root of the implicit equation

$$\alpha = \int \mathbf{1}(t \leq EVaR_{n,\alpha}) g\left(\frac{t}{A+By}, y\right) \frac{dt\, dy}{A+By}. \tag{11}$$

The latter equation can be solved numerically. Some closed-form formulas of $EVaR_{n,\alpha}$ could be found under some particular distributions g, for instance when $p(\mathbf{X})$ and $q(\mathbf{X})$ are independent and the law of $p(\mathbf{X})$ is uniform.

To calculate GAs (the point (iii) in Section 2), we need to evaluate analytically the quantities $\mathcal{I}_{a,b}(y) := \mathbb{E}[p(\mathbf{X})^a q(\mathbf{X})^b \,|\, \mu(\mathbf{X}) = y]$, for several couples of integers (a, b). With our notations, we have

$$\mathcal{I}_{a,b}(y) = \int \frac{y^a t^b}{(A+Bt)^{a+1}} g\left(\frac{y}{A+Bt}, t\right) dt / f_\mu(y). \tag{12}$$

Deduce

$$\mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X}) \,|\, \mu(\mathbf{X}) = y\right] = \sum_{k=1}^{2} \sum_{l=0}^{2} \gamma_{i,k,l} \mathcal{I}_{k,l}(y), \tag{13}$$

$$\gamma_{i,1,0} = \pi_i \nu_i^2 e_i,\ \gamma_{i,1,1} = 2\pi_i \nu_i \omega_i e_i,\ \gamma_{i,1,2} = \pi_i \omega_i^2 e_i,$$

$$\gamma_{i,2,0} = -\pi_i^2 \nu_i^2 b_i^2,\ \gamma_{i,2,1} = -2\pi_i^2 \nu_i \omega_i b_i^2,\ \gamma_{i,2,2} = -\pi_i^2 \omega_i^2 b_i^2.$$

Note that the calculations of κ_i and its derivatives are a lot simplified by the fact the density of $\mu(\mathbf{X})$ disappears:

$$\kappa_i(y) = \sum_{k=1}^{2} \sum_{l=0}^{2} \gamma_{i,k,l} y^k \int \frac{t^l}{(A+Bt)^{k+1}} g\left(\frac{y}{A+Bt}, t\right) dt. \tag{14}$$

GA formulas are obtained through (1), once we differentiate $\kappa_i(\cdot)$ and calculate the density of $\mu(\mathbf{X})$. This will be detailed for some particular models in Subsection 3.5.2.

We have considered some functions $p(\mathbf{x})$ and $q(\mathbf{x})$ that summarize the effect of possibly a lot of systematic factors, for instance through two univariate indices. This idea can be extended as $\mu_i(\mathbf{x}) = \nu_i + \sum_{j=1}^{\bar{m}} \omega_{i,j} q_j(\mathbf{x})$, introducing several functions $q_j(\cdot)$. The same methodology applies as long as we are able to evaluate the joint density of $(p(\mathbf{X}), q_1(\mathbf{X}), \ldots, q_{\bar{m}}(\mathbf{X}))$, without the need of explicitly calculating the density of $\mathbb{E}[L_n|\mathbf{X}]$. This would provide a $\bar{m}+1$-factor model, that is particularly relevant in the case of heterogenous portfolios. In technical terms, this requires the tedious calculation of $EVaR_{n,\alpha}$ as the root of a $\bar{m}+1$-dimensional integral equation (generalization of (11)).

3.4 Model extensions

It is well-known that asset returns and loss distributions exhibit fat tails and/or skewed distributions. Such features may induce larger VaR or expected shortfall values than expected (by a naive model), particularly at high levels. Our initial model assumptions (2) and (3) about random losses may be seen rather restrictive, because they are based on conditional gaussian losses implicitly. Actually, this is not really true. Indeed, in a factor model and under the conditional independence property, we are essentially free to specify the laws of $\mathbf{X}$ *and* the laws of the idiosyncratic noises given $\mathbf{X}$. The unconditional laws of losses are given by mixture models, that can generate fat tails easily.

The previous framework allows a high degree of flexibility by choosing different distributions of the systematic random factors, possibly fat-tailed or skewed. Through the specification of the first two conditional moments of individual losses, we build realistic one-period models. Another more direct way of getting such features is to replace the truncated gaussian conditional distributions of individual losses in Section 3.1 by other distributions, $\mathbf{X}$ keeping the same distribution.

For instance, instead of assuming (2) in the case of bonds, assume that Z_i knowing that $\mathbf{X} = \mathbf{x}$ follows a Beta distribution $Z_i|\mathbf{X} = \mathbf{x} \sim B(\alpha_i(\mathbf{x}), \beta_i(\mathbf{x}))$, with $\mathbb{E}[Z_i|\mathbf{X} = \mathbf{x}] = \alpha_i(\mathbf{x})/(\alpha_i(\mathbf{x}) + \beta_i(\mathbf{x})) = \mu_i(\mathbf{x})$ and some functions $\alpha_i(\cdot)$ and $\beta_i(\cdot)$. Beta distributions are particularly well-suited for LGDs, as several empirical studies have shown: see Calabrese and Zenga (2010), e.g. With Beta distributions, we are able to generate a significant percentage of LGDs that are close to zero or to one, even given $\mathbf{X}$. This is in line with some stylized empirical facts, that show the large amount of heterogeneity among corporate bond recovery rates, even after controlling for the situation inside the credit cycle. Some authors have linked this feature to differences in terms of defaulted firm debt structures (Carey and Gordy 2005), or to fire sales in some distressed industries (Acharya *et al.* 2007), particularly.

Additionally, let us reconsider (3). Instead of "gaussian-type" losses, assume we live in the larger and more flexible class of elliptical distributions (see Gomez *et al.* 2003). If the random variable Y follows an $\mathcal{E}_1(\theta,\sigma^2,g)$, then the law of $(Y-\theta)/\sigma$ is entirely specified by the density generator g. We denote by F_g and f_g its cdf and its density respectively. Therefore, replace (3) by

$$Z_i|\mathbf{X} \sim \begin{cases} \max\left(\mathcal{E}_1(\mu_i(\mathbf{X}),\sigma_i(\mathbf{X})^2,g_i(\mathbf{X})),0\right) & \text{with probability} \quad p_i(\mathbf{X}), \\ 0 & \text{else.} \end{cases} \tag{15}$$

The calculations above apply similarly, replacing Φ (resp. ϕ) by $F_{g_i(\mathbf{X})}$ (resp. $f_{g_i(\mathbf{X})}$). However, to get nice formulas, it is necessary to assume that the generator $g_i(\mathbf{X})$ does not depend on $\mathbf{X}$.

3.5 Empirical illustrations

Let us evaluate the performances of our GAs for counterparty risks numerically. We consider some simple, but not unrealistic, portfolios. To simplify and unless it is specified differently, we assume balanced portfolios, i.e. $A_{in} = 1/n$ for every i and different n values. We will compare the empirically estimated (500,000 simulations of portfolio losses) value-at-risk $VaR_{n,\alpha}$ with its first-order approximation $EVaR_{n,\alpha}$ and its granularity adjustment approximation $VaRGA_{n,\alpha}$. Hereafter, the value-at-risk level will be $\alpha = 0.99\%$. The standard deviations around the approximated $VaR_{n,\alpha}$ are estimated by nonparametric bootstrap (200 replications): See Shao and Tu (1995) for technical details. When $n = 1000$, the infinitely granular case should not be far away and we expect $EVaR_{n,\alpha}$ provide convenient approximations. When n is very small, this is no longer the case: idiosyncratic risk dominates and then, any technique based on analytic approximations is questionable. The most favourable situation for GAs should correspond to intermediate portfolio sizes, for which it makes sense to calculate asymptotic expansions of loss distributions with some correcting terms for some remaining significant idiosyncratic risks.

3.5.1 A family of models under (B.1).

Under Assumption (B.1), the main technical remaining point is the calculation of the so-called functions ζ_i, where $\zeta_i(x_1) = \mathbb{E}[1/p_i(\mathbf{X})|X_1 = x_1]$. To keep things simple, let us assume that $\mathbf{X}$ is a random vector in $\mathbb{R}^m_+$. As a consequence, all $\mu_i(\mathbf{x})$ are non-negative, $s_i(\mathbf{x}) = 1$ and the functions $\beta(\mathbf{x})$, $e_i(\mathbf{x})$, $b_i(\mathbf{x})$ and $\beta(\mathbf{x})$ take unique values.

In this example, we assume that, for every i, $p_i(\mathbf{X}) = \frac{X_1}{\xi_{i,0}+\sum_{k=1}^m \xi_{i,k}X_k}$, for some non-negative constants $\xi_{i,k}$, $k = 0, \ldots, m$. These constants have to be chosen so that $p_i(\mathbf{X})$ is less than one almost surely. For instance, if X_1 is uniform on $(0, 1)$, set $\xi_{i,0} = 1$. In every case, we can chose $\xi_{i,1} \geq 1$ to ensure such a condition. In practice, the constant $\xi_{i,k}$ can be estimated by maximum likelihood (under the historical measure), or by calibration w.r.t. prices of multi-asset credit derivatives. We deduce $\mu_i(\mathbf{x}) = c_i x_1/p_i(\mathbf{x}) = c_i\xi_{i,0} + c_i \sum_{k=1}^m \xi_{i,k}x_k$, and $\sigma_i(\mathbf{x}) = a_i\mu_i(\mathbf{x})$. This model is well-suited to bond/stock portfolios but not swaps, because $\mu_i(\mathbf{X})$ is always positive by construction. Once the law of $\mathbf{X}$ is stated, the model is fully specified. Indeed, once $\mathbf{X}$ is drawn, we can simulate default events and random losses independently, and we obtain portfolio loss realizations.

To fix the ideas and w.l.o.g., let us assume that $\mathbf{X}$ is a vector of correlated lognormal distributions: for some m-dimensional gaussian random vector $\mathbf{Y} \sim \mathcal{N}(\theta, \Sigma)$, $\Sigma = [\sigma_{i,j}]$, and some positive constants ν_k, we have $X_k = \exp(\nu_k Y_k)$, $k = 1, \ldots, m$. Therefore,

$$\begin{aligned}
\zeta_i(x_1) &= \frac{1}{x_1}\mathbb{E}\left[\xi_{i,0} + \sum_{k=1}^{m} \xi_{i,k} X_k | X_1 = x_1\right] \\
&= \frac{\xi_{i,0}}{x_1} + \xi_{i,1} + \sum_{k=2}^{m} \frac{\xi_{i,k}}{x_1}\mathbb{E}\left[\exp(\nu_k Y_k) | Y_1 = \ln(x_1)/\nu_1\right] \\
&= \frac{\xi_{i,0}}{x_1} + \xi_{i,1} + \sum_{k=2}^{m} \frac{\xi_{i,k}}{x_1}\exp\left(\nu_k\theta_k + \frac{\nu_k\sigma_{1,k}}{\sigma_{1,1}}(\frac{\ln(x_1)}{\nu_1} - \theta_1) + \frac{\nu_k^2\sigma_{k,k}}{2}(1-\rho_{1,k}^2)\right),
\end{aligned}$$

where $\rho_{1,k} = \sigma_{1,k}/(\sigma_{1,1}\sigma_{k,k})^{1/2}$ is the correlation between Y_1 and Y_k.

The density of the portfolio expected loss $\mu(\mathbf{X}) = \beta X_1$ is

$$f_\mu(y) = \frac{1}{y\nu_1\sqrt{\sigma_{1,1}}}\phi\left(\frac{\ln(y/\beta)/\nu_1 - \theta_1}{\sqrt{\sigma_{1,1}}}\right).$$

We deduce from (6) and (8) that

$$\begin{aligned}
\kappa_i(y) &= \mathbb{E}\left[\mathbb{V}(Z_i|\mathbf{X})|\mu(\mathbf{X}) = y\right] f_\mu(y) = \mathbb{E}\left[\mathbb{V}(Z_i|X)|X_1 = \frac{y}{\beta}\right] f_\mu(y) \\
&= f_\mu(y)\cdot\left(\frac{c_i y}{\beta}\right)^2 \cdot \Bigg[e_i\Bigg(\frac{\xi_{i,0}\beta}{y} + \xi_{i,1} \\
&+ \sum_{k=2}^{m}\frac{\xi_{i,k}\beta}{y}\exp\left(\nu_k\theta_k + \frac{\nu_k\sigma_{1,k}}{\sigma_{1,1}}(\ln(y/\beta)/\nu_1 - \theta_1) + \frac{\nu_k^2\sigma_{k,k}}{2}(1-\rho_{1,k}^2)\right)\Bigg) - b_i^2\Bigg], \text{ and}
\end{aligned}$$

$$\begin{aligned}
\kappa_i'(y) &= \left(\frac{\theta_1 - \ln(y/\beta)/\nu_1}{y\nu_1\sigma_{1,1}} + \frac{1}{y}\right)\kappa_i(y) + f_\mu(y)\cdot\left(\frac{c_i^2 e_i}{\beta}\right) \\
&\cdot \left[-\xi_{i,0} + \sum_{k=2}^{m}\xi_{i,k}\exp\left(\nu_k\theta_k + \frac{\nu_k\sigma_{1,k}}{\sigma_{1,1}}(\ln(y/\beta)/\nu_1 - \theta_1) + \frac{\nu_k^2\sigma_{k,k}}{2}(1-\rho_{1,k}^2)\right)\cdot\left(\frac{\nu_k\sigma_{1,k}}{\sigma_{1,1}\nu_1} - 1\right)\right].
\end{aligned}$$

As usual, the corresponding GA formula is given by

$$VaRGA_{n,\alpha} = EVaR_{n,\alpha} - \frac{T_{n,\infty}(EVaR_{n,\alpha})}{f_\mu(EVaR_{n,\alpha})} = EVaR_{n,\alpha} - \frac{\sum_{i=1}^{n}\kappa_i'(EVaR_{n,\alpha})}{2n^2 f_\mu(EVaR_{n,\alpha})}\cdot \quad (16)$$

but, under the latter model specification, the denominator of the formula (16) simplifies because $\kappa_i'(\cdot)$ is proportional to $f_\mu(\cdot)$. Finally, we obtain

$$
\begin{aligned}
VaRGA_{n,\alpha} = EVaR_{n,\alpha} - \frac{1}{2n^2}\sum_{i=1}^{n}\left(\frac{c_i y}{\beta}\right)^2 \Bigg\{ & \left(\frac{\theta_1 - \ln(y/\beta)/\nu_1}{y\nu_1\sigma_{1,1}} + \frac{1}{y}\right) \\
\cdot \quad & \left[e_i\left(\frac{\xi_{i,0}\beta}{y} + \xi_{i,1} + \sum_{k=2}^{m}\frac{\xi_{i,k}\beta}{y}\psi_k(y)\right) - b_i^2\right] \\
+ \quad & \frac{e_i\beta}{y^2}\left[\sum_{k=2}^{m}\xi_{i,k}\left(\frac{\nu_k\sigma_{1,k}}{\sigma_{1,1}\nu_1} - 1\right)\psi_k(y) - \xi_{i,0}\right]\Bigg\}_{|y=EVaR_{n,\alpha}},
\end{aligned}
$$

$$
\psi_k(y) := \exp\left(\nu_k\theta_k + \frac{\nu_k\sigma_{1,k}}{\sigma_{1,1}}\left(\frac{\ln(y/\beta)}{\nu_1} - \theta_1\right) + \frac{\nu_k^2\sigma_{k,k}}{2}(1-\rho_{1,k}^2)\right).
$$

In this experiment, we consider random factors $\mathbf{X}$ with different dimensions $m \in \{2,3,5\}$. The portfolio size is $n \in \{10, 50, 100, 500, 1000\}$. For every i, $a_i = 1$, $c_i = 1$, and then $b_i = \beta = 1.0833$ and $e_i = 1.9246$. Moreover, every component of θ and ν is one. The extra-diagonal coefficients of the correlation matrix Σ are 0.3 and $\xi_i = (0, 2, 1, 1, \ldots, 1)$ for all i.

The results appear in Table 1 and 2. Clearly, GAs improve the $EVaR$-approximations significantly, particularly for small portfolio sizes. This result is robust w.r.t. the number of systematic factors. Even with very small portfolio sizes, $VaRGA_{n,\alpha}$ is relatively close to the right VaR. On the other side, when n is large, the additional terms of $VaRGA_{n,\alpha}$ with respect to $EVaR_{n,\alpha}$ do not deteriorate the analytic approximation, but do not provide improvements, because the portfolios are close to the "infinitely granular" case.

n	VaR (stdev)	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	35.97 (0.22)	30.16	37.54	1.62 $\times 10^{-1}$	-4.37 $\times 10^{-2}$
50	31.50 (0.15)	30.16	31.63	4.28$\times 10^{-2}$	-4.08 $\times 10^{-3}$
100	30.84 (0.15)	30.16	30.89	2.22 $\times 10^{-2}$	-1.70$\times 10^{-3}$
500	30.25 (0.14)	30.16	30.30	3.09 $\times 10^{-3}$	-1.79$\times 10^{-3}$
1000	30.19 (0.14)	30.16	30.23	1.11 $\times 10^{-3}$	-1.34$\times 10^{-3}$

Table 1: Comparison of $VaR_{99\%}$ calculations for counterparty risk under (A) and (B.1). Two factor model ($m = 2$)

	$m = 3$		$m = 5$	
n	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	1.94 $\times 10^{-1}$	-5.31 $\times 10^{-2}$	2.47 $\times 10^{-1}$	-7.82 $\times 10^{-2}$
50	4.93 $\times 10^{-2}$	-9.02 $\times 10^{-3}$	6.20 $\times 10^{-2}$	-1.89 $\times 10^{-2}$
100	3.39 $\times 10^{-2}$	4.26 $\times 10^{-3}$	3.31 $\times 10^{-2}$	-8.60 $\times 10^{-3}$
500	9.70 $\times 10^{-3}$	3.62 $\times 10^{-3}$	4.42 $\times 10^{-3}$	-3.17 $\times 10^{-3}$
1000	4.05 $\times 10^{-3}$	9.96 $\times 10^{-4}$	7.23 $\times 10^{-3}$	2.95 $\times 10^{-3}$

Table 2: Comparison of $VaR_{99\%}$ calculations for counterparty risk under (A) and (B.1), and for three-factor and five-factor models.

3.5.2 A family of models under (B.2)

Under Assumption (B.2), the model specifications depend uniquely on the joint law of the "systematic" driver of default events $p(\mathbf{X})$ and the "systematic" driver of random losses $q(\mathbf{X})$.

Let us consider a bivariate gaussian random vector (Y_1, Y_2), $\mathbb{E}[Y_1] = \mathbb{E}[Y_2] = 0$, $\mathbb{E}[Y_1^2] = \mathbb{E}[Y_2^2] = 1$, and $\mathbb{E}[Y_1.Y_2] = \rho$. Set $p(\mathbf{X}) = \Phi(\nu_p Y_1 + \pi_p)$, with some constants

ν_p and π_p, $\nu_p \geq 0$ by convention. For a book of derivatives, set $q(\mathbf{X}) = Y_2$ directly. For a portfolio of bonds and/or stocks, whose market values keep constant signs, set $q(\mathbf{X}) = \exp(\nu_q Y_2 + \pi_q)$, introducing some constants ν_q and π_q, $\nu_q \geq 0$.

Then, it is easy to calculate the joint law of $(p(\mathbf{X}), q(\mathbf{X}))$. When $q(\mathbf{X}) = Y_2$, we obtain, for every $u \in (0,1)$ and $v \in \mathbb{R}$,

$$G(u,v) := \mathbb{P}(p(\mathbf{X}) \leq u, q(\mathbf{X}) \leq v) = \Phi_\rho\left((\Phi^{-1}(u) - \pi_p)/\nu_p, v\right), \tag{17}$$

where Φ_ρ is the joint cdf of (Y_1, Y_2). When $q(\mathbf{X}) = \exp(\nu_q Y_2 + \pi_q)$, we have, for every $u \in (0,1)$ and $v \in \mathbb{R}^+$,

$$G(u,v) := \mathbb{P}(p(\mathbf{X}) \leq u, q(\mathbf{X}) \leq v) = \Phi_\rho\left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{\ln(v) - \pi_q}{\nu_q}\right). \tag{18}$$

As a consequence, we can evaluate EVaRs by solving (14) numerically now.

Note that, in the case of a portfolio of derivatives, $q(\mathbf{X}) = Y_2$ can be positive or negative randomly, contrary to the simplifying assumption we made in Subsection 3.3. Nonetheless, it is easy to extend our formulas when all couples of coefficients (ν_i, ω_i) are the same (the case in our empirical illustration below). Therefore, as in Subsection 3.2, $b_i(\cdot)$ and $e_i(\cdot)$ take only two different values:

$$e_i(\mathbf{X}) = \bar{e}_1 \mathbf{1}(Y_2 > 0) + \bar{e}_2 \mathbf{1}(Y_2 < 0), \text{ and } b_i(\mathbf{X}) = \bar{b}_1 \mathbf{1}(Y_2 > 0) + \bar{b}_2 \mathbf{1}(Y_2 < 0).$$

We deduce $\mu(\mathbf{X}) = 2\sum_{i=1}^n A_{in}\pi_i.\left(\bar{b}_1 \mathbf{1}(Y_2 > 0) + \bar{b}_2 \mathbf{1}(Y_2 < 0)\right) Y_2 \Phi(Y_1 - 1)$ a.e.

Note that, since $\alpha > 1/2$, $EVaR_{n,\alpha}$ does not depend on b_2. Therefore, $EVaR_{n,\alpha}$ can be obtained as if $\bar{b}_2 = \bar{b}_1$, through Equation (11) as above.

We get GAs through the derivatives of (14). The calculations of GA formulas are detailed in Subsection A.1.

For this experiment, choose $a_i = 1$ for every i, and then $b_i = \beta = 1.0833$ and $e_i = 1.9246$, $\nu_i = 0$ and $\omega_i = 2$ for every i, $\nu_p = 1$ and $\pi_p = -1$. The π_i are randomly chosen in the interval $(0, 1)$. For a book of stocks/bonds, choose $q(\mathbf{X}) = \exp(Y_2)$, i.e. $\nu_q = 1$ and $\pi_q = 0$. The correlation parameter ρ of (Y_1, Y_2) is 0.5. Since $(\nu_i, \omega_i) = (0, 2)$ for every i, the sign of $\mu_i(\mathbf{x})$ is simply the sign of Y_2. Then, $\bar{b}_1 = 1.0833$, $\bar{b}_2 = 0.4006$, $\bar{e}_1 = 1.9246$ and $\bar{e}_2 = 0.5592$.

The simulation results appear in Tables 3 and 4. Globally, they confirm our previous findings. Granularity adjustment calculations are very accurate for small/medium portfolio sizes, up to $n = 500$. In every case, they never provide a significantly worse work than $EVaR_{n,\alpha}$.

n	VaR (stdev)	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	15.07 (0.09)	13.95	15.09	7.43×10^{-2}	-1.33×10^{-3}
50	14.25 (0.11)	13.95	14.17	2.10×10^{-2}	5.61×10^{-3}
100	14.20(0.10)	13.95	14.07	1.76×10^{-2}	9.86×10^{-3}
500	14.01(0.10)	13.95	13.98	3.95×10^{-3}	2.32×10^{-3}
1000	13.87(0.09)	13.95	13.96	-5.50×10^{-3}	-6.32×10^{-3}

Table 3: Comparison of $VaR_{99\%}$ calculations for counterparty risk. We consider a book of stocks and/or bonds under (A) and (B.2).

n	VaR (stdev)	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	3.88 (0.012)	3.49	3.88	9.82×10^{-2}	-1.27×10^{-3}
50	3.59 (0.014)	3.49	3.57	2.55×10^{-2}	3.97×10^{-3}
100	3.54 (0.013)	3.49	3.53	1.28×10^{-2}	1.88×10^{-3}
500	3.51 (0.016)	3.49	3.50	5.56×10^{-3}	3.36×10^{-3}
1000	3.51(0.014)	3.49	3.50	4.02×10^{-3}	2.92×10^{-3}

Table 4: Comparison of $VaR_{99\%}$ calculations for counterparty risk. We consider a book of derivatives, under (A) and (B.2).

4 Granularity Adjustment formulas for market risk

The "market risk" associated to a portfolio is the risk of losses that may result from the fluctuations of the prices of some financial instruments. Technically, its main difference with default/counterparty risk is the continuous shape of individual loss profiles, while jump-to-default events induce large and sudden market value jumps. Moreover, exposures are always non-negative by definition in the case of counterparty risk. On the contrary, the loss function L_n can be positive or negative when it relates to market risk. Indeed, L_n measures the opposite of the so-called "profit and loss" between $t = 0$ and $t = T$, assuming the underlying portfolio is frozen between both dates. Default risk is no longer key, i.e. we imagine that the positions are no longer exposed to default.

4.1 Granularity adjustments with exponential-type conditional volatilities

As usual, the loss variables Z_i will be mutually independent given $\mathbf{X}$. Assume that

$$\mathbb{E}[Z_i|\mathbf{X}] = w_i'\mathbf{X} + c_i, \text{ and } \mathbb{V}[Z_i|\mathbf{X}] = \exp(\beta_i'\mathbf{X} + d_i), \tag{19}$$

for some fixed quantities w_i, β_i, c_i and d_i. The portfolio conditional expected loss is then $\mu(\mathbf{x}) := \mathbb{E}[L_n|\mathbf{X} = \mathbf{x}] = \sum_{i=1}^n A_{in}(w_i'\mathbf{x} + c_i) := w'\mathbf{x} + c$. Without a lack of generality, set $c = 0$. Therefore, $\mu(\mathbf{x}) = w'\mathbf{x}, \quad w = \sum_{i=1}^n A_{in}w_i$. To obtain GA formulas, the key technical question is to calculate $\mathbb{E}[\exp(\beta_i'\mathbf{X})|w'\mathbf{X} = v]$ for any v.

A simple and natural model specifications is

$$Z_i = \mathbb{E}[Z_i|\mathbf{X}] + \mathbb{V}[Z_i|\mathbf{X}]^{1/2}\eta_i, \quad i = 1, \ldots, n, \tag{20}$$

for some random variables η_i such that $\mathbb{E}[\eta_i|\mathbf{X}] = 0$, $\mathbb{E}[\eta_i^2|\mathbf{X}] = 1$. Typically, the systematic factor $\mathbf{X}$ and the variables η_i are mutually independent, as in most GARCH-type models. In this case, the law of η_i does not influence $EVaR$ and $VaRGA$ calculations. Indeed, the variables η_i are related to idiosyncratic risks only, that are diversified away through our expansions. Therefore, we are free of choosing arbitrarily complex η_i-law, for instance skewed and fat-tailed distributions, as long as the second conditional moment of η_i given $\mathbf{X}$ is finite.

4.1.1 Elliptical systematic random vectors

Under (20), our risk measures can be calculated once the law of $\mathbf{X}$ is specified, and without knowing the law of the idiosyncratic noises η_i, $i = 1, \ldots, n$. Here, we assume that $\mathbf{X}$ is a m-dimensional elliptical vector $\mathcal{E}_m(\theta, \Sigma, g_{\mathbf{X}})$. In particular, this covers the case of a gaussian $\mathbf{X}$-random vector $\mathcal{N}(\theta, \Sigma)$. Then, any couple $(\beta_i'\mathbf{X}, w'\mathbf{X})$ is a bivariate elliptical vector. We stress that, even if $\mathbf{X}$ is gaussian, we do not evaluate a parametric gaussian VaR. Indeed, only the *conditional distributions* of losses given $\mathbf{X}$ are gaussian/elliptical. And the *"true" underlying* loss distributions can be a lot more complex. To lighten notations, set $(Y_i, Z) := (\beta_i'\mathbf{X}, w'\mathbf{X})$. Its expectation is $[\mu_i, \mu_Z] := [\beta_i'\theta, w'\theta]'$, and its variance-covariance matrix is

$$Cov(Y_i, Z) = Cov(\beta_i'\mathbf{X}, w'\mathbf{X}) = \begin{bmatrix} \beta_i'\Sigma\beta_i \; (:= \sigma_i^2) & \beta_i'\Sigma w \; (:= \rho_i\sigma_i\sigma_Z) \\ w'\Sigma\beta_i \; (:= \rho_i\sigma_i\sigma_Z) & w'\Sigma w \; (:= \sigma_Z^2) \end{bmatrix}.$$

Actually, the distribution of (Y_i, Z) is elliptical: $(Y_i, Z) \sim \mathcal{E}_2([\mu_i, \mu_Z]', Cov(Y_i, Z), g_i)$, with $g_i(v) = \int_0^\infty w^{n/2-2} g_{\mathbf{X}}(v + w)\, dw$. We deduce, for any $z \in \mathbb{R}$,

$$Y_i | Z = z \sim \mathcal{E}_1\left(\frac{\rho_i\sigma_i}{\sigma_Z}(z - \mu_Z) + \mu_i, (1 - \rho_i^2)\sigma_i^2, g_{i|z}\right), \tag{21}$$

where $g_{i|z}(v) = g_i(v + (z - \mu_Z)^2/\sigma_Z^2)$. With obvious notations, deduce

$$\begin{aligned} & \mathbb{E}(\exp(\beta_i'\mathbf{X} + d_i)|w'\mathbf{X} = z) = \exp(d_i)\mathbb{E}[\exp(Y_i)|Z = z] \\ = \; & \exp(d_i)\Psi_{i|z}(1) = \exp(d_i)\int_{-\infty}^{+\infty} \exp(ix) g_{i|z}(x^2)\, dx. \end{aligned}$$

For a gaussian vector $\mathbf{X} \sim \mathcal{N}(\theta, \Sigma)$, we get

$$Y_i | Z = z \sim \mathcal{N}\left(\frac{\rho_i\sigma_i}{\sigma_Z}(z - \mu_Z) + \mu_i, (1 - \rho_i^2)\sigma_i^2\right), \text{ and} \tag{22}$$

$$\mathbb{E}(\exp(\beta_i'\mathbf{X} + d_i)|w'\mathbf{X} = z) = \exp\left(\frac{\rho_i\sigma_i}{\sigma_Z}(z - \mu_Z) + \mu_i + d_i + \frac{(1 - \rho_i^2)\sigma_i^2}{2}\right).$$

Simple calculations provide

$$\kappa_i(z) = \exp\left(d_i + (1 - \rho_i^2)\frac{\sigma_i^2}{2} + \left(\frac{z - \mu_Z}{\sigma_Z}\right)\rho_i\sigma_i + \mu_i\right) f_{\mathcal{N}(\mu_z, \sigma_z^2)}(z), \text{ and}$$

$$\kappa_i'(z) = \kappa_i(z) \cdot \left(\frac{\sigma_i\rho_i}{\sigma_Z} - \frac{z - \mu_Z}{\sigma_Z^2}\right).$$

Note that $T_{n,\infty}(z) = \frac{1}{2}\sum_{i=1}^n A_{in}^2 \kappa_i'(z)$ is proportional to the density of $w'\mathbf{X}$ at z.

With obvious notations, we get simply

$$
\begin{aligned}
VaRGA_{n,\alpha} = EVaR_{n,\alpha} - \frac{T_{n,\infty}(EVaR_{n,\alpha})}{f_\mu(EVaR_{n,\alpha})} \\
= \quad EVaR_{n,\alpha} - \frac{1}{2}\sum_{i=1}^{n} A_{in}^2 \exp\left(\frac{\beta_i'\Sigma w}{w'\Sigma w}(EVaR_{n,\alpha} - w'\theta) + \beta_i'\theta + d_i\right. \\
+ \quad \left.(1 - \frac{(\beta_i'\sigma w)^2}{\beta_i'\Sigma\beta_i w'\Sigma w})\frac{\beta_i'\Sigma\beta_i}{2}\right) \cdot \left(\frac{\beta_i'\Sigma w}{w'\Sigma w} - \frac{EVaR_{n,\alpha} - w'\theta}{w'\Sigma w}\right).
\end{aligned}
$$

4.1.2 Empirical illustration

Now, let us illustrate the relevance of such formulas with a simulation exercise. To induce a certain amount of heterogeneity in the portfolio, a proportion h of the exposures are K times higher than the others. In this experiment, we choose $m = 2$, $d_i := d = 5$, $w_i := w = (4, 0)$ and $\beta_i := \beta = (0.01, 0.3)$ for all i, $K = 4$, $h = 20\%$, $\theta = (0, 0)$, $\Sigma =: Diag(\tau_1, \tau_2) = Id$.

Under Equation (19), the individual random losses are drawn as $Z_i \sim \mathbb{E}[Z_i|\mathbf{X}] + \mathbb{V}[Z_i|\mathbf{X}]^{1/2}W_i$, where $(W_i)_{i=1,\dots,n}$ denotes a gaussian white noise.

The results are detailed in Table 5. Clearly, granularity adjustments provide very significant improvements w.r.t. $EVaR$ approximations, even when the portfolio size is large. More heterogeneity in the portfolio (through a larger d value) increases the importance of measuring individual characteristics finely, because the total loss is more sensitive to some idiosyncratic risks. Even without this feature (homogeneous portfolios), GAs provide useful and accurate results in every case.

Moreover, the impact of the number of systematic factors seems to be relatively weak. For instance, when $m = 5$, $d \in \{5, 7\}$, $\Sigma = Id$, $w_i = (4, 0, 0, 0, 0)$, $\beta_i = (0.01, 0.3, 0.1, -0.1, 0.5)$, the performances of $VaRGA$ that appear in Table 6 are still good. They are comparable with those obtained in Table 5.

n	VaR	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	-14.52 (0.027)	-9.31	-13.55	3.59 $\times 10^{-1}$	6.64 $\times 10^{-2}$
50	-10.56 (0.026)	-9.31	-10.15	1.19 $\times 10^{-1}$	3.84 $\times 10^{-2}$
100	-9.97 (0.026)	-9.31	-9.73	6.63 $\times 10^{-2}$	2.38 $\times 10^{-2}$
500	-9.44 (0.028)	-9.31	-9.39	1.47$\times 10^{-2}$	5.69 $\times 10^{-3}$
1000	-9.41 (0.027)	-9.31	-9.35	1.16$\times 10^{-2}$	7.08 $\times 10^{-3}$

Table 5: Comparison of $VaR_{99\%}$ calculations for market risk and gaussian systematic factors.

	$m = 5, d = 5$		$m = 5, d = 7$	
n	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	4.17 $\times 10^{-1}$	9.90 $\times 10^{-2}$	7.51 $\times 10^{-1}$	-2.51 $\times 10^{-1}$
50	1.39 $\times 10^{-1}$	4.55 $\times 10^{-2}$	4.93 $\times 10^{-1}$	8.45 $\times 10^{-2}$
100	7.32 $\times 10^{-2}$	2.27 $\times 10^{-2}$	3.56 $\times 10^{-1}$	9.73 $\times 10^{-2}$
500	1.37 $\times 10^{-2}$	2.99 $\times 10^{-3}$	1.07 $\times 10^{-1}$	3.56 $\times 10^{-2}$
1000	6.65 $\times 10^{-3}$	1.24 $\times 10^{-3}$	5.77 $\times 10^{-2}$	1.97 $\times 10^{-2}$

Table 6: Comparison of $VaR_{99\%}$ calculations for market risk and gaussian systematic factors.

4.2 Granularity adjustments with "quadratic-type" conditional volatilities

The previous family of models was based on an exponential form of conditional volatilities $\mathbb{V}(Z_i|\mathbf{X})$. Depending on the $\mathbf{X}$-law and the coefficients (β_i, d_i), this assumption could generate large uncertainties of realized losses among the names in the portfolio. Sometimes, this could be seen as a drawback. In this section, we present an alternative family of models of market risk that should not suffer from such feature.

Now, the conditional idiosyncratic variances are quadratic functions of the systematic factor $\mathbf{X}$, instead of an exponential function. The model specification is

$$\mathbb{E}[Z_k|\mathbf{X}] = w_k'\mathbf{X}, \text{ and } \mathbb{V}(Z_k|\mathbf{X}) = \mathbf{X}'\Omega_k\mathbf{X} = \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} X_i X_j, \; k = 1, \dots, n, \tag{23}$$

for some positive definite matrices $\Omega_k := [\alpha_{i,j}^{(k)}]$ and deterministic vectors w_k. Therefore, $\mathbb{E}[L_n|\mathbf{X}] = w'\mathbf{X}$, $w := \sum_{k=1}^n A_{k,n} w_k$. And we get explicit GA formulas by calculating $\mathbb{E}[X_i X_j | w'\mathbf{X} = v]$, $1 \leq i, j \leq m$.

4.2.1 GA formulas with elliptically-distributed systematic factors

As previously, let us consider an elliptical vector $\mathbf{X} \sim \mathcal{E}_m(\theta, \Sigma, g_{\mathbf{X}})$, $\mathbb{E}[\mathbf{X}] = \theta$, $\mathbb{V}(\mathbf{X}) = \Sigma$. For every indices i, j in $\{1, \dots, n\}$,

$$\mathbb{E}[X_i X_j | w'\mathbf{X} = z] = \mathbb{E}\left[\left(\frac{X_i + X_j}{2} \right)^2 - \left(\frac{X_i - X_j}{2} \right)^2 | w'\mathbf{X} = z \right].$$

Set $Y_{ij} = (X_i + X_j)/2$, $\bar{Y}_{ij} = (X_i - X_j)/2$ et $Z = w'\mathbf{X}$. Therefore, to get GA formulas, it will be sufficient to calculate $\mathbb{E}(Y_{ij}^2 | Z = z)$ and $\mathbb{E}(\bar{Y}_{ij}^2 | Z = z)$.

Let us detail these calculations in the case of Y_{ij}. We can lead the same reasoning as in Subsection 4.1.1, replacing β_i by $\gamma_{i,j} := (0, \dots, 0, 1/2, 0, \dots, 0, 1/2, 0, \dots, 0)$, when $i \neq j$ (1/2 appears at the coordinates i and j only), or by $\gamma_{i,i} := (0, \dots, 0, 1, 0, \dots, 0)$, with 1 at the i-th position. Moreover, the first two moments of $Y_{i,j}$ given Z are the same as in (22), due to the properties of elliptical vectors (see Theorems 5 and 8 in Gomez *et al.* 2003): for every couple (i, j), we have

$$\mathbb{E}(Y_{ij} | Z = z) = \rho_{ij} \left(\frac{z - \mu_Z}{\sigma_Z} \right) \sigma_{ij} + \mu_{ij},$$

$$\mathbb{E}(Y_{ij}^2 | Z = z) = (1 - \rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij} \left(\frac{z - \mu_Z}{\sigma_Z} \right) \sigma_{ij} + \mu_{ij} \right)^2,$$

where $\sigma_{ij}^2 = \gamma_{i,j}'\Sigma\gamma_{i,j}$, $\sigma_Z^2 = w'\Sigma w$, $\rho_{ij} = \gamma_{i,j}'\Sigma w/(\sigma_{ij}\sigma_Z)$, $\mu_{ij} = \gamma_{i,j}'\theta$ and $\mu_Z = w'\theta$.

The same calculations can be done with $\bar{Y}_{ij}$. The single difference with Y_{ij} comes from a coefficient $-1/2$ instead of $1/2$, for the j-th component of the vectors $\gamma_{i,j}$, $i \neq j$, providing $\bar{\gamma}_{ij}$ and the associated quantities $\bar{\sigma}_{ij}^2 := \bar{\gamma}_{i,j}'\Sigma\bar{\gamma}_{i,j}$, $\bar{\rho}_{ij} := \bar{\gamma}_{i,j}'\Sigma w/(\bar{\sigma}_{ij}\sigma_Z)$, $\bar{\mu}_{ij} := \bar{\gamma}_{i,j}'\theta$. Obviously, we get $\mathbb{E}[\bar{Y}_{i,j}|Z]$ and $\mathbb{E}[\bar{Y}_{i,j}^2|Z]$ as above, replacing $(\sigma_{ij}^2, \rho_{ij}, \mu_{ij})$ by $(\bar{\sigma}_{ij}^2, \bar{\rho}_{ij}, \bar{\mu}_{ij})$. We obtain, for every $k = 1, \ldots, n$,

$$\mathbb{E}[\mathbb{V}(Z_k|\mathbf{X}) \,|\, w'\mathbf{X} = z] = \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} \left\{ \mathbb{E}[Y_{ij}^2|w'\mathbf{X} = z] - \mathbb{E}[\bar{Y}_{ij}^2|w'\mathbf{X} = z] \right\}, \qquad (24)$$

and the GAs follow relatively easily. Indeed, $\mu(\mathbf{X}) = w'\mathbf{X}$ follows an elliptical distribution $\mathcal{E}_1(w'\theta, w'\Sigma w, g_{\mu(\mathbf{X})})$, where the density generator of $E[L_n|\mathbf{X}]$ is given by $g_{\mu(\mathbf{X})}(t) = \int_0^{+\infty} s^{-1/2} g_{\mathbf{X}}(t+s)\,ds$. Therefore, the density of $\mu(\mathbf{X})$ is

$$f_\mu(z) = g_{\mu(\mathbf{X})}\left(\frac{(z-\mu_Z)^2}{\sigma_Z^2}\right)/c_\mu, = g_{\mu(\mathbf{X})}\left(\frac{(z-w'\theta)^2}{w'\Sigma w}\right)/c_\mu \qquad (25)$$

$$c_\mu = \frac{\sqrt{\pi w'\Sigma w}}{\Gamma(1/2)} \int_0^{+\infty} v^{-1/2} g_{\mu(\mathbf{X})}(v)\,dv.$$

The latter constant has to be estimated numerically. Thanks to formulas (24) and (25), the associated GA terms are obtained by deriving the functions $\kappa_k(z) = \mathbb{E}[\mathbb{V}(Z_k|\mathbf{X}) \,|\, w'\mathbf{X} = z] f_\mu(z)$, for any $k = 1, \ldots, n$. To be specific, we obtain

$$\begin{aligned}\kappa_k(z) = f_\mu(z) \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} \Bigg(&(1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}(\frac{z-\mu_Z}{\sigma_Z})\sigma_{ij} + \mu_{ij}\right)^2 \\ - \quad &(1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2 - \left(\bar{\rho}_{ij}(\frac{z-\mu_Z}{\sigma_Z})\bar{\sigma}_{ij} + \bar{\mu}_{ij}\right)^2 \Bigg), \text{ and}\end{aligned}$$

$$\kappa_k'(z) = \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)}. \left\{ 2f_\mu(z) \left[\left(\rho_{ij}(\frac{z-\mu_Z}{\sigma_Z})\sigma_{ij} + \mu_{ij} \right) \frac{\rho_{ij}\sigma_{ij}}{\sigma_Z} \right. \right.$$

$$- \quad \left. \left(\bar{\rho}_{ij}(\frac{z-\mu_Z}{\sigma_Z})\bar{\sigma}_{ij} + \bar{\mu}_{ij} \right) \frac{\bar{\rho}_{ij}\bar{\sigma}_{ij}}{\sigma_Z} \right] + f_\mu'(z) \left[(1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}(\frac{z-\mu_Z}{\sigma_Z})\sigma_{ij} + \mu_{ij} \right)^2 \right.$$

$$- \quad \left. \left. (1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2 - \left(\bar{\rho}_{ij}(\frac{z-\mu_Z}{\sigma_Z})\bar{\sigma}_{ij} + \bar{\mu}_{ij} \right)^2 \right] \right\}, \text{ where}$$

$$f_\mu'(z) = \frac{2(z-\mu_Z)}{c_\mu \sigma_Z^2} g_{\mu(\mathbf{X})}' \left(\frac{(z-\mu_Z)^2}{\sigma_Z^2} \right).$$

It is difficult to specify these GA formulas further without particularizing some generators $g_{\mathbf{X}}$. In the next subsections, we will study the numerical performances of such specifications, when $\mathbf{X}$ is bivariate gaussian and when $\mathbf{X}$ follows a fat-tailed distribution.

4.2.2 Empirical illustration when X is gaussian

We have particularized the previous model by assuming that $\mathbf{X}$ follows a bivariate gaussian vector: $m = 2$, $\mathbf{X} \sim \mathcal{N}(\theta, \Sigma)$ and $g_{\mathbf{X}}(t) = \exp(-t/2)/\sqrt{2\pi}$. The GA formula is provided in Subsection A.2.

Let us evaluate the performances of GAs numerically, with a simple example. As in Subsection 4.1.2 and rather than considering uniform asset exposures, some portion h of the portfolio exposures will be K times higher than the others. In this experiment, we set $w_i := w = (1, 0)$, $K = 4$, $h = 0.2$, $\theta = (2, 2)$, $\Sigma = Diag(64, 4)$, $\Omega = [1.60.1, 0.10.4]$. The results appear in Table 7. GAs perform very well for every portfolio size. Such observations confirm and strengthen our findings in Section 4.1.2.

n	VaR	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	-20.91 (0.077)	-16.61	-22.44	2.06 $\times 10^{-1}$	-7.29 $\times 10^{-3}$
50	-17.69 (0.055)	-16.61	-17.77	6.12 $\times 10^{-2}$	-4.68 $\times 10^{-3}$
100	-17.16 (0.055)	-16.61	-17.19	3.21 $\times 10^{-2}$	-1.83 $\times 10^{-3}$
500	-16.72 (0.046)	-16.61	-16.73	6.55$\times 10^{-3}$	-4.24 $\times 10^{-4}$
1000	-16.67 (0.044)	-16.61	-16.67	3.49$\times 10^{-3}$	-5.46 $\times 10^{-6}$

Table 7: Comparison of $VaR_{99\%}$ calculations for market risk and gaussian systematic factors.

4.2.3 Application when X is a non-gaussian elliptical vector

To provide complementary results, and to challenge the current framework, we consider a bivariate elliptical vector $\mathbf{X} \sim \mathcal{E}_2(0, \Sigma, g_{\mathbf{X}})$, $g_{\mathbf{X}}(t) = 1/(\pi(t^2+1))$. Note that the second order moments of $\mathbf{X}$ are not finite, due to the fat tails of $\mathbf{X}$. We are interested in checking whether the GA approximations are suffering from such a feature. Indeed, Fermanian (2014) noticed that fat-tailed loss distributions can disturb GA approximations.

We reiterate that $\mu(\mathbf{X}) = w'\mathbf{X}$ is a linear transform of $\mathbf{X}$. Following Theorem 5 in Gomez *et al.* (2003), the density generator of $E[L_n|\mathbf{X}]$ is

$$
\begin{aligned}
g_{\mu(\mathbf{X})}(t) &= \int_0^{+\infty} s^{-1/2} g_{\mathbf{X}}(t+s)ds = \int_0^{+\infty} \frac{2\,dv}{\pi((t+v^2)^2+1)} \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+t^2}+t}\right)^{1/2} \cdot \frac{1}{(1+t^2)^{3/4}}.
\end{aligned}
$$

Therefore, $E[L_n|\mathbf{X}] \sim \mathcal{E}_1(w'\theta, w'\Sigma w, g_{\mu(\mathbf{X})})$, and the density of $\mu(\mathbf{X})$ is

$$
f_\mu(z) = g_{\mu(\mathbf{X})}\left(\frac{(z-w'\theta)^2}{w'\Sigma w}\right)/c_\mu,\; c_\mu = \frac{\sqrt{\pi w'\Sigma w}}{\Gamma(1/2)} \int_0^{+\infty} v^{-1/2} g_{\mu(\mathbf{X})}(v)\,dv,
$$

the latter constant being estimated numerically. The corresponding GA formulas are detailed in Subsection A.2.

The parameters of this experiment are $K = 4$, $h = 0.2$, $w_i := w = (1, 0)$, $\Sigma = Diag(1, 1/16)$, $\Omega = [1.60.1, 0.10.4]$. The results are detailed in Table 8. As previously, VaRGAs improve the EVaR approximations in every case, particularly for medium-sized portfolios. Nonetheless, the VaRGA performances are less striking than in the gaussian case (Table 7). The fatter tail behavior of $\mathbf{X}$ (and then of the associated losses) in the current case could be one explanation of these relatively poorer approximations.

n	VaR	EVaR	VaRGA	(VaR-EVaR)/VaR	(VaR-VaRGA)/VaR
10	-4.36 (0.023)	-3.87	-4.84	1.13×10^{-1}	-1.08×10^{-1}
50	-3.98 (0.023)	-3.87	-4.07	2.75×10^{-2}	-2.16×10^{-2}
100	-3.94 (0.017)	-3.87	-3.97	1.78×10^{-2}	-7.37×10^{-3}
500	-3.90 (0.018)	-3.87	-3.89	7.81×10^{-3}	-2.14×10^{-3}
1000	-3.90 (0.026)	-3.87	-3.88	7.17×10^{-3}	-2.97×10^{-3}

Table 8: Comparison of $VaR_{99\%}$ calculations for market risk and an elliptical systematic factor.

5 Conclusion

We have explained why granularity adjustment formulas for risk measure calculations were so scarce in a multi-factor framework, by pointing the associated technical difficulties out. We have proposed several flexible families of models to obtain such formulas for some portfolios that are exposed to counterparty and/or market risk. Therefore, we have extended significantly the scope of multi-factor granularity adjustments, particularly for VaR calculations. A complementary work could be to provide the corresponding formulas and empirical illustrations in the case of expected shortfalls.

We have showed the relevance of such multi-factor GAs empirically, for some families of models and some sets of parameters. To check the robustness of our conclusions, we have calculated VaRs, EvaRs and VaRGAs for a lot more model parameters: VaRGAs have provided better approximations than EVaRs in all cases virtually, often by a factor of ten. Due to the large number of model parameters, to the calculation times and to space limitations, we have not reported these additional results here. They can be provided under request. Nonetheless, more extensive simulations and some real data experiments are surely necessary to identify under which circumstances such GA techniques reach their limits.

A Detailed calculations of some $VaRGA_{n,\alpha}$

A.1 Granularity adjustments under Assumption B.2

Let us detail GA formulas for the models in Subsection 3.5.2.

In the case of our portfolio of derivatives, the joint law of the systematic drivers is given by (17). The chosen model specifications imply that $\mu(\mathbf{X}) = p(\mathbf{X})[A(\mathbf{X}) + B(\mathbf{X})q(\mathbf{X})]$, where $A(\cdot)$ and $B(\cdot)$ takes only two values: almost everywhere,

$$A(\mathbf{X}) = A_1\mathbf{1}(Y_2 \geq 0) + A_2\mathbf{1}(Y_2 < 0),\ B(\mathbf{X}) = B_1\mathbf{1}(Y_2 \geq 0) + B_2\mathbf{1}(Y_2 < 0),$$

$$A_k = \bar{b}_k \sum_{i=1}^{n} A_{in}\pi_i\nu_i,\ B_k = \bar{b}_k \sum_{i=1}^{n} A_{in}\pi_i\omega_i,\ k = 1, 2.$$

Then, the density of $\mathbb{E}[L_n|\mathbf{X}]$ is, for every $y \in \mathbb{R}$,

$$\begin{aligned} f_\mu(y) = & \int_0^1 \mathbf{1}(y \geq A_1 u)\phi_\rho\left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{y - A_1 u}{B_1 u}\right) \cdot \frac{du}{B_1\nu_p u\, \phi\circ\Phi^{-1}(u)} \\ + & \int_0^1 \mathbf{1}(y < A_2 u)\phi_\rho\left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{y - A_2 u}{B_2 u}\right) \cdot \frac{du}{B_2\nu_p u\, \phi\circ\Phi^{-1}(u)}. \end{aligned} \tag{26}$$

Note that, in our particular case, $A_1 = A_2 = 0$ since $\nu_i = 0$ for all i. And the density of the conditional expected loss is simply

$$f_\mu(y) = \mathbf{1}(y \geq 0) \int_0^1 \phi_\rho \left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{y}{B_1 u} \right) \cdot \frac{du}{B_1 \nu_p u \, \phi \circ \Phi^{-1}(u)}$$
$$+ \quad \mathbf{1}(y < 0) \int_0^1 \phi_\rho \left(\frac{\Phi^{-1}(u) - \pi_p}{\nu_p}, \frac{y}{B_2 u} \right) \cdot \frac{du}{B_2 \nu_p u \, \phi \circ \Phi^{-1}(u)}. \tag{27}$$

When the portfolio components are stocks and/or bonds, we have assumed (18) and things are simpler. In particular, for every $y \in \mathbb{R}^+$,

$$f_\mu(y) = \int_0^{+\infty} \phi_\rho \left(\frac{\Phi^{-1}(y/[A+Bv]) - \pi_p}{\nu_p}, \frac{\ln(v) - \pi_q}{\nu_q} \right) \cdot \frac{\mathbf{1}(\max(0,(y-A)/B) \leq v))\, dv}{\nu_p \nu_q v [A+Bv] \, \phi \circ \Phi^{-1}(y/[A+Bv])}. \tag{28}$$

Under (17), the joint density of $(p(\mathbf{X}), q(\mathbf{X}))$ is

$$g(u,v) = \phi_\rho \left((\Phi^{-1}(u) - \pi_p)/\nu_p, v \right) \cdot \frac{\mathbf{1}(u \in (0,1))}{\phi \circ \Phi^{-1}(u) \nu_p}, \tag{29}$$

and under (18), it is

$$g(u,v) = \phi_\rho \left((\Phi^{-1}(u) - \pi_p)/\nu_p, (\ln(v) - \pi_q)/\nu_q \right) \cdot \frac{\mathbf{1}(u \in (0,1), v \geq 0)}{v \phi \circ \Phi^{-1}(u) \nu_p \nu_q}. \tag{30}$$

Under (18) (a portfolio of stock/bonds), $\kappa_i(\cdot)$ is obtained by invoking Equations (13) and (14). We get easily

$$\kappa_i'(y) = \sum_{k=1}^{2} \sum_{l=0}^{2} \gamma_{i,k,l} \left\{ k y^{k-1} \int \frac{t^l}{(A+Bt)^{k+1}} g\left(\frac{y}{A+Bt}, t \right) dt \right.$$
$$+ \quad \left. y^k \int \frac{t^l}{(A+Bt)^{k+2}} \partial_1 g \left(\frac{y}{A+Bt}, t \right) dt \right\}, \tag{31}$$

$$\partial_1 g(u,v) = g(u,v) \cdot \left(\frac{\rho(\ln(v) - \pi_q)}{\nu_p \nu_q (1-\rho^2)} - \frac{\Phi^{-1}(u) - \pi_p}{(1-\rho^2)\nu_p^2} + \Phi^{-1}(u) \right) \cdot \frac{1}{\phi \circ \Phi^{-1}(u)}. \tag{32}$$

When the sign of $\mu_i(\mathbf{X})$ is arbitrary (portfolio of derivatives), the calculations of the κ_i functions may be tedious. Fortunately, all coefficients ν_i are zero and then only the sign of Y_2 determines the sign of $\mu_i(\mathbf{X})$. Therefore, $\kappa_i(\cdot)$ is obtained by invoking Equations (13) and (14), simply by replacing A, B, b_i and e_i by A_1, B_1, $\bar{b}_1$ and $\bar{e}_1$ respectively. With obvious notations, we deduce that, under (17) and when $y > 0$,

$$\kappa_i'(y) = \sum_{k=1}^{2}\sum_{l=0}^{2}\gamma_{i,k,l}\left\{ky^{k-1}\int \frac{t^l}{(A_1+B_1t)^{k+1}}g\left(\frac{y}{A_1+B_1t},t\right)dt\right.$$
$$+ \quad \left. y^k\int \frac{t^l}{(A_1+B_1t)^{k+2}}\partial_1 g\left(\frac{y}{A_1+B_1t},t\right)dt\right\}. \qquad (33)$$

$$\partial_1 g(u,v) = g(u,v)\cdot\left(\frac{\rho v}{\nu_p(1-\rho^2)} - \frac{\Phi^{-1}(u)-\pi_p}{(1-\rho^2)\nu_p^2} + \Phi^{-1}(u)\right)\cdot\frac{1}{\phi\circ\Phi^{-1}(u)}. \qquad (34)$$

Since

$$VaRGA_{n,\alpha} = EVaR_{n,\alpha} - \frac{\sum_{i=1}^n A_{i,n}^2\kappa_i'(EVaR_{n,\alpha})}{2f_\mu(EVaR_{n,\alpha})},$$

we obtain the corresponding GA formula under (17) (resp. under (18)) invoking Equations (33), (34) and (26) (resp. (31), (32) and (28)).

A.2 Granularity adjustments under a "quadratic form" conditional variance

Under (23), let us provide the exact formula when $\mathbf{X}$ is normal. Using the notations of Subsection 4.2.1, the conditional expected loss follows a gaussian distribution $\mathcal{N}(\mu_Z,\sigma_Z^2)$, with $\mu_Z = w'\theta$, $\sigma_Z^2 = w'\Sigma w$. We obtain

$$\kappa_k(z) = f_{\mathcal{N}(\mu_Z,\sigma_Z^2)}(z)\sum_{i,j=1}^{m}\alpha_{i,j}^{(k)}\left\{(1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)^2\right.$$
$$- \quad \left.(1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2 - \left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)^2\right\}, \text{ and}$$

$$
\begin{aligned}
\frac{\kappa_k'(z)}{f_{\mathcal{N}(\mu_Z,\sigma_Z^2)}(z)} &= \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} \left\{ 2\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)\frac{\rho_{ij}\sigma_{ij}}{\sigma_Z} \right. \\
&- 2\bar\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar\sigma_{ij}+\bar\mu_{ij}\Big)\frac{\bar\rho_{ij}\bar\sigma_{ij}}{\sigma_Z} \\
&- \left(\frac{z-\mu_Z}{\sigma_Z^2}\right)\cdot\left[(1-\rho_{ij}^2)\sigma_{ij}^2+\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)^2\right. \\
&- \left.\left.(1-\bar\rho_{ij}^2)\bar\sigma_{ij}^2-\left(\bar\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar\sigma_{ij}+\bar\mu_{ij}\right)^2\right]\right\}.
\end{aligned}
$$

We deduce

$$
\begin{aligned}
VaRGA_{n,\alpha} &= EVaR_{n,\alpha} - \frac{1}{2}\sum_{k=1}^{n} A_{k,n}^2 \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} \\
&\left\{ 2\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)\frac{\rho_{ij}\sigma_{ij}}{\sigma_Z} - 2\left(\bar\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar\sigma_{ij}+\bar\mu_{ij}\right)\frac{\bar\rho_{ij}\bar\sigma_{ij}}{\sigma_Z} \right. \\
&- \left(\frac{z-\mu_Z}{\sigma_Z^2}\right)\cdot\left[(1-\rho_{ij}^2)\sigma_{ij}^2+\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)^2\right. \\
&- \left.\left.(1-\bar\rho_{ij}^2)\bar\sigma_{ij}^2-\left(\bar\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar\sigma_{ij}+\bar\mu_{ij}\right)^2\right]\right\}_{|z=EVaR_{n,\alpha}}.
\end{aligned}
$$

Let us lead similar calculations in the case of the elliptical vector $\mathbf{X}$ of Subsection 4.2.3. Given the functional form of the density generator, the relevant expressions are

$$
f_\mu(z) = g_{\mu(\mathbf{X})}\left(\frac{(z-\mu_Z)^2}{\sigma_Z^2}\right)/c_\mu, \;\; f_\mu'(z) = \frac{2(z-\mu_Z)}{\sigma_Z^2} g_{\mu(\mathbf{X})}'\left(\frac{(z-\mu_Z)^2}{\sigma_Z^2}\right)/c_\mu,
$$

$$
g_{\mu(\mathbf{X})}(t) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+t^2}+t}\right)^{1/2}\cdot\frac{1}{(1+t^2)^{3/4}},
$$

$$
\begin{aligned}
g_{\mu(\mathbf{X})}'(t) &= \frac{(-1)}{2\sqrt{2}}\left(\frac{1}{\sqrt{1+t^2}+t}\right)^{1/2}\cdot\frac{1}{(1+t^2)^{5/4}} - \frac{3t}{2\sqrt{2}(1+t^2)^{7/4}}\cdot\left(\frac{1}{\sqrt{1+t^2}+t}\right)^{1/2} \\
&= \frac{(-1)}{2\sqrt{1+t^2}}\left[1+\frac{3t}{\sqrt{1+t^2}}\right] g_{\mu(\mathbf{X})}(t).
\end{aligned}
$$

We deduce

$$
\begin{aligned}
VaRGA_{n,\alpha} &= EVaR_{n,\alpha} - \frac{1}{2}\sum_{k=1}^{n} A_{k,n}^2 \sum_{i,j=1}^{m} \alpha_{i,j}^{(k)} \\
& \left\{ 2\left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)\frac{\rho_{ij}\sigma_{ij}}{\sigma_Z} - 2\left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)\frac{\bar{\rho}_{ij}\bar{\sigma}_{ij}}{\sigma_Z} \right. \\
+ & \quad \frac{f'_\mu}{f_\mu}(z)\cdot\left[(1-\rho_{ij}^2)\sigma_{ij}^2 + \left(\rho_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\sigma_{ij}+\mu_{ij}\right)^2\right. \\
- & \quad \left.\left.(1-\bar{\rho}_{ij}^2)\bar{\sigma}_{ij}^2 - \left(\bar{\rho}_{ij}\left(\frac{z-\mu_Z}{\sigma_Z}\right)\bar{\sigma}_{ij}+\bar{\mu}_{ij}\right)^2\right]\right\}_{|z=EVaR_{n,\alpha}}, \quad \text{where}
\end{aligned}
$$

$$
\frac{f'_\mu}{f_\mu}(z) = -\frac{z-\mu_Z}{\sqrt{\sigma_Z^2+(z-\mu_Z)^2}}\left[1+\frac{3(z-\mu_Z)}{\sqrt{\sigma_Z^2+(z-\mu_Z)^2}}\right].
$$

Since we have imposed $\theta = 0$ in the numerical experiment, $\mu_Z = w'\theta = 0$ and $\sigma_Z^2 = w'\Sigma w$.

Declarations of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

References

Acharya, V.V., Bharath, S.T., and Srinivasan, A. (2007). Does industry-wide distress affect defaulted firms ? Evidence from creditor recoveries. *Journal of Financial Economics*, **85**, 787-821. doi:10.1016/j.jfineco.2006.05.011

Altman, E., Brady, B., Resti, A., and Sironi, A. (2005). The Link between Default and Recovery Rates: Theory, Empirical Evidence, and Implications. *Journal of Business*, **78**, 2203-2228. doi:10.1086/497044

Amraoui, S., Cousot, L., Hitier, S., Laurent, J.-P., 2012. Pricing CDOs with State Dependent Stochastic Recovery Rates. Quantitative Finance 12 (8), 1219-1240.

Calabrese, R., and Zenga, M. (2010). Bank loan recovery rates: Measuring and nonparametric density estimation. *Journal of Banking and Finance*, **34**, 903-911. doi:10.1016/j.jbankfin.2009.10.001

Carey, M., and Gordy, M. (2005). Measuring Systematic Risk in Recoveries on Defaulted Debt I: Firm-Level Ultimate LGDs. Working paper, FDIC Center for Financial Research. doi:10.2139/ssrn.394522

Credit Suisse Financial Products, 1997. *CreditRisk+: A Credit Risk Management Framework*. Credit Suisse, London.

Emmer, S., and Tasche, D. (2005). Calculating credit risk capital charges with the onefactor model. *Journal of Risk*, **7**, 85-101. doi:10.21314/jor.2005.110

Fermanian, J.-D. (2014). The limits of granularity adjustments. *Journal of Banking and Finance*, **45**, 9-25. doi:10.2139/ssrn.2236267

Gagliardini, P., and Gouriéroux, C. (2013). Granularity adjustment for risk measures: Systematic vs unsystematic risks. *Journal of Approximate Reasoning*, **54**, 717-747. doi:10.1016/j.ijar.2013.01.003

Garcia Cespedes, J. C., de Juan Herrero, J. A., Kreinin, A. and Rosen, D. (2006). A Simple Multi-Factor "Factor Adjustment" for the Treatment of Credit Capital Diversification. *Journal of Credit Risk*, **2**, 57-85. doi:10.21314/jcr.2006.039

Gómez E., Gómez-Villegas, M., and Marín, J.-M. (2003). A survey on continuous elliptical vector distributions. *Revista Matematica Complutense*, **16**, 345-361. doi:10.5209/rev rema.2003.v16.n1.16889

Gordy, M.B. (2003). A risk-factor model foundation for ratings-based bank capital rules. *Journal of Financial Intermediation*, **12**, 199-232. doi:10.1016/s1042-9573(03)00040-8

Gordy, M.B. (2004). Granularity adjustment in portfolio credit risk measurement. In: Szegö, G.P. (Ed.), *Risk Measures for the 21st Century.* Wiley.

Gordy, M.B., and Marrone, J. (2012). Granularity adjustment for mark-to-market credit risk models. *Journal of Banking and Finance*, **36**, 1896-1910.

Gordy, M.B., and Lütkebohmert, E. (2013). Granularity adjustment for Regulatory Capital Assessment. *International Journal of Central Banking*, **9**, 33-71. doi:10.1016/j.jbankfin.2012.02.010

Gupton, G., Finger, C., and Bhatia, M. (1997). *CreditMetrics Technical Document.* J.P. Morgan & Co., New-York.

Martin, R., and Wilde, T. (2002). Unsystematic credit risk. *Risk* **15**, 123-128.

Pykhtin, M. (2004). Multi-factor adjustment. *Risk*, **17**, 85-90.

Shao, J., and Tu, D. (1995). *The Jackknife and Bootstrap.* Springer. doi:10.1007/978-1-4612-0795-5

Tasche, D. (2006). Measuring sectoral diversification in an asymptotic multi-factor. *Journal of Credit Risk*, **2**, 33-55. doi:10.21314/jcr.2006.040

Voropaev, M. (2011). An analytical framework for credit portfolio risk measures. *Risk*, **24**(5), 72-78.

Wilde, T. (2001a). IRB approach explained. *Risk* **14**(5), 87-90.

Wilde, T. (2001b). Probing granularity. *Risk* **14**(8), 103-106.