CHAPTER 7

A Comparative Analysis of Dependence Levels in Intensity-Based and Merton-Style Credit Risk Models

Jean-David Fermanian and Mohammed Sbai

7.1 INTRODUCTION

In finance, especially for credit portfolio modeling, basket credit derivatives (CDOs, *n*-th to default) pricing and hedging, the building of an accurate measure of the dependence between the underlying default events is becoming a key-challenge (see Crouhy, Galai and Mark, 2002; Koyluoglu and Hickman, 1998, for a review of the current credit risk portfolio models). This new frontier has induced a huge amount of literature for several years: Nyfeler (2000), Frey and McNeil (2001), Schönbucher and Schubert (2001), Das, Geng and Kapadia (2002), Elizalde (2003), Turnbull (2003), Yu (2003), among others.

There are mainly two usual approaches to simulate dependent default events (Schlögl, 2002, for example): in the structural framework (Merton, 1974) a firm is falling into default when its asset value falls below its debt level. In its multidimensional version, the default process of all the underlying obligors is directly deduced from the joint process of asset values. Most of the time, the increments of the asset process are assumed Gaussian. Thus, a correlation matrix allows a full description of the dependence between the default events.

In the intensity-based (or reduced-form) approach (Jarrow, Lando and Turnbull, 1997; Duffie and Singleton, 1999), we focus directly on the joint law of defaults, conditionally on some factors, without trying to explain the firm behaviors. Sometimes, such models seek to exhibit some observable variables for explaining the defaults, or consider defaults simpler as exogenous processes. They are trying to answer the following questions: "How and when do rating transitions happen", or "how do the spread curves behave", rather than "why".

Such a distinction may appear to be a bit artificial. As every duration model, Merton-style models can be rewritten in terms of intensities.[1] Moreover, when dealing with portfolios, the dependence structures obtained by both approaches are induced most of the time by some extra-random factors. Thus, most of the models that are built in practice can be considered as factor-models (Schönbucher, 2001). Nonetheless, we keep the distinction between structural and intensity models because it is now a type of common language in the credit risk arena.

The aim of this chapter is to exhibit simple intensity models that induce a sufficient amount of dependence. To be more specific, we would like that some dependence indicators cover a large scope of values. We prove the intensity-based approach is as flexible as the Merton-style one, in terms of dependence between obligors. It is just necessary to adopt the right point of view, and to specify conveniently such intensity-based models.

In sections 7.2 and 7.3, we detail both frameworks, and compare the respective loss distributions. Subsequently, some dependence indicators are provided and compared in section 7.4. In section 7.5, we extend the previous basic intensity-based model towards two directions : correlated frailty models and α-stable distributions.

7.2 MERTON-STYLE MODELS

In such approaches, a value A_i is associated with any firm i. An obligor is defaulting when its asset value falls below a barrier, generally representing its debt. Given these barrier levels and the dynamic of the asset values, we are able to draw the loss distribution for a whole portfolio. Thus, we consider a portfolio of k obligors and we set a fixed time horizon T, typically $T = 1$ year. The default probability for firm $i = 1, \ldots, k$ is:

$$p_i = P(A_i < D_i)$$

In this model, the correlation between default events is related to the correlation between assets values. Here, the latter correlation coefficient is equal to,

$$corr_{ij} = corr(A_i,\ A_j)$$

Even if there exist many alternative models for setting the dynamic of the asset value, we will consider in this paper the usual simple one factor model:

$$A_i = \rho V + \sqrt{1-\rho^2}\varepsilon_i, \tag{7.1}$$

where V follows a standard normally distributed random variable. It may be seen as an overall macro-economic factor that influences all the firm values. ρ is a constant between -1 and 1. We will consider positive ρ only because it is the case most of the time in practice.[2] ε_i is a standard normally distributed random variable, specific to the obligor i. As usual, we assume that all the ε_i are mutually independent and independent from V.

Therefore, the firm's value is also normally distributed and

$$corr_{ij} = corr(A_i,\ A_j) = \rho^2. \tag{7.2}$$

In order to simulate the portfolio loss distribution, we follow these successive steps:

1 For any firm i, we get its mean historical default probability p_i at the horizon T, as given by the rating agencies (here Standard & Poor's).

2 We calculate the barrier $l_i = \Phi^{-1}(p_i)$ where Φ is the cumulated distribution function of a N(0, 1) (see (7.1)).

3 We generate some random variables V for the whole portfolio and ε_i for every firm. Both are N(0, 1). Then, we compare $\rho V + \sqrt{1-\rho^2}\varepsilon_i$ with l_i and record if a i's default is triggered or not.

4 We finally cumulate the losses and repeat the same procedure many times in order to get the loss distribution.

The calibration will be done on ρ. Below is an example of what we get with the following parameters:

- $\rho = \sqrt{0.2}$ (the choice promoted by Basel 2).
- A time-horizon $T = 1$ year.
- One year default probabilities given by Standard & Poor's in Table 7.1:
- A portfolio of 100 firms:[3]
  - 10 firms rated AAA
  - 20 firms rated AA

Table 7.1 Average default rates over 1981–2002

Rating	CCC	B	BB	BBB	A	AA	AAA
PD (%) (1 year)	27.87	6.20	1.38	0.37	0.05	0.01	0.00

Source: Standard & Poor's.

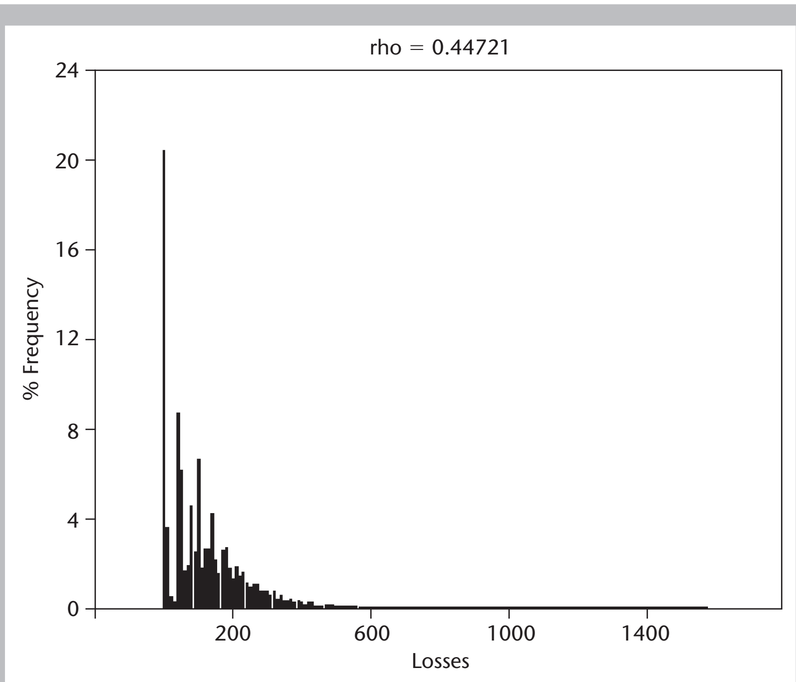

Figure 7.1 Histogram of losses in the Merton model

- 20 firms rated A
- 20 firms rated BBB
- 15 firms rated BB
- 10 firms rated B
- 5 firms rated CCC

- Constant exposure levels drawn randomly between 0 and 100.[4] Once they have been simulated, these exposure levels will be kept constant during the whole study. Their maturities are assumed infinite: when a default event is simulated, it always induces a non zero loss (whose value is the previous level associated with the defaulted counterparty). With such choices, we obtain Figure 7.1.

Table 7.2 One-year default events correlations between firms as a function of their ratings (%), with $\rho = \sqrt{0.2}$

	AAA	AA	A	BBB	BB	B	CCC
AAA	0.27	0.27	0.32	0.58	0.80	1.04	1.09
AA	0.27	0.27	0.32	0.58	0.80	1.04	1.09
A	0.32	0.32	0.38	0.69	0.96	1.27	1.35
BBB	0.58	0.58	0.69	1.33	1.94	2.70	3.06
BB	0.80	0.80	0.96	1.94	2.90	4.20	5.02
B	1.04	1.04	1.27	2.70	4.20	6.42	8.23
CCC	1.09	1.09	1.35	3.06	5.02	8.23	11.65

For $\rho = \sqrt{0.2}$, we also calculate the linear correlation between the default events for couples of firms that belong to pre-specified rating classes. The results are gathered in Table 7.2. In the Appendix we explain how we calculate such correlations. As empirically measured previously, the correlation levels we get among speculative grade firms are higher than those obtained with investment firms. They cover a range between 0.7 percent up to 11.6 percent, which is coherent with the empirical literature (de Servigny and Renault, 2002).

7.3 INTENSITY BASED MODELS

Such models are based on a direct evaluation of the intensity processes themselves. We are reminded that the default intensity is the instantaneous arrival rate of default:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(\tau \in [t, t + \Delta t] | \tau > t)$$

denoting by τ the default time. Let f be the probability density function of τ and S its survival function. For every time t, we have obviously

$$\lambda(t) = \frac{f(t)}{S(t)}$$

Just as the density f, the functions λ and S determine the law of τ, because

$$S(t) = \exp\left(-\int_0^t \lambda(s)ds\right)$$

The model we consider now belongs to the well-known frailty models family (Clayton and Cuzick, 1985). It has been used extensively in Survival Analysis (Hougaard, 2000). Frailty models are extensions of the Cox model

(Cox, 1972), where the (conditionally on the covariates) default intensities are multiplied by some unobservable random effects. Thus, in the basic version of frailty models, we set for every time t and every firm i,

$$\lambda_i(t, X_i, Z) = Z\lambda_0(t)\exp(\beta^T X_i), \tag{7.3}$$

where β is a vector-valued parameter of interest. X_i is the vector of observable covariates of the firm i. They may be firm specific and/or systemic (macro-economic indices). λ_0 is the deterministic baseline hazard function. Z is a frailty, an unobservable gamma distributed random variable. We assume it is the same for every obligor.

The random variable Z can be interpreted as a synthetic macro-economic factor that has not been included into the observable covariates X_i. For the sake of simplicity, we assume that λ_0 is a constant function and that β is equal to 0 (no observable covariates). Thus, the dependence is driven by Z only. Moreover, the Z realizations are assumed constant. This constancy is clearly a strong assumption, but it is realistic when we restrict ourselves to a one or two year horizon. This is indeed the case in this section. Then:

$$\lambda_i(t) = \lambda_i = Z\lambda_{0,i} \quad \text{where } Z \text{ is following a gamma law } G(\alpha, \theta) \tag{7.4}$$

This implies that the expectation of Z is α/θ and that its variance is α/θ^2. The default probabilities are taken from the same source as in the Merton model. We consider one year as the time unit, say T is expressed in years. Thus, λ_i can be identified with the yearly default intensity. We get the random default probability at time T as:

$$p_i(T|\lambda_i) = P(\tau \leq T|\lambda_i) = 1 - \exp(-\lambda_i T) \tag{7.5}$$

When we take the expectation with respects to Z, we have:

$$E(1 - \exp(-T\lambda_i)) = 1 - \left(\frac{\theta}{\theta + T\lambda_{0,i}}\right)^{\alpha} = \overline{p_i}(T) \tag{7.6}$$

This provides a first condition on the parameters (α, θ) and $\lambda_{0,i}$ since we know the mean historical probabilities $\overline{p}_i(T)$. In order to make the baseline hazard function $\lambda_{0,i}$ identifiable, we normalize the frailty variable : $E(Z) = 1$, i.e $\alpha = \theta$. In this case, Var(Z) $= 1/\alpha$. Now, the key parameter is α.

We consider the same portfolio as in the Merton model and we follow the following steps to get the loss distribution: for every time T,

1 we invoke $\overline{p}_i(T)$, the mean default probability (see(7.6)) to deduce $\lambda_{0,i}$;

2 we simulate Z and deduce $\lambda_{0,i}$ for each obligor i (see (7.4));

3 we draw a uniform random variable and we compare it to $\overline{p}_i(T|\lambda_i)$ to see if a default is triggered or not; see (7.5); and finally,

4 we cumulate the losses and we repeat the same procedure many times.

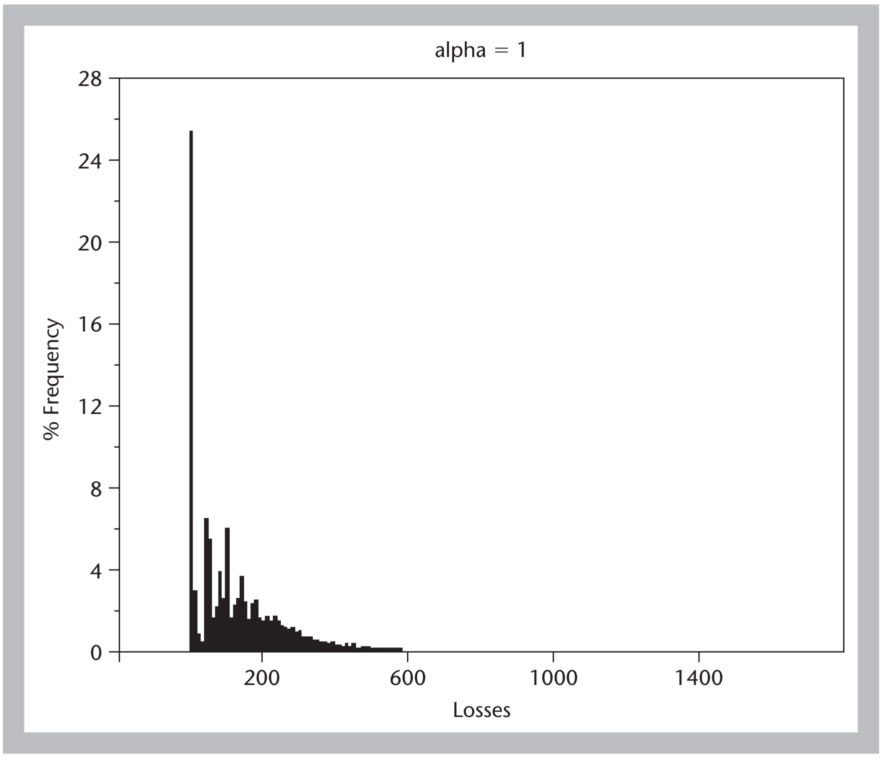

Figure 7.2 Histogram of the losses in an intensity based model

Table 7.3 One-year default events correlations between firms with different ratings (%), with $\alpha = 1$

	AAA	AA	A	BBB	BB	B	CCC
AAA	0.03	0.03	0.04	0.10	0.20	0.42	0.78
AA	0.03	0.03	0.04	0.10	0.20	0.42	0.78
A	0.04	0.04	0.05	0.13	0.26	0.54	1.00
BBB	0.10	0.10	0.13	0.37	0.71	1.46	2.72
BB	0.2	0.2	0.26	0.71	1.36	2.81	5.25
B	0.42	0.42	0.54	1.46	2.81	5.84	11.00
CCC	0.78	0.78	1.00	2.72	5.25	11.00	21.79

For example, for $\alpha = 1$ and $T = 1$, we get the histogram of the losses in Figure 7.2. Such empirical distribution looks like the one obtained with the Merton-style model (graph 1), especially in the right tail.

Again, for $\alpha = 1$ we calculate the default events correlations between firms with different ratings: Table 7.3. We get levels that are comparable with those obtained in Table 7.2, especially for speculative grade firms. Nonetheless, the differences by rating classes seem to be even stronger in the intensity framework. In other words, it is not easy to get significant correlation levels for couple of investment grade firms.

The same tabulars have been calculated with larger time horizons $T = 5$ and $T = 20$ years. See Appendix B. The conclusions are broadly the same, in terms of comparison between Merton-style and intensity-style models. Nonetheless, it is difficult to draw any general conclusions by focusing on some particular values for ρ and α.

7.4 COMPARISONS BETWEEN SOME DEPENDENCE INDICATORS

For several years, there has been a debate in the financial literature and among practitioners to compare the advantages and the drawbacks of both the previous approaches. Some authors[5] have come to the conclusion that realistic dependence levels between obligors cannot be easily obtained with intensity models. Notably, Schönbucher (2003) argues that, under some hypotheses, the strongest possible default correlation in an intensity-based model is of the same order of magnitude as the default probabilities. We briefly detail his technical argument.

Consider two firms A and B. For a fixed time horizon T, let

- p_A and p_B be the two individual default probabilities of A and B;
- λ_A and λ_B their random default intensities. For every realization ω, the functions $\lambda_A(\omega)$ and $\lambda_B(\omega)$ are assumed constant between 0 and T for the sake of simplicity;
- p_{AB} their joint default probability;
- ρ_{AB} the correlation coefficient between both default events.

By simple calculations, we obtain:

$$\begin{aligned}
p_{AB} &= \mathbb{E}(1_{\{A\}}1_{\{B\}}) \\
&= \mathbb{E}(\mathbb{E}(1_{\{A\}}1_{\{B\}}|\lambda)) \\
&= \mathbb{E}\left(1 - \exp\left(-\int_0^T \lambda_A(s)ds\right)\right)\left(1 - \exp\left(-\int_0^T \lambda_B(s)ds\right)\right) \\
&= 1 - (1 - p_A) - (1 - p_B) + \mathbb{E}\left(\exp\left(-\int_0^T \lambda_A(s) + \lambda_B(s)ds\right)\right) \\
&= p_A + p_B + \mathbb{E}\left(\exp\left(-\int_0^T \lambda_A(s) + \lambda_B(s)ds\right)\right) - 1
\end{aligned}$$

If both intensities are perfectly correlated: $\lambda_A = \lambda_B = \lambda$, then:

$$p_{AB} = 2p + \mathbb{E}\left(\exp\left(-2\int_0^T \lambda(s)ds\right)\right) - 1, \quad \text{where } p = p_A = p_B$$

The correlation between the two default events is then:

$$\begin{aligned} \rho_{AB} &\stackrel{def}{=} \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}} && (7.7) \\ &= \frac{2p + \mathbb{E}(\exp(-2\int_0^T \lambda ds)) - 1 - p^2}{p(1-p)} \\ &= \frac{\mathbb{E}(\exp(-2\int_0^T \lambda ds)) - (1-p)^2}{p(1-p)} \\ &= \frac{\mathbb{V}ar(\exp(-\int_0^T \lambda ds))}{p(1-p)} && (7.8) \end{aligned}$$

If we assume that the variance of the survival probability is at most of order p^2, then the correlation is of order p. Nonetheless, we argue that this is far from being satisfied usually.

To justify his assumption, Schönbucher (2003) suggested a normally distributed integrated intensity, for which we assume that the integrated hazard function between 0 and T is following a normal law N (μ, σ^2).

Note that such an assumption does not generate a "true" intensity process because some values of the integrated intensity may be negative. Nonetheless, forgetting such a detail, we get:

$$\mathbb{E}\left(\exp\left(-\int_0^T \lambda(s)ds\right)\right) = 1 - p = \exp\left(-\mu + \frac{1}{2}\sigma^2\right)$$

$$\mathbb{E}\left(\exp\left(-2\int_0^T \lambda(s)ds\right)\right) = \exp(-2\mu + 2\sigma^2)$$

and we deduce

$$\rho = (e^{-2\mu+2\sigma^2} - e^{-2\mu+\sigma^2})/(p - p^2) \approx (1-p)(e^{\sigma^2} - 1)/p \qquad (7.9)$$

If $\sigma \approx \lambda T$, we get that ρ and p are of the same order with this normal intensities specification. Clearly, it is a very crude approximation. A more careful approximation provides:

$$\exp(\sigma^2) - 1 \approx \sigma^2 \approx 2(\mu - p),$$

because $1 - p = \exp(-\mu + \sigma^2/2) \approx 1 - \mu + \sigma^2/2$. Thus, we get $\rho \approx 2(\mu - p)/p$, but we have no ideas (a priori) concerning the size of the latter ratio. To conclude, it seems that no strong argument has been done to conclude that the correlation levels ρ induced by intensity-based models are most of the time insufficient in practice.

In our previous setting, it would be more realistic to assume the random intensities follow the usual log-normal assumption:

$$\lambda = \lambda_0 \exp(-\sigma^2/2 + \sigma\varepsilon), \quad \varepsilon \sim N(0, 1)$$

In this case, the intensities are positive and they can be dealt as usual market factors in pricing formulas. Thus, we can evaluate the variance of the survival probability in equation (7.8). Remind that, if a random variable X follows a lognormal law, say $X = \exp(Z)$ with Z following a N(0,1), then

$$\mathbb{E}(\exp(-tX)) = \sum_{p=0}^{\infty} \frac{(-t)^p}{p!} \exp\left(p\mu + \frac{p^2\sigma^2}{2}\right)$$

Here, λ is assumed constant between 0 and T. Thus,

$$\mathbb{E}\left(\exp\left(-t\int_0^T \lambda\right)\right) = \sum_{p=0}^{\infty} \frac{(-t)^p}{p!} (\lambda_0 T)^p \exp\left(-\frac{p\sigma^2}{2} + \frac{p^2\sigma^2}{2}\right).$$

By a limited expansion, we get:

$$\begin{aligned} Var\left(\exp\left(-\int_0^T \lambda\right)\right) &= E\left[\exp\left(-2\int_0^T \lambda\right)\right] - E\left[\exp\left(-\int_0^T \lambda\right)\right]^2 \\ &\approx (\lambda_0 T)^2 \left(\exp(\sigma^2 - 1)\right) \end{aligned}$$

Thus, from equation (7.8), the correlation level between the two default times of the obligors A and B is approximately:

$$\rho_{AB} \approx p(\exp(\sigma^2) - 1)$$

Note that the coefficient σ has not the same meaning as in (7.9). Moreover, $Var(\lambda) = \lambda_0^2\,(\exp(\sigma^2) - 1)$. It is reasonable to assume that the standard deviation of the variations of λ is two or three times λ_0 (see Figure 7.3).

Thus, $\exp(\sigma^2) - 1$ is easily 4, 9 or more. For instance, if the default rate of the obligors is 1 percent between 0 and T, then the correlation level can reasonably be of the order 5 percent or 10 percent. Higher correlation levels can even be reached when assuming more volatility for the random intensities. In our current framework,[6] we can remind the following useful rule-of-thumb: when the standard deviation of the changes in random intensities is q times the mean level of these intensities, then the correlation levels are of order q^2 times the mean probability of default.

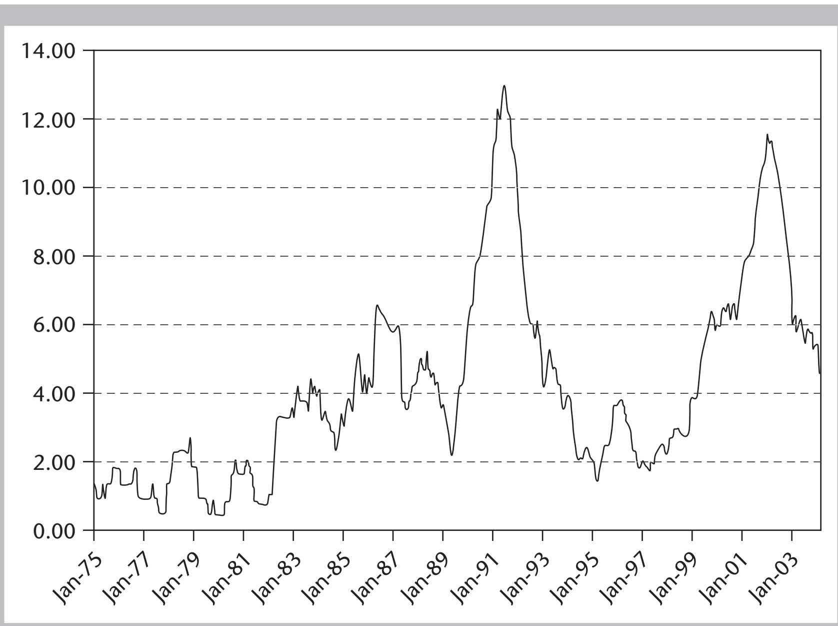

Figure 7.3 Monthly default rates, US bonds speculative grade, trailing 12 months, in percents
Source: Moody's.

Table 7.4 Features of the loss distribution for different ρ values (Merton model), $T = 1$ year

ρ	**0.01**	**0.1**	**0.3**	**0.4**	**0.6**	**0.7**	**0.9**	**0.95**
Quantile of order 99%	320	328	413	473	679	876	1163	1297
E(losses | losses > q99%)	356	367	482	571	858	1266	1615	1945
Skewness	0.64	0.70	1.15	1.50	2.47	4.13	4.33	5.47
K kurtosis	3.19	3.27	4.80	6.25	13.49	35.67	29.82	49.76
Average correlation (%)	10^{-4}	0.04	0.46	0.94	3.31	6.06	20.82	28.93

We led many simulations for different values of the parameters ρ and α. Tables 7.4 and 7.5 summarize the results we obtained. We took the same default probabilities and the same exposure in the two cases in order to have the same mean distribution.

We note that the dependence indicators between default events take some values of the same order of magnitude in the two cases. Empirically, default event correlations are varying from 0 percent to 30 percent for the Merton model, and from 0 percent to 20 percent for the reduced-form model. For some "reasonable" ρ and α levels ($\rho = 0.4$ and $\alpha = 2$, for instance), the

Table 7.5 Features of the loss distribution for different α values (intensity-based model), $T=1$ year

Var(Z) = 1/α	0.01	0.1	0.5	2	5	10	50	100
Quantile of order 99%	331	350	414	592	783	946	1278	1401
E(losses | losses > q99%)	368	392	477	685	912	1112	1638	1803
Skewness	0.69	0.79	1.09	1.60	2.03	2.46	3.73	4.06
k kurtosis	3.36	3.54	4.22	5.74	7.50	9.87	19.84	23.50
Average correlation (%)	0.01	0.08	0.39	1.37	2.80	4.46	10.78	14.56

sizes of the dependence indicators are the same. These levels are consistent with those obtained by de Servigny and Renault (2002): the latter authors report intra industries empirical correlation levels between one-year default events less than 10 percent, with typical levels around 2–3 percent for the speculative grade firms.

We note that the values of α considered in Table 7.5 are not unrealistic: they correspond to a standard deviation of the frailty variable Z varying from 0.1 to 10. Historically, important variations of default rates from one year to another have been met: see Figure 7.3. For instance, the mean default rate for US speculative grade bonds was more than 12 percent at the mid-year 1991, and fell below 2 percent in 1995.[7]

We have calculated the same indicators for $T=10$ years: see Appendix B. The Merton model seems to generate relatively more dependence in this case, especially under some extreme conditions (small or large ρ).

7.5 EXTENSIONS OF THE BASIC INTENSITY-BASED MODEL

7.5.1 A multi-factor model

The main idea here is to introduce an additional idiosyncratic unobservable random variable that summarizes the effect of an unobservable micro-economic factor.[8] We keep the same notations as in the first intensity model. We choose the correlated frailty model framework (Yashin and Iachine, 1995) whose asymptotic theory has been studied in Parner (1998). Such models allow taking into account simultaneously systematic and idiosyncratic random effects. In this case, we assume that

$$\lambda_i(t, X_i, Z) = (Z_0 + Z_i)\lambda_0 \exp(\beta^T X_i) \tag{7.10}$$

where Z_0 is an unobservable systemic gamma random variable, and Z_i is an unobservable gamma random variable that is specific to the obligor i.

The random variable Z_i's are mutually independent and Z_0 is independent from all the Z_i. The simulation method is almost the same as in the

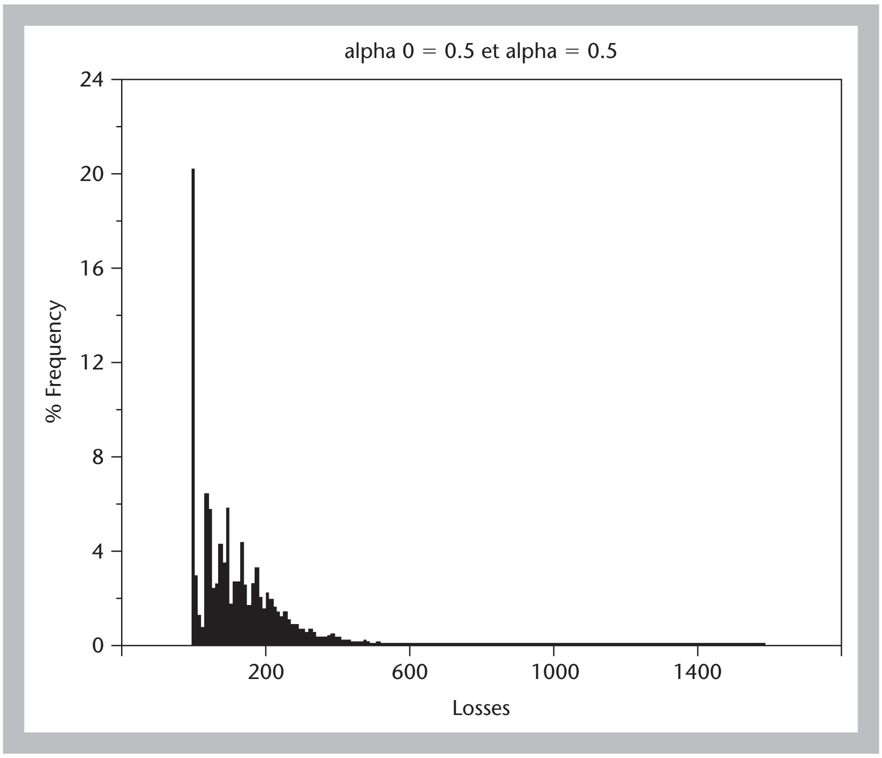

Figure 7.4 Histogram of the losses in the multi-factor intensity-based model ($T = 1$ year)

first model. We just have to draw the realizations of additional gamma random variables (one for each obligor). In practice, there are now two free parameters α_0 and α_i, related to Z_0 and Z_i respectively. This may cause some estimation complications, even if the log-likelihood of the observations can be written in closed form (Parner, 1998). In Figure 7.4, we draw the histogram of the losses obtained with model (7.10). Since we impose that the expectation of the global frailty component $Z_0 + Z_i$ equals one, we draw $Z_0 \sim \mathcal{G}(\alpha_0, \alpha_0 + \alpha)$ and $Z_0 \sim$ G(α, $\alpha_0 + \alpha$). We have chosen the parameter values $\alpha_0 = 0.5$ and $\alpha = 0.5$ for every i in Figure 7.4.

In this case,

$$Var(Z_0 + Z_i) = \alpha_0/(\alpha_0 + \alpha)^2 + \alpha/(\alpha_0 + \alpha)^2 = 1$$

and the correlated frailty $Z_0 + Z_i$ has the same two first moments as in Figure 7.2. The loss distributions seem to be very similar. At first glance, the introduction of specific components does not lessen too much the dependence between defaults.[9]

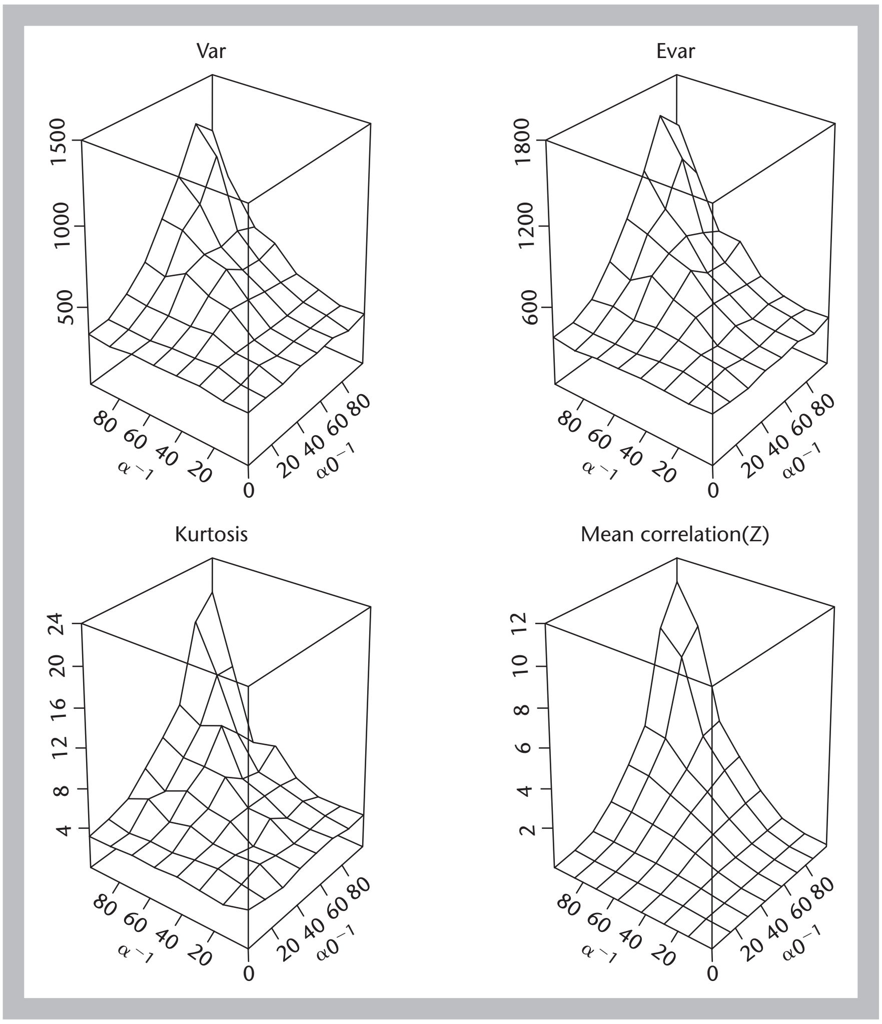

Figure 7.5 Combined effect of the parameters α_0 and α in the multi-factor intensity model

We lead many simulations with different values of the parameters α_0 and α_i in order to study their combined effects on the loss distribution (see Figure 7.5).

The variance of $Z_0 + Z_i$ varies from 0.005 to 50 when (α_0, α) varies from 0.01 to 100, which seems to be reasonable. The levels of our dependence indicators seem to be in line with those obtained in section 7.3. Note that we lose some dependence when the relative importance between Z_0 and Z_i is balanced. This is due to a diversification effect inside both components of the frailty factors. Globally, adding an idiosyncratic frailty allow more flexibility in the model, without losing the ability to reach realistic dependence levels.

Actually, the ratio $r = \alpha/\alpha_0$ provides a good measure of the dependence we obtain: the larger r, the higher the dependence indicators.

7.5.2 α-stable distributions

Properties of the family

Stable distributions allow building a rich class of probability distributions. They induce highly skewed and heavy tails features and have many interesting mathematical properties: see the survey of Samorodnitsky and Taqqu (1994), Hougaard (1986), or Mittnik and Rachev (1999) and Carr and Wu (2002) for financial applications. However, the lack of closed-form formulas for their densities and their cumulative distribution functions, despite a few exceptions, has been a major drawback that has limited their use by practitioners. To correct the ideas, we recall some basic theoretical results concerning such distributions.

Definition 1 A random variable X is said to be α-stable if for any X_1 and X_2, some independent copies of X, and for any positive numbers c_1 and c_2, there exist $c \in \mathbb{R}^+$ and $d \in \mathbb{R}$ such that:

$$cX + d \overset{d}{=} c_1 X_1 + c_2 X_2$$

If $d = 0$, X is said to be strictly stable.

There are other equivalent definitions of α-stable distributions (see Nolan, 2004, for a more detailed presentation of this distribution family) and we are going to invoke the following one because it is much more tractable:

Definition 2 A random variable X is said to be α-stable if its characteristic function takes the form:

$$\Phi_X(t) \overset{def}{=} \mathbb{E}(e^{itX}) = \begin{cases} \exp(-\gamma^\alpha\, |t|^\alpha\, (1 - i\beta \tan\left(\frac{\pi\alpha}{2}\right) sign(t)) + i\delta t) & \text{if } \alpha \neq 1, \\ \exp(-\gamma\, |t|\, (1 + i\beta \frac{2}{\pi}\, sign(t) \ln(|t|)) + i\delta t) & \text{if } \alpha = 1, \end{cases} \tag{7.11}$$

where $\alpha \in [0, 2]$, $\beta \in [-1, 1]$, $\gamma \geq 0$ and $\delta \in \mathbb{R}$.

This definition shows that an α-stable distribution generally requires four parameters as inputs:

- α, the index of stability. It is related to the tail behavior of the distribution. The smaller α, the stronger the leptokurtic feature of the distribution.
- β, the skewness parameter. If $\beta = 0$ then the distribution is symmetrical. If $\beta > 0$ then it is right skewed. Otherwise, it is left skewed.
- γ, the scale parameter.

- δ, the location parameter (When $\alpha > 1$, it measures the mean of the distribution).

There are multiple parameterizations for α-stable laws which may lead to some confusion. We keep the previous one, and we denote the α-stable distribution by $S(\alpha, \beta, \gamma, \delta)$ and its probability distribution function by f.

Definition 3 The support of an α-stable distribution is:

$$support\ (f(x)) = \begin{cases} [\delta, +\infty] & if\ \alpha < 1 \quad and \quad \beta = 1, \\ [-\infty, \delta] & if\ \alpha < 1 \quad and \quad \beta = -1, \\ \mathbb{R} & otherwise. \end{cases} \tag{7.12}$$

Because of the presence of heavy tails, all moments do not exist. Actually, we have:

Definition 4 Let $X \sim S(\alpha, \beta, \gamma, \delta)$.

$$\mathbb{E}(|X|^r) < +\infty \quad \text{if and only if } 0 < r < \alpha.$$

As far as we are concerned, for example, within the framework of frailty models the Laplace transforms are key tools.

Definition 5 Let $X \sim S(\alpha, \beta, \gamma, \delta)$. Its Laplace transform is defined if and only if $\beta = 1$, in which case it equals:

$$L_X(t) \equiv E\left(e^{-tX}\right) = \exp\left(-t\delta - t^\alpha \gamma^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\right), \quad t \geq 0, \tag{7.13}$$

by denoting $\sec(x) = 1/\cos(x)$.We will also need the following property:

Definition 6 Let $X \sim S(\alpha, \beta, \gamma, \delta)$ where $\alpha \neq 1$. Then for all $\alpha \neq 0$ and $b \in \mathbb{R}$ we have $aX + b \sim S(\alpha, \text{sign}(a)\, \beta, |a|\gamma, a\delta + b)$.

In particular, if $Z \sim S(\alpha, \beta, 1, 0)$ and

$$X = \begin{cases} \gamma Z + \delta & if \alpha \neq 1, \\ \gamma Z + (\delta + \frac{2\beta}{\pi}\gamma \ln(\gamma)) & if \alpha = 1, \end{cases}$$

then $X \sim S(\alpha, \beta, \gamma, \delta)$. We will simply note $S(\alpha, \beta)$ instead of $S(\alpha, \beta, 1, 0)$. Thus, by some linear transformations, we get all the α-stable laws starting from the family $S(\alpha, \beta)$.

7.5.3 Simulation of an α-stable distribution

As mentioned earlier, α-stable density functions do not admit closed forms. The usual method to obtain these functions is to inverse their characteristic functions $f(x) = \frac{1}{2\pi}\int \exp(-itx)\Phi_X(t)\, dt$. Except in a few cases,[10] the estimation of the latter expression is difficult, and will rather use the method in Chambers, Mallows and Stuck (1996). Let W be a random variable exponentially distributed with parameter $\lambda = 1$, and U a random variable uniformly distributed on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and let $\xi = \arctan(\beta \tan(\pi\alpha/2))/\alpha$ and the random variable

$$Z = \begin{cases} \dfrac{\sin(\alpha(\xi + U))}{\sqrt[\alpha]{\cos(\alpha\xi)\cos(U)}} \left(\dfrac{\cos(\alpha\xi + (\alpha - 1)U)}{W}\right)^{\frac{1-\alpha}{\alpha}}, & if\, \alpha \neq 1 \\ \dfrac{2}{\pi}\left(\left(\dfrac{\pi}{2} + \beta U\right)\tan(U) - \beta \ln\left(\dfrac{\frac{\pi}{2} W \cos U}{\frac{\pi}{2} + \beta U}\right)\right), & if\, \alpha = 1. \end{cases} \quad (7.14)$$

Then $Z \sim S(\alpha, \beta)$. To get $S(\alpha, \beta, \gamma, \delta)$, we invoke the linear transform of Definition 6.

7.5.4 α-stable intensity-based model

To simulate more heavy tailed random intensities, we are going to replace the gamma frailty random variable in (7.3) by an α-stable distributed frailty. As an intensity process is always positive and according to (7.12), we impose that $\alpha < 1$, $\beta = 1$ and $\delta = 0$ in order that the support of the frailty is $[0, +\infty]$. We keep the same simple specification as in our first intensity model: for every obligor i and every time t,

$$\lambda_i(t) = \lambda_i = Z\lambda_{0,i}$$

Therefore, $Z \sim S(\alpha, 1, \gamma, 0)$ where $\alpha \in [0, 1]$. Indeed, as the frailty variable has a multiplicative effect on the intensity, its baseline hazard function plays the role of a scale parameter. Thus, the parameter γ is unnecessary. In fact, we identify $\lambda_{0,i}$ by using the Laplace transform of the α-stable distribution (7.13), which leads to the one-year default probability:

$$\overline{p}_i = 1 - \exp\left(-\gamma^\alpha \sec\left(\frac{\pi\alpha}{2}\right)\gamma^\alpha\, \lambda_{0,i}^\alpha\right)$$

This implies:

$$\lambda_0 = \frac{1}{\gamma}\left(\frac{\ln\left(\frac{1}{1-\overline{p}_i}\right)}{\sec\left(\frac{\pi\alpha}{2}\right)}\right)^{\frac{1}{\alpha}}$$

Hence

$$\begin{aligned} \lambda &\overset{d}{=} \lambda_0 Z \\ &= \frac{1}{\gamma}\left(\frac{\ln\left(\frac{1}{1-\overline{p}_i}\right)}{\sec\left(\frac{\pi\alpha}{2}\right)}\right)^{\frac{1}{\alpha}} \gamma S(\alpha, 1) \qquad (7.15) \\ &= \left(\frac{\ln\left(\frac{1}{1-\overline{p}_i}\right)}{\sec\left(\frac{\pi\alpha}{2}\right)}\right)^{\frac{1}{\alpha}} S(\alpha, 1) \end{aligned}$$

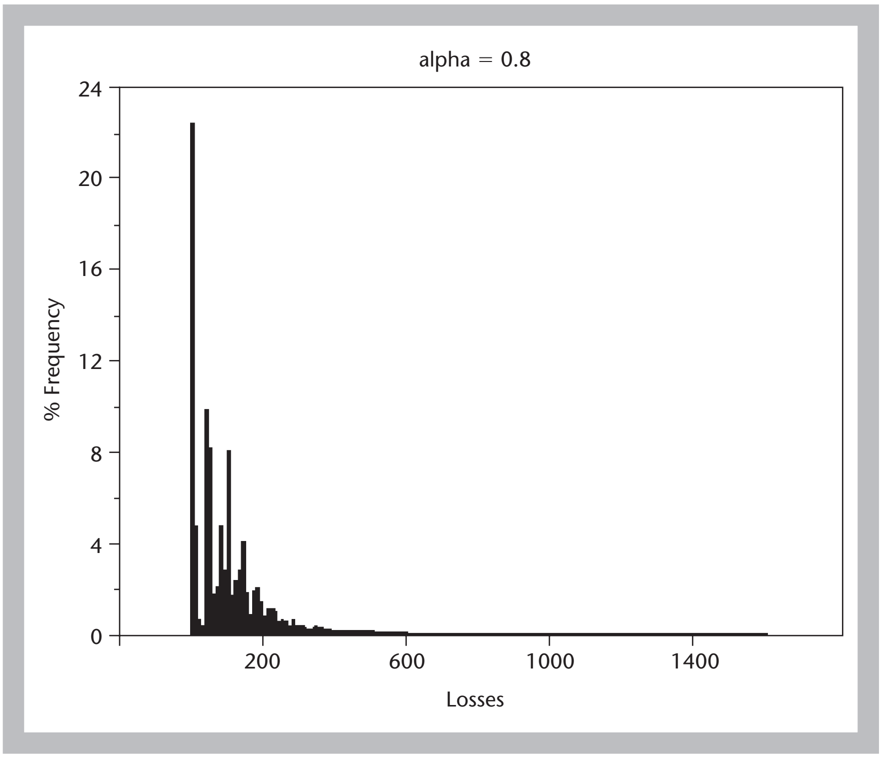

Figure 7.6 Histogram of the losses in the α – stable intensity based model

Obviously, the random intensities, and so the whole model, depend on α. In order to simulate the loss distribution, we draw a random variable $Z \sim S(\alpha,1)$ (see (7.14)) and we deduce λ from (7.15). We then follow the same steps as with the other models. For example, setting $\alpha = 0.8$ we obtain the histogram of portfolio[11] losses in Figure 7.6.

As expected, it is now easier to get large dependence levels between individual defaults inside the portfolio. Actually, the correlation between default events is even stronger than in our previous Merton-type model. Thus, α-stable frailties are a simple way to induce a strongly dependent credit-risky portfolio.

Table 7.6 presents the characteristics of the distribution for different values of the parameter α. The smaller the α, the larger the dependence between default events. The dependence indicators we get with α-stable laws are stronger than previously. Thus, it is a relatively simple way to generate highly dependent defaults, without modifying the intensity-based framework. Surprisingly, the kurtosis is increasing when the VaR and Expected Shortfall are decreasing. This can be explained by a type of degeneracy of the loss distributions: when α is very small, the losses are concentrated near the origin and very far towards the right. The implicit reference to the

Table 7.6 α-stable intensity-based model, $T = 1$ year

α	**0.1**	**0.3**	**0.5**	**0.7**	**0.8**	**0.9**	**0.95**
Quantile of order 99%	1588	1416	1170	1108	990	663	505
E(losses | losses > q99%)	2048	1905	1766	1654	1510	1066	844
Skewness	5.33	5.22	5.51	6.81	6.98	6.74	7.07
k kurtosis	45.59	46.10	51.11	87.51	90.26	95.60	115.48
Average correlation (%)	44.33	40.19	33.72	23.86	17.26	9.35	4.86

Gaussian distribution (when dealing with kurtosis) has no more sense in such situations.

7.6 CONCLUSION

We find some evidence that realistic and comparable dependence levels can be obtained by both intensity-style models and Merton-style models. With long time horizons, the latter approach gains a relative advantage, but the former can be strengthened by some extensions towards α-stable frailty models. Thus, the issue is not really to choose between both approaches but rather to specify conveniently a model, an intensity-based one or a Merton-style one. In practice, it is important to solve the following issues:

- What is the correlation scope that the model needs to cover?
- Observable and/or unobservable exogenous factors?
- Which distribution for such factors?
- Constant or time dependent frailties? If yes, whose process is best suited?

Moreover, one of the main practical issues concerns the estimation of the key dependence parameters, typically ρ and α in our previous frameworks. Such an issue may become a hurdle for the implementation of such models. For instance, clean estimations of the simplistic frailty model (7.3) are far from trivial (see Andersen, Gill, Borgan and Keiding, 1997, for the theory, and Metayer, 2004, for a financial application). And, even more, the introduction of dynamic frailties[12] induces likelihoods without any closed form, which imposes some delicate numerical optimization procedures (simulated maximum likelihood, EM algorithm, and so for).

APPENDIX A: CALCULATION OF CORRELATION BETWEEN DEFAULT EVENTS

Our goal is the calculation of the correlation between default events and between the dates $t = 0$ and $t = T$, controlling eventually by the rating categories. Technically speaking, it is equivalent to the calculation of joint default probabilities.

To calculate the joint default probability of two obligors, say A and BB, with different ratings in the intensity-based model, we note that:

$$
\begin{aligned}
P(\tau_A < T, \tau_{BB} < T) &= E\left[E[1(\tau_A < T)1(\tau_{BB} < T)\,|\lambda]\right] \\
&= \mathrm{E}\left[\left(1 - \exp\left(-\int_0^T \lambda_A\right)\right) \cdot \left(1 - \exp\left(-\int_0^T \lambda_{BB}\right)\right)\right] \\
&= 1 - (1 - \mathrm{p_A}) - (1 - p_{BB}) \\
&\quad + E\left[\exp(-\int_0^T (\lambda_A + \lambda_{BB}))\right] \\
&= \mathrm{p_A} + p_{BB} - 1 + E\left[\exp\left(-T\left(\lambda_A^0 Z + \lambda_{BB}^0 Z\right)\right)\right] \\
&= \mathrm{p_A} + p_{BB} - 1 + L_{G(\alpha,\alpha)}\left(T(\lambda_A^0 + \lambda_{BB}^0)Z\right) \\
&= \mathrm{p_A} + p_{BB} - 1 + \left(\frac{\alpha}{\alpha + T(\lambda_A^0 + \lambda_{BB}^0)}\right)^{\alpha}
\end{aligned}
$$

where $\mathcal{L}_{\mathcal{G}(\alpha,\theta)}(t)$ is the Laplace transform of a gamma-distributed random variable with parameter (α, θ).

From (7.9), we deduce the default correlation coefficient between default events for firms that are rated A and BB. Finally, to get an average correlation, we calculate a mean over all the possible couples of different firms. To be specific, we calculate:

$$
\rho_m = \frac{1}{\sum\limits_{i,j=1}^{7} n_i n_j} \sum_{i,j=1}^{7} n_i n_j \rho_{i,j}
$$

where n_i is the number of firms of rating i, and $\rho_{i,j}$ is the correlation coefficient obtained as previously explained.

To calculate the joint default probability of two obligors with different ratings in the Merton-style model, for example A and BB, we use the usual technique. According to (7.1) and (7.2) we have:

$$
\begin{pmatrix} A_A \\ A_{BB} \end{pmatrix} \sim \mathrm{N}\left(0, \begin{bmatrix} 1 & \rho^2 \\ \rho^2 & 1 \end{bmatrix}\right),
$$

which provides:

$$
\begin{aligned}
&P(\tau_A < 1\text{year}, \tau_{\mathrm{BB}} < 1\text{year}) \\
&= \frac{1}{2\pi\sqrt{1-\rho^4}} \int_{-\infty}^{D_A} \int_{-\infty}^{D_{BB}} \exp\left(-\frac{x^2+y^2-2\rho^2 xy}{2(1-\rho^4)}\right) \mathrm{dx\,dy}.
\end{aligned}
$$

We estimate numerically the latter double integral and deduce the average correlation between default events for every couple of ratings, as we made in the intensity-based model. The average correlation level is obtained by weighting conveniently such quantities.

APPENDIX B: EXTENSIONS OF THE RESULTS TO LARGE TIME HORIZONS

We use the same method as in sections 7.2 and 7.3. We choose the following default probabilities:

Average default rates over 1981–2002							
Rating	**CCC**	**B**	**BB**	**BBB**	**A**	**AA**	**AAA**
PD (%) (5 years)	61.35	33.02	14.45	3.83	0.75	0.27	0,11
PD (%) (20 years)	73.94	67.68	48.09	19.48	6.78	4.38	1.13

Source: Standard & Poor's Credit Pro.

We obtain default event correlations for the time horizons $T=5$ years and $T=20$ years (Tables 7.7, 7.8, 7.10 and 7.11), and the usual dependence indicators with $T=10$ years (Tables 7.9 and 7.12). In the latter case, particularly, the scope of values obtained in both cases is similar.

Note that when the time horizon T is increasing, it is surely questionable to assume the same values ρ and α as when $T=1$ year apply. Indeed, in the Merton-style models there is some empirical evidence that the asset correlations depend on T (see the discussion in de Servigny and Renault, 2002, for example).

Moreover, since we assumed the random default intensities λ_i are constant functions between 0 and T, their (random) levels should be less and less variable when T is increasing.[13] It should be more relevant to simulate an annual process (Z_t) for the frailty, but this does not belong in our simple framework. Thus, a realistic range of α-values is

Table 7.7 5-years default events correlations in the Merton model, with $\rho=\sqrt{0.2}$ (%)

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
AAA	0.63	0.82	1.10	1.59	1.89	1.85	1.51
AA	0.82	1.10	1.48	2.21	2.68	2.67	2.22
A	1.10	1.48	2.04	3.12	3.90	3.98	3.40
BBB	1.59	2.21	3.12	5.05	6.67	7.11	6.37
BB	1.89	2.68	3.90	6.67	9.32	10.43	9.84
B	1.85	2.67	3.98	7.12	10.43	12.15	12.01
CCC	1.51	2.22	3.40	6.37	9.84	12.01	12.53

Table 7.8 20-years default events correlations in the Merton model, with $\rho=\sqrt{0.2}$ (%).

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
AAA	2.60	3.68	4.02	4.64	4.41	3.78	3.50
AA	3.68	5.41	6.01	7.22	7.20	6.35	5.92
A	4.02	6.01	6.70	8.17	8.30	7.40	6.93
BBB	4.64	7.22	8.17	10.41	11.16	10.27	9.73
BB	4.41	7.20	8.30	11.16	12.81	12.28	11.81
B	3.78	6.35	7.40	10.27	12.28	12.09	11.74
CCC	3.50	5.92	6.93	9.73	11.81	11.74	11.43

Table 7.9 10-years Merton model

ρ	0.01	0.1	0.3	0.4	0.6	0.7	0.9	0.95
Q_{99}% quantile of order 99%	1,193	1,254	1,543	1,827	2,357	2,664	3,537	3,757
E(losses | losses > *q99%)*	1,253	1,339	1,663	2,020	2,704	3,089	4,145	4,426
skewness	0.15	0.22	0.46	0.75	1.05	1.20	1.53	1.5
k kurtosis	2.95	3.08	3.15	3.68	4.50	4.78	5.87	6.42
average correlation (%)	$10^{"3}$	0.24	2.26	4.19	10.62	15.62	31.60	37.72

Table 7.10 5-years default events correlations in the intensity model, with $\alpha = 1$ (%)

	AAA	AA	A	BBB	BB	B	CCC
AAA	0.04	0.05	0.08	0.16	0.30	0.36	0.31
AA	0.05	0.08	0.12	0.24	0.44	0.55	0.53
A	0.08	0.12	0.18	0.39	0.71	0.93	0.94
BBB	0.16	0.24	0.39	0.84	1.57	2.14	2.22
BB	0.30	0.44	0.71	1.57	3.02	4.19	4.31
B	0.36	0.55	0.93	2.14	4.19	6.24	7.34
CCC	0.31	0.53	0.94	2.22	4.31	7.34	13.01

Table 7.11 20-years default events correlations in the intensity model, with $\alpha = 1$ (%)

	AAA	AA	A	BBB	BB	B	CCC
AAA	0.13	0.07	0.12	0.23	0.29	0.33	0.26
AA	0.07	0.49	0.44	0.57	1.10	0.93	0.27
A	0.12	0.44	0.48	0.67	1.07	1.02	0.47
BBB	0.23	0.57	0.67	1.06	1.59	1.65	1.11
BB	0.29	1.10	1.07	1.59	3.03	2.99	2.04
B	0.33	0.93	1.02	1.65	2.99	3.46	3.41
CCC	0.26	0.27	0.47	1.11	2.04	3.41	8.39

Table 7.12 10-years intensity-based model

Var(Z) = $1/\alpha$	0.01	0.1	0.5	2	5	10	50	100
quantile of order 99%	1190	1209	1283	1508	1766	2015	2722	3026
E(losses | losses > q99%)	1256	1275	1364	1621	1897	2178	3003	3333
skewness	0.16	0.14	0.15	0.31	0.41	0.52	0.87	0.99
k kurtosis	2.99	2.98	2.96	2.89	2.72	2.63	2.87	3.16
average correlation (%)	0.01	0.09	0.44	1.68	3.82	6.72	18.24	24.17

becoming thinner and thinner when the time horizon is growing. That is why a straight comparison between Tables 7.8 and 7.11 particularly is not fully satisfying, because $\alpha = 1$ is probably too high is such a case.

NOTES

1. See Luciano (2004) for a discussion in the finance field. In a more general context, there is no issue to rewrite the marginal laws of the default times with intensities. At the opposite, it is more challenging to rewrite the full joint law of defaults because one needs to invoke multivariate hazard rates (Dabrowska, 1988; Fermanian, 1997). For example, a large number of intensities has to be modelized: $2^m - 1$ when m denotes the number of firms in the portfolio. In practice, such a number is unrealistic when dealing with more than 2 or 3 obligors.
2. Even if some firms or more generally some industries may be considered as negatively correlated with the "market", or rather with the vast majority of other corporates.
3. Since most of bank portfolios are composed mainly with investment grade debts, we overweight such firms.
4. The recovery rate is assumed to be zero here. This is not a limitation of our purpose. Indeed, in this chapter we do not try to study the internal source of randomness given by the exposure amounts.
5. Particularly Hull and White (2001), Schönbucher and Schubert (2003).
6. A random intensity model with constant levels between 0 and T and the same frailty for all obligors.
7. The relative sizes of the monthly default rates in Figure 7.3 are comparable with annual default rates because the former are trailed over 12 months.
8. The unobservable explanatory variables that are specific to i and that have not been taken into account previously in the vector X_i.
9. At least when we keep the balance between both Z_0 and Z_i.
10. For example, $\alpha = 2$ provides a Gaussian law and $\alpha = 1$, $\beta = 0$ provides a Cauchy distribution.
11. We are always dealing with the same portfolio from the beginning.
12. Paik *et al*. (1994) or Yue and Chan (1997), for instance.
13. because such λ_i are comparable with mean monthly default rates over a period T.

REFERENCES

Andersen, P.K., Gill, R., Borgan, O. and Keiding, N. (1997) *Statistical Models Based on Counting Processes* (New York: Springer).

Carr, P. and Wu, L. (2002) "The Finite Moment Log Stable Process and Option Pricing", Working Paper.

Chambers, J.M., Mallows, C.L. and Stuck, B.W. (1976) "A Method for Simulating Stable Random Variables", *Journal of the American Statistical Association*, 71(2): 340–4.

Clayton, D. and Cuzick, J. (1985) "Multivariate Generalizations of the Proportional Hazards Model", *Journal of the Royal Statistical Society*, *A*, 148(1): 82–117.

Crouhy, M., Galai, D. and Mark, R. (2000) "A Comparative Analysis of Current Credit Risk Models", *Journal of Banking and Finance*, 24(1–2): 59–117.

Dabrowska, D. (1988) "Kaplan–Meier on the Plane", *Annals of Statistics*, 16(4): 1475–89.

Das, S., Freed, L., Geng, G. and Kapadia, N. (2002) "Correlated Default Risk", Working Paper.

Duffie, D. and Singleton, K. (1999) "Modeling Term Structure of Defaultable Bonds", *Review of Financial Studies*, 12(4): 687–720.
Elizalde, A. (2003) "Credit Risk Models I: Default Correlation in Intensity Models", Working Paper CEMFI.
Fermanian, J.D. (1997) "Multivariate Hazard Rates Under Random Censorship", *Journal of Multivariate Analysis*, 62(2): 273–309.
Frey, R. and McNeil, A. (2001) Modelling Dependent Defaults. ETH E-collection.
Hougaard, P. (1986) "Survival Models for Heterogeneous Populations Derived from Stable Distributions", *Biometrika*, 73(2): 387–96.
Hougaard, P. (2000) *Analysis of Multivariate Survival Data* (New York, NY: Statistics for Biology and Health, Springer).
Hull, J. and White, A. (2001) "Valuing Credit Default Swaps II: Modeling Default Correlations", *Journal of Derivatives*, 8(3): 12–22.
Jarrow, R., Lando, D. and Turnbull, S. (1997) "A Markov Model for the Term Structure of Credit Risk Spreads", *Review of Financial Studies*, 10(3): 481–523.
Koyluoglu, H. and Hickman, A. (1998) "A Generalized Framework for Credit Risk Portfolio Models", Working Paper.
Luciano, E. (2004) "Credit Risk Assessment via Copulas: Association In-Variance and Risk-Neutrality", Working Paper, University of Turin, Italy.
Metayer, B. (2005) "Shared Frailty Model for Rating Transitions", Working Paper.
Mittnik, S. and Rachev, S.T. (1999) *Stable Models in Finance* (New York, NY: John Wiley & Sons).
Nolan, J.P. (2004) "Stable Distributions: Models for Heavy Tailed Data" Working Paper, American University, Washington, DC.
Nyfeler, M. (2000) "Modeling Dependencies in Credit Risk Management", Thesis, ETH, Zürich.
Paik, M.C., Tsai, W. and Ottman, R. (1994) "Multivariate Survival Analysis Using Piecewise Gamma Frailty", *Biometrics*, 50(4): 975–88.
Parner, E. (1998) "Asymptotic Theory for the Correlated Gamma-Frailty Model", *Annals of Statistics*, 26(1): 183–214.
Samorodnitsky, G. and Taqqu, M.S. (1994) *Stable Non-Gaussian Random Variables* (New York, NY: Chapman and Hall).
Schlögl, E. (2002) "Default Correlation Modeling", Working Paper, University of Technology Sydney, Australia.
Schönbucher, P.J. (2001) "Factor Models for Portfolio Credit Risk", Bonn Economics Discussion Paper, No. 16.
Schönbucher, P.J. (2003) *Credit Derivatives Pricing Models: Models, Pricing and Implementation* (New York, NY: John Wiley & Sons).
Schönbucher, P.J. and Schubert, D. (2001) "Copula-Dependent Default Risk in Intensity Models", Working Paper.
de Servigny, A. and Renault, O. (2002) "Default Correlation: Empirical Results", Working Paper, Standard and Poor's solutions.
Standard & Poor's (2003) Special Report. Rating performances 2002. Février 2003.
Turnbull, S. (2003) "Practical Issues in Modeling Default Dependence", Working Paper, Houston.
Yashin, A.I. and Iachine, I.A. (1995) "Genetic Analysis of Durations: Correlated Frailty Model Applied to Survival of Danish Twins", *Genetic Epidemiology*, 12(3): 529–38.
Yu, F. (2003) "Dependent Default in Intensity-based Models" Working Paper, University of California, Irvine.
Yue, H. and Chan, K.S. (1997) "A Dynamic Frailty Model for Multivariate Survival Data", *Biometrics*, 53(3): 785–93.