Quantitative Finance

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/rquf20

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Published online: 22 Jul 2013.

To cite this article: Jean-David Fermanian & Olivier Vigneron (2015) On break-even correlation: the way to price structured credit derivatives by replication, Quantitative Finance, 15:5, 829-840, DOI: 10.1080/14697688.2013.812233

To link to this article: http://dx.doi.org/10.1080/14697688.2013.812233

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On break-even correlation: the way to price structured credit derivatives by replication

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(Received 7 May 2012; in final form 29 May 2013)

We consider the pricing of European-style structured credit pay-off under the Gaussian Copula Model (GCM). When no sudden jump-to-default events occur, the perfect replication of these pay-offs under the GCM is obtained if and only if the underlying single-name credit spreads follow a particular family of dynamics and if the pricing parameters are given by so-called ‘break-even’ correlations. We exhibit a class of Merton-style models that are consistent with this result. We calculate break-even correlations explicitly to price nth-to-default baskets under the GCM. Finally, we illustrate the usefulness of this concept as a relative-value tool.

Keywords: Collateralized debt obligation; Dynamic hedging; Gaussian copula; Structural Merton models

JEL Classifications: G12, G13

1. Introduction

The risk management of structured credit products has become a key issue for many financial institutions. Even if their payoffs are only driven by the realization of default events, the values of these products are exposed to credit spread volatility. The recent credit crisis has highlighted the need for pricing models which can produce dynamic hedging strategies that would replicate the marked-to-market moves of these products.

The most common structured credit products are synthetic CDO tranches. A standard modelling approach to value CDO tranches is the Gaussian copula model (GCM), most often under its one-factor setting (1FGCM). This model specifies directly the joint law of the underlying default events (Li 2000), where the dependence between the underlying default times is measured through a correlation parameter. With the marking of correlation levels of base tranches $[0, K]$, $K \in [0, 100\%]$, the 1FGCM has become a market standard known as the ‘base correlation’ model (see O’Kane and Livasey 2004, for instance). In the structured credit market, it has been playing the same role as the Black–Scholes model for standard options.

For a long time, the Gaussian copula approach has been criticized for several reasons. First, the model does not define any credit spread or default intensity dynamics. It only takes as a set of inputs current credit spread curves, assumed constant recoveries and a correlation parameter, but no volatilities. Second, the common opinion is that CDO tranches are not priced through a replication argument, contrary to the Black–Scholes model. The GCM relies on a ‘largely ad-hoc’ pricing formula, based on an actuarial reasoning. Third, the model does not provide any way of finding the ‘right’ correlation parameter value, based on sound theoretical arguments. This induces delicate calibration issues, and increases model risk and the uncertainty around the available market quotes. Fourth, a single number (a correlation level) is not able to reflect conveniently the complexity of the dependence between dozens of assets or underlying risks. Therefore, after the collapse of the structured credit business in 2007, the GCM was even stronger blamed. Its main formula was called ‘the formula that killed Wall-Street’ (Salmon 2009).

Nonetheless, despite its atypical features and its lack of structure, the GCM remains the market standard for structured credit products. Li (2008) pointed out that ‘the current copula framework gains its popularity owing to its simplicity. However, there is little theoretical justification of the current framework from financial economics. We essentially have a credit portfolio model without solid credit portfolio theory’. In our opinion, the previous criticisms are partly due to a lack of understanding of the properties and limitations of the GCM. In this article, in a continuous spread model, we show that the pay-off of any arbitrary European basket credit derivative can be perfectly hedged under the GCM, but only under a particular family of credit spread dynamics. We explain to what extent and under which conditions such copula models can be seen

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as pricing by replication models, as in the standard Black–Scholes theory. As an important sub-product, we exhibit the ‘right’ correlation matrix that should be used to price such products under the multi-factor version of the GCM.

The replication of CDO-type cash flows has generated a significant amount of literature in the last years. Some authors have tried to write explicit hedging strategies for some particular top-down models (pure jump dynamics with contagion): Arnsdorf and Halperin (2008), Frey and Backhaus (2008, 2010), Herbertsson (2008) and Laurent et al. (2011), among others. There, the relevant hedging instruments are most often instantaneous CDS, that are virtual instruments, and/or credit indices. See the survey of Cousin et al. (2010).

Recently, Cousin and Jeanblanc (2011) have obtained general theoretical results when the underlying default events are ordered. None of these authors deals with the case of static factor models, particularly the market standard GCM. Others have studied empirically the performances of various delta-hedging strategies, including the current practice: Cont and Kan (2008), Ammann and Brommundt (2009), Cont et al. (2010), Cousin et al. (2012), etc. In this article, we choose an opposite point of view: we restrict ourselves to spread risk, but under standard market models and with individual CDS as hedging instruments.

In section 2, we specify the framework and the notations. We explain how the price of a credit structured product, as given by a GCM model, can become a martingale. Here, this property is equivalent to its perfect hedging with individual Credit-Default Swaps in continuous time. We introduce the concept of ‘break-even’ correlation level and matrix. These ‘right’ pricing parameters are written explicitly as some functions of spread volatilities and the correlations between spread moves. In section 3, we show the one-to-one correspondence between our spreads dynamics and a class of structural models. We calculate explicit hedging errors in the case of First p-th-to-default securities in section 4, p ≥ 1. Finally, we illustrate these results by showing the performances of break-even correlations as relative-value indicators.

2. Under the GCM, CDO cash-flows can be replicated

First, before stating our main results, we need some preliminaries and assumptions. To simplify, we will assume we are not exposed to any interest rate risk: interest rates are zero now and in the future. For every name i and every maturity T, let its survival probability be 

\[ Q_{it}(T) = Q(t_i > T | F_t), \]

where \( t_i \) denotes its default time. When there will be no ambiguity, such survival probabilities will be denoted by \( Q_{it} \), or even \( Q_i \). In the equation above, \( Q \) denotes a risk-neutral probability measure, since we are placing ourselves within the framework of a usual arbitrage-free financial market model, as in Bielecki and Rutkowski (2002) for instance. Due to the existence of T-maturity Credit-Default Swaps and assuming deterministic recovery rates, \( Q_{it} \), can be considered as the price of a traded asset. Thus, our market will be defined as \((Q_{it}(T), \ldots, Q_{it}(T))_{t \in [0,T]}\), endowed with its natural filtration \( \mathcal{F} := (\mathcal{F}_t)_{t \in [0,T]}, \mathcal{F}_t = \sigma(Q_{ia}(T), u \leq t, i = 1, \ldots, n). \)

Assumption (A): no default event will occur in the time interval \([0, T)\), but only at time \( T \) possibly.

Apparently, the latter assumption is a bit provocative. However, it does not preclude that the spread of any name becomes so wide that it trades close to default within that period. Also, for short-medium term horizons and for investment grade portfolios (like Master Credit indices), the assumption (A) may be acceptable. It is equivalent to considering that structured credit products, while in a trading book, are exposed to spread risks mainly. It could be stressed that, historically, most of the default events have been anticipated by the main market dealers some weeks before their official announcements, with the price of debt trading near its recovery value.

Assumption (Q): Under the historical measure \( P \), the survival probability process of name i up to maturity T follows the Ito process

\[ dQ_{it}(T) = \mu_i \cdot T dt + \sigma_{i,t} \cdot T dW_i, \]

where \( t_i \) is not defaulted, and \( Q_{it}(T) = 0 \) else. The volatility and drift processes above are \( F \)-adapted. The processes \((W_i)_{i \in [0,T]}\) are correlated \( F \)-Brownian motions under \( P: E_P[dW_i dW_j] = \rho_{ij} dt, i \neq j \).

For the moment, we do not assume anything concerning the processes \((\mu_i(t,T))_{t \in [0,T]}\) and \((\sigma_{i,t})_{t \in [0,T]}\), except measurability. In particular, we do not know whether some solutions \((Q_{it}(T))_{t \in [0,T]}\) of the PDE above exist and belong to \([0, 1]\). Actually, this will be the case below, but as a consequence of the particular dynamics, we will exhibit in Theorem 2.2: see equation (11). Then, under the conditions (A) and (Q) together, the individual survival probabilities will follow random continuous paths between \( t = 0 \) and \( t = T \), and \( \lim_{t \to T} Q_{it}(T) \in [0, 1] \) (default or survival at \( T \)).

We will call \( \rho_{ij} \) the ‘spread correlation’ between the names \( i \) and \( j \). Under (A) and (Q), we are living in a usual ‘Black–Scholes’ world when \( t < T \), because default events can occur at \( t = T \) only. As deduced from standard arguments, the risk neutral \( Q \) is given by the Girsanov’s theorem. Therefore, under \( Q \), the survival probabilities follow the processes \( dQ_{it}(T) = \sigma_{i,t} dW_i, \) when \( i \) is not defaulted, and \( Q_{it}(T) = 0 \) else. Here, the processes \((W_i)_{i \in [0,T]}\) are correlated \( F \)-Brownian motions under \( Q \), and \( E_Q[dW_i dW_j] = \rho_{ij} dt, i \neq j \). Since there will be no ambiguity, \( \sigma_{i,t,T} \) is denoted by \( \sigma_i \) simply. We denote \( \Sigma^q := [\rho_{ij}]_{i,j \leq n} \), where \( \rho_{ii} = 1 \).

Second, let us consider a credit basket product of \( n \) underlying names, \( n \geq 2 \). Its pay-off depends only on the realization of default events in the underlying basket up to its maturity \( T \). We will further assume that all the default payments are made at maturity and that the premium to be paid to enter this derivative contract is an upfront payment. Mathematically, we can write this pay-off using indicator functions:

\[ \text{Pay-off}(T) := \psi(\delta_1(T), \ldots, \delta_n(T)), \]

\[ \delta_i(T) = 1(t_i \leq T), i = 1, \ldots, n, \]  \( (1) \)

where \( \psi \) denotes an arbitrary function from \([0, 1]^n \) to \( \mathbb{R} \). This framework encompasses potentially all the European credit basket derivatives, through an Arrow-Debreu-type pay-off breakdown. For convenience, set \( \delta(t) := (\delta_1(t), \ldots, \delta_n(t)) \). The \( t \)-value of this structured product will be denoted by \( V_t \).

In our arbitrage-free financial market, the \( t \)-value of any attainable product should be the expected value of its future cash-flows under a risk-neutral measure \( Q \). In other words, \((V_t)_{t \in [0,T]}\) has to be a \( F \)-martingale under \( Q \) and we should be able to write
where the summation is w.r.t. all the n-vectors of default indicators.

Now, let us describe the way practitioners price such structured products: at any time t, conditionally on some unobserved random factor $X \in \mathbb{R}^p$ and the current market information, the underlying default events ($t_i \leq T$), $i = 1, \ldots, n$, are assumed to be independent. This factor $X$ has a density $f_X$ w.r.t. the Lebesgue measure in $\mathbb{R}^p$. Note that this 'systemic' factor $X$ depends on the current time $t$ implicitly, but its dynamics are not specified. Most often, practitioners specify one-factor models, i.e. $p = 1$.

In practice, the risk-neutral joint law of the default events ($t_i \leq T$), $i = 1, \ldots, n$, is defined by some intermediary quantities called 'conditional default probabilities' $p_{ij|X} = Q(t_i \leq T | X = x, F_t)$, that depend on $Q_{it}$, $t$ and $x$ only. Similarly, the 'conditional survival probabilities' will be $q_{ij|X} := 1 - p_{ij|X}$. It is proved in section 3 that $(p_{ij|X})_{i,j \in \{0,1\}}$ is a martingale under $Q$ and the GCM, but for a convenient extension of the filtration $\mathcal{F}$. There, $x$ will be the proper $t$ realization of an adapted process in this extended filtration.

Most of the pricing models of structured credit products that are used in practice (see for instance Bartsch et al. 2005; Laurent and Gregory 2005) invoke the conditional independence assumption to postulate that

$$V_{t,m} = \sum_{\delta(T)} \psi(\delta(T)) \int \prod_{j=1}^n p_{ij|X}^{\delta_i} f_X(x) \, dx$$

is the $t$-price of our structured product. Such a pricing formula is then specified entirely by the analytical forms of $p_{ij|X}$ and $f_X$. Note that $V_{t,m} = V_{t,m}(Q_{1t}, \ldots, Q_{nt})$, and $V_{t,m}$ does not depend on time or remaining maturity, except through the survival probabilities themselves. Equivalently, there is no 'theta' effect.

The key issue we are interested in is the following one: under which conditions is an 'ad-hoc' pricing formula (3) a $\mathcal{F}$-martingale ? In other words, can we write $V_{t,m} = E_Q \left[ \psi(\delta(T)) \right]$, as it is valid for any attainable claim?

**Third**, to answer such a question, we restrict ourselves to the market standard, the GCM.

Assumption (Cop): to price any European structured product, the framework is the $p$-factor GCM, for some $p \in \{1, \ldots, n\}$.

In this case, there exist vectors $\rho_i \in \mathbb{R}^p$, $i = 1, \ldots, n$, such that the 'conditional default probabilities' are

$$p_{ij|X} = Q(t_i \leq T | X = x, F_t) = \Phi \left( \Phi^{-1}(Q_{it}) - \rho_i^T x \right) : = \Phi(d_i),$$

for all $i = 1, \ldots, n$ and $x \in \mathbb{R}^p$. In the equation above, $|\rho_i|$ denotes the Euclidian norm of the $p$-vector $\rho_i$. As usual, $\Phi$ and $\Phi$ denote respectively the density, the cdf and the survival function of a standard normal r.v. Obviously, the probabilities of joint default events are obtained by an integration w.r.t. the law of the factor $X$, whose density is $f_X : \mathbb{R}^p \to \mathbb{R}^+$, $x \mapsto \prod_{j=1}^n \phi(x_j) := \phi(x)$. In this framework, the pricing parameters are the vectors $\rho_i \in \mathbb{R}^p$, $i = 1, \ldots, n$. They allow to build a correlation matrix $\Sigma$ of rank $p$, for pricing purpose. It is defined by $\Sigma = [\Sigma_{ij}], \Sigma_{ij} = \rho_i^T \rho_j$, $i \neq j$ and $\Sigma_{ii} = 1$.

By applying Ito’s formula and by a smart integration by part, we obtain:

**Lemma 2.1 Under (A) (Q) (Cop),**

$$dV_{t,m} = (\cdots) \cdot d\tilde{W}_{t} + \frac{dt}{2} \sum_{\delta(T)} \psi(\delta(T))$$

$$\cdot \sum_{i < j} \int \prod_{k \neq i,j} p_{ik|X}^{\delta_k} f_X(x) \, dx \cdot \frac{\phi(d_i) \phi(d_j)}{\sqrt{1 - |\rho_i|^2 \sqrt{1 - |\rho_j|^2}} \prod_{l=1}^p \phi(x_l) \, dx_l},$$

where

$$\rho_{it} := \frac{\sigma_{it}}{\phi(\Phi^{-1}(Q_{it}))}.$$  (5)

See the proof in Appendix A. Therefore, as long as there are no default events, $(V_t)$ is a $\mathcal{F}$-martingale for every pay-off $\psi$ if and only if, for all $t$, $i, j = 1, \ldots, n$.

$$2\beta_{ij} \rho_{ij} - \rho_{i}^T \rho_{j} \beta_{ij}^2 - \rho_{i}^T \rho_{j} \beta_{ij}^2 = 0, \quad i \neq j, \quad j = i, \ldots, n. \quad (6)$$

These key identities will induce some constraints on the dynamics that can be followed by the survival probabilities $Q_{it}$.

**Fourth**, before stating the main result, we need to define the names in the basket that have really an influence on the realized pay-offs of the structured product.

Assumption (P): for every name $i$ in the basket, there exist another index $j \neq i$ and some indicators $\delta^*_k$, $k \neq i, j$, such that

$$\psi \left( 1, 1, \delta^*_k, 0, 0 \right) = \psi \left( 0, 1, \delta^*_k, 0, 0 \right) \neq 0.$$  (7)

In the equation above, the first (resp. second) index is related to name $i$ (resp. $j$), and $\delta^*_k$ is the $n - 2$-vector of indicators $\delta_{ik}, k \neq i, j$.

The equation (7) means: the knowledge of i’s default is informative to evaluate the change of the pay-off induced by j’s behaviour, under some scenario of other default events. This technical condition is satisfied in practice by all the usual credit derivatives, particularly First-To-Default securities.

Assumption (B): for every name $i$ in the basket, the function $Q_{it} \mapsto \sigma_{it}(Q_{it})/\phi(\Phi^{-1}(Q_{it}))$ from $0, 1$ to $\mathbb{R}^+$ is bounded from above.

The latter technical assumption avoids some pathological behaviours, when survival probabilities become close to zero or one.

**Now**, we can state a striking result: we are able to exhibit the single dynamics of $Q_{it}$, that allow perfect replication in our framework. Moreover, we exhibit the ‘right’ values of pricing parameters, i.e. the values under which $(V_{t,m})$ becomes a $\mathcal{F}$-martingale under $Q$.

**Theorem 2.2 Under the assumptions (A) (Q) (Cop) (P) (B), consider a European credit basket derivative. Its 'model price' process $(V_{t,m})_{t \in [0,T]}$ is a $\mathcal{F}$-martingale under $Q$ if and only if:**
\( \text{(i) The dynamics of survival probabilities under } Q \text{ are given by} \)
\[
dQ_{it} = \tilde{\sigma}_i \phi(\Phi^{-1}(Q_{it}))dW_{it}, \quad (8)
\]
\( \text{for every index } i. \text{ Above, } \tilde{\sigma}_i \text{ denotes a non-negative constant and } \xi \text{ some deterministic positive function such that } \lim_{t \to T} \int_0^t \tilde{\sigma}_i^2 du = +\infty. \)
\( \text{Indeed, they often} \)
\[
\rho_{ij} = \exp \left[ \frac{\tilde{\sigma}_i^2 + \tilde{\sigma}_j^2}{2\tilde{\sigma}_i \tilde{\sigma}_j} \right] (\rho_i^S)^{\prime} \rho_j^S, \quad \forall \ i \neq j. \quad (9)
\]
\( \text{(iii) For every index } i, \text{ the correlations that are used for the pricing of this structured product are given by the relations } \rho_i = \rho_i^S, i = 1, \ldots, n. \)

Therefore, a CDO tranche can be perfectly hedged with individual CDS continuously, by pricing and calculating hedge ratios under the GCM, but if and only if the spread dynamics are given by (8) and if the pricing parameters \( \rho_i \) are chosen as above. Indeed, in our framework, perfect hedging is equivalent to the martingale property above.

A \( p \)-factor correlation matrix \( \Sigma = [\Sigma_{ij}] = [\rho_i^S \rho_j] \), that is used for pricing purpose, and that induces a flat P&L in a given time interval and for a realized trajectory of individual credit spreads is called a ‘break-even’ correlation matrix. When its value is independent of the (random) credit spread trajectories, it will be called a ‘universal break-even’ correlation matrix.

The existence of a ‘universal break-even’ matrix \( \Sigma \) is not guaranteed for arbitrary spread correlations \( \rho_{ij} \) and arbitrary volatilities \( \tilde{\sigma}_{ij} \) even under (8). To be specific, the matrix \( \Sigma^S = \left[ \frac{2\tilde{\sigma}_i \tilde{\sigma}_j}{\tilde{\sigma}_i^2 + \tilde{\sigma}_j^2} \right] \rho_{ij} \) is always a correlation matrix (see section 3). When it comes from a \( p \)-factor correlation structure, then the Gaussian copula pricing model should be \( p \)-factor too. And the corresponding vectors \( \rho_i \), given by a factor-decomposition of \( \Sigma^S \), generate a universal break-even correlation matrix.

This is clearly in contrast with the current practice, where practitioners use only the 1FGCM by far. Indeed, they often assume flat correlation structures, where a single-number \( \rho \) summaries dependencies: \( p = 1 \) and \( \rho_i = \rho \) for all \( i \). In this case, there exists a ‘universal break-even’ correlation if and only if the credit spread correlations \( \rho_{ij} \) are proportional to \( \tilde{\sigma}_j / \tilde{\sigma}_i + \tilde{\sigma}_i / \tilde{\sigma}_j \), implying a one- or two-factor correlation structure between spread moves.

A break-even correlation is comparable to a base correlation, i.e. the square of a ‘beta-factor’ \( \rho \). Invoking Lemma 2.1, the univariate break-even correlation \( \rho_{BE,i}^2 \) in the infinitesimal time interval \([t, t + dt]\) is a linear combination of the pairwise spread correlations: \( \rho_{BE,i}^2 = \sum_{j \neq i} \rho_{ij} \tilde{\sigma}_j w_{ij}, \) where the weights \( w_{ij} \) involve individual survival probabilities at \( t \), instantaneous volatilities \( \tilde{\sigma}_i \) and the pay-off functional. Actually, the break-even correlation itself is hidden in \( w_{ij} \) through \( d_i \) and \( d_j \). This means we have to solve an implicit function to calculate break-even correlations. Note that the weights \( w_{ij} \) could be negative because of the default indicator functions \( \delta_i \). Thus, the existence of \( \rho_{BE,i}^2 \) is not guaranteed. For a given derivative, the only way to be sure that there always exists a break-even correlation is to satisfy (6). If it can be done, we find a universal break-even correlation.

Note that the pricing formula in equation (3) did not depend on the volatilities of the default intensities and/or probabilities explicitly, due to the static nature of the underlying reasoning, and contrary to the Black–Scholes model. It does not mean that the volatilities \( \sigma_{it} \) do not matter to calculate the price \( V_{t,m} \).

Indeed, we have seen in the theorem above that they influence the choice of the ‘right’ pricing parameters \( \rho_i \).

Nicely, equation (8) can be solved explicitly. Indeed, it can be checked that the solution of the latter equation is
\[
Q_{it} = \Phi \left( \frac{\tilde{\sigma}_i^2 \int_0^t \tilde{\sigma}_j^2 ds}{2} \right) \Phi^{-1}(Q_{it}) + \int_0^t \exp \left( \frac{\tilde{\sigma}_i^2 \int_0^t \tilde{\sigma}_j^2 ds}{2} \right) \tilde{\sigma}_{it} dW_{it}, \quad (10)
\]
where \( (W_{it}) \) is a Brownian motion under \( Q \). Under \( P \), the PDE followed by \( (Q_{it}(T)) \) is unchanged, except through an additional drift \( \mu_{it} \) (Assumption (Q)). Similarly, we check that
\[
Q_{it} = \Phi \left( \frac{\tilde{\sigma}_i^2 \int_0^t \tilde{\sigma}_j^2 ds}{2} \right) \Phi^{-1}(Q_{it}) + \int_0^t \exp \left( \frac{\tilde{\sigma}_i^2 \int_0^t \tilde{\sigma}_j^2 ds}{2} \right) \tilde{\sigma}_{it} dW_{it} + \frac{\mu_{it}}{\tilde{\sigma}_i^2} \Phi^{-1}(Q_{it}) du, \quad (11)
\]
where \( (W_{it}) \) is a Brownian motion under \( P \). We can assume that \( (\mu_{it}) \) is an \( \mathcal{F} \)-adapted process such that
\[
\int_0^T \exp \left( -\frac{\tilde{\sigma}_i^2 \int_0^u \tilde{\sigma}_j^2 ds}{2} \right) \frac{|\mu_{it}|}{\tilde{\sigma}_i^2} \Phi^{-1}(Q_{it}) du < \infty, \ a.e. \quad (12)
\]
This insures that the integrals in (11) exist a.e. Note that the consistency condition \( \lim_{t \to T} Q_{it}(T) \in [0, 1] \) for every trajectory is satisfied because \( \lim_{t \to T} \int_0^t \tilde{\sigma}_i^2 du = +\infty. \)

3. Consistency with structural models

It is fruitful to link the copula framework to structural models, following the seminal paper of Merton (1974). Classically, they are defined in terms of asset value processes \( (A_{it}) \) that follow some particular dynamics in continuous time. The default time of a name \( i \) is defined as a functional of \( i \)’s asset value trajectory. The simplest case is given by the usual one-period Merton model, where
\[
(\tau_i > T) := (A_{iT} \leq b_{iT}), \quad (13)
\]
with deterministic boundaries \( b_{iT} \) that are calibrated to be consistent with the current observed default probabilities. Since the knowledge of \( (A_{iT})_{0 \leq t \leq T} \) is equivalent to the knowledge of \( i \)’s default likelihoods, our asset values depend on the same drivers as the survival probabilities \( Q_{it} \). Typically, it is assumed that the (risk-neutral) asset values follow some Ito processes
\[
dA_{it} = \sigma(t, A_{it}) dW_{it} + \nu(t, A_{it}) dt, \quad (14)
\]
where \( (W_{it}) \) are Brownian motions under \( Q \). For some classical families of diffusion processes (geometric Brownian motion, Ornstein–Uhlenbeck process, e.g.), the survival events can be rewritten
\[
(A_{iT} \leq b_{iT}) = \left( \int_0^T \psi_i(u) dW_{it} \leq \bar{b}_{iT} \right), \quad (14)
\]
for some deterministic functions $\psi_i$ and other default boundaries $b_iT$. In this section, we assume we can write (14). We deduce the survival probabilities that are implied by this structural approach:

$$Q_{it}(T) = Q(A_{it} \leq b_iT | F_{it}) = \Phi \left( \frac{b_iT - f_0^T \psi_i(u)du}{\sqrt{f_0^T \psi_i^2}} \right).$$

(15)

Now, in this Merton framework, assume that the correlation structure among the underlying names is one-factor:

$$W_{it} = \theta_i W_t + \sqrt{1 - \theta_i^2} W_{it}^*, \quad i = 1, \ldots, n,$$

(16)

with independent Brownian motions $(W_{it}^*)$ under $Q$, and a common $Q$-Brownian motion $(W_t)$. Thus, the default events $(\tau_i \leq T)$ are independent given the Gaussian r.v. $Z_i := f_0^T \psi_i(u) du$, $i = 1, \ldots, n$. By defining our default events through (14), we get back the GCM, but generally in a multifactor version.

Let us detail this mapping between structural models and the framework introduced in section 2. Obviously, at time $t \leq T$, the available information is the current survival probabilities $Q_{it}(T)$, or equivalently the current asset values, or even the $f_0^T \psi_i(u) du$, $i = 1, \ldots, n$. Thus, $F_{it} = \sigma \left( f_0^T \psi_i(u) du, u \in t \right), i = 1, \ldots, n; v \leq t$. Moreover, the common factor at time $t$ (the ‘famous’ $X$ in section 2) is now a summary of the trajectory of the common Brownian motion $(W_t)$ between $t$ and $T$. To be specific, $X$ is reduced to the knowledge of the random variables $f_0^T \psi_i(u) du, i = 1, \ldots, n$. For instance, in the standard IFGCM, the common factor $X$ at time $t$ is univariate and is simply $W_t - W_{it}$, the increment of the common Brownian motion between $t$ and $T$. In this case, $\psi_i = 1$ for every $i$.

Actually, the previous quantities $p_{ij}$ are really martingales in a convenient filtration, even if they have been defined in an ad-hoc way. Let us introduce the filtration $\mathcal{G} = (G_t)_{t \in [0, T]}$, $G_t := \sigma(f_0^t \psi_i(u) du, i = 1, \ldots, n; v \leq t)$, and the extended filtration $\mathcal{H} = (F_t \vee G_t)_{t \in [0, T]}$. We prove in Appendix C:

**Theorem 3.1** In the IFGCM, the conditional probabilities $(p_{ij}|W_{it} - W_t)_{i \in [0, T]}$, as given by equation (4), are $\mathcal{H}$-martingales under $Q$.

Now, to continue the analogy, we are interested in the $Q_{it}$-dynamics that are implied by the previous Merton model, and in comparing them with those that have been obtained in Theorem 2.2, through fully different arguments. We would like to see whether one of these families of dynamics encompasses the other, or whether they have a non-empty intersection.

From (15), we deduce the dynamics of the Merton-based survival probabilities under $Q$:

$$dQ_{it} = -\beta_{it} \Phi^{-1}(Q_{it})dW_{it}, \quad \beta_{it} := \frac{\psi_i(t)}{\sqrt{f_0^T \psi_i^2}}.$$ (17)

Note that $\beta_{it}$ is not random. Therefore, when $t_0 < t$, we get easily

$$\int_{t_0}^t \beta_{ia}^2 du = \ln \left( \int_{t_0}^T \psi_i^2 \right) - \ln \left( \int_{t_0}^T \psi_i^2 \right).$$

(18)

that tends to $+\infty$ when $t$ tends to $T$. Actually, there is a one-to-one mapping between the deterministic functions $\beta_i$ and $\psi_i,

$$\psi_i(t) := C\beta_i \exp \left( -\int_0^t \frac{\beta_i^2}{2} du \right),$$

(19)

for some arbitrary positive constant $C$. Therefore, if the latter functions are of the type $\beta_i = \delta_i \xi_i$, or equivalently

$$\psi_i(t) := C_i \xi_i \exp \left( -\frac{\delta_i^2}{2} \int_0^t \frac{\xi_i^2}{2} du \right),$$

(20)

then there is identity between the spread dynamics induced by the Merton approach (see (14) and (17)) and those obtained in Theorem 2.2 by reverse engineering the GCM.

For example, if the underlying asset values follow Brownian motions, then $A_{it} := W_{it}$ and, with the previous notations, $\psi_i(u) = 1$ and $b_iT = \sqrt{T} \Phi^{-1}(Q_{it}(T))$. This is the most usual specification of a one-period Merton model. Then, in this case, $\beta_i = \xi_i = 1/\sqrt{T - t}$ for all the names. With this very simple specification, the spread dynamics do not involve any firm-specific ‘volatility-type’ coefficient $\delta_i; \delta_i = 1$ for all $i$.

Actually, the latter couples of functions $(\beta_i, \psi_i)$ are not the single ones that generate the dynamics obtained in Theorem 2.2. For instance, we could propose the model

$$(\beta_{it}, \psi_{it}) := \left( \frac{\sigma_i}{\sqrt{T - t}}, \psi_i(t) \left( \frac{1}{T} - \frac{t}{T} \right) \right).$$

(21)

Now, an heterogeneity in terms of spread dynamics appears through different coefficients $\sigma_i$. Therefore, to get heterogeneity (or ‘volatility’) effects, we can define the underlying survival events by

$$(\tau_i > T) := \left( \int_0^T (T - u)^{\delta_i^2/2} du \right) \leq \beta_iT.$$

(22)

Nonetheless, to price a basket in this model, it is necessary to integrate the conditional default probabilities with respect to several factors.

In the specification (21), it would be nice to recover the results of section 2. With the notations of the latter section, the correlation $\rho_{ij}$ between the survival probabilities (or the ‘spread’ moves rather) of $i$ and $j$ is the correlation between the increments of the previous Brownian motions $(W_{it})$ and $(W_{jt})$. Moreover, the correlations between the pricing factors were $\rho_i \neq \rho_j, i \neq j$. Since the asset values at maturity are given by $A_{ij} = \int_0^T \psi_i du$, $i, j = 1, \ldots, n$, by definition of the vectors $\psi_i$, we should satisfy $\rho_i \rho_j = \text{Corr}(A_{ij}, A_{jT}).$ But, simple calculations provide

$$\text{Corr}(A_{ij}, A_{jT}) = \frac{\rho_{ij} \int_0^T \psi_i \psi_j \rho_{ij}}{\sqrt{\int_0^T \psi_i^2} \sqrt{\int_0^T \psi_j^2}} = \frac{\rho_{ij} \psi_i \psi_j}{\sigma_i^2 + \sigma_j^2}.$$ (23)

Thus, we recover the key relation of Theorem 2.2:

$$\rho_i \rho_j = \frac{\rho_{ij} \psi_i \psi_j}{\sigma_i^2 + \sigma_j^2}.$$ (24)

Note that, through the equation (23), we have proved that, for an arbitrary correlation matrix $[\rho_{ij}]$, every symmetrical matrix of the type $\left[ 2\sigma_i \sigma_j \rho_{ij} / (\sigma_i^2 + \sigma_j^2) \right]$ is a correlation matrix. This point was far from obvious.
Another model could be to set

\[
(p_{it}^{(a)}) \cdot \psi_{it} := \left( \frac{\sigma_i}{(1 - t/T)^{\alpha}} \cdot (1 - t/T) \right) \cdot \exp \left( -\frac{\sigma_i^2 T}{2(2\alpha - 1)} \left( (1 - t/T)^{1-2\alpha} - 1 \right) \right),
\]

for some constants \( \alpha < 0.5, \sigma_i > 0 \) and \( \psi_i > 0 \). Under this Merton-style specification, the asset value processes are now different from usual Brownian motions, but follow the PDE (no drift, for instance) \( dA_{ij} = \psi_{it}^{(a)} dW_{it} \). Even in this case, we are still able to exhibit some spread dynamics that allow perfect replication of CDO tranche. Through Theorem 2.2 and under (Cop), these are

\[
dQ_{it} = p_{it}^{(a)} \cdot \psi_{it}^{(a)} \cdot \phi^{-1}(Q_{it}) dW_{it}.
\]

Simple calculations show that, with the latter specification, we recover the usual relation (24).

To summarize this analysis, we have found that some one-period Merton-style model specifications provide the same spread dynamics and the same pricing formulas those obtained in Theorem 2.2, to value CDOs under (Cop).

We have restricted our analysis to default events that are induced by structural models but through the relation (12). Alternatively, we could define a default event as the first hitting time of a boundary, i.e. \( (t_i > T) = (A_{ij} < b_i T, \forall i \in [0, T]) \). An open question is to study such alternative structural models, and to exhibit the spread dynamics that would be consistent with a 'perfect' replication. Since multivariate extensions of First-hitting time models are very complicated in analytic terms (Zhou 2001, Hull et al. 2010), this avenue is left for further research.

4. Analysis of a First \( p \)-th-to-default

It is useful to exhibit explicit formulas of break-even correlations directly, by relying on Theorem 2.2. This can be done in a few cases, particularly First \( p \)-th-to-default options. This family of securities is important because every structured credit product can be replicated as a linear combination of First-To-Default options, that are built on some sub-portfolio (Brausch 2006). Therefore, First \( p \)-th-to-default options may be seen as the basic building blocks of every European structured credit product.

The simplest basket credit derivative is a First-To-Default, where its pay-off is non-zero, except when all the names have survived until the maturity date \( T \). By assuming that all the names share the same recovery rate \( R \) and with our previous notations, it means

\[
\psi_{F_{itD}}(\delta) = (1 - R) \cdot \left[ 1 - \prod_{j=1}^n (1 - \delta_j) \right].
\]

Invoking Lemma 2.1, between \( t \) and \( t + dt \) and under the one factor copula model, the drift of an infinitesimal variation of a First-To-Default value is

\[
\left( \frac{(1 - R)dt}{2(1 - \rho_{ij}^2)} \sum_{i,j \leq j} \left[ (\beta_{ii}^2 + \beta_{jj}^2) \rho_{ij}^2 - 2\beta_{ii} \beta_{jj} \rho_{ij} \right] \right) \cdot \int \prod_{k \neq i,j} q_{ik}(x) \cdot \phi(d_i) \phi(d_j) \phi(x) \ dx,
\]

if no default occurs in this time interval. More generally, it is possible to find a similar formula with First \( p \)-th-to-default, \( p = 1, \ldots, n \). The default leg of such securities is the amount of loss in the portfolio, but capped at the \( p \)-th default. In other words, the pay-off of a First \( p \)-th-to-default is \( \psi_p = \min \left( \sum_{i=1}^p \delta_i, p \right) \cdot (1 - R) \).

**Theorem 4.1** Under the assumptions (A) (Q) (Cop) (Q), the value of a First \( p \)-th-to-default will change between \( t \) and \( t + dt \) by the amount

\[
dV_{t,m}^{(p)} (\cdots) \cdot dW_t + \left( \frac{1 - R}{2} \right) dt \cdot \sum_{i < j} \left[ \rho_{ij}^2 \beta_{ii}^2 + \rho_{ij}^2 \beta_{jj}^2 - 2 \beta_{ii} \beta_{jj} \rho_{ij} \right] \cdot \beta_{ij}.
\]

where

\[
\beta_{ij} := \frac{\sigma_{ij}}{\phi^{-1}(Q_{it})}, \quad \beta_{ij}^* := \sum_{\delta_i = p - 1} \sum_{\delta_j = 0} A_{ij}(\delta), \quad \text{and}
\]

\[
A_{ij}(\delta) := \int \prod_{k \neq i,j} \rho_{ik}^{|\delta_i| - \delta_i} \cdot \phi(d_i) \phi(d_j) \phi(x) \ dx.
\]

See the proof in Appendix D. Check that we recover equation (28) when \( p = 1 \). The (instantaneous) break-even correlation of a First \( p \)-th-to-default is the flat correlation level that cancels the drift of \( dV_{t,m}^{(p)} \). In the case of identical spread correlations \( \rho_{ij}^2 \) and identical pricing correlations, the break-even correlation level lies between zero and \( \rho_{ij}^2 \). This result is true for arbitrary spread volatilities and initial credit spread levels.

At first glance, Theorem 4.1 allows to write the break-even correlation as a linear combination of spread correlations, with positive coefficients. Indeed, since all the quantities \( \beta_{ij}^* \) above are positive, we can rewrite

\[
\rho_{BE}^2 := \frac{2 \sum_{i < j} \beta_{ij} \beta_{ii} \beta_{jj} \rho_{ij}}{\sum_{i < j} \beta_{ij}^* \beta_{ii}^* \rho_{ij}} := \sum_{i \neq j} w_{ij} \beta_{ij},
\]

where all the \( w_{ij} \) belong to \([0, 1]\). Nonetheless, the previous coefficients \( w_{ij} \) depend on \( \rho_{BE} \) itself, through the so-called \( d_i \) terms. Moreover, \( w_{ij} \) depends on the current values of the survival probabilities \( Q_{it} \) and \( Q_{jt} \). Thus, the calculation of the break-even correlation of a First \( p \)-th-to-default involves solving an implicit equation.

If all the \( Q_{it} \) are equal, then the previous terms \( \beta_{ij}^* \) are equal too. Therefore, in this case, the (instantaneous) break-even correlation is given by

\[
\rho_{BE}^2 = \frac{1}{\sum_{i < j} (\beta_{ii}^* + \beta_{jj}^*)} \cdot \sum_{i < j} \beta_{ii} \beta_{jj} \rho_{ij}.
\]

Note that this equation is always true formally when there are only two names in our basket. In (31), the break-even correlation depends explicitly on the empirical correlations \( \rho_{ij} \) and on some 'volatility-type' coefficients \( \beta_{ij} \). If all the spread volatilities and all the survival probabilities are the same, then \( \rho_{BE}^2 = 2 \sum_{i < j} \beta_{ij} / [n(n - 1)] \).
5. Empirical analysis of break-even correlations

In practice, it is difficult to evaluate the theoretical break-even correlations of synthetic CDO tranches, as given by Theorem 2.2. Indeed, in real life, liquid and reliable quotes are available for the standard tranches of the main indices iTraxx and CDX only. Moreover, to price/hedge these tranches cleanly, we move away from our idealized framework. Particularly, we have to manage the term structure of the underlying default events, heterogenous recovery rates and non-zero interest rates. Moreover, the CDO prices integrate likely additional risk premia due to the likelihood of sudden (unanticipated) default events.

Nonetheless, there are two ways of applying our results empirically. The first one is classical in asset pricing: after (unanticipated) default events, likely additional risk premia due to the likelihood of sudden non-zero interest rates. Moreover, the CDO prices integrate underlying default events, heterogeneous recovery rates and clean. We move away from our idealized framework.

Available for the standard tranches of the main indices iTraxx rem2.2. Indeed, in real life, liquid and reliable quotes are correlated with synthetic CDO tranches, as given by Theo-

In practice, it is difficult to evaluate the theoretical break-even concept for relative-value analysis. Every day, we can ‘delta-hedge’ a CDO tranche with individual Credit-Default Swaps. Hedge ratios are calculated under the IFGCM model, thus with a single correlation number. Instead of estimating the model price of a CDO tranche, we can find implicitly its ‘break-even’ correlations for some given periods of time, i.e. the flat correlation levels that induce no P&L moves. This quantity cannot be calculated analytically, except in a few homogenous cases (see section 4). In this section, we choose the second approach, because pricing with non-flat correlation structures is a lot more demanding in practice. That is why traders use general correlation matrices rarely, but rather one-factor matrices at most. In other words, we favour market practice in this experiment.

We have calculated historical series of ‘empirical’ break-even correlations on the standard credit index iTraxx S8 Master 5Y, between its launch in September 2007 and June 2009. In practice, we have calculated weekly P&Ls that are generated by continuously delta-hedging a base tranche [0, x%]. The hedging instruments are usual CDS on every name of the basket, and every relevant maturity. The CDO tranche calculation is led for a grid of constant correlation levels, from 0% to 95%. We calculate P&L increments with a rolling window of six weeks. For every window, the ‘empirical’ break-even correlation is defined as the level of flat correlation that induces a zero P&L variation. Moreover, to simplify, we assume that no interest is generated in the cash account, and that the running premia of all the instruments (CDS and tranches) are zero. Thus, they are managed fully with upfront payments.

For the sake of comparison, on our graphs, we have shown the break-even correlations and the associated base correlations, as deduced from the market and calculated by the BNP-Paribas analytics: see figures D1 and D2. It can be observed that the two series move consistently, even if the break-even series generate more volatile moves in general. But in average, the market base correlations are not so different than the from break-even ones, as expected.

A same experiment with CDX has provided similar results globally (the results can be given under request), except for senior CDX tranches. It may be possible that investors in the latter tranches require an additional risk premium, or, in other words, overestimated systemic risk. Actually, the CDX experiment is more challenging than the iTraxx one, because some components of the CDX have defaulted during this period of time (Fannie Mae, Freddy Mac, Washington Mutual), and these events have not been fully anticipated by the market.

In this practical experiment, several features differ from the theoretical framework: the CDO tranches are not European; the recovery rates of some names differ; at the end of 2007, it has sometimes been difficult to find implied levels of base correlations from market quotes. Nonetheless, especially for iTraxx, it appears that the concept of break-even correlation can be a very useful relative-value tool. Indeed, theoretically, its moves should be in line with those of the market base correlations. It is what we observe empirically, broadly speaking. Thus, significant and persistent gaps between both types of correlations should help to forecast future moves of base correlations, or to indicate trade opportunities. Indeed, both series should converge towards each other, or at least should move similarly.

6. Conclusion

We have shown under which conditions the GCM, apparently 'static', can be seen and used as a replication model. In particular, we have investigated the one-factor GCM, the market standard. We have exhibited the unique family of dynamics (written in terms of survival probabilities), that is consistent with a pricing by replication. We have shown that the matrix of spread correlations $\rho_{ij}$ and some spread 'volatility-type' coefficients $\sigma_i$ should be combined in a particular way to get break-even correlations. Broadly speaking, the matrix $[2\sigma_i^2 + \sigma_j^2]$ should be definite positive. When it is the case, this matrix is our break-even correlation matrix and should be used for pricing and hedging purpose. We have shown the one-to-one correspondence between our spreads dynamics and a class of structural models. We have calculated explicit hedging errors for all First $p$-th-to-default securities. Finally, we have illustrated these results by showing the performances of break-even correlations as relative-value indicators.

Inside the standard family of ‘static’ models, for which the pricing formulas are given by (3), is it possible to exhibit alternative model specifications that would share the same properties? In other words, besides the GCM, can we get other price martingales by defining the quantities $p_{ij}^{(x)}$ above conveniently? Surprisingly, it is absolutely not easy. Particularly, it can be proved that no Archimedean copula model can be seen as a replication model. Thus, the difficulty in finding alternative models can be seen as an argument in favour of the GCM. Therefore, this specification has some nice analytical properties, that cannot be found elsewhere easily. This may explain the GCM ‘robustness’ in market practices, despite the current crisis. The latter point should be stressed, when a lot of people criticize the GCM (see Lipton and Rennie 2008, for instance).

Due to its simplicity, our framework is not fully in line with the specification of true CDO tranches. In particular, we do not deal with random recoveries (Amraoui and Hitier 2008, Krekel 2008, Amraoui et al. 2012) that are more and more
frequent. Moreover, it would be necessary to model jointly all the $Q_{ij}(T)$-dynamics for different maturities $T$, in order to get arbitrage-free specification of term structures. Indeed, our analysis has been led with a single maturity $T$. With two maturities $T < T'$, it is not possible to satisfy the arbitrage relation $Q_{ii}(T) \geq Q_{ij}(T')$, for all times $t < T$, under our spread dynamics. To find other model specifications, similar to first-hitting time models for instance, is the main challenge to tackle the multi-period formulation of the problem.

Acknowledgements

The authors thank their colleagues and former colleagues of BNP-Paribas and JP-Morgan for their help and fruitful discussions. Bruno Bouchard, Stéphane Crépey, Peter Jäckel, Monique Jeanblanc, Jean-Paul Laurent, Dong Li and Marek Rutkowski have provided highly valuable comments.

References


Appendix A: Proof of Lemma 2.1

By the conditional independence property, the price of our structured product is

$$V_{t,m} = \sum_{l \in \{1, \ldots, n\}} \psi(\delta(T)) \int \prod_{j=1}^{n} \delta_{j} p_{j|x_{j}} f(x) \, dx \, (A1)$$

where our notations have.

$$\frac{\partial^{2} V_{t,m}}{\partial Q_{i}^{2}} = \sum_{l \in \{1, \ldots, n\}} \psi(\delta(T)) \int \prod_{j \neq i} \delta_{j} \frac{\partial p_{j|x_{j}}}{\partial Q_{i}} f(x) \, dx \, (A2)$$

and, for every couple $(i, j), i \neq j,$

$$\frac{\partial^{2} V_{t,m}}{\partial Q_{i} \partial Q_{j}} = \sum_{l \in \{1, \ldots, n\}} \psi(\delta(T)) \int \prod_{k \neq i, j} \delta_{k} \frac{\partial p_{k|x_{k}}}{\partial Q_{i}} \frac{\partial p_{j|x_{j}}}{\partial Q_{j}} f(x) \, dx \, (A3)$$

Then, by applying Ito’s formula, we get

$$dV_{t,m} = (\ldots) \cdot d\tilde{W}_{t} + \frac{1}{2} \sum_{i} \frac{\partial^{2} V_{t,m}}{\partial Q_{i}^{2}} \sigma_{i} \sigma_{j} \rho_{ij} \, dt$$

$$= (\ldots) \cdot d\tilde{W}_{t} + \sum_{i} \psi(\delta(T)) \int dt + \sum_{i} \delta_{i} \frac{\partial p_{i|x_{i}}}{\partial Q_{i}} f(x) \, dx \, (A4)$$

$$= (\ldots) \cdot d\tilde{W}_{t} + \frac{1}{2} \sum_{i} \sum_{j \neq i} \psi(\delta(T)) \int \prod_{k \neq i, j} \delta_{k} \frac{\partial p_{k|x_{k}}}{\partial Q_{i}} \frac{\partial p_{j|x_{j}}}{\partial Q_{j}} f(x) \, dx$$
Under the $p$-factor Gaussian copula framework, these quantities can be rewritten:

$$
\frac{d\pi_1}{dt} = \sum_{\delta(T)} \frac{1}{\sqrt{T}} \sum_{i=1}^{n} \int \prod_{j \neq i} p_{ij} \phi_{ij} \left( 1 - |\rho_{ij}|^2 \right) \left( 2\delta_j - 1 \right) \phi(x_k) dx_k,
$$

where we have set $\bar{\delta}_j := \Phi^{-1}(Q_i)$. Moreover,

$$
\frac{d\pi_2}{dt} = \frac{1}{2} \sum_{i \neq j} \phi(\bar{\delta}(T)) \int \prod_{k \neq i,j} p_{ik} q_{kj} \phi_{ij} \left( 1 - |\rho_{ij}|^2 \right) \left( 2\delta_i - 1 \right) \phi(x_k) dx_k.
$$

Thus, by an integration by parts w.r.t. $x_r$, we have

$$
\frac{d\pi_1}{dt} = \sum_{\delta(T)} \frac{1}{\sqrt{T}} \sum_{i=1}^{n} \int \prod_{j \neq i} p_{ij} \phi_{ij} \left( 1 - |\rho_{ij}|^2 \right) \left( 2\delta_j - 1 \right) \phi(x_k) dx_k,
$$

$$
\cdot \frac{1}{2} \sum_{i \neq j} \phi(\bar{\delta}(T)) \int \prod_{k \neq i,j} p_{ik} q_{kj} \phi_{ij} \left( 1 - |\rho_{ij}|^2 \right) \left( 2\delta_i - 1 \right) \phi(x_k) dx_k.
$$

The key point of the proof is to exhibit the right variables to lead some integration by parts. Let us introduce some additional

$$
\phi(d_i) \phi(x_r) = \phi \left( \frac{1 - |\rho_{ir}|^2}{1 - |\rho_{ir}|^2} \right) \left( 1 - |\rho_{ir}|^2 \right) \phi(\bar{x}_r) \left( 1 - |\rho_{ir}|^2 \right)
$$

where $\bar{x}_r = \bar{x} - \rho_i x_r$. Now, it can be checked that we can rewrite

$$
\rho_i x_r - \rho_i |\bar{x}|^2 = \sum_{r=1}^{p} \left[ \rho_{ir} x_r (1 - |\rho_{ir}|^2) - c_{ir} \right]
$$

where we have set $c_{ir} := \rho_i^2 \bar{x}_r - \rho_i^2 \rho_{ir} x_r$. We deduce

$$
\frac{d\pi_1}{dt} = \sum_{\delta(T)} \frac{1}{\sqrt{T}} \sum_{i=1}^{n} \int \prod_{j \neq i} p_{ij} \phi_{ij} \left( 1 - |\rho_{ij}|^2 \right) \left( 2\delta_j - 1 \right) \phi(x_k) dx_k.
$$

By integrating w.r.t. the variable $x_r$ and by invoking equation (A6), we get

$$
\int_{-\infty}^{\bar{x}} \phi(d_i) \phi(x_r) dx_r = \phi \left( \frac{1 - |\rho_{ir}|^2}{1 - |\rho_{ir}|^2} \right) \left( 1 - |\rho_{ir}|^2 \right) \phi(\bar{x}_r) \left( 1 - |\rho_{ir}|^2 \right).
$$

By summing the equations (A7) and (A5) up, and noting that $\phi(\bar{x}) = \Phi^{-1}(Q_i)$, we get the result.

## Appendix B: Proof of Theorem 2.2

Conditions (i), (ii) and (iii) imply (6). Therefore, invoking Lemma 2.1, they are sufficient to obtain that $(V_{i,m})_{i \in [0,T]}$ is a $\mathcal{F}_{-}$martingale under $Q$.

Now, let us prove the necessity of such conditions. Without a lack of generality, consider the name 1, and assume that the name 2 is associated to 1 through (7). We want to find the convenient processes $(Q_{it})_{i \in [0,T]}$ that satisfy (Q) and for which the drift of $dV_{i,m}$ can be cancelled. We are free to consider increments between $t$ and $t + dt$. In this proof, we will consider trajectories where the quantities $Q_{it}$ take very large or very small values, $i \neq 1, 2$. It can be justified, as we will check once the right dynamics (8) will be exhibited. Indeed, some realizations of the underlying Brownian motions always exist to fulfill such requests.

To be specific, we consider $(Q_{it})$ trajectories such that $\sup_x \phi(d_i) \phi(x) < \varepsilon$, for a given $\varepsilon > 0$, and when $i \neq 1, 2$. Thus, invoking Lemma 2.1, the drift of $V_{i,m}$ is

$$
\frac{dt}{2} \left[ 2\rho_{i1} \beta_{i1} - \rho_{i1}^2 \beta_{i1}^2 \right] \cdot \sum_{\delta(T)} \frac{1}{\sqrt{T}} \left( 2\delta_j - 1 \right) \phi(x_k) dx_k.
$$

By summing the equations (A7) and (A5) up, and noting that $\phi(\bar{x}) = \Phi^{-1}(Q_i)$, we get the result.
We have assumed that there exist some indicators $\delta^*_k, k \neq 1, 2$ such that
\[
\psi(1, 1, \tilde{e}^{*}_{(1,2)}) - \psi(1, 0, \tilde{e}^{*}_{(1,2)}) - \psi(0, 1, \tilde{e}^{*}_{(1,2)})
+ \psi(0, 0, \tilde{e}^{*}_{(1,2)}) \neq 0.
\]
Now, consider a compact subset $A$ such that $\int 1(x \notin A) \phi(x) \, dx < \epsilon$. We can particularize our trajectories $(Q_{it})$ even more, so that
\[
\sup_{x \in A, k \neq 1, 2} \int p_{k|x}^i \delta^*_k \leq \epsilon,
\]
when $\delta_k \neq \delta^*_k$ for at least one index $k \neq 1, 2$, and
\[
\inf_{x \in A, k \neq 1, 2} \int p_{k|x}^i \delta^*_k \geq 1/2.
\]
The point (ii) is feasible, because, for every index $k \neq 1, 2$, we can find $Q_{kt}$ such that $(ii)$ provides the existence of the pricing parameters $q_{k|x}$ is closed to one when $\delta_k \neq \delta^*_k$, so (i).

Therefore, the sum defining the hedging error is 'reduced' even more: for this particular choice of $Q_{it}, k \neq i, j$, the drift of $dV_{t,m}$ is
\[
\frac{dt}{2} \left( \psi \left( 1, 1, \tilde{e}^{*}_{(1,2)} \right) - \psi \left( 1, 0, \tilde{e}^{*}_{(1,2)} \right) - \psi \left( 0, 1, \tilde{e}^{*}_{(1,2)} \right) + \psi \left( 0, 0, \tilde{e}^{*}_{(1,2)} \right) \right)
\cdot \left[ 2 \beta_{11}\beta_{21}\rho_{12} - \rho_{11}\rho_{22} - \rho_{12}\rho_{21} \right]
\cdot \int p_{k|x}^i \delta^*_k \leq \epsilon,
\]
then we have
\[
Q_{it} = \Phi \left( \exp \left( \frac{\tilde{\sigma}^{2}_{i} \int_{0}^{t} \xi_{u} \psi_{u} \, du}{2} \right) \right) \Phi^{-1} (Q_{0i})
+ \int_{0}^{t} \exp \left( \frac{\tilde{\sigma}^{2}_{i} \int_{0}^{t} \xi_{u} \psi_{u} \, du}{2} \right) \tilde{\sigma}_{i} \xi_{u} \, dW_{iu}
= \Phi \left( \exp \left( \frac{\tilde{\sigma}^{2}_{i} \int_{0}^{t} \xi_{u} \psi_{u} \, du}{2} \right) \right) \Phi^{-1} (Q_{0i})
+ \int_{0}^{t} \exp \left( \frac{\tilde{\sigma}^{2}_{i} \int_{0}^{t} \xi_{u} \psi_{u} \, du}{2} \right) \tilde{\sigma}_{i} \xi_{u} \, dW_{iu}
\]
where $(W_{iu})$ denotes a Brownian motion under $Q$. Clearly, the range of possible values of $Q_{it}$ is $(0, 1)$ for every $i$ and every $t$, except when $\tilde{\sigma}_{i}$ is zero, that is excluded.

Note that $Q_{it}(T)$ tends to zero or one when $t$ tends to $T$. Assume that $\lim_{t \to T} \int_{0}^{t} \xi_{u} \psi_{u} \, du = \ell < +\infty$. Then, since $\int_{0}^{T} \exp(-\sigma_{i}^{2} \int_{0}^{T} s^{2}/2 \xi_{u} \, dW_{iu})$ is a non-degenerate Gaussian r.v., it will be impossible to get lim$_{t \to T} Q_{it} \in \{0, 1\}$ almost surely. On the opposite, if $\lim_{t \to T} \int_{0}^{t} \xi_{u} \psi_{u} \, du = +\infty$, then lim$_{t \to T} Q_{it} \in \{0, 1\}$ a.e., due to equation (B2).

\section*{Appendix C: Proof of Theorem 3.1}

In the one-period Merton model, with our notations and given $\mathcal{H}_t$, $p_{j,t|\mathcal{H}_t} = Q (\tau_j \leq T | \mathcal{H}_t) = Q \left( \int_{0}^{T} \psi_j(u) \, dW_{ju} < \tilde{b}_j T | \mathcal{H}_t \right)
= Q \left( \sqrt{1 - \rho_{j}^{2}} \int_{0}^{T} \psi_j(u) \, dW_{ju} < \tilde{b}_j T \right)
= \Phi \left( \tilde{b}_j T - \sqrt{1 - \rho_{j}^{2}} \int_{0}^{T} \psi_j(u) \, dW_{ju} \right)
= \Phi \left( \tilde{b}_j T - \sqrt{1 - \rho_{j}^{2}} T \psi_j \right)
\]
When $\psi_j(u) = 1$ for any $j$ and $u$, we recover the standard Merton model and the pricing formula (4). In this case, $\mathcal{H}_t = \sigma (W_{ju}, W_T - W_u, u \leq t)$. For any $t, t', T \geq t > t' \geq 0$, we obtain
\[
E[p_{j,t|t'} | \mathcal{H}_t'] = E \left[ \Phi \left( \frac{\tilde{b}_j T - W_{ju} - \rho_{j} (W_T - W_u)}{\sqrt{1 - \rho_{j}^{2} T - t}} \right) | \mathcal{H}_t' \right]
= \Phi \left( \tilde{b}_j T - \sqrt{1 - \rho_{j}^{2} T - t} Z + (W_{ju} - W_{j' u}) \rho_{j} (W_T - W_u) \right)
\leq \tilde{b}_j T - W_{j' u} - \rho_{j} (W_T - W_u) \mathbb{I}_{\mathcal{H}_t'}
\]
for some gaussian r.v. $Z$, independent of $(W_{ju})$ and $(W_{j'u})$. We get
\[
E[p_{j,t|t'} | \mathcal{H}_t'] = \Phi \left( \tilde{b}_j T - W_{ju} - \rho_{j} (W_T - W_u) \sqrt{1 - \rho_{j}^{2} T - t} \right) = p_{j,t|t'}.
\]
Therefore, the process $(p_{jt}|_{t=\tau-\omega, \tau}]_{t=0,T})$ is an $\mathcal{H}$-martingale under $Q$. □

Appendix D: Proof of Theorem 4.1

Let $\psi_p(\delta) = \min(\sum_{i=1}^{n} \delta_i, p)(1 - R)$, and $\theta_p(\delta) = 1(\sum_{i=1}^{n} \delta_i \leq p). (1 - R)$. We can check easily that, for every $p \geq 1$,

$$\psi_p = \psi_{p-1} + 1 - R - \theta_{p-1}.$$ 

The drift of the value process associated with a particular payoff $\psi$ (i.e. the drift of the P&L of an hedged position) will be denoted by $d\pi(\psi)$. Since $\pi$ is a linear functional, we deduce

$$d\pi(\psi_p) = d\pi(\psi_{p-1}) - d\pi(\theta_{p-1}).$$ 

Now, let us specify $d\pi(\theta_p)$ for $p \leq n$. To lighten the notations, we set

$$\tilde{A}_{ij} = \frac{1}{2} \frac{2\beta \beta_j \rho_i \rho_j - \rho_i^2 \rho_j^2}{\sqrt{1 - |\rho_i|^2} \sqrt{1 - |\rho_j|^2}} \phi(d_i) \phi(d_j) \phi(x).$$

Let us apply Lemma 2.1. By distinguishing between the different values that are taken by the couples $(\delta_i, \delta_j)$, we get

$$d\pi_{l}(\theta_p) = dt \sum_{i=1}^{n} \left( \frac{1}{\sum_{l \neq i,j}^{n} \delta_l \leq p} \right) \sum_{i < j}^{(2 \delta_i - 1)(2 \delta_j - 1)} 1 \cdot \int \prod_{k \neq i,j}^{n} p_{k|x}^{k} \tilde{A}_{ij} \, dx$$

$$= dt \sum_{i < j}^{n} \left[ 1 \left( \sum_{l \neq i,j}^{n} \delta_l \leq p - 2 \right) - 21 \left( \sum_{l \neq i,j}^{n} \delta_l \leq p - 1 \right) + 1 \left( \sum_{l \neq i,j}^{n} \delta_l \leq p \right) \right]$$

$$\cdot \int \prod_{k \neq i,j}^{n} p_{k|x}^{k} \tilde{A}_{ij} \, dx,$$

where $\delta_{(i,j)}$ denotes the vector of $(\delta_k)$, with $k \in \{1, \ldots, n\}$ but $k \neq i, j$. To simplify, let us denote

$$\alpha_p := dt \sum_{i < j}^{n} \sum_{l \neq i,j}^{n} \left( \sum_{l \neq i,j}^{n} \delta_l = p \right) \int \prod_{k \neq i,j}^{n} p_{k|x}^{k} \tilde{A}_{ij} \, dx.$$
If $p \geq 1$, we have got

$$d\pi_{t}(\theta_{p}) = dt \sum_{i < j} \sum_{\delta_{ij}(i,j)} \left[ 1 \left( \sum_{l \neq i,j} \delta_{l} = p \right) - 1 \cdot \left( \sum_{l \neq i,j} \delta_{l} = p - 1 \right) \right] \int \prod_{k \neq i,j} p_{k|x} q_{k|x}^{1-\delta_{k}} A_{ij} \, dx,$$

$$= \alpha_{p} - \alpha_{p-1}.$$