Chapter 1
An overview of the goodness-of-fit test problem for copulas.

Jean-David Fermanian

Abstract We review the main “omnibus procedures” for goodness-of-fit testing for copulas: tests based on the empirical copula process, on probability integral transformations, on Kendall’s dependence function, etc, and some corresponding reductions of dimension techniques. The problems of finding asymptotic distribution-free test statistics and the calculation of reliable p-values are discussed. Some particular cases, like convenient tests for time-dependent copulas, for Archimedean or extreme-value copulas, etc, are dealt with. Finally, the practical performances of the proposed approaches are briefly summarized.

1.1 Introduction

Once a model has been stated and estimated, a key question is to check whether the initial model assumptions are realistic. In other words, and even it is sometimes eluted, every modeler is faced with the so-called “goodness-of-fit” (GOF) problem. This is an old-dated statistical problem, that can be rewritten as: denoting by F the cumulative distribution function (cdf hereafter) of every observation, we would like to test

$$\mathscr{H}_0 : F = F_0, \text{ against } \mathscr{H}_a : F \neq F_0,$$

for a given cdf F_0, or, more commonly,

$$\mathscr{H}_0 : F \in \mathscr{F}, \text{ against } \mathscr{H}_a : F \notin \mathscr{F},$$

for a given family of distributions $\mathscr{F} := \{F_\theta, \theta \in \Theta\}$. This distinction between simple and composite assumptions is traditional and we keep it. Nonetheless, except in

---

Jean-David Fermanian
Department of Finance, Ensae-Crest, Malakoff, France, e-mail: `jean-david.fermanian@ensae.fr`

some particular cases (test of independence, e.g.), the latter framework is a lot more useful than the former in practice.

Some testing procedures are "universal" (or "omnibus"), in the sense they can be applied whatever the underlying distribution. In other terms, they do not depend on some particular properties of F_0 or of the assumed family $\mathscr{F}$. Such tests are of primary interest for us. Note that we will not consider Bayesian testing procedures, as proposed in [54], for instance.

To fix the ideas, consider an i.i.d. sample $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ of a d-dimensional random vector $\mathbf{X}$. Its joint cdf is denoted by F, and the associated marginal cdfs' by F_j, $j = 1, \dots, d$. Traditional key quantities are provided by the empirical distribution functions of the previous sample: for every $\mathbf{x} \in \mathbb{R}^d$, set d marginal cdfs'

$$F_{n,k}(x_k) := n^{-1} \sum_{i=1}^{n} \mathbf{1}(X_{i,k} \leq x_k),\ k = 1, \dots, d,$$

and the joint empirical cdf $F_n(\mathbf{x}) := n^{-1} \sum_{i=1}^{n} \mathbf{1}(\mathbf{X}_i \leq \mathbf{x})$. The latter inequality has to be understood componentwise. Most of the "omnibus" tests are based on transformations of the underlying empirical distribution function, or of the empirical process $\mathbb{F}_n := \sqrt{n}(F_n - F_0)$ itself: $T_n = \psi_n(F_n)$ or $T_n = \psi_n(\mathbb{F}_n)$. It is the case of the famous Kolmogorov-Smirnov (KS), Anderson-Darling (AD), Cramer-von-Mises (CvM) and chi-squared tests, for example.

Naively, it could be thought the picture is the same for copulas, and that straightforward modifications of standard GOF tests should do the job. Indeed, the problem for copulas can be simply written as testing

$$\mathscr{H}_0 : C = C_0, \text{ against } \mathscr{H}_a : C \neq C_0,$$

$$\mathscr{H}_0 : C \in \mathscr{C}, \text{ against } \mathscr{H}_a : C \notin \mathscr{C},$$

for some copula family $\mathscr{C} := \{C_\theta, \theta \in \Theta\}$. Moreover, empirical copulas, introduced by Deheuvels in the 80's (see [23], [24], [25]) play the same role for copulas as standard empirical cdfs' for general distributions. For any $\mathbf{u} \in [0,1]^d$, they can be defined by

$$C_n(\mathbf{u}) := F_n(F_{n,1}^{(-1)}(u_1), \dots, F_{n,d}^{(-1)}(u_d)),$$

with the help of generalized inverse functions, or by

$$\bar{C}_n(\mathbf{u}) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(F_{n,1}(X_{i,1}) \leq u_1, \dots, F_{n,d}(X_{i,d}) \leq u_d).$$

It can be proved easily that $\|C_n - \bar{C}_n\|_\infty \leq dn^{-1}$ (see [35]). Then, for the purpose of GOF testing, working with C_n or $\bar{C}_n$ does not make any difference asymptotically. In every case, empirical copulas are explicit functionals of the underlying empirical cdf: $C_n = \zeta(F_n)$. Thus, any previous GOF test statistics for copulas could be defined as $T_n = \psi_n(C_n) = \psi_n \circ \zeta(F_n)$. But this functional ζ is sufficient to induce significant technical difficulties, when applied to standard statistical procedures.

Actually, the latter parallel applies formally, but strong differences appear in terms of the limiting laws of the "copula-related" GOF test statistics. Indeed, some of them are distribution-free in the standard case, i.e., their limiting laws under the null do not depend on the true underlying law F, and then, they can be tabulated: KS (in the univariate case), chi-squared tests, for example. Unfortunately, it is almost impossible to get such nice results for copulas, due to their multivariate nature and due to the complexity of the previous mapping between F_n and C_n. Only a few GOF test techniques for copulas induce distribution-free limiting laws. Therefore, most of the time, some simulation-based procedures have been proposed for this task.

In section 1.2, we discuss the "brute-force" approaches based on some distances between the empirical copula C_n and the assumed copula (under the null), and we review the associated bootstrap-like techniques. We detail how to get asymptotically distribution-free test statistics in section 1.3, and we explain some testing procedures that exploit the particular features of copulas. We discuss some ways of testing the belonging to some "large" infinite-dimensional families of copulas like Archimedean, extreme-value, vine, or HAC copulas in section 1.4. Tests adapted to time-dependent copulas are introduced in section 1.5. Finally, empirical performances of these GOF tests are discussed in section 1.6.

1.2 The "brute-force" approach: the empirical copula process and the bootstrap

1.2.1 Some tests based on empirical copula processes

Such copula GOF tests are the parallels of the most standard GOF tests in the literature, replacing F_n (resp. F_0) by C_n (resp. C_0). These statistics are based on distances between the empirical copula C_n and the true copula C_0 (simple zero assumption), or between C_n and $C_{\hat{\theta}_n}$ (composite zero assumption), for some convergent and convenient estimator $\hat{\theta}_n$ of the "true" copula parameter θ_0. It is often reduced simply to the evaluation of norms of the empirical copula process $\mathbb{C}_n := \sqrt{n}(C_n - C_0)$, or one of its approximations $\widehat{\mathbb{C}}_n := \sqrt{n}(C_n - C_{\hat{\theta}_n})$.

In this family, let us cite the Kolmogorov-Smirnov type statistics

$$T_n^{KS} := \|\mathbb{C}_n\|_\infty = \sup_{\mathbf{u}\in[0,1]^d} |\sqrt{n}(C_n - C_0)(\mathbf{u})|,$$

and the Anderson-Darling type statistics

$$T_n^{AD} := \|\mathbb{C}_n\|_{L^2} = n \int (C_n - C_0)^2(\mathbf{u}) w_n(\mathbf{u})\, d\mathbf{u},$$

for some positive (possibly random) weight function w_n, and their composite versions. By smoothing conveniently the empirical copula process, [70] defined alternative versions of the latter tests.

In practice, the statistics T_n^{KS} seem to be less powerful than a lot of competitors, particularly of the type T_n^{AD} (see [45]). Therefore, a "total variation" version of T_n^{KS} has been proposed in [36], that appears significantly more powerful than the classical T_n^{KS}:

$$T_n^{ATV} := \sup_{B_1,\ldots,B_{L_n}} \sum_{k=1}^{L_n} |\mathbb{C}_n(B_k)|, \text{ or } \hat{T}_n^{ATV} := \sup_{B_1,\ldots,B_{L_n}} \sum_{k=1}^{L_n} |\widehat{\mathbb{C}}_n(B_k)|,$$

for simple or composite assumptions respectively. Above, the supremum is taken over all disjoint rectangles $B_1,\ldots,B_{L_n} \subset [0,1]^d$, and $L_n \sim \ln n$.

Another example of distance is proposed in [71]: let two functions f_1 and f_2 in $\mathbb{R}^d$. Typically, they represent copula densities. Set a positive definite bilinear form as

$$< f_1, f_2 >:= \int \kappa_d(\mathbf{x}_1,\mathbf{x}_2)\, f_1(\mathbf{x}_1) f_2(\mathbf{x}_2)\, d\mathbf{x}_1\, d\mathbf{x}_2,$$

where $\kappa_d(\mathbf{x}_1,\mathbf{x}_2) := \exp(-\|\mathbf{x}_1-\mathbf{x}_2\|^2/(2dh^2))$, for some Euclidian norm $\|\cdot\|$ in $\mathbb{R}^d$ and a bandwidth $h>0$. A squared distance between f_1 and f_2 is given simply by $\mu(f_1,f_2) :=< f_1-f_2, f_1-f_2 >=< f_1,f_1 > -2 < f_1,f_2 > + < f_2,f_2 >$. When f_1 and f_2 are the copula densities of C_1 and C_2 respectively, the three latter terms can be rewritten in terms of copula directly. For instance, $< f_1, f_2 >= \int \kappa_d(\mathbf{x}_1,\mathbf{x}_2)\, C_1(d\mathbf{x}_1) C_2(d\mathbf{x}_2)$. Since such expressions have simple empirical counterparts, a GOF test for copulas can be built easily: typically, replace C_1 by the empirical copula C_n and C_2 by the true copula C_0 (or $C_{\hat{\theta}_n}$).

Closely connected to this family of tests are statistics T_n that are zero when the associated copula processes are zero, but not the opposite. Strictly speaking, this is the case of the Cramer-von Mises statistics

$$T_n^{CvM} := n \int (C_n - C_0)^2(\mathbf{u})\, C_n(d\mathbf{u}),$$

and of chi-squared type test statistics, like

$$T_n^{Chi} := n \sum_{k=1}^{p} w_k (C_n - C_0)^2(B_k),$$

where $B_1,\ldots,B_p$ denote disjoint boxes in $[0,1]^d$ and w_k, $k=1,\ldots,p$ are convenient weights (possibly random). More generally, we can consider

$$T_n^{\mu} := \sum_{k=1}^{p} \mu(C_n(E_k), C_0(E_k)), \text{ or } T_n^{\mu} := \sum_{k=1}^{p} \mu(C_n(E_k), C_{\hat{\theta}_n}(E_k)),$$

for any metric μ on the real line, and arbitrary subsets $E_1,\ldots,E_p$ in $[0,1]^d$. This is the idea of the chi-square test detailed in [30]: set the vectors of pseudo-observations $\hat{\mathbf{U}}_i := (F_{n,1}(X_{i,1}),\ldots,F_{n,d}(X_{i,d}))$, and a partition of $[0,1]^d$ into p disjoint rectangles B_j. The natural chi-square-style test statistics is

$$T_n^{\chi} := \sum_{k=1}^{p} \frac{\left(\hat{N}_k - p_k(\hat{\theta}_n)\right)^2}{n p_k(\hat{\theta}_n)}$$

where $\hat{N}_k$ denotes the number of vectors $\hat{\mathbf{U}}_i$, $i=1,\ldots,n$ that belong to B_k, and $p_k(\theta)$ denotes the probability of the event $\{\mathbf{U}\in B_k\}$ under the copula C_θ. This idea of applying an arbitrary categorization of the data into contingency tables $[0,1]^d$ has been applied more or less fruitfully in a lot of papers: [46], [59], [33], [4], [58], etc.

Finally, note that a likelihood ratio test has been proposed in [30], based on a Kullback-Leibler pseudo distance between a "discrete" version of C_n and the corresponding estimated copula under the null:

$$T_n^{LR} := \sum_{k=1}^{p} N_k \ln p_k(\hat{\theta}_n).$$

To compare the fit of two potential parametric copulas, the same information criterion has been used in [28] to build a similar test statistics, but based on copula densities directly.

The convergence of all these tests relies crucially on the fact that the empirical copula processes $\mathbb{C}_n$ and $\widehat{\mathbb{C}}_n$ are weakly convergent under the null, and for convenient sequences of estimates $\hat{\theta}_n$: see [81], [38], [35]. Particularly, it has been proved that $\mathbb{C}_n$ tends weakly in $\ell^\infty([0,1]^d)$ (equipped with the metric induced by the sup-norm) to a Gaussian process $\mathbb{G}_{C_0}$, where

$$\mathbb{G}_{C_0}(\mathbf{u}) := \mathbb{B}_{C_0}(\mathbf{u}) - \sum_{j=1}^{d} \partial_j C_0(\mathbf{u})\mathbb{B}_{C_0}(u_j,\mathbf{1}_{-j}),\ \forall \mathbf{u}\in[0,1]^d,$$

with obvious notations and for some d-dimensional Brownian bridge $\mathbb{B}$ in $[0,1]^d$, whose covariance is

$$\mathbb{E}\left[\mathbb{G}_{C_0}(\mathbf{u})\mathbb{G}_{C_0}(\mathbf{v})\right] = C_0(\mathbf{u}\wedge\mathbf{v}) - C_0(\mathbf{u})C_0(\mathbf{v}),\ \ \forall(\mathbf{u},\mathbf{v})\in[0,1]^{2d}.$$

To get this weak convergence result, it is not necessary to assume that C_0 is continuously differentiable on the whole hypercube $[0,1]^d$, a condition that is often not fulfilled in practice. Recently, [87] has shown that such a result is true when, for every $j=1,\ldots,d$, $\partial_j C_0$ exists and is continuous on the set $\{\mathbf{u}\in[0,1]^d, 0<u_j<1\}$.

Clearly, the law of $\mathbb{G}$ involves the particular underlying copula C_0 strongly, contrary to usual Brownian bridges. Therefore, the tabulation of the limiting laws of T_n GOF statistics appears difficult. A natural idea is to rely on computer intensive methods to approximate these law numerically. The bootstrap appeared as a natural tool for doing this task

1.2.2 Bootstrap techniques

The standard nonparametric bootstrap is based on resampling with replacement inside an original i.i.d. $\mathbf{X}$-sample $S_{\mathbf{X}}$. We get new samples $S^*_{\mathbf{X}} = (\mathbf{X}^*_1, \ldots, \mathbf{X}^*_n)$. Associate to every new sample $S^*_{\mathbf{X}}$ its "bootstrapped" empirical copula C^*_n and its bootstrapped empirical process $\mathbb{C}^*_n := \sqrt{n}(C^*_n - C_n)$. In [35], it is proved that, under mild conditions, this bootstrapped process $\mathbb{C}^*_n$ is weakly convergent in $\ell^\infty([0,1]^d)$ towards the previous Gaussian process $\mathbb{G}_{C_0}$. Therefore, in the case of simple null assumptions, we can get easily some critical values or p-values of the previous GOF tests: resample M times, $M >> 1$, and calculate the empirical quantiles of the obtained bootstrapped test statistics. Nonetheless, this task has to be done for every zero assumption. This can become a tedious and rather long task, especially when d is "large" (> 3 in practice) and/or with large datasets (> 1000, typically).

When dealing with composite assumptions, some versions of the parametric bootstrap are advocated, depending on the limiting behavior of $\hat{\theta}_n - \theta_0$: see the theory in [44], and the appendices in [45] for detailed examples. To summarize these ideas in typical cases, it is now necessary to draw random samples from $C_{\hat{\theta}_n}$. For every bootstrapped sample, calculate the associated empirical copula C^*_n and a new estimated value $\hat{\theta}^*_n$ of the parameter. Since the weak limit of $\sqrt{n}(C^*_n - C_{\hat{\theta}^*_n})$ is the same as the limit of $\widehat{\mathbb{C}}_n = \sqrt{n}(C_n - C_{\hat{\theta}_n})$, the law of every functional of $\widehat{\mathbb{C}}_n$ can be approximated. When the cdf $C_{\hat{\theta}_n}$ cannot be evaluated explicitly (in closed-form), a two-level parametric bootstrap has been proposed in [44], by bootstrapping first a approximated version of $C_{\hat{\theta}_n}$.

Instead of resampling with replacement, a multiplier bootstrap procedure can approximate the limiting process $\mathbb{G}_{C_0}$ (or one of its functionals), as in [86]: consider $Z_1, \ldots, Z_n$ i.i.d. real centered random variables with variance one, independent of the data $\mathbf{X}_1, \ldots, \mathbf{X}_n$. A new bootstrapped empirical copula is defined by

$$C^*_n(\mathbf{u}) := \frac{1}{n} \sum_{i=1}^n Z_i.\mathbf{1}(F_{n,1}(X_{i,1}) \leq u_1, \ldots, F_{n,d}(X_{i,d}) \leq u_d),$$

for every $\mathbf{u} \in [0,1]^d$. Setting $\bar{Z}_n := n^{-1} \sum_{i=1}^n Z_i$, the process $\beta_n := \sqrt{n}(C^*_n - \bar{Z}_n C_n)$ tends weakly to the Brownian bridge $\mathbb{B}_{C_0}$. By approximating (by finite differences) the derivatives of the true copula function, it is shown in [86] how to modify β_n to get an approximation of $\mathbb{G}_{C_0}$. To avoid this last stage, another bootstrap procedure has been proposed in [14]. It applies the multiplier idea to the underlying joint and marginal cdfs', and invoke classical delta method arguments. Nonetheless, despite more attractive theoretical properties, the latter technique does not seem to improve the initial multiplier bootstrap of [86]. In [62], the multiplier approach is extended to deal with parametric copula families of any dimension, and the finite-sample performance of the associated Cramer-von-Mises test statistics has been studied. A variant of the multiplier approach has been proposed in [60]. It is shown that the use of multiplier approaches instead of the parametric bootstrap leads to a strong reduction in the computing time. Note that both methods have been implemented in the copula R package.

Recently, in [36], a modified nonparametric bootstrap technique has been introduced to evaluate the limiting law of the previous Komogorov-Smirnov type test statistics T_n^{ATV} in the case of composite zero assumptions. In this case, the key process is still

$$\widehat{\mathbb{C}}_n := \sqrt{n}(C_n - C_{\hat{\theta}_n}) = \mathbb{C}_n - \sqrt{n}(C_{\hat{\theta}_n} - C_{\theta_0}).$$

Generate a usual nonparametric bootstrap sample, obtained after resampling with replacement from the original sample. This allows the calculation of the bootstrapped empirical copula C_n^* and a new parameter estimate $\hat{\theta}_n^*$. Instead of considering the "intuitive" bootstrapped empirical copula process $\sqrt{n}(C_n^* - C_{\hat{\theta}_n^*})$, a new bootstrapped process is introduced:

$$\mathbb{Y}_n^* := \sqrt{n}(C_n^* - C_n) - \sqrt{n}(C_{\hat{\theta}_n^*} - C_{\hat{\theta}_n}).$$

Indeed, the process $\sqrt{n}(C_n^* - C_{\widehat{\theta}^*})$, while perhaps a natural candidate, does not yield a consistent estimate of the distribution of $\widehat{\mathbb{C}}_n$, contrary to $\mathbb{Y}_n^*$. For the moment, the performances of this new bootstrapped process have to be studied more in depth.

1.3 Copula GOF test statistics: alternative approaches

1.3.1 Working with copula densities

Even if the limiting laws of the empirical copula processes $\mathbb{C}_n$ and $\widehat{\mathbb{C}}_n$ involve the underlying (true) copula in a rather complex way, it is still possible to get asymptotically distribution-free test statistics. Unfortunately, the price to be paid is an additional level of complexity.

To the best of our knowledge, there exists a single strategy. The idea is to rely on copula densities themselves, rather than copulas (cdfs'). Indeed, testing the identity $C = C_0$ is equivalent to studying the closeness between the true copula density τ_0 (w.r.t. the Lebesgue measure on $[0,1]^d$, that is assumed to exist) and one of its estimates τ_n. In [33], a L^2-distance between τ_n and τ_0 allows to build convenient test statistics. To be specific, a kernel estimator of a copula density τ at point $\mathbf{u}$ is defined by

$$\tau_n(\mathbf{u}) = \frac{1}{h^d}\int K\left(\frac{\mathbf{u}-\mathbf{v}}{h}\right) C_n(d\mathbf{v}) = \frac{1}{nh^d}\sum_{i=1}^{n} K\left(\frac{\mathbf{u}-\hat{\mathbf{U}}_i}{h}\right),$$

where $\hat{\mathbf{U}}_i := (F_{n,1}(X_{i,1}),\ldots,F_{n,d}(X_{i,d}))$ for all $i=1,\ldots,n$. Moreover, K is a d-dimensional kernel and $h=h(n)$ is a bandwidth sequence, chosen conveniently. Under some regularity assumptions, for every m and every vectors $\mathbf{u}_1,\ldots,\mathbf{u}_m$ in $]0,1[^d$, such that $\tau_0(\mathbf{u}_k)>0$ for every k, the vector $(nh^d)^{1/2}\left((\tau_n-\tau_0)(\mathbf{u}_1),\ldots,(\tau_n-\tau_0)(\mathbf{u}_m)\right)$ tends weakly to a Gaussian random vector, whose components are independent. Therefore, under the null, the test statistics

$$T_n^{\tau,0} = \frac{nh^d}{\int K^2}\sum_{k=1}^m \frac{(\tau_n(\mathbf{u}_k)-\tau_0(\mathbf{u}_k))^2}{\tau_0(\mathbf{u}_k)^2}.$$

tends in law towards a m-dimensional chi-squared distribution. This can be adapted easily for composite assumptions. The previous test statistics depend on a finite and arbitrary set of points $\mathbf{u}_k$, $k=1,\ldots,m$. To avoid this drawback, [33] has introduced

$$J_n = \int (\tau_n - K_h * \hat{\tau})^2(\mathbf{u})\omega(\mathbf{u})\,d\mathbf{u},$$

for some nonnegative weight function ω. Here, $\hat{\tau}$ denotes τ_0 (simple assumption) or $\tau(\cdot,\hat{\theta}_n)$ (composite assumption), for sufficiently regular estimates $\hat{\theta}_n$ of θ_0. It is proved that

$$T_n^{\tau,1} := \frac{n^2h^d\left(J_n - (nh^d)^{-1}\int K^2(\mathbf{t}).(\hat{\tau}\omega)(\mathbf{u}-h\mathbf{t})\,d\mathbf{t}\,d\mathbf{u} + (nh)^{-1}\int \hat{\tau}^2\omega.\sum_{r=1}^d \int K_r^2\right)^2}{2\int \hat{\tau}^2\omega\cdot\int\left\{\int K(\mathbf{u})K(\mathbf{u}+\mathbf{v})\,d\mathbf{u}\right\}^2 d\mathbf{v}}$$

tends to a $\chi^2(1)$ under the null.

Even if the previous test statistics are pivotal, they are rather complex and require the choice of smoothing parameters and kernels. Nonetheless, such ideas have been extended in [85] to deal with the fixed design case. Moreover, the properties of these tests under fixed alternatives are studied in [13]. The impact of several choices of parameter estimates $\hat{\theta}_n$ on the asymptotic behavior of J_n is detailed too. Apparently, for small sample sizes, the normal approximation does not provide sufficiently exact critical values (in line with [51] or [32]), but it is still possible to use a parametric bootstrap procedure to evaluate the limiting law of T_n^τ in this case. Apparently, in the latter case, the results are as good as the main competitors (see [13], section 5).

Since copula densities have a compact support, kernel smoothing can generate some undesirable boundary effects. One solution is to use improved kernel estimators that take care of the typical corner bias problem, as in [70]. Another solution is to estimate copula densities through wavelets, for which the border effects are handled automatically, due to the good localization properties of the wavelet basis: see [43]. This idea has been developed in [39], in a minimax theory framework, to determine the largest alternative for which the decision remains feasible. Here, the copula densities under consideration are supposed to belong to a range of Besov balls. According to the minimax approach, the testing problem is then solved in an adaptive framework.

1.3.2 The probability integral transformation (PIT)

A rather simple result of probability theory, proposed initially in [80], has attracted the attention of authors for copula GOF testing purpose. Indeed, this transformation maps a general d-dimensional random vector $\mathbf{X}$ into a vector of d independent uniform random variables on $[0,1]$ in a one-to-one way. It is known as Rosenblatt's or probability integral transformation (PIT). Once the joint law of $\mathbf{X}$ is known and analytically tractable, this is a universal way of generating independent and uniform random vectors without losing statistical information. Note that other transformations of the same type exist (see [22]).

To be specific, the copula C is the joint cdf of $\mathbf{U} := (F_1(X_1),\ldots,F_d(X_d))$. We define the d-dimensional random vector $\mathbf{V}$ by

$$V_1 := U_1 = F_1(Z_1),\, V_2 := C(U_2|U_1),\cdots,V_d := C(U_d|U_1,\ldots,U_{d-1}), \tag{1.1}$$

where $C(\cdot|u_1,\ldots,u_{k-1})$ is the law of U_k given $U_1 = u_1$,...,$U_{k-1} = u_{k-1}$, $k = 2,\ldots,d$. Then, the variables V_k, $k = 1,\ldots,d$ are uniformly and independently distributed on $[0,1]$. In other words, $\mathbf{U} \sim C$ iff $\mathbf{V} = \mathscr{R}(\mathbf{U})$ follows the d-variate independence copula $C_\perp(\mathbf{u}) = u_1.\cdots.u_d$.

The main advantage of this transformation is the simplicity of the transformed vector $\mathbf{V}$. This implies that the zero assumptions of a GOF test based on $\mathbf{V}$ are always the same: test the i.i.d. feature of $\mathbf{V}$, that is satisfied when C is the true underlying copula. A drawback is the arbitrariness in the choice of the successive margins. Indeed, there are at most $d!$ different PITs', that induce generally different test statistics. Another disadvantage is the necessity of potentially tedious calculations. Indeed, typically, the conditional joint distributions are calculated through the formulas

$$C(u_k|u_1,\ldots,u_{k-1}) = \partial^{k-1}_{1,2,\ldots,k-1}C(u_1,\ldots,u_k,1,\ldots,1)/\partial^{k-1}_{1,2,\ldots,k-1}C(u_1,\ldots,u_{k-1},1,\ldots,1),$$

for every $k = 2,\ldots,d$ and every $\mathbf{u} \in [0,1]^d$. Therefore, with some copula families and/or with large dimensions d, the explicit calculation (and coding!) of the PIT can become unfeasible.

The application of such transformations for copula GOF testing appeared first in [12]. This idea has been reworked and extended in several papers afterwards: see [31], [10], [40], [8], etc. Several applications of such techniques to financial series modelling and risk management has emerged, notably [65], [27], [19], [63], [92], among others.

For copula GOF testing, we are only interested in the copula itself, and the marginal distributions F_k, $k = 1,\ldots,d$ are seen as nuisance parameters. Therefore, they are usually replaced by the marginal empirical cdfs' $F_{n,k}$. Equivalently, the observations $\mathbf{X}_i$, $i = 1,\ldots,n$ are often replaced by their pseudo-observations $\hat{\mathbf{U}}_i := (F_{n,1}(X_{i,1}),\ldots,F_{n,d}(X_{i,d}))$. Moreover, for composite zero assumptions, the chosen estimator $\hat{\theta}_n$ disturbs the limiting law of the test statistics most of the time. This difficulty is typical of the statistics of copulas, and it is a common source of mistakes, as pointed out in [34]. For instance, in [12], these problems were not tackled conveniently and the reported p-values are incorrect. [12] noticed that the r.v. $\sum_{k=1}^d [\Phi^{-1}(V_k)]^2$ follows a $\chi^2(d)$. But it is no more the case of $\sum_{k=1}^d [\Phi^{-1}(\hat{V}_{n,k})]^2$, where $\hat{\mathbf{V}} = \mathscr{R}(\hat{\mathbf{U}})$. This point has been pointed out in [44]. A corrected test statistics with reliable p-values has been introduced in [31]. An extension of these tests has been introduced in [10]. It implies data-driven weight functions, to emphasize some regions of underlying the copula possibly. Its comparative performances are studied in [9] and [8].

Thus, to the best of our knowledge, all the previous proposed tests procedures have to rely on bootstrap procedures to evaluate the corresponding limiting laws under the null. This is clearly a shame, keeping in mind the simplicity of the law of $\mathbf{V}$, after a PIT of the original dataset (but with *known* margins). In practice, we have to work with (transformed) pseudo-observations $\hat{\mathbf{V}}_i$, $i = 1,\ldots,n$. As we said, they are calculated from formulas (1.1), replacing unobservable uniformly distributed vectors $\mathbf{U}_i$ by pseudo-observations $\hat{\mathbf{U}}_i$, $i = 1,\ldots,n$. The vectors $\hat{\mathbf{V}}_i$ are no longer independent and only approximately uniform on $[0,1]^d$. Nonetheless, test statistics $T_n^{\psi,PIT} = \psi(\hat{\mathbf{V}}_1,\ldots,\hat{\mathbf{V}}_n)$ may be relevant, for convenient real functions ψ. In general and for composite zero assumptions, we are not insured that the law of $\hat{\mathbf{V}}$, denoted by $C_{\infty,\mathbf{V}}$, tends to the independence copula. If we were able to evaluate $C_{\infty,\mathbf{V}}$, a "brute-force" approach would still be possible, as in section 1.2. For instance and naively, we could introduce the Kolmogorov-type statistics

$$T_n^{KM,PIT} := \sup_{\mathbf{u}\in(0,1)^d} |\frac{1}{n}\sum_{i=1}^n \mathbf{1}(\hat{\mathbf{V}}_i \leq \mathbf{u}) - C_{\infty,\mathbf{V}}(\mathbf{u})|.$$

Nonetheless, due to the difficulty to evaluate precisely $C_{\infty,\mathbf{V}}$ (by Monte-Carlo, in practice), most authors have preferred to reduce the dimensionality of the problem. By this way, they are able to tackle more easily the case $d \geq 3$.

1.3.3 Reductions of dimension

Generally speaking, in a GOF test, it is tempting to reduce the dimensionality of the underlying distributions, for instance from d to one. Indeed, especially when $d >> 1$, the "brute-force" procedures based on empirical processes involve significant analytical or numerical difficulties in practice. For instance, a Cramer-von-Mises necessitates the calculation of a d-dimensional integral.

Formally, a reduction of dimension means replacing the initial GOF problem "$\mathscr{H}_0$: the copula of $\mathbf{X}$ is C_0" by "$\mathscr{H}_0^*$: the law of $\psi(\mathbf{X})$ is $G_{\psi,0}$", for some transformation $\psi : \mathbb{R}^d \to \mathbb{R}^p$, with $p << d$, and for some p-dimensional cdf $G_{\psi,0}$. As $\mathscr{H}_0$ implies $\mathscr{H}_0^*$, we decide to reject $\mathscr{H}_0$ when $\mathscr{H}_0^*$ is not satisfied. Obviously, this reduction of the available information induces a loss of power, but the practical advantages of this trick often dominate its drawbacks.

For instance, when $p=1$ and if we are able to identify $G_{\psi,0}$, it becomes possible to invoke standard univariate GOF test statistics, or even to use ad-hoc visual procedures like QQ-plots. Thus, by reducing a multivariate GOF problem to a univariate problem, we rely on numerically efficient procedures, even for high dimensional underlying distributions. However, we still depend on Monte-Carlo methods to evaluate the corresponding p-values. Inspired by [84], we get one of the most naive method of dimension reduction: replace T_n^{KS} above by

$$\tilde{T}_n^{KS} := \sum_{\alpha\in(0,1)} |C_n(A_\alpha) - C_0(A_\alpha)|, \text{ or } \tilde{T}_n^{KS} := \sum_{\alpha\in(0,1)} |C_n(\hat{A}_\alpha) - C_{\hat{\theta}_n}(\hat{A}_\alpha)|,$$

where $(A_\alpha)_{\alpha\in(0,1)}$ is an increasing sequence of subsets in $[0,1]^d$ s.t. $A_\alpha = \{\mathbf{u} \in [0,1]^d | C_0(\mathbf{u}) \leq \alpha\}$ and $\hat{A}_\alpha = \{\mathbf{u} \in [0,1]^d | C_{\hat{\theta}_n}(\mathbf{u}) \leq \alpha\}$.

To revisit a previous example and with the same notations, [31] considered particular test statistics $T_n^{\psi,PIT}$ based on the variables $\hat{Z}_i := \sum_{k=1}^d \Phi(\hat{V}_{i,k})^{-1}$, $i=1,\ldots,n$. If the margins F_k, $k=1,\ldots,d$, and the true copula C_0 were known, then we were be able to calculate $Z_i := \sum_{k=1}^d \Phi(V_{i,k})^{-1}$ that tends in law towards a chi-square law of dimension d under the null. Since it is not the case in practice, the limiting law of $\hat{Z}_i$ is unknown, and it has to be evaluated numerically by simulations. It is denoted by $F_{\hat{Z}}$. Therefore, [31] propose to test

$$\mathscr{H}_0^* : \text{the asymptotic law of } T_n^{\psi,PIT} \text{ is a given cdf } F_\psi \text{ (to be estimated)},$$

where $T_n^{\psi,PIT}$ is defined by usual (univariate) Kolmogorov-Smirnov, Anderson-Darling or Cramer-von-Mises test statistics. For instance,

$$T_n^{AD,PIT} := n \int \frac{(F_{n,\hat{Z}} - F_{0,\hat{Z}})^2}{F_{0,\hat{Z}}(1-F_{0,\hat{Z}})},$$

where $F_{n,\hat{Z}}$ is the empirical cdf of the pseudo sample $\hat{Z}_1,\ldots,\hat{Z}_n$. Note that $F_{n,\hat{Z}}$ and $F_{0,\hat{Z}}$ depend strongly on the underlying cdf of $\mathbf{X}$, its true copula C_0, the way marginal cdfs' have been estimated to get pseudo-observations (empirical or parametric estimates) and possibly the particular estimate $\hat{\theta}_n$.

Beside the PIT idea, there exist a lot of possibilities of dimension reductions potentially. They will provide more or less relevant test statistics, depending on the particular underlying parametric family and on the empirical features of the data. For instance, in the bivariate case, Kendall's tau τ_K or Spearman's rho ρ_S may appear as nice "average" measures of dependence. They are just single numbers, instead of a true 2-dimensional function like C_n. Therefore, such a GOF test may be simply

$$\mathscr{H}_0^* : \hat{\tau}_K = \tau_{K,C_0},$$

where $\tau_{K,C_0} = 4\mathbb{E}_{C_0}[C_0(\mathbf{U})] - 1$ is the Kendall's tau of the true copula C_0, and $\hat{\tau}_K$ is an estimate of this measure of dependence, for instance its empirical counterpart

$$\hat{\tau}_{K,n} := \frac{2}{n(n-1)}[\text{number of concordant pairs of observations} - \text{number of discordant pairs}].$$

Here, we can set $T_n^{KTau} := n(\hat{\tau}_{K,n} - \tau_{C_0})^2$, or $T_n^{KTau} := n(\hat{\tau}_{K,n} - \tau_{C_{\hat{\theta}_n}})^2$ in the case of composite assumption. Clearly, the performances of all these tests in terms of power will be very different and there is no hope to get a clear hierarchy between all of them. Sometimes, it will be relevant to discriminate between several distributions depending on the behaviors in the tails. Thus, some adapted summaries of the information provided by the underlying copula C are required, like tail-indices for instance (see [68] e.g.). But in every case, their main weakness is a lack of convergence against a large family of alternatives. For instance, the previous test T_n^{KTau} will not be able to discriminate between all copulas that have the same Kendall's tau τ_{K,C_0}. In other words, this dimension reduction is probably too strong, most of the time: we reduce a d-dimensional problem to a real number. It is more fruitful to keep the idea of generating a univariate process, i.e., going from a dimension d to a dimension one. This is the idea of Kendall's process (see below).

Another closely related family of tests is based on the comparison between several parameter estimates. They have been called "moment-based" GOF test statistics (see [88], [41], [11]). In their simplest form, assume a univariate unknown copula parameter θ, and two estimation equations ("moments") such that $m_1 = r_1(\theta)$ and $m_2 = r_2(\theta)$ (one-to-one mappings). Given empirical counterparts $\hat{m}_k$ of m_k, $k = 1,2$, [88] has proposed the copula GOF test

$$T_n^{moment} := \sqrt{n}\left\{r_1^{-1}(\hat{m}_1) - r_2^{-1}(\hat{m}_2)\right\}.$$

Typically, some estimating equations are provided by Kendall's tau and Spearman's rho, that have well-known empirical counterparts. Nonetheless, other estimates have been proposed, as the pseudo-maximum likelihood (also called "canonical maximum likelihood"). To deal with multi-dimensional parameters θ, estimating equations can be obtained by the equality between the hessian matrix and minus the expected outer product of the score function. This is the idea of White's specification test (see [93]), adapted to copulas in [76].

1.3.4 Kendall's process

This is another and well-known example of dimension reduction related to copula problems. Let C be the copula of an arbitrary random vector $\mathbf{X} \in \mathbb{R}^d$. Define the univariate cdf

$$K(t) := \mathbb{P}(C(\mathbf{U}) \leq t), \;\; \forall t \in \mathbb{R},$$

where, as usual, we set $\mathbf{U} = (F_1(X_1), \ldots, F_d(X_d))$. The function K depends on C only. Therefore, this univariate function is a "summary" of the underlying dependence structure given by C. It is called the Kendall's dependence function of C. An empirical counterpart of K is the empirical Kendall's function

$$K_n(t) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(C_n(\hat{\mathbf{U}}_i) \leq t),$$

with pseudo-observations $\hat{\mathbf{U}}_1, \ldots, \hat{\mathbf{U}}_n$. The associated Kendall's process is simply given by $\mathbb{K}_n = \sqrt{n}(K_n - K)$, or $\hat{\mathbb{K}}_n = \sqrt{n}(K_n - K(\hat{\theta}_n, \cdot))$ when the true copula is unknown but belongs to a given parametric family. The properties of Kendall's processes has been studied in depth in [5], [49], and [40] particularly. In the latter papers, the weak convergence of $\mathbb{K}_n$ towards a continuous centered Gaussian process in the Skorohod space of cadlag functions is proved, for convenient consistent sequences of estimates $\hat{\theta}_n$. Its variance-covariance function is complex and copula dependent. It depends on the derivatives of K w.r.t. the parameter θ and the limiting law of $\sqrt{n}(\hat{\theta}_n - \theta_0)$.

Then, there are a lot of possibilities of GOF tests based on the univariate function K_n or the associated process $\mathbb{K}_n$. For instance, [90] introduced a test statistics based on the L^2 norm of $\mathbb{K}_n$. To be specific, they restrict themselves to bivariate Archimedean copulas, but allow censoring. That is why their GOF test statistics $T_n^{L2,Kendall} = \int_\xi^1 |\mathbb{K}_n|^2$ involves an arbitrary cut-off point $\xi > 0$. Nonetheless, the idea of such a statistics is still valid for arbitrary dimensions and copulas. It has been extended in [40], that considers

$$T_n^{L2,Kendall} := \int_0^1 |\mathbb{K}_n(t)|^2 k(\hat{\theta}_n, t)\, dt, \text{ and } T_n^{KS,Kendall} := \sup_{t \in [0,1]} |\mathbb{K}_n(t)|,$$

where $k(\theta, \cdot)$ denotes the density of $C(\mathbf{U})$ w.r.t. to the Lebesgue measure (i.e. the derivative of K), and $\hat{\theta}_n$ is a consistent estimate of the true parameter under the null.

Nonetheless, working with $\mathbb{K}_n$ or $\hat{\mathbb{K}}_n$ instead of $\mathbb{C}_n$ or $\hat{\mathbb{C}}_n$ respectively is not the panacea. As we said, the dimension reduction is not free of charge, and testing $\mathscr{H}_0^*$ instead of $\mathscr{H}_0$ reduces the ability to discriminate between copula alternatives. For instance, consider two extreme-value copulas C_1 and C_2, i.e., in the bivariate case,

$$C_j(u, v) = \exp\left(\ln(uv) A_j(\frac{\ln u}{\ln uv})\right), \; j = 1, 2,$$

for some Pickands functions A_1 and A_2 (convex functions on $[0,1]$, such that $\max(t, (1-t)) \leq A_j(t) \leq 1$ for all $t \in [0,1]$). As noticed in [49], the associated Kendall's functions are

$$K_j(t) = t - (1 - \tau_{K,j}) t \ln t, \; t \in (0, 1),$$

where $\tau_{K,j}$ denotes the Kendall's tau of C_j. Then, if the two Kendall's tau are the same, the corresponding Kendall's functions K_1 and K_2 are identical. Thus, a test of $\mathscr{H}_0^* : K = K_0$ will appear worthless if the underlying copulas are of the extreme-value type.

In practice, the evaluation of the true Kendall function K_0 under the null may become tedious, or even unfeasible for a lot of copula families. Therefore, [9] proposed to apply the previous Kendall process methodology to random vectors obtained through a PIT in a preliminary stage, to "stabilize" the limiting law under the null. In this case, K_0 is always the same: the Kendall function associated to the independence copula $C_\perp$. This idea has been implemented in [45], under the form of Cramer-von-Mises GOF test statistics of the type

$$T_n^{CvM,PIT} := n \int (D_n(\mathbf{u}) - C_\perp(\mathbf{u}))^2 \, dD_n(\mathbf{u}) = \sum_{i=1}^n \left(D_n(\hat{\mathbf{U}}_i) - C_\perp(\hat{\mathbf{U}}_i) \right)^2 ,$$

were $D_n(\mathbf{u}) = n^{-1} \sum_{i=1}^n \mathbf{1}(\hat{\mathbf{U}}_i \leq \mathbf{u})$ is the empirical cdf associated to the pseudo-observations of the sample. Nonetheless, the limiting behavior of all these test statistics are not distribution-free for composite zero assumptions, and limiting laws have to be evaluated numerically by Monte-Carlo methods (as usual).

Note that [77] have proposed a similar idea, but based on Spearman's dependence function L instead of Kendall's dependence function. Formally, L is defined by

$$L(u) := \mathbb{P}(C_\perp(\mathbf{U}) \leq u) = \mathbb{P}\left(\prod_{k=1}^d F_k(X_k) \leq u \right), \;\; \forall u \in [0,1].$$

When working with a random sample, the empirical counterpart of L is then

$$\hat{L}_n(u) := \frac{1}{n} \sum_{i=1}^n \mathbf{1}\left(C_\perp(\hat{\mathbf{U}}_i) \leq u \right),$$

and all the previous GOF test statistics may be applied. For instance, [8] proposed to use the Cramer-von-Mises statistic

$$T_n^{L,CvM} := \int_0^1 \left(\hat{L}_n - L_{\hat{\theta}_n} \right)^2 \hat{L}_n(du),$$

where $L(\theta)$ is the Spearman's dependence function of an assumed copula C_θ, and $\hat{\theta}_n$ is an estimate of the true parameter under the zero assumption.

1.4 GOF tests for some particular classes of copulas

Beside omnibus GOF tests, there exist other test statistics that are related to particular families of copulas only. We will not study such GOF tests when they are related to particular finite-dimensional parametric families (to decide whether C_0 is a Gaussian copula, for instance). Nonetheless, in this section, we will be interested in a rather unusual GOF problem: to say whether C_0 belongs to a particular infinite-dimensional parametric family of copulas. Among such large families, some of them are important in practice: the Archimedean family, the elliptical one, extreme-value copulas, vines, hierarchical Archimedean copulas etc.

1.4.1 Testing the Archimedeanity

All the previously proposed test statistics can be applied when $\mathscr{C}$ is an assumed particular Archimedean family, as in [90], [83]... Other test statistics, that are based on some analytical properties of Archimedean copulas, have been proposed too ([52], for instance). Interestingly, [46] proposed a graphical procedure for selecting a Archimedean copula (among several competitors), through a visual comparison between the empirical Kendall's function K_n and an estimated Kendall function obtained under a composite null hypothesis $\mathscr{H}_0$.

Now, we would like to test "$\mathscr{H}_0 : C$ is Archimedean" against the opposite, i.e., without any assumption concerning a particular parametric family. This problem has not received a lot of attention in the literature, despite its practical importance.

Consider first the (unknown) generator ϕ of the underlying bivariate copula C, i.e. $C(\mathbf{u}) = \phi^{-1}(\phi(u_1) + \phi(u_2))$ for every $\mathbf{u} = (u_1, u_2) \in [0,1]^2$. [46] proved that $V_1 := \phi(F_1(X_1))/\{\phi(F_1(X_1)) + \phi(F_2(X_2))\}$ is uniformly distributed on $(0,1)$ and that $V_2 := C(F_1(X_1)), F_2(X_2))$ is distributed as the Kendall's dependence function $K(t) = t - \phi(t)/\phi'(t)$. Moreover, V_1 and V_2 are independent. Since K can be estimated empirically, these properties provide a way of estimating ϕ itself (by ϕ_n). Therefore, as noticed in the conclusion of [46], if the underlying copula is Archimedean, then the r.v.

$$\hat{V}_1 := \phi_n(F_{1,n}(X_1))/\{\phi(F_{1,n}(X_1)) + \phi(F_{2,n}(X_2))\}$$

should be distributed uniformly on $(0,1)$ asymptotically. This observation can lead to some obvious GOF test procedures.

Another testing strategy starts from the following property, proved in [68]: a bivariate copula C is Archimedean iff it is associative (i.e. $C(u_1, C(u_2, u_3)) = C(C(u_1, u_2), u_3)$ for every triplet (u_1, u_2, u_3) in $[0,1]^3$) and satisfies the inequality $C(u,u) < u$ for all $u \in (0,1)$. This property, known as Ling's Theorem (see [64]), has been extended in an arbitrary dimension $d > 2$ by [89]. Then, [56] proposed to test the associativity of C to check the validity of the Archimedean zero assumption. For every couple (u_1, u_2) in $(0,1)^2$, he defined the test statistics

$$\mathscr{T}_n^J(u_1, u_2) := \sqrt{n}\{C_n(u_1, C_n(u_2, u_2)) - C_n(C_n(u_1, u_2), u_2)\}.$$

Despite its simplicity, the latter pointwise approach is not consistent against a large class of alternatives. For instance, there exist copulas that are associative but not Archimedean. Therefore, [15] revisited this idea, by invoking fully the previous characterization of Archimedean copulas. To deal with associativity, they introduced the trivariate process

$$\mathscr{T}_n(u_1,u_2,u_3) := \sqrt{n}\left\{C_n(u_1,C_n(u_2,u_3)) - C_n(C_n(u_1,u_2),u_3)\right\},$$

and proved its weak convergence in $\ell^\infty([0,1]^3)$. Cramer-von-Mises T_n^{CvM} and Kolmogorov-Smirnov T_n^{KS} test statistics can be build on $\mathscr{T}_n$. To reject associative copulas that are not Archimedean, these statistics are slightly modified to get

$$\tilde{T}_n^{CvM} := T_n^{CvM} + n^\alpha \psi\left(\max\left\{\frac{i}{n}(1-\frac{i}{n}) : C_n(\frac{i}{n},\frac{i}{n}) = \frac{i}{n}\right\}\right),$$

for some chosen constant $\alpha \in (0,1/2)$ and some increasing function ψ, $\psi(0)=0$. Therefore, such final tests are consistent against all departures from Archimedeanity.

Unfortunately, the two previous procedures are limited to bivariate copulas, and their generalization to higher dimensions d seems to be problematic.

1.4.2 Extreme-value dependence

As we have seen previously, bivariate extreme-value copulas are written as

$$C(u,v) = \exp\left\{\ln(uv)A(\frac{\ln(v)}{\ln(uv)})\right\}, \quad (1.2)$$

for every u,v in $(0,1)$, where $A:[0,1]\to[1/2,1]$ is convex and satisfies $\max(t,1-t)\leq A(t)\leq 1$ for every $t\in[0,1]$. Therefore, such copulas are fully parameterized by the so-called Pickands dependence function A, that is univariate. Extreme-value copulas are important in a lot of fields because they characterize the large-sample limits of copulas of componentwise maxima of strongly mixing stationary sequences ([26], [53], and the recent survey [50]). Then, it should be of interest to test whether whether the underlying copula can be represented by (1.2), for some unspecified dependence function A.

Studying the Kendall's process associated to an extreme-value copula C, [48] have noticed that, by setting $W := C(U_1,U_2)$, we have $K(t) = P(W\leq t) = t-(1-\tau)t\ln(t)$, for every $t\in(0,1)$, where τ is the underlying Kendall's tau. Moreover, they show that the moments of W are $E[W^i] = (i\tau+1)/(i+1)^2$, for all $i\geq 1$. Therefore, under $\mathscr{H}_0$, $-1+8E[W]-9E[W^2]=0$. Then they proposed a test (that the underlying copula is extreme-value) based on an empirical counterpart of the latter relation: set

$$T_n := -1 + \frac{8}{n(n-1)} \sum_{i \neq j} I_{ij} - \frac{9}{n(n-1)(n-2)} \sum_{i \neq j \neq k} I_{ij} I_{kj},$$

where $I_{ij} := \mathbf{1}(X_{i,1} \leq X_{j,1}, X_{i,2} \leq X_{j,2})$, for all $i, j \in \{1, \ldots, n\}$. Under $\mathcal{H}_0$, the latter test statistic is asymptotically normal. Its asymptotic variance has been evaluated in [7]. [78] has provided extensions of this idea towards more higher order moments of W.

These approaches rely on the so-called "reduction of dimension" techniques (see Section...). To improve the power of GOF tests, it would be necessary to work in functional spaces, i.e. concentrate on empirical counterparts of extreme-value copulas, or, equivalently, of the functions A themselves. For instance, [78] proposed a Cramer-von-Mises GOF test, based on the Kendall's function K above. More generally, several estimates of the Pickands dependence function are available, but most of them rely on the estimation of marginal distributions: see section 9.3 in [6] or [2]. Nonetheless, [42] have built "pure" copula GOF test statistics, i.e. independent from margins, by invoking empirical counterparts of the Pickands function introduced in [47]: given our previous notations,

1. define the pseudo-observations

$$\tilde{U}_i := nF_{n,1}(X_{i,1})/(n+1), \; \tilde{V}_i := nF_{n,1}(X_{i,2})/(n+1);$$

2. define the r.v. $\hat{S}_i := -\ln \tilde{U}_i$ and $\hat{T}_i := -\ln \tilde{V}_i$;
3. for every $i = 1, \ldots, n$, set $\hat{\xi}(0) := \hat{S}_i$, and $\hat{\xi}(1) := \hat{T}_i$. Moreover, for every $t \in (0,1)$, set

$$\hat{\xi}_i(t) := \min\left(\frac{\hat{S}_i}{1-t}, \frac{\hat{T}_i}{t}\right).$$

4. Two estimates of A are given by

$$A_n^P(t) := \left[n^{-1} \sum_{i=1}^n \hat{\xi}_i(t)\right]^{-1} \text{ and } A_n^{CFG}(t) := \exp\left(-\gamma - n^{-1} \sum_i^n \ln \hat{\xi}_i(t)\right),$$

where γ denotes the Euler constant.

The two latter estimates are the "rank-based" version of those proposed in [75] and [16] respectively.

There is an explicit one-to-one mapping between A_n^P (resp. A_n^{CFG}) and the empirical copula C_n. Therefore, after endpoint corrections, [47] have exhibited the weak limit of the corresponding processes $\mathbb{A}_n^P := \sqrt{n}(A_n^P - A)$ and $\mathbb{A}_n^{CFG} := \sqrt{n}(A_n^{CFG} - A)$. Working with the two latter processes instead of $\mathbb{C}_n$, a lot of GOF tests can be built. For instance , [42] have detailed an Anderson-Darling type test based on the L^2 norm of $\mathbb{A}_n^P$ and $\mathbb{A}_n^{CFG}$, even under composite null assumptions.

In the same vein, another strategy has been proposed in [61]: there is an equivalence between extreme-value copula C and max-stable copulas, i.e. copulas for which $C(\mathbf{u})^r = C(\mathbf{u}^r)$, for every $\mathbf{u} \in [0,1]^d$ and $r \in \mathbb{R}^+$. By setting $\mathbb{D}_n(\mathbf{u}) := \sqrt{n}(\{C_n(\mathbf{u}^{1/r})\}^r - C_n(\mathbf{u}))$, for all $\mathbf{u} \in [0,1]^d$ and every $r > 0$, [61] have built some tests based on the limiting law of the joint process $(\mathbb{D}_{n,r_1}, \ldots, \mathbb{D}_{n,r_p})$ for an arbitrary integer p.

1.4.3 Pair-copula constructions

In the recent years, a lot of effort has been devoted to the construction of d-dimensional copulas, $d > 2$, as combinations of several 2-dimensional copulas. Some authors have enriched the Archimedean copula class: Hierarchical, nested or multiplicative Archimedean copulas. Among others, see [57], [94], [67], [82], [69]. Other authors have studied the large class of vines: D-vines, C-vines, regular vines more generally (see [1], [20], e.g.). Inference, simulation and specification techniques have made significant progress to deal with these families of models $\mathscr{F}$. This advances provide large classes of very flexible copulas.

We will not discuss in depth the way of choosing the best Hierarchical Archimedean copula or the best D-vine, for a given data. Apparently, every proposition in this stream of the literature follows the same steps:

(i) Assume an underlying class of models $\mathscr{F}$ (D-vine, for instance);
(ii) Choose the potential bivariate families of copulas that may appear in the construction;
(iii) Evaluate the best structure (a network, or a tree), and estimate the associated bivariate copulas (simultaneously, in general).

Mathematically, we can nest this methodology inside the previous general GOF copula framework detailed above. Indeed, the copula candidates belong to a finite dimensional parametric family, even if the dimension of the unknown parameter θ can be very large. Obviously, authors have developed ad-hoc procedures to avoid such a violent approach of GOF testing: see [21] or [29] for vine selection, for instance.

At the opposite, there is no test of the slightly different and more difficult GOF problem

$$\mathscr{H}_0 : C \text{ belongs to a given class } \mathscr{F}.$$

For instance, a natural question would be to test whether an underlying copula belongs to the large (and infinite dimensional!) class of Hierarchical Archimedean copulas. To the best of our knowledge, this way of testing is still a fully open problem.

1.5 GOF copula tests for multivariate time series

One limiting feature of copulas is the difficulty to use them in the presence of multivariate dependent vectors $(\mathbf{X}_n)_{n\in\mathbb{Z}}$, with $\mathbf{X}_n \in \mathbb{R}^d$. In general, the "modeler problem" is to specify the full law of this process, i.e., the joint laws $(\mathbf{X}_{n_1},\ldots,\mathbf{X}_{n_p})$ for every p and every indices $n_1,\ldots,n_p$ and in a consistent way. Applying the copula ideas to such a problem seems to be rather natural (see [74] for a survey). Nonetheless, even if we restrict ourselves to stationary processes, the latter task is far from easy.

A first idea is to describe the law of the vectors $(\mathbf{X}_m,\mathbf{X}_{m+1},\ldots,\mathbf{X}_n)$ with copulas directly, for every couple (m,n), $m<n$. This can be done by modeling separately (but consistently) $d(n-m+1)$ unconditional margins plus a $d(n-m+1)$-dimensional copula. This approach seems particularly useful when the underlying process is stationary and Markov (see [17] for the general procedure). But the conditions of Markov coherence are complex (see [55]), and there is no general GOF strategy in this framework, to the best of our knowledge.

A more usual procedure in econometrics is to specify a multivariate time-series model, typically a linear regression, and to estimate residuals, assumed serially independent: see [18], that deals with a GARCH-like model with diagonal innovation matrix. They showed that estimating the copula parameters using rank-based pseudo-likelihood methods with the ranks of the residuals instead of the (non-observable) ranks of innovations, leads to the same asymptotic distribution. In particular, the limiting law of the estimated copula parameters does not depend on the unknown parameters used to estimate the conditional means and the conditional variances. This is very useful to develop goodness-of-fit tests for the copula family of the innovations. [79] extended these results: under similar technical assumptions, the empirical copula process has the same limiting distribution as if one would have started with the innovations instead of the residuals. As a consequence, a lot of tools developed for the serially independent case remain valid for the residuals. However, that is not true if the stochastic volatility is genuinely non-diagonal.

A third approach would be to use information on the marginal processes themselves. This requires to specify conditional marginal distributions, instead of unconditional margins as above in the first idea. This would induce a richer application of the two-step basic copula idea i.e., use "standard" univariate processes as inputs of more complicated multivariate models:

1. for every $j=1,\ldots,d$, specify the law of $X_{n,j}$ knowing the past values $X_{n-1,j},X_{n-2,j},\ldots$;
2. specify (and/or estimate) relevant dependence structures, "knowing" these univariate underlying processes, to recover the entire process $(\mathbf{X}_n)_{n\in\mathbb{Z}}$.

Using similar motivations, Patton ([72], [73]) introduced so-called conditional copulas, which are associated with conditional laws in a particular way. Specifically, let $\mathbf{X}=(X_1,\ldots,X_d)$ be a random vector from $(\Omega,\mathscr{A}_0,\mathbb{P})$ to $\mathbb{R}^d$. Consider some arbitrary sub-σ-algebra $\mathscr{A}\subset\mathscr{A}_0$. A conditional copula associated to $(\mathbf{X},\mathscr{A})$ is a $\mathscr{B}([0,1]^d)\otimes\mathscr{A}$ measurable function C such that, for any $x_1,\ldots,x_d\in\mathbb{R}$,

$$\mathbb{P}(\mathbf{X} \leq \mathbf{x}|\mathscr{A}) = C\left\{\mathbb{P}(X_1 \leq x_1|\mathscr{A}),\ldots,\mathbb{P}(X_d \leq x_d|\mathscr{A})|\mathscr{A}\right\}.$$

The random function $C(\cdot|\mathscr{A})$ is uniquely defined on the product of the values taken by $x_j \mapsto \mathbb{P}(X_j \leq x_j \mid \mathscr{A})(\omega)$, $j=1,\ldots,d$, for every realization $\omega \in \mathscr{A}$. As in the proof of Sklar's theorem, $C(\cdot|A)$ can be extended on $[0,1]^d$ as a copula, for every conditioning subset of events $A \subset \mathscr{A}$.

In Patton's approach, it is necessary to know/model each margin, knowing all the past information, and not only the past observations of each particular margin. Nonetheless, practitioners often have good estimates of the conditional distribution of each margin, conditionally given its own past, i.e., $\mathbb{P}(X_{n,j} \leq x_j|\mathscr{A}_{n,j})$, $j=1,\ldots,d$, by setting $\mathscr{A}_{n,j} = \sigma(X_{n-1,j}, X_{n-2,j},\ldots)$. To link these quantities with the (joint) law of $\mathbf{X}_n$ knowing its own past, it is tempting to write

$$\mathbb{P}(\mathbf{X}_n \leq \mathbf{x}|\mathscr{A}_n) = C^*\left\{\mathbb{P}(X_{1,n} \leq x_1|\mathscr{A}_{n,1}),\ldots,\mathbb{P}(X_{d,n} \leq x_d|\mathscr{A}_{n,d})\right\},$$

for some random function $C^* : [0,1]^d \longrightarrow [0,1]$ whose measurability would depend on $\mathscr{A}_n$ and on the $\mathscr{A}_{n,j}$, $j=1,\ldots,d$. Actually, the latter function is a copula only if the process $(X_{k,n}, k \neq j)_{n\in\mathbb{Z}}$ does not "Granger-cause" the process $(X_{j,n})_{n\in\mathbb{Z}}$, for every $j=1,\ldots,d$. This assumption that each variable depends on its own lags, but not on the lags of any other variable, is clearly strong, even though it can be accepted empirically; see the discussion in [74], pp. 772–773. Thus, [37] has extended Patton's conditional copula concept, by defining so-called pseudo-copulas, that are simply cdf on $[0,1]^d$ with arbitrary margins. They prove:

Theorem 1.5.1. *For any sub-algebras $\mathscr{B},\mathscr{A}_1,\ldots,\mathscr{A}_d$ such that $\mathscr{A}_j \subset \mathscr{B}$, $j=1,\ldots,d$, there exists a random function $C : [0,1]^d \times \Omega \longrightarrow [0,1]$ such that*

$$\begin{aligned}\mathbb{P}(\mathbf{X} \leq \mathbf{x} \mid \mathscr{B})(\omega) &= C\left\{\mathbb{P}(X_1 \leq x_1 \mid \mathscr{A}_1)(\omega),\ldots,\mathbb{P}(X_d \leq x_d \mid \mathscr{A}_d)(\omega), \omega\right\} \\ &\equiv C\left\{\mathbb{P}(X_1 \leq x_1 \mid \mathscr{A}_1),\ldots,\mathbb{P}(X_d \leq x_d \mid \mathscr{A}_d)\right\}(\omega),\end{aligned}$$

for every $\mathbf{x} = (x_1,\ldots,x_d) \in \mathbb{R}^d$ and almost every $\omega \in \Omega$. This function C is $\mathscr{B}([0,1]^d) \otimes \mathscr{B}$ measurable. For almost every $\omega \in \Omega$, $C(\cdot,\omega)$ is a pseudo-copula and is uniquely defined on the product of the values taken by $x_j \mapsto \mathbb{P}(X_j \leq x_j \mid \mathscr{A}_j)(\omega)$, $j=1,\ldots,d$.

If C is unique, it is called the conditional $(\mathscr{A},\mathscr{B})$-pseudo-copula associated with $\mathscr{X}$ and denoted by $C(\cdot|\mathscr{A},\mathscr{B})$. Actually, $C(\cdot|\ \mathscr{A},\mathscr{B})$ is a copula iff

$$\mathbb{P}(X_j \leq x_j \mid \mathscr{B}) = \mathbb{P}(X_j \leq x_j \mid \mathscr{A}_j) \quad \text{a.e.} \tag{1.3}$$

for all $j=1,\ldots,d$ and $\mathbf{x} \in \mathbb{R}^d$. This means that $\mathscr{B}$ cannot provide more information about X_j than $\mathscr{A}_j$, for every j. Patton's conditional copula corresponds to the particular case $\mathscr{B} = \mathscr{A}_1 = \cdots = \mathscr{A}_d$, for which (1.3) is clearly satisfied.

One key issue is to state if pseudo-copulas depend really on the past values of the underlying process, i.e., to test their constancy, an assumption often made in practice. In [37], they estimate nonparametrically conditional pseudo-copulas, including

Patton's conditional copulas as a special case, and test their constancy with respect to their conditioning subsets. Here, we specify their technique.

For a stationary and strongly mixing process $(\mathbf{X}_n)_{n\in\mathbb{Z}}$, we restrict ourselves to conditional sub-algebras $\mathscr{A}_n$ and $\mathscr{B}_n$ that are defined by a finite number of past values of the process, typically $(\mathbf{X}_{n-1}, \mathbf{X}_{n-2}, \ldots, \mathbf{X}_{n-p})$ for some $p \geq 1$. The dependence of $\mathscr{A}$ and $\mathscr{B}$ with respect to past values $\mathbf{y}$ will be implicit hereafter. Formally, [37] consider the test of several null hypothesis:

(a)

$$\mathscr{H}_0^{(1)}: \text{ For every } \mathbf{y},\, C(\cdot \mid \mathscr{A}, \mathscr{B}) = C_0(\cdot),$$

against

$$\mathscr{H}_a: \text{ For some } \mathbf{y},\, C(\cdot \mid \mathscr{A}, \mathscr{B}) \neq C_0(\cdot),$$

where C_0 denotes a fixed pseudo-copula function. In this case, $\mathscr{H}_0^{(1)}$ means that the underlying conditional $(\mathscr{A}, \mathscr{B})$-pseudo-copula is in fact a true copula, independent of the past values of the process.

(b)

$$\mathscr{H}_0^{(2)}: \text{ There exists a parameter } \theta_0 \text{ such that } C(\cdot|\mathscr{A}, \mathscr{B}) = C_{\theta_0} \in \mathscr{C}, \text{ for every } \mathbf{y},$$

where $\mathscr{C} = \{C_\theta, \theta \in \Theta\}$ denotes some parametric family of pseudo-copulas.

(c)

$$\mathscr{H}_0^{(3)}: \text{ For some function } \theta(\mathbf{y}) = \theta(\mathscr{A}, \mathscr{B}), \text{ we have}$$
$$C(\cdot|\mathscr{A}, \mathscr{B}) = C_{\theta(\mathbf{y})} \in \mathscr{C}, \text{ for every } \mathbf{y}.$$

The latter assumption says that the conditional pseudo-copulas stay inside the same pre-specified parametric family of pseudo-copulas (possibly copulas), for different observed values in the past. [37] proposed a fully nonparametric estimator of the conditional pseudo-copulas, and derived its limiting distribution. This provides a framework for "brute-force" GOF tests of multivariate dynamic dependence structures (conditional copulas, or even pseudo-copulas), similarly to what has been done in section 1.2.

[37] stated the equivalent of the empirical processes $\mathbb{C}_n$ or $\hat{\mathbb{C}}_n$. Use the short-hand notation $\mathbf{X}_m^n$ for the vector $(\mathbf{X}_m, \mathbf{X}_{m+1}, \ldots, \mathbf{X}_n)$. Similarly, write $\mathbf{X}_{m,j}^n = (X_{m,j}, \ldots, X_{n,j})$. Assume that every conditioning set $\mathscr{A}_{n,j}$ (resp. $\mathscr{B}_n$) is related to the vector $\mathbf{X}_{n-p,j}^{n-1}$ (resp. $\mathbf{X}_{n-p}^{n-1}$). Specifically, consider the events $(\mathbf{X}_{n-p}^{n-1} = \mathbf{y}^*) \in \mathscr{B}_n$, with $\mathbf{y}^* = (\mathbf{y}_1, \ldots, \mathbf{y}_p)$, and $(\mathbf{X}_{n-p,j}^{n-1} = \mathbf{y}_j^*) \in \mathscr{A}_{n,j}$, with $\mathbf{y}_j^* = (y_{1j}, \ldots, y_{pj})$. Their nonparametric estimator of the pseudo-copula is based on a standard plug-in technique that requires estimates of the joint conditional distribution

$$m(\mathbf{x} \mid \mathbf{y}^*) = \mathbb{P}\left(\mathbf{X}_p \leq \mathbf{x} \mid \mathbf{X}_0^{p-1} = \mathbf{y}^*\right),$$

and of conditional marginal cdf's

$$m_j(x_j \mid \mathbf{y}_j^*) = \mathbb{P}\left(X_{pj} \leq x_j \mid \mathbf{X}_{0,j}^{p-1} = \mathbf{y}_j^*\right), \quad j = 1, \ldots, d.$$

Let F_{nj} be the (marginal) empirical distribution function of X_j, based on the $(X_{1,j}, \ldots, X_{n,j})$. For convenient kernels K and $\bar{K}$, set

$$K_h(\mathbf{x}) = h^{-pd} K\left(\frac{x_1}{h}, \cdots, \frac{x_{pd}}{h}\right), \text{ and } \quad \bar{K}_{\bar{h}}(\mathbf{x}) = \bar{h}^{-p} \bar{K}\left(\frac{x_1}{\bar{h}}, \cdots, \frac{x_p}{\bar{h}}\right).$$

For every $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y}^* \in \mathbb{R}^{pd}$, estimate the conditional distribution $m(\mathbf{x} \mid \mathbf{y}^*) = \mathbb{P}\left(\mathbf{X}_p \leq \mathbf{x} \mid \mathbf{X}_0^{p-1} = \mathbf{y}^*\right)$ by

$$m_n(\mathbf{x} \mid \mathbf{y}^*) = \frac{1}{n-p} \sum_{\ell=0}^{n-p} K_n(\mathbf{X}_\ell^{\ell+p-1}) \mathbf{1}(\mathbf{X}_{\ell+p} \leq \mathbf{x}),$$

where

$$K_n(\mathbf{X}_\ell^{\ell+p-1}) = K_h\{F_{n1}(X_{\ell 1}) - F_{n1}(y_{11}), \ldots, F_{nd}(X_{\ell d}) - F_{nd}(y_{1d}), \ldots,$$
$$\ldots, F_{n1}(X_{(\ell+p-1),1}) - F_{n1}(y_{p1}), \ldots, F_{nd}(X_{(\ell+p-1),d}) - F_{nd}(y_{pd})\}.$$

Similarly, for all $x_j \in \mathbb{R}$ and $\mathbf{y}_j^* \in \mathbb{R}^p$, the conditional marginal cdf's $m_j(x_j \mid \mathbf{y}_j^*)$ is estimated in a nonparametric way by

$$m_{n,j}(x_j \mid \mathbf{y}_j^*) = \frac{1}{n-p} \sum_{\ell=1}^{n-p} \bar{K}_{\bar{h}}\{F_{nj}(X_{\ell,j}) - F_{nj}(y_{1j}), \ldots,$$
$$F_{nj}(X_{\ell+p-1,j}) - F_{nj}(y_{pj})\} \mathbf{1}(\mathbf{X}_{\ell+p,j} \leq x_j),$$

for every $j = 1, \ldots, d$. [37] proposed to estimate the underlying conditional pseudo-copula by

$$\widehat{C}(\mathbf{u} \mid \mathbf{X}_{n-1}^{n-p} = \mathbf{y}^*) = m_n\{m_{n,1}^{(-1)}(u_1 \mid \mathbf{y}_1^*), \ldots, m_{n,d}^{(-1)}(u_d \mid \mathbf{y}_d^*) \mid \mathbf{y}^*\},$$

with the use of pseudo-inverse functions. Then, under $\mathscr{H}_0^{(1)}$, for all $\mathbf{u} \in [0,1]^d$ and $\mathbf{y}^* = (\mathbf{y}_1, \ldots, \mathbf{y}_p) \in \mathbb{R}^{dp}$,

$$\sqrt{nh_n^{pd}}\,\{\widehat{C}(\mathbf{u} \mid \mathbf{X}_{n-1}^{n-p} = \mathbf{y}^*) - C_0(\mathbf{u})\} \xrightarrow{\mathrm{d}} \mathscr{N}[0, \sigma(\mathbf{u})]$$

as $n \to \infty$, where $\sigma(\mathbf{u}) = C_0(\mathbf{u})\{1 - C_0(\mathbf{u})\} \int K^2(\mathbf{v})\, d\mathbf{v}$. This result can be extended to deal with different vectors $\mathbf{y}^*$ simultaneously, and with the null hypotheses $\mathscr{H}_0^{(2)}$ and $\mathscr{H}_0^{(3)}$: for all $\mathbf{u} \in \mathbb{R}^d$,

$$\sqrt{nh_n^{pd}}\,\{\widehat{C}(\mathbf{u} \mid \mathbf{y}_1^*) - C_{\hat{\theta}_1}(\mathbf{u}), \ldots, \widehat{C}(\mathbf{u} \mid \mathbf{y}_q^*) - C_{\hat{\theta}_q}(\mathbf{u})\} \xrightarrow{\mathrm{d}} \mathscr{N}[0, \Sigma(\mathbf{u}, \mathbf{y}_1^*, \ldots, \mathbf{y}_q^*)],$$

as $n \to \infty$, where

$$\Sigma(\mathbf{u},\mathbf{y}_1^*,\ldots,\mathbf{y}_q^*) = \text{diag}\left(C_{\theta(\mathbf{y}_k^*)}(\mathbf{u})\{1 - C_{\theta(\mathbf{y}_k^*)}(\mathbf{u})\} \int K^2(\mathbf{v})\,d\mathbf{v},\ 1 \le k \le q \right),$$

for some consistent estimators $\hat{\theta}_k$ such that $\hat{\theta}_k = \theta(\mathbf{y}_k^*) + O_P(n^{-1/2}),\ k = 1,\ldots,q$.

Each kth term on the diagonal of Σ can be consistently estimated by

$$\hat{\sigma}_k^2(\mathbf{u}) = C_{\hat{\theta}_k}(\mathbf{u})\{1 - C_{\hat{\theta}_k}(\mathbf{u})\} \int K^2(\mathbf{v})\,d\mathbf{v}.$$

Note that, in the corollary above, the limiting correlation matrix is diagonal because we are considering different conditioning values $\mathbf{y}_1^*,\ldots,\mathbf{y}_q^*$ but the same argument $\mathbf{u}$. At the opposite, an identical conditioning event but different arguments $\mathbf{u}_1,\mathbf{u}_2,\ldots$ would lead to a complex (non diagonal) correlation matrix, as explained in [33]. The latter weak convergence result of random vectors allows the building of GOF tests as in section 1.2. For instance, as in [33], a simple test procedure may be

$$T(\mathbf{u},\mathbf{y}_1^*,\ldots,\mathbf{y}_q^*) = (nh_n^{pd}) \sum_{k=1}^{q} \frac{\{\widehat{C}(\mathbf{u} \mid \mathbf{X}_{n-1}^{n-p} = \mathbf{y}_k^*) - C_{\hat{\theta}_k}(\mathbf{u})\}^2}{\hat{\sigma}_{\mathbf{y}_k^*}^2(\mathbf{u})},$$

for different choices of $\mathbf{u}$ and conditioning values $\mathbf{y}_k^*$. Under $\mathscr{H}_0^{(1)}$, the term on the right-hand-side tends to a $\chi^2(q)$ distribution under the null hypothesis. Note that this test is “local” since it depends strongly on the choice of a single $\mathbf{u}$. An interesting extension would be to build a “global” test, based on the behavior of the full process

$$\sqrt{nh_n^{pd}}\,\{\widehat{C}(\cdot \mid \mathbf{X}_{n-1}^{n-p} = \mathbf{y}_k^*) - C_{\hat{\theta}_k}(\cdot)\}.$$

But the task of getting pivotal limiting laws is far from easy, as illustrated in [33].

In practice, authors often restrict themselves to the case of time-dependent copula parameters instead of managing time-dependent multivariate cdfs’ nonparametrically. For instance, every conditional copula or pseudo-copula is assumed to belong to the Clayton family, and their random parameters θ depend on the past observations. [3] has proposed a non-parametric estimate $\hat{\theta}(\cdot)$ of the function θ, in the case of a univariate conditioning variable. It seems possible to build some GOF tests based on this estimate and its limiting behavior, at least for simple null hypothesis, but the theory requires more developments.

1.6 Practical performances of GOF copula tests

Once a paper introduces one or several new copula GOF tests, it is rather usual to include an illustrative section. Typically, two characteristics are of interest for some tests in competition: their ability to maintain the theoretical levels powers, and their power performances under several alternatives. Nonetheless, these empirical elements, even useful, are often partial and insufficient to found a clear judgement. Actually, only a few papers have studied and compared the performances of the main previous tests in depth. Indeed, the calculation power required for such a large analysis is significant. That is why a lot of simulation studies restrict themselves to bivariate copulas and small or moderate sample sizes (from $n = 50$ to $n = 500$, typically). The most extensive studies of finite sample performances are probably those of [8] and [45]. In both papers, the set of tests under scrutiny contains the three main approaches:

1. "brute-force" proposals like T_n^{KS} and/or T_n^{CvM}, as in section 1.2;
2. Kendall's process based tests;
3. test statistics invoking the PIT (see section 1.3).

These works found that a lot of tests perform rather well, even for small samples (from $n = 50$, e.g.). Moreover, it is difficult to exhibit clear hierarchy among all of these tests in terms of power performances. As pointed out by [45],

> No single test is preferable to all others, irrespective of the circumstances.

In their experiments, [45] restricted themselves to bivariate copulas and small sample sizes $n \in \{50, 150\}$. The statistics based on Kendall's dependence function are promoted, particularly when the underlying copula is assumed to be Archimedean. It appeared that Cramer-von-Mises style test statistics are preferable to Kolmogorov-Smirnov ones, all other things being equal, and whatever the possible transformations of the data and/or the reductions of information. Among the tests based on a Cramer-von-Mises statistic, it is difficult to discriminate between the three main approaches.

The latter fact is confirmed in [8], that led some simulated experiments with higher dimensions $d \in \{2, 4, 8\}$ and larger sample sizes $n \in \{100, 500\}$. [8] observed the particularly good performances of a new test statistic, calculated as the average of the three approaches. Moreover, he studied to impact of the variables ordering in the PIT. Even if estimated p-values may be different, depending on which permutation order is chosen, this does not seem to create worrying discrepancies.

Notably [11] led an extensive simulated experiment of the same type, but their main focus was related to detecting small departures from the null hypothesis. Thus, they studied the asymptotic behavior of some GOF test statistics under sequences of alternatives of the type

$$\mathscr{H}_{a,n} : C = (1 - \delta_n) C_0 + \delta_n D,$$

where $\delta_n = n^{-1/2}\delta$, $\delta > 0$, and D is another copula. They computed local power curves and compared them for different test statistics. They showed that the estimation strategy can have a significant impact on the power of Cramer-von-Mises statistics and that some "moment-based" statistics provide very powerful tests under many distributional scenarios.

Despite the number of available tests in the literature, the usefulness of all these procedures in practice has to be proved more convincingly. Apparently, some authors have raised doubts about the latter point. For instance, [91] has evaluated the performances of Value-at-Risk or VaR (quantiles of loss) and Expected Shortfall or ES (average losses above a VaR level) forecasts, for a large set of portfolios of two financial assets and different copula models. They estimate static copula models on couples of asset return residuals, once GARCH(1,1) dynamics have been fitted for every asset independently. They applied three families of GOF tests (empirical copula process, PIT, Kendall's function) and five copula models. They found that,

> Although copula models with GARCH margins yield considerably better estimates than correlation-based models, the identification of the optimal parametric copula form is a serious unsolved problem.

Indeed, none of the GOF tests is able to select the copula family that yields the best VaR- or ES-forecasts. This points out the difficulty of finding relevant and stable multivariate dynamics models, especially related to joint extreme moves. But, such results highlight the fact that it remains a significant the gap between good performances with simulated experiments and trustworthy multivariate models, even validated formally by statistical tests.

Indeed, contrary to studies based on simulated samples drawn from an assumed copula family (the standard case, as in [45] or [8]), real data can suffer from outliers or measurement errors. This is magnified by the fact that most realistic copulas are actually time-dependent ([91]) and/or are mixtures or copulas ([63]). Therefore, [92] showed that even minor contamination of a dataset can lead to significant power decreases of copula GOF tests. He applied several outlier detection methods from the theory of robust statistics, as in [66], before leading the formal GOF test of any parametric copula family. [92] concluded that the exclusion of outliers can have a beneficial effect on the power of copula GOF tests.

References

1. Aas, K., Czado, C., Frigessi, A., Bakken, H.: Pair-copula construction of multiple dependence. Insurance: Mathematics and Economics **44**, 182-198 (2009)
2. Abdous, B., Ghoudi, K.: Non-parametric estimators of multivariate extreme dependence functions. Journal of Nonparametric Statistics **17**, 915–935 (2005)
3. Abegaz, F., Gijbels, I., Veraverbeke, N.: Semiparametric estimation of conditional copulas. J. Multivariate Anal. **110**, 43–73 (2012)
4. Andersen, P.K., Ekstrøm, C.T., Klein, J.P., Shu, Y., Zhang, M.-J.: A class of goodness of fit tests for a copula based on bivariate right-censored data. Biometrical J. **47**, 815–824 (2005)
5. Barbe, P., Genest, C., Ghoudi, K., Rémillard, B.: On Kendall's process. J. Multivariate Anal. **58**, 197–229 (1996)
6. Beirlant, J., Goegebeur, Y., Segers, J., Teugels, J.: Statistics of Extremes: Theory and Applications. Wiley, Chichester (2004)
7. Ben Ghorbal, N., Genest, C., Nešlehová, J.: On the Ghoudi, Koudraji and Rivest test for extreme-value dependence. Can. J. Statistics **37**, 534–552 (2009)

8. Berg, D.: Copula goodness-of-fit testing: an overview and power comparison. European J. Finance **15**, 675–705 (2009)
9. Berg, D., Bakken, H.: Copula goodness-of-fit tests: A comparative study. Working paper, Norwegian Computing Centre (2006)
10. Berg, D., Bakken, H.: A copula goodness-of-fit approach based on the conditional probability integral transformation. Norwegian Computing Centre, SAMBA/41/05 (2007)
11. Berg, D., Quessy, J.-F.: Local sensitivity analysis of goodness-of-fit tests for copulas. Scandinavian J. Statist. **36**, 389–412 (2009)
12. Breymann, W., Dias, A., Embrechts, P.: Dependence structures for multivariate high-frequency data in finance. Quantitative Finance **1**, 1–14 (2003).
13. Bücher, A., Dette, H.: Some comments on goodness-of-fit tests for the parametric form of the copula based on L^2-distances. J. Multivariate Anal. **101**, 749–763 (2010)
14. Bücher, A., Dette, H.: A note on bootstrap approximations for the empirical copula process. Stat. Prob. Letters **80**, 1925–1932 (2010)
15. Bücher, A., Dette, H., Volgushev, S.: A test for Archimedeanity in bivariate copula models. J. Multivariate Anal. **110**, 121–132 (2012)
16. Capéraà, P., Fougères, A.-L., Genest, C.: A nonparametric estimation procedure for bivariate extreme value copulas. Biometrika **84**, 567–577 (1997)
17. Chen, X., Fan, Y.: Estimation of copula-based semiparametric time series models. J. Econometrics **130**, 307-335 (2006a)
18. Chen, X., Fan, Y.: Estimation and model selection of semiparametric copula-based multivariate dynamic models under copula misspecification. J. Econometrics **135**, 125-154 (2006b)
19. Chen, X., Fan, Y., Patton, A.: Simple tests for models of dependence between multiple financial time series, with applications to U.S. equity returns and exchange rates. London School of Economics, Discussion paper 483 (2004)
20. Czado, C.: Pair-copula constructions of multivariate copulas. In: Jaworski, P., Durante, F., Härdle, W., Rychlik, T. (eds.): Copula Theory and Its Applications, Proceedings of the Workshop Held in Warsaw 25-26 September 2009, Lecture Notes in Statistics – Proceedings, vol. 198. Springer, Dordrecht (2010).
21. Czado, C., Schepsmeier, U., Min, A.: Maximum likelihood estimation of mixed C-vines with applications to exchange rates. Statistical Modelling: An International Journal **12**, 229–255 (2012)
22. D'Agostino, R.B., Stephens, M.A.: Goodness-of-Fit Techniques. Marcel Dekker, New-York (1986)
23. Deheuvels, P.: La fonction de dépendance empirique et ses propriétés. Un test non paramétrique d'indépendance. Acad. R. Belg., Bull. Cl. Sci., 5 Série **65**, 274–292 (1979)
24. Deheuvels, P.: A Kolmogorov-Smirnov type test for independence and multivariate samples. Rev. Roum. Math. Pures et Appl. XXVI **2**, 213–226 (1981a)
25. Deheuvels, P.: A Nonparametric Test of Independence. Publications de l'ISUP **26**, 29–50 (1981b)
26. Deheuvels, P.: Probabilistic aspects of multivariate extremes. In: Statistical Extremes and Applications (J. Tiago de Oliveira, ed.) 117-130. Reidel, Dordrecht (1984)
27. Dias, A., Embrechts, P.: Dynamic copula models for multivariate high-frequency data in finance. Working paper, New University of Lisbon, Portugal (2004)
28. Diks, C., Panchenko, V., van Dijk, D.: Out-of-sample comparison of copula specifications in multivariate density forecasts. J. Econ. Dynamics & Control **34**, 1596–1609 (2010)
29. Dissmann, J., Brechmann, E. C., Czado, C., Kurowicka, D.: Selecting and estimating regular vine copulae and application to financial returns. Working paper (2012)
30. Dobric, J., Schmid, F.: Testing goodness-of-fit for parametric families of copulas. Applications to financial data. Comm. Statist., Simulation and Computation **34**, 1053–1068 (2005)
31. Dobric, J., Schmid, F.: A goodness of fit test for copulas based on Rosenblatt's transformation. Comput. Statist. Data Anal. **51**, 4633–4642 (2007)
32. Fan, Y., Linton, O.: Some higher order theory for a consistent nonparametric model specification test. J. Statist. Planning Inference **109**, 125–154 (2003)

33. Fermanian, J-D.: Goodness-of-fit tests for copulas. J. Multivariate Anal. **95**, 119-152 (2005)
34. Fermanian, J.-D., Scaillet, O.: Some statistical pitfalls in copula modelling for financial applications. In Klein, E. (ed.): Capital formation, governance and banking, Nova Pub. 59–74 (2005)
35. Fermanian, J.-D., Radulovic, D., Wegkamp, M.: Weak convergence of empirical copula processes. Bernoulli **10**, 847–860 (2004)
36. Fermanian, J.-D., Radulovic, D., Wegkamp, M.: An asymptotc total variation test for copulas. Working paper (2012)
37. Fermanian, J.-D., Wegkamp, M.: Time-dependent copulas. J. Multivariate Anal. **110**, 19–29 (2012)
38. Gaënsler, P., Stute, W.: Seminar on empirical processes. Birkhäuser Verlag (1987)
39. Gayraud, G.,Tribouley, K.: A test of goodness-of-fit for copulas. Test **20**, 549–573 (2011)
40. Genest, C., Quessy J.-F., Rémillard, B.: Goodness-of-fit procedures for copula models based on the probability integral transform. Scandin. J. Statist. **33**, 337–366 (2006)
41. Genest, C., Quessy J.-F., Rémillard, B.: On the joint asymptotic behavior of two rank-based estimators of the association parameterin the gamma fraily model. Statist. Probab. Letters **76**, 10–18 (2006)
42. Genest, C., Kojadinovic, I., Nešlehová, J., Yan, J.: Bernoulli **17**, 253-275 (2011)
43. Genest, C., Masiello, E., Tribouley, K.: Estimating copula densities through wavelets. Insurance: Mathematics & Economics **44**, 170–181 (2009)
44. Genest, C., Rémillard, B.: Validity of the parametric bootstrap for goodness-of-fit testing in semiparametric models. Ann. Inst. Henri Poincaré, Prob. et Stat. **44**, 1096–1127 (2008)
45. Genest, C., Rémillard, B., Beaudoin, D.: Goodness-of-fit tests for copulas: a review and a power study. Insurance: Mathematics and Economics **44**, 199–213 (2009)
46. Genest C., Rivest, L.: Statistical inference procedures for bivariate archimedean copulas. J. American Statist. Ass. **88**, 1034-1043 (1993)
47. Genest, C., Segers, J.: Rank-based inference for bivariate extreme-value copulas. Ann. Statist. **37**, 2990-3022 (2009)
48. Ghoudi, K., Khoudraji, A., Rivest, L.-P.: Propriétés statistiques des copules de valeurs extrêmes bidimensionnelles. Canad. J. Statist. **26**, 187–197 (1998)
49. Ghoudi, K., Rémillard, B.: Empirical processes based on pseudo-observations. In: Szyskowitz, B. (ed.): Asymptotic methods in probability and statistics: a volume in honor of Miklos Csörgö, 171–197, Elsevier, Amsterdam (1998)
50. Gudendorf, G., Segers, J.: Extreme-value copulas. In: Jaworski, P., Durante, F., Härdle, W., Rychlik, T. (eds.): Copula Theory and Its Applications, Proceedings of the Workshop Held in Warsaw 25-26 September 2009, Lecture Notes in Statistics – Proceedings, vol. 198. Springer, Dordrecht (2010).
51. Härdle, W., Mammen, E.: Testing parametric versus nonparametric regression. Ann. Statist. **21**, 1926–1947 (1993)
52. Hering, C., Hofert, M.: Goodness-of-fit tests for Archimedean copulas in large dimensions. Working paper (2010)
53. Hsing, T.: Extreme value theory for multivariate stationary sequences. J. Multivariate Anal. **29**, 274–291 (1989)
54. Huard, D., Evin, G., Favre, A.-C.: Bayesian copula selection. Comput. Statist. & Data Anal. **51**, 809-822 (2006)
55. Ibragimov, R.: Copula-based characterizations for higher-order Markov processes. Econometric Theory **25**, 819–846 (2009)
56. Jaworski, P.: Testing Archimedeanity. In: Borglet, C., Gonzalez-Rodrigez, G., Trutschnig, W., Lubiano, M., Gil, M., Grzegorzewski, P., Hryniewicz, O. (eds.): Combining soft comuting and statistical methods in data analysis, Advances in Intelligent and Soft Computing **77** 353–360, Springer, Berlin (2010)
57. Joe, H.: Multivariate Models and Dependence Concepts. Chapman & Hall, London (1997)
58. Junker, M., May, A.: Measurement of aggregate risk with copulas. Econometrics J. **8**, 428–454 (2005)

59. Klugman, S., Parsa, R.: Fitting bivariate loss distributions with copulas. Insurance: Math. & Econom. **24**, 139–148 (1999)
60. Kojadinovic, I., Yan, J.: Fast large-sample goodness-of-fit for copulas. Statist. Sinica **21**, 841–871 (2011)
61. Kojadinovic, I., Segers, J., Yan, J.: Large-sample tests of extreme-value dependence for multivariate copulas. Canad. J. Statist. **39**, 703–720 (2011)
62. Kojadinovic, I., Yan, J.: A goodness-of-fit test for multivariate multiparameter copulas based on multiplier central limit theorems. Stat. Computing **21**, 17–30 (2011).
63. Kole, E., Koedijk, K.C.G., Verbeek, M.: Selecting copulas for risk management. J. Banking & Finance **31**, 2405–2423 (2007)
64. Ling, C.M.: Representation of associative functions. Publ. Math. Debrecen **12**, 189–212 (1965)
65. Malevergne, Y., Sornette, D.: Testing the Gaussian copula hypothesis for financial assets dependence. Quantitative Finance **3**, 231–250 (2003)
66. Mendes, B., de Melo, E., Nelsen, R.: Robust fits for copula models Comm. Statist, Simulation & Computation. **36**, 997–1017 (2007)
67. Morillas, P.: A method to obtain new copulas from a given one. Metrika **61**, 169-184 (2005)
68. Nelsen, R.B.: An introduction to copulas, Lecture Notes in Statistics, vol. 139. Springer-Verlag, New York (1999)
69. Okhrin, O., Okhrin, Y., Schmid, W.: On the structure and estimation of hierarchical Archimedean copulas. Discussion Paper 2009-014, Humbolt-Univerzität zu Berlin, Germany (2009)
70. Omelka, M. Gijbels, I., Veraverbeke, N.: Improved kernel estimation of copulas: weak convergence and goodness-of-fit testing. Ann. statist. **37**, 3023–3058 (2009)
71. Panchenko, V.: Goodness-of-fit test for copulas. Physica A **335**, 176–182 (2005)
72. Patton, A.: Modelling Asymmetric Exchange Rate Dependence. International Economic Review, **47**, 527–556 (2006)
73. Patton A.: Estimation of multivariate models for time series of possibly different lengths. Journal of Applied Econometrics, **21**, 147–173 (2006)
74. Patton, A.: Copula based models for financial time series. In: T.G. Andersen, R.A. Davis, J.-P. Kreiss, T. Mikosch (Eds.): Handbook of Financial Time Series, Springer, 767–785 (2009).
75. Pickands, J.: Multivariate extreme value distributions (with a discussion). In Proceedings of the 43rd Session of the International Statistical Institute. Bull. Inst. Internat. Statist. **49**, 859–878, 894–902 (1981)
76. Prokhorov, A.: A goodness-of-fit test for copulas. Working paper, Columbia University (2008)
77. Quessy, J.-F., Mesfioui, M., Toupin, M.-H.: On a new goodness-of-fit process for families of copulas. Canad. J. Statist. **37**, 80–101 (2009)
78. Quessy, J.-F.: Testing for Bivariate Extreme Dependence Using Kendalls Process. Scandinavian J. Statist. **39**, 497–514 (2011)
79. Rémillard, B.: Goodness-of-fit tests for copulas of multivariate time series. Working paper (2010)
80. Rosenblatt, M.: Remarks on a multivariate transformation. Ann. Math. Statist. **23**, 470–472 (1952)
81. Rüschendorf, L.: Asymptotic distributions of multivariate rank order statistics. Ann. Statist. **4**, 912–923 (1976)
82. Savu, C., Trede, M.: Hierarchical Archimedean copulas. In: International Conference on High Frequency Finance, Konstanz, Germany, May (2006)
83. Savu, C., Trede, M.: Goodness-of-fit tests for parametric families of Archimedean copulas. Quantitative Finance **8**, 109–116 (2008)
84. Saunders, R., Laud, P.: The multidimensional Kolmogorov-Smirnov test. Biometrika **67**, 237 (1980)
85. Scaillet, O.: Kernel based goodness-of-fit tests for copulas with fixed smoothing parameters. J. Multivariate Anal. **98**, 533–543 (2007)
86. Rémillard, B., Scaillet, O.: Testing the equality between two copulas. J. Multivariate Anal. **100**, 377–386 (2009)

87. Segers, J.: Asymptotics of empirical copula processes under nonrestrictive smoothness assumptions. Bernoulli **26**, 288–305 (2012)
88. Shih, J.H.: A goodness-of-fit test for association in a bivariate survival model. Biometrika **85**, 189–200 (1998)
89. Stupňanová, A., Kolesárová, A.: Associative *n*-dimensional copulas. Kybernetika **47**, 93–99 (2011)
90. Wang,W.,Wells, M.T.: Model selection and semiparametric inference for bivariate failure-time data. J. Amer. Statist. Ass. **95**, 62–72 (2000)
91. Weiss, G.: On the robustness of goodness-of-fit tests for copulas. Discussion paper, Dortmund (2011)
92. Weiss, G.: Are copula-GOF-test of any practical use ? Empirical evidence for stocks, commodities and FX futures. Quarterly Review Econ. Finance **51**, 173–188 (2011)
93. White, H.: Maximum likelihood estimation in misspecified models. Econometrica **50**, 1–26 (1982)
94. Whelan, N.: Sampling from Archimedean copulas. Quantitative Finance **4**, 339–352 (2004)