Strategic asset allocation with switching dependence.

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Abstract

In this paper, we revisit the problem of the strategic asset allocation between stocks and bonds. The novelty of our analyze is to consider this issue when marginal distributions of benchmarks tracked by assets, and their dependence structure both depend on the same unobservable Markov chain. Benchmarks chosen in this work are the CAC 40 and the SGI Bond 10 years. In this setting, we investigate the performance of static and dynamic strategies. The dynamic investment policy proposed in this paper, is based upon the estimated probabilities of sojourn in each state of the Markov chain. Even if the Markov chain ruling the assets dynamics is hidden, a Bayesian procedure can indeed be applied to infer the probabilities of being in a certain state. The asset allocation is then adapted to provide the highest yield in the most likely state. Results of Monte-Carlo simulation show that dynamic strategies perform better than static policies with a limited risk and an acceptable number of reallocations.

Keywords : copula, switching regime.
JEL Classification: C6

1 Introduction.

Strategic asset allocation is a matter of concerns for all institutional investors, such insurance companies or pension funds, which have long term commitments with respect to insureds. Ideally, the investment policy should optimize the percentage of stocks and bonds that the company hold to achieve a reasonable return compared to the minimal guarantee of products sold to customers, whatsoever the trends followed by the markets. The strategic asset allocation usually set up by the asset-liability management (ALM) department and consists in guidelines in terms of minimal and maximal percentages per asset classes. The ALM department also recommends the benchmarks tracked by each category of assets. The department in charge of managing investments optimizes the assets allocation but without crossing the borders delimited by the ALM.

Modeling correctly the benchmarks driving the categories of investment and their dependence is a central issue to determine an assets allocation policy. To achieve this goal, the use of copulas allows to separate the dependence model from the marginal distributions. The interest reader may refer to Nelsen (1999) for an introduction on this topic. Recent publications have focused on the modeling of non linear dependence with copulas, driven by switching regime. E.g. Guidolin & Timmermann (2005, 2006) have modeled the joint distribution of US stock and bond returns in the presence of regime switching dynamics. In a similar framework, Hennessy and Lapan (2002) have studied the asset allocation problem when the dependence is modeled by an Archimedian copula. Switching regimes are particularly well adapted to model change of interdependence level such as those observed during the last major crisis of 2008. Rodriguez (2007) have modeled the financial contagion occurring during crisis by a two states switching copula. Pelletier (2006) has studied the change of correlation regimes in a Gaussian framework. More recently, Okimoto (2008) has brought new evidences of the existence of distinct switching regimes for the US and UK equity

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markets.

In this paper, we revisit the problem of the strategic asset allocation between stocks and bonds. The novelty of our approach is to consider this issue when marginal distributions of benchmarks and their dependence structure both depend on the same unobservable Markov chain. To illustrate our work, we have chosen as benchmark for stocks, the French index, CAC 40, of the forties biggest French companies. Whilst the benchmark tracked by bonds is the SGI Bond 10 years, which is an index computed by the French bank "Société Générale". This index duplicates a portfolio of Euro bonds continuously rebalanced to keep a constant maturity of 10 years.

After the presentation of the calibration procedure, we investigate the one year return of static and dynamic investment strategies. Monte-Carlo simulations reveal that the static investment strategy is widely influenced by the current state of the hidden Markov chain. The dynamic investment policy proposed in this work, takes benefit from the estimated state probabilities of the Markov chain. Indeed, even if the Markov chain ruling the assets dynamics is hidden, a Bayesian procedure may be used to infer the probabilities of being in a certain state. If the Markov chain is in a certain state with a high probability, the investment policy is adapted to provide the highest yield in this state. Monte-Carlo simulations tend to show that the performance of such policies is above the return achieved by static strategies, with a limited risk and acceptable number of portfolio reallocations.

2 The states of the world.

In this work, we assume that the behavior of all assets is driven by a hidden Markov process, which defines a finite number of states of the world. The main motivation to work within this framework is to model the fact that returns of financial assets switch from time to time, between a stable low-volatility state and a more unstable high volatility regime. In the next sections, each state of the Markov process will be linked to a set of parameters for assets returns densities and dependence structure. The non observable process driving financial markets is noted α_t and takes its value in the set $\mathcal{N} = \{1, 2, ...N\}$. The generator of α_t is a $N \times N$ matrix Q, whose elements, noted $q_{i,j}$, satisfy the following conditions:

$$q_{i,j} \geq 0 \quad \forall\, i \neq j \qquad\qquad \sum_{j=1}^{N} q_{i,j} = 0 \quad \forall i \in \mathcal{N}\ . \tag{2.1}$$

The probabilities of transition between states between times t and $s \geq t$ are obtained as the (matrix) exponential of Q :

$$P(t,s) = \exp\left(Q(s-t)\right)\ . \tag{2.2}$$

The elements of the matrix $P(t,s)$ are noted $p_{i,j}(t,s)$ $i,j \in \mathcal{N}$. And $p_{i,j}(t,s)$ is the probability of jumping from state i at time t to state j at time s :

$$p_{i,j}(t,s) \quad = \quad P(\alpha_s = j \,|\, \alpha_t = i) \quad i,j \in \mathcal{N}\,. \tag{2.3}$$

The probability of being in state i at time t, noted $p_i(t)$ depends upon the initial probabilities $p_{k=1..N}(0)$ at time $t = 0$, as follows:

$$p_i(t) \quad = P(\alpha_t = i) = \quad \sum_{k=1}^{N} p_k(0) p_{k,i}(0,t) \quad \forall i \in \mathcal{N}\ . \tag{2.4}$$

However, if the Markov process has been running for a sufficiently long enough period of time, we can show that this probability is independent from the initial state:

$$p_i = \lim_{t \to +\infty} p_i(t) \qquad \forall i \in \mathcal{N}\ . \tag{2.5}$$

This is a standard property of Markov chains (often termed as the stationary property). The probabilities $p_{i=1...N}$ can be computed from the limit of equation (2.2). In the calibration procedure, we have chosen to work with three states $N = 3$. Working with more states is theoretically possible, but it increases widely the number of parameters to estimate and is a source of numerical instability.

3 Marginal distribution of assets.

As mentioned earlier, our aim is to revisit the problem of strategic allocation between two classes of assets: stocks and bonds. The theory of copula allows us to define separately the marginal distributions of assets and their structure of dependence. For this reason, we first briefly describe the main features of marginal distributions chosen to model assets log-returns in the forthcoming numerical application: namely the Student t and the Weibull distributions. The copulas chosen to model the dependence are detailed in the following section. Note that, in our framework, both parameters of marginal distributions and copulas depend on the state in which the hidden Markov chain, α_t, lies. The main consequence of this approach is that the calibration of marginal laws and of the related copula must be done at the same time. We will come back on this point later. In the remainder, we assume that stocks and bonds track two benchmarks whose the values are respectively noted S^1 and S^2. In the forthcoming application, values of those indices are observed at equidistant times $t_i \quad i = 1, ..., T$. The constant size of intervals $[t_i, t_{i+1}]$ is noted Δt. On the interval of time $[t_i,\, t_{i+1}]$, the log-returns are noted $Y^k_{t_{i+1}}$, for $k = 1, 2$.

3.1 Student t distribution.

The first marginal distribution considered in this work, is the Student t distribution whose the parameters $\mu_k(.)\,, \sigma_k(.)\,, n_k(.)$ depend on the state of world at time t_i, α_{t_i}:

$$Y^k_{t_{i+1}} = \log \frac{S^k_{t_{i+1}}}{S^k_{t_i}} \quad = \quad \mu_k(\alpha_{t_i}) + \sigma_k(\alpha_{t_i})\, T^k_{n_k(\alpha_{t_i})} \qquad k = 1, 2$$

and where $T^k_{n_k(\alpha_{t_i})}$ is a Student t random variable having $n_k(\alpha_{t_i})$ degrees of freedom. The student t distribution includes the normal distribution (whether $n_k(.)$ tends to $+\infty$) and the Cauchy distribution (whether $n_k(.) = 1$) as special cases. The choice of this distribution is mainly motivated by its ability to capture the excess kurtosis of stocks returns. The marginal density function of the k^{th} asset return on the period $[t_i, t_{i+1}]$, $Y^k_{t_{i+1}}$, is noted $f_k(t_{i+1}, .)$ and is given by the following expression :

$$f_k(t_{i+1}\,,\, y) \quad = \quad \frac{\Gamma((n_k(\alpha_{t_i}) + 1)/2)}{\sqrt{\pi n_k(\alpha_{t_i})\sigma_k^2(\alpha_{t_i})}\Gamma(n_k(\alpha_{t_i})/2)} \left(1 + \frac{\left(\frac{y-\mu_k(\alpha_{t_i})}{\sigma_k(\alpha_{t_i})}\right)^2}{n_k(\alpha_{t_i})}\right)^{-\frac{n_k(\alpha_{t_i})+1}{2}} \tag{3.1}$$

where $\Gamma(.)$ is the gamma function. The standard moments of returns in state α_t are:

$$\begin{aligned}
\mathbb{E}\left(Y^k_{t_{i+1}}\right) &= \mu_k(\alpha_{t_i}) \\
Std\left(Y^k_{t_{i+1}}\right) &= \sqrt{\frac{n_k(\alpha_{t_i})}{n_k(\alpha_{t_i}) - 2}}\sigma \\
Sk\left(Y^k_{t_{i+1}}\right) &= 0 \\
Ku\left(Y^k_{t_{i+1}}\right) &= 3 + \frac{6}{n_k(\alpha_{t_i}) - 4}
\end{aligned}$$

As the Gaussian, The Student t distribution is a symmetric function and in this sense does not perfectly fit distribution of returns which are slightly asymmetric. To remedy this situation, we have also tested the Weibull distribution to model marginal returns.

3.2 Weibull distribution.

The Weibull distribution is also said Stretched-exponential distribution because it can decay more slowly than an exponential distribution. As for the student t distribution, we have assumed that parameters of depends on the Markov process α_t. In this case, the log-returns are assumed to be equal to

$$Y^k_{t_{i+1}} = \log \frac{S^k_{t_{i+1}}}{S^k_{t_i}} \quad = \quad \mu_k(\alpha_{t_i}) + W^k_{\lambda_k(\alpha_{t_i}),n_k(\alpha_{t_i})} \qquad k = 1, 2$$

where $W^k_{\lambda_k(\alpha_{t_i}),n_k(\alpha_{t_i})}$ is a Weibull distribution whose the parameters $\lambda_k(.)$, $n_k(.)$ depend upon the state of world at time t_i, α_{t_i}. The marginal density function of the asset k, $f_k(t_{i+1}, .)$, is given by the following expression:

$$f(t_{i+1}\,,\, y) \quad = \quad \frac{n_k(\alpha_{t_i})}{\lambda_k(\alpha_{t_i})} \left(\frac{y - \mu_k(\alpha_{t_i})}{\lambda_k(\alpha_{t_i})} \right)^{n_k(\alpha_{t_i})-1} e^{-((y-\mu_k(\alpha_{t_i}))/\lambda_k(\alpha_{t_i}))^{n_k(\alpha_{t_i})}}$$

Notwithstanding their fat tallness, the Weibull distribution has all their moments finite.The standard moments of $Y^k_{t_{i+1}}$ in state α_t are:

$$\begin{aligned}
\mathbb{E}\left(Y^k_{t_{i+1}}\right) &= \mu_k(\alpha_{t_i}) + \lambda_k(\alpha_{t_i})\Gamma\left(1 + \frac{1}{n_k(\alpha_{t_i})}\right) \\
Std\left(Y^k_{t_{i+1}}\right) &= \sqrt{\lambda^2_k(\alpha_{t_i})\Gamma\left(1 + \frac{2}{n_k(\alpha_{t_i})}\right) - \left(\mathbb{E}\left(Y^k_{t_{i+1}}\right)\right)^2}
\end{aligned}$$

$$\begin{aligned}
Sk\left(Y^k_{t_{i+1}}\right) &= \frac{\lambda^3_k(\alpha_{t_i})\Gamma(1 + \frac{3}{n_k(\alpha_{t_i})}) - 3\mathbb{E}\left(Y^k_{t_{i+1}}\right) Std\left(Y^k_{t_{i+1}}\right)^2 - \mathbb{E}\left(Y^k_{t_{i+1}}\right)^3}{Std\left(Y^k_{t_{i+1}}\right)^3} \\
Ku\left(Y^k_{t_{i+1}}\right) &= \frac{\lambda^3_k(\alpha_{t_i})\Gamma(1 + \frac{4}{n_k(\alpha_{t_i})}) - 4\,Sk\left(Y^k_{t_{i+1}}\right) Std\left(Y^k_{t_{i+1}}\right)^3 \mathbb{E}\left(Y^k_{t_{i+1}}\right)}{Std\left(Y^k_{t_{i+1}}\right)^4} \\
&\quad - \frac{6\,\mathbb{E}\left(Y^k_{t_{i+1}}\right)^2 Std\left(Y^k_{t_{i+1}}\right)^2 + \mathbb{E}\left(Y^k_{t_{i+1}}\right)^4}{Std\left(Y^k_{t_{i+1}}\right)^4}
\end{aligned}$$

As mentioned in Malevergne & Sornette (2006), from a theoretical point of view, the use of this class of distributions is motivated in part by the fact that the large deviations of multiplicative processes are generically distributed with stretched exponential distributions. In the next section, we present the different structures of dependence between marginal distributions and that are tested in the sequel of this work.

4 The copulas.

A copula is a function linking the multivariate distribution of benchmarks returns to their marginal distributions. In our model, the copula parameters are as marginal densities directly influenced by the state in which α_t stays. We introduce in this section the five copulas tested in the forthcoming numerical application. Three of them belong to the family of Archimedean copulas. We also focus on the Gaussian and the Student t copulas, which are directly related to multivariate distributions of the same name. But before some additional notations are needed. In particular, we denote the marginal cumulative distributions, Student t or Weibull, of benchmarks log-returns by $F_{k=1,2}(t_{i+1}, y)$. While the copula is noted $C(.\,|\,\alpha_{t_i})$ and depends on the state of α_{t_i}. This copula is related to the multivariate distribution of log-returns, noted $H(.\,|\,\alpha_{t_i})$ as follows:

$$P\left(Y^1_{t_{i+1}} \leq y^1,\, Y^2_{t_{i+1}} \leq y^2\right) = H(y^1,\, y^2\,|\,\alpha_{t_i}) = C\left(F_1(t_{i+1}, y^1)\, F_2(t_{i+1}, y^2)\,|\,\alpha_{t_i}\right) \tag{4.1}$$

For additional details about copulas, we refer the interest reader to Nelsen (1999).

4.1 Gaussian copula.

The Gaussian copula is the dependence function related to the multidimensional Gaussian distribution. Let $\rho(\alpha_{t_i})$ be the parameter of correlation depending on the state of the Markov process α_t at time t_i. We define the symmetric positive matrix $\Sigma(\alpha_{t_i})$ as follows:

$$\Sigma(\alpha_{t_i}) = \begin{pmatrix} 1 & \rho(\alpha_{t_i}) \\ \rho(\alpha_{t_i}) & 1 \end{pmatrix}$$

Let us note $\Phi^{-1}(.)$, the inverse of the cumulative probability function of an univariate $N(0,1)$ random variable. We also write $\Phi_\Sigma(.,.)$, the cumulative probability function of a 2D $N(0,\Sigma(\alpha_t))$. The Gaussian copula, linking benchmarks returns observed on the interval of time $[t_i,\, t_{i+1}]$, is then defined by the following expression:

$$C(u_1, u_2\,|\,\alpha_{t_i}) \quad = \quad \Phi_{\Sigma(\alpha_{ti})}\left(\Phi^{-1}(u_1)\,,\, \Phi^{-1}(u_2)\right)$$

The Kendall's τ is known on the interval of time $[t_i,\, t_{i+1}]$ is given by:

$$\tau(\alpha_{t_i}) \quad = \quad \frac{2}{\pi} \arcsin \rho(\alpha_{t_i})$$

As shown in Roncalli (2004), the density of the Gaussian copula is given by the following expression:

$$c((u_1, u_2\,|\,\alpha_{t_i})) \quad = \quad \frac{1}{\sqrt{1-\rho(\alpha_{t_i})^2}} \exp\left(-\frac{1}{2(1-\rho(\alpha_{t_i})^2)}\left(x_1^2 + x_2^2 - 2\rho(\alpha_{t_i})x_1x_2\right) + \frac{1}{2}\left(x_1^2 + x_2^2\right)\right)$$

where $x_1 = F_1^{-1}(t_i, u_1)$ $x_2 = F_2^{-1}(t_i, u_2)$. The density will be particularly useful during the phase of calibration. Note that whether $\rho(\alpha_{t_i}) < 1$, the upper and lower tail dependence are null:

$$\lambda_U(\alpha_{t_i}) = \lambda_L(\alpha_{t_i}) \quad = \quad 0$$

In this sense, the Gaussian copula does not allow to correlate extremal values.

4.2 Student t copula.

The student t copula is the dependence function related to the Student t multidimensional distribution. Let $\rho(\alpha_{t_i})$ be the parameter of correlation depending on the state of the Markov process α_t at time t_i. As for the Gaussian copula, we define a symmetric positive matrix $\Sigma(\alpha_{t_i})$:

$$\Sigma(\alpha_{t_i}) = \begin{pmatrix} 1 & \rho(\alpha_{t_i}) \\ \rho(\alpha_{t_i}) & 1 \end{pmatrix}$$

Let us note $F^{-1}_{\nu(\alpha_{t_i})}(.)$, the inverse of the cumulative probability function of an univariate Student t , having $\nu(\alpha_{t_i})$ degrees of freedom at time t_i. Furthermore, let $F_{\Sigma(\alpha_{t_i}),\nu(\alpha_{t_i})}(.,.)$ be the cumulative probability function of a 2D Student t, of parameters $\Sigma(\alpha_{t_i})$, $\nu(\alpha_{t_i})$. The Student copula, linking stocks and bonds returns observed on the interval of time $[t_i,\, t_{i+1}]$, is defined by the following expression:

$$C(u_1, u_2\,|\,\alpha_{t_i}) \quad = \quad F_{\Sigma(\alpha_{ti}),\nu(\alpha_{ti})}\left(F^{-1}_{\nu(\alpha_{ti})}(u_1)\,,\, F^{-1}_{\nu(\alpha_{ti})}(u_2)\right)$$

The density of the Student t copula is given by the following expression:

$$c((u_1, u_2 \,|\, \alpha_{t_i})) = \frac{1}{\sqrt{1-\rho(\alpha_{t_i})^2}} \frac{\Gamma\left(\frac{\nu(\alpha_{ti})+2}{2}\right)\Gamma\left(\frac{\nu(\alpha_{ti})}{2}\right)}{\Gamma\left(\frac{\nu(\alpha_{ti})+1}{2}\right)^2} \frac{\left(1+\frac{1}{\nu(\alpha_{ti})}\left(x_1^2+x_2^2-2\rho(\alpha_{t_i})x_1x_2\right)\right)^{-\frac{\nu(\alpha_{ti})+2}{2}}}{\prod_{k=1}^{2}\left(1+\frac{x_k^2}{\nu(\alpha_{ti})}\right)^{\frac{\nu(\alpha_{ti})+1}{2}}}$$

At our knowledge, there does not exist any closed form expression of the Kendall's τ (even if practitioners tend to observe that the numerical values of the Kendall's τ are close to those of a Gaussian copula, having the same matrix of correlations Σ). As the student t copula is symmetrical, its upper and lower tail dependence are identical:

$$\lambda_U(\alpha_{t_i}) = \lambda_L(\alpha_{t_i}) = 2 - 2\, t_{\nu(\alpha_{t_i})+1}\left(\left(\frac{(\nu(\alpha_{t_i})+1))\,(1-\rho(\alpha_{t_i}))}{(1+\rho(\alpha_{t_i}))}\right)^{\frac{1}{2}}\right) \geq 0$$

Note that if the parameter $\nu(\alpha_{t_i})$ tend to infinity, the Student t copula tend to a Gaussian copula.

4.3 Archimedian copulas.

The class of Archimedian copulas is particularly popular because the structures belonging to this family are rather easy to handle and fit extreme dependence events. We assume that the copula defining the multivariate distribution of returns belongs to the family of Archimedian copulas, indexed by the process α_t. Those are defined as follows:

$$C(u_1, u_2 \,|\, \alpha_t) = \begin{cases} \varphi^{-1}\left(\varphi(u_1, \alpha_t) + \varphi(u_2, \alpha_t)\,,\, \alpha_t\right) & if\ \sum_{k=1}^{M2} \varphi(u_k, \alpha_t) \leq \varphi(0, \alpha_t) \\ 0 & else \end{cases} \quad (4.2)$$

where $\varphi(u, \alpha_t)$ is a continuous strictly decreasing and convex function such $\varphi(1, \alpha_t) = 0$ and $\varphi(0, \alpha_t) = \infty$ in all states of the world. $\varphi(.,.)$ is called the generator of the copula. Furthermore $\varphi^{-1}(u, \alpha_t)$ must be completely monotonic on $[0, +\infty[$:

$$(-1)^d \frac{\partial^k}{\partial u^k}\varphi^{-1}(u) \geq 0 \qquad for\, k = 1, 2, ...$$

The function φ is called the generator of the copula. Genest and Mackay (1986) have proved that the Kendall's tau of those distributions in each state of the world is easily computed by the formula:

$$\tau = 1 + 4\int_0^1 \frac{\varphi(u, \alpha_t)}{\varphi'(u, \alpha_t)} du\,.$$

They have also shown that the density of this category of copula has the following expression:

$$c(u_1, u_M \,|\, \alpha_t) = -\frac{\varphi''(C(u_1, u_2 \,|\, \alpha_t), \alpha_t)\varphi'(u_1, \alpha_t)\varphi'(u_2, \alpha_t)}{\left[\varphi'(C(u_1, u_2 \,|\, \alpha_t), \alpha_t)\right]^3} \quad (4.3)$$

In particular, we focus on three copulas presented in table 4.1 :

	$\varphi(.)$	$C(u,v)$
Gumbel	$(-\ln u)^{\theta(\alpha_t)}$	$\exp\left(-\left((-\ln u)^{\theta(\alpha_t)} + (-\ln v)^{\theta(\alpha_t)}\right)^{1/\theta(\alpha_t)}\right)$
Franck	$-\ln \frac{e^{-\theta(\alpha_t)u}-1}{e^{-\theta(\alpha_t)}-1}$	$-\frac{1}{\theta(\alpha_t)} \ln\left(1 + \frac{(e^{-\theta(\alpha_t)u}-1)(e^{-\theta(\alpha_t)v}-1)}{e^{-\theta(\alpha_t)}-1}\right)$
Clayton	$\frac{1}{\theta(\alpha_t)}\left(u^{-\theta(\alpha_t)}-1\right)$	$\left(u^{-\theta(\alpha_t)} + v^{-\theta(\alpha_t)} - 1\right)^{-1/\theta(\alpha_t)}$

Table 4.1: Archimedian copulas

where $\theta(\alpha_t)$ is a positive process depending upon the state of the world and taking its value in the set $\{\theta_1, \theta_2, \ldots, \theta_N\}$. The Kendall tau of those copulas, the upper and lower tail dependencies and the range of parameters[1] are provided in table 4.2.

	Parameter range	τ	λ_u	λ_l
Gumbel	$\theta(\alpha_t) \geq 1$	$1 - 1/\theta(\alpha_t)$	$2 - 2^{1/\theta(\alpha_t)}$	0
Franck	$\theta(\alpha_t) \in \mathbb{R}$	$1 - 4\theta(\alpha_t)^{-1}(1 - D_1(\theta(\alpha_t)))$	0	0
Clayton	$\theta(\alpha_t) \geq -1$	$\frac{\theta(\alpha_t)}{\theta(\alpha_t)+2}$	0	$2^{-1/\theta(\alpha_t)}$ $\theta(\alpha_t) > 0$ 0 $\theta(\alpha_t) \leq 0$

Table 4.2: Kendall tau, upper & lower tail dependencies

5 Calibration of the switching regime model.

The common calibration method for regime switching models is the Hamilton filter (1989), which estimates parameters, by maximum likelihood estimation (MLE), from past returns. Let O_1, O_2, ...,O_T be T observed log-returns of benchmarks:

$$O_i = \left(y^1_{t_i} \, ; \, y^2_{t_i}\right) \quad i = 1, ..., T\,.$$

We denote Θ the set of parameters of our model:

$$\Theta = \{\mu^{k=1,2}_{i=1\ldots N}, \, \sigma^{k=1,2}_{i=1\ldots N}\,, \, n^{k=1,2}_{i=1\ldots N}, \, q_{i=1\ldots N, j=1\ldots N}\}$$

The log-likelihood function is defined as follows:

$$\begin{aligned} \log \mathcal{L} &= \log \quad f(O_1|\Theta) + \log f(O_2|\Theta, O_1) + \log f(O_3|\Theta, O_1, O_2) \\ &\quad + \ldots + \log f(O_T|\Theta, O_1, \ldots, O_{T-1}) \end{aligned}$$

where $f(O_k|\Theta, O_1, \ldots O_{k-1})$ is the multivariate density function of log-returns on the k^{th} period, conditionally to a set of given model parameters Θ and to previous observations $O_1, \ldots O_{k-1}$. The parameters fitting the switching regime model are the one maximizing the log-likelihood function. The conditional density, involved in the calculation of $f(O_k|\Theta, O_1, \ldots O_{k-1})$, may be recursively calculated as follows:

$$\begin{aligned} & f(O_k|\Theta, O_1, \ldots O_{k-1}) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} p_i(t_{k-1}|\Theta, O_1, \ldots O_{k-1})\, p_{i,j}(t_{k-1}, t_k|\Theta)\, f(O_k|\Theta, \alpha_{t_{k-1}} = i\,,\, \alpha_{t_k} = j) \end{aligned}$$

where

- $f(O_k|\Theta, \alpha_{t_{k-1}} = i\,,\, \alpha_{t_k} = j)$ is the multivariate density of $\left(Y^1_{t_k}, Y^2_{t_k}\right)$ in state j,
- $p_{i,j}(t_{k-1}, t_k|\Theta)$ is the probability of transition, as defined by equation (2.3), from state i at time t_{k-1} to state j at time t_k for the set of parameters Θ,
- $p_i(t_{k-1}|\Theta, O_1, \ldots O_{k-1})$ is the probability of being in state i at time t_{k-1}, conditionally to previous observations.

The multivariate density of $\left(Y^1_{t_k}, Y^2_{t_k}\right)$ in state j is equal to the following expression:

$$(O_k|\Theta, \alpha_{t_{k-1}} = i\,,\, \alpha_{t_k} = j) = c(F_1(t_k, y^1_{t_k}), F_2(t_k, y^1_{t_k}))\, f_1(t_k, y^1_{t_k})\, f_2(t_k, y^2_{t_k})$$

[1] $D_1(\theta)$ is the Debye function $D_1(\theta) = \theta^{-1} \int_0^{\theta} t/(exp(t) - 1)dt$

where $c(.)$, $F_{1,2}(.)$, $f_{1,2}(.)$ are respectively the density of copula, the marginal cumulative and probability density functions of returns. The probability $p_i(t_{k-1}|\Theta, O_1, \dots O_{k-1})$ are inferred recursively from $f(O_{k-1}|\Theta, O_1, \dots O_{k-2})$ as follows:

$$p_i(t_{k-1}|\Theta, O_1, \dots O_{k-1}) = \frac{\sum_{j=1}^{N} p_j(t_{k-2}|\Theta, O_1, \dots O_{k-2})\, p_{j,i}(t_{k-2}, t_{k-1}|\Theta)\, f(O_{k-1}|\Theta, \alpha_{t_{k-1}}, \alpha_{t_{k-2}})}{f(O_{k-1}|\Theta, O_1, \dots O_{k-2})} \quad (5.1)$$

In order to initiate the recursion, we need to determine $f(O_1|\Theta)$. Hamilton assumes that the Markov chain has been running for a sufficiently long enough period of time, so as to apply the stationary property of Markov chains, mentioned in section 3. In particular, we get that:

$$f(O_1|\Theta) = \sum_{i=1}^{N} p_i(\Theta)\, f(O_1|\Theta, \alpha_{t_0} = i),$$

where $p_i(\Theta)$ are the stationary probabilities of the Markov chain $\alpha(t)$, as defined by equation. (2.5), for the set of parameters Θ.

6 Numerical application.

In this numerical application, we investigate the performance of static and dynamic asset allocation in stocks and bonds. We have chosen as benchmark for stocks, the French indice, CAC 40, of the forties biggest French companies. The benchmark of bonds is the SGI Bond 10 years (SGIXBE10), which is an indice computed by the bank "Société Générale". This indice duplicates a portfolio of Euro bonds continuously rebalanced to keep a constant maturity of 10 years. Before presentation of asset allocation strategies, we discuss the calibration results. The marginal distributions and copulas are fitted to daily returns collected from the 2/1/2003 to the 30/11/2009. The statistics of daily returns are presented in table 6.1. The number of hidden states of the Markov chain α_t is set to three ($N = 3$).

Daily returns	CAC 40	SGIXBE10
Mean	0,0117%	0,0097%
Standard Deviation	1,4741%	0,1596%
Skewness	0,0900	-0,3237
Kurtosis	7,9470	7,3183

Table 6.1: Statistics of daily returns

The log-likelihoods obtained after calibration of switching regime models are presented in table 6.2. Those results reveal that modeling the marginal returns of assets by a 3 states Student t leads to a higher log-likelihood, whatsoever the copula. Despite its ability to capture asymmetry and leptokurticity, the Weibull distribution is less efficient than the Student t to model daily returns. For this reason, we have decided to focus on Student t distribution in the remainder of this work.

	Student t	Weibull
Gumbel	14 362	14 330
Clayton	14 363	14 331
Franck	**14 391**	14 359
Gauss	**14 392**	14 359
Student	14 391	14 358

Table 6.2: Log likelihood

The copula modeling at best the dependence between the CAC 40 and the SGIXBE10 is the Gaussian one. The parameters of Gumbel and Clayton copulas are respectively equal to 1 and 0. As such values correspond to the degenerate case of statistical independence between assets, this indicates that Gumbel and Clayton copulas are not relevant to model the joint dynamics of CAC 40 and SGIXBE10. The degrees of freedom of the Student copula tend to infinity in each state. This reveals that this copula converges toward a Gaussian one. For this reason, we only consider the Franck and Gauss copulas in next paragraphs.

6.1 Gaussian copula and Student t marginal distributions.

This section presents the results of the calibration procedure whether the index returns are assumed Student t distributed and dependent trough a Gaussian copula. The table 6.3 indicates that in two states on three, the daily average return of the CAC 40 is negative. while the return of the SGIXBE10 is on average positive whatsoever the state. The states 1, 2 and 3 correspond respectively to bear, bull markets and to a deep crisis. The standard deviation reaches a peak of 3.25% in state 3. The kurtosis of both categories of assets is close to 3 (which is the kurtosis of a Gaussian) in states 1 and 2. In the state of crisis, the kurtosis of the stocks index reaches 4. This indicates that the distribution of stocks return has fat tails in this case. The bonds index has an opposite behavior.

		$\mu(\alpha_t)$	$\sigma(\alpha_t)$	$\nu(\alpha_t)$	Kurtosis $\kappa(\alpha_t)$
State $\alpha_t = 1$	CAC 40	-0.04%	1.47%	911.99	3.0066
	SGIXBE10	0.01%	0.15%	11.45	3.8052
State $\alpha_t = 2$	CAC 40	0.09%	0.70%	13.80	3.6125
	SGIXBE10	0.01%	0.11%	26.81	3.2630
State $\alpha_t = 3$	CAC 40	-0.31%	3.25%	9.09	4.1795
	SGIXBE10	0.05%	0.21%	2.93	-2.6317

Table 6.3: Parameters of marginal distributions

The table 6.4 of daily probabilities of transition $p_{i,j}(t, t+1\,day)$, shows that there is no way to go from state 2, a bull market, to state 3, the situation of crisis, without transiting by a bear market, state 2. This result is rather intuitively satisfying: as seen during the internet bubble crash and in credit crunch, markets go first through a decreasing phase before falling in a deep crisis. Note that the probability of remaining in the same state, the next day, is above 94%, whatsoever the state. This observation is particularly useful to build a dynamic strategy.

	State $\alpha_t = 1$	State $\alpha_t = 2$	State $\alpha_t = 3$
State $\alpha_t = 1$	0.9838	0.0063	0.0099
State $\alpha_t = 2$	0.0057	0.9943	0
State $\alpha_t = 3$	0.0577	0	0.9423

Table 6.4: Daily probabilities of transition

The table 6.5 reveals that the dependence, measured by the Kendall τ , is negative and nearly identical in state 1 and 3. In a bull market, the dependence is twice less important, but still negative. This negative dependence confirms that a small diversification effect exists between bonds and stocks.

	Gaussian Copula $\rho(\alpha_t)$	$\tau(\alpha_t)$
State $\alpha_t = 1$	-0.2192	-0.1407
State $\alpha_t = 2$	-0.1349	-0.0861
State $\alpha_t = 3$	-0.2296	-0.1475

Table 6.5: Parameters of the Gaussian copula

The figures 6.1 , 6.2 , 6.3 present the probability of being in states 1, 2 and 3 at time t_{k-1}, conditionally to previous observations, noted $p_i(t_{k-1}|\Theta, O_1, \ldots O_{k-1})$ in section 5. Those graphs are particularly interesting because they indicate the periods during which the market was depressed or bullish. An analysis of probabilities of sojourn in state 3 allows us to detect financial crisis related to the Irak war, in 2003, and to the recent credit crunch, started in 2008.

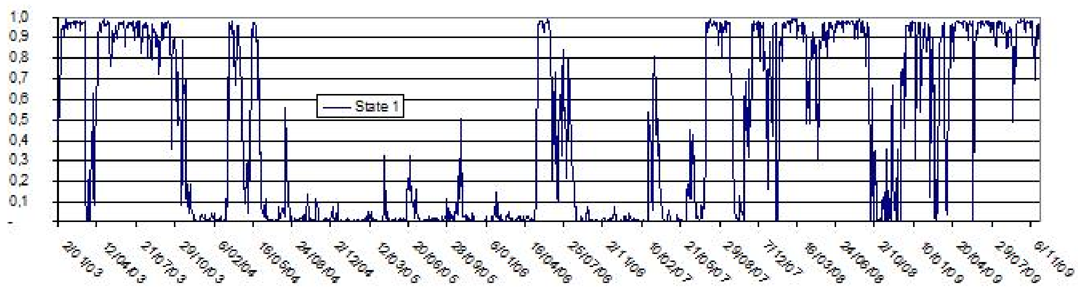

Figure 6.1: Probabilities of being in state 1.

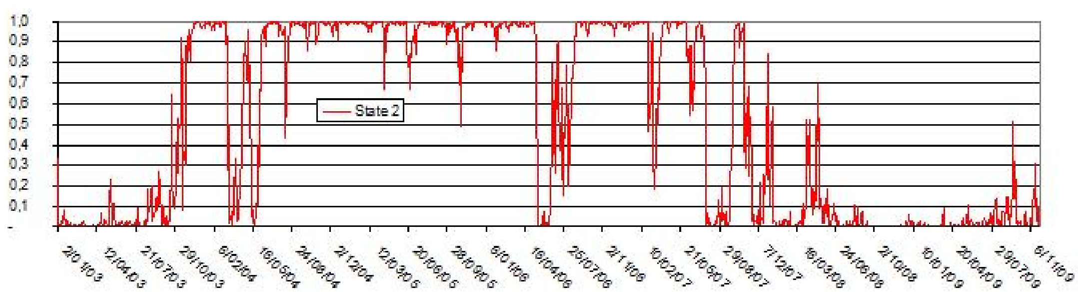

Figure 6.2: Probabilities of being in state 2.

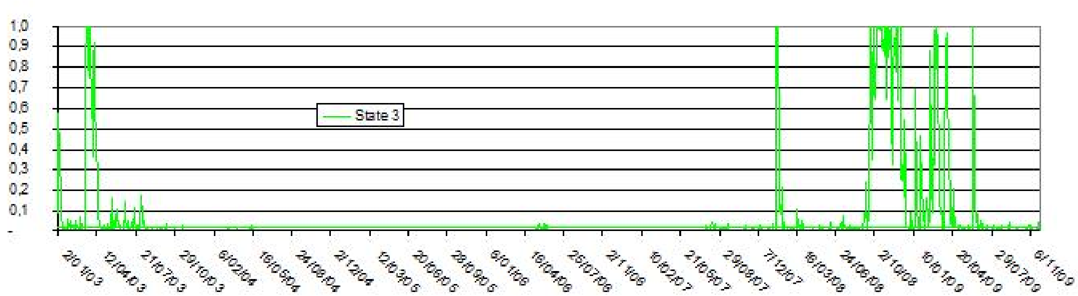

Figure 6.3: Probabilities of being in state 3.

6.2 Frank copula and Student t marginal distributions.

This section presents the results of the calibration procedure whether the dependence between Student t marginal returns is modeled by a Frank copula. The parameters of marginal returns, in table 6.6, are very close to those obtained when the dependence is modeled by a Gaussian copula. The transition probabilities in table 6.7 are also nearly identical to those obtained with a Gaussian dependence. This resemblance is intuitively satisfying and reveals a certain robustness of the calibration procedure.

		$\mu(\alpha_t)$	$\sigma(\alpha_t)$	$\nu(\alpha_t)$	Kurtosis $\kappa(\alpha_t)$
State $\alpha_t = 1$	CAC 40	-0.04%	1.474%	911.98	3.0066
	SGIXBE10	0.01%	0.14%	11.41	3.8052
State $\alpha_t = 2$	CAC 40	0.09%	0.70%	13.84	3.6125
	SGIXBE10	0.01%	0.11%	30.46	3.2630
State $\alpha_t = 3$	CAC 40	-0.27%	2.88%	6.01	4.1795
	SGIXBE10	0.04%	0.21%	3.42	-2.6317

Table 6.6: Parameters of marginal distributions

	State $\alpha_t = 1$	State $\alpha_t = 2$	State $\alpha_t = 3$
State $\alpha_t = 1$	0.9844	0.0065	0.0091
State $\alpha_t = 2$	0.0059	0.9941	0
State $\alpha_t = 3$	0.0466	0	0.9554

Table 6.7: Daily probabilities of transition

The table 6.8 shows that the dependence, measured by the Kendall τ , is negative in each state. Contrary to what we have observed, the dependence is slightly more negative in period of crisis than in a bear market. We also notice that the diversification effect is less important in state 2.

	Franck Copula $\theta(\alpha_t)$	$\tau(\alpha_t)$
State $\alpha_t = 1$	-1.2607	-0.1379
State $\alpha_t = 2$	-0.7888	-0.0871
State $\alpha_t = 3$	-1.4936	-0.1624

Table 6.8: Parameters of the Franck copula

We don't reproduce the plots of past states probabilities because they are very close to those presented in figures 6.1, 6.2, 6.3. So as to compare the copulas, we have plotted them in figure 6.4, when the Markov chain is in state 3. Both copulas looks like a saddle, excepted that the curvature of the Frank copula in the central area is higher than those of the Gauss copula.

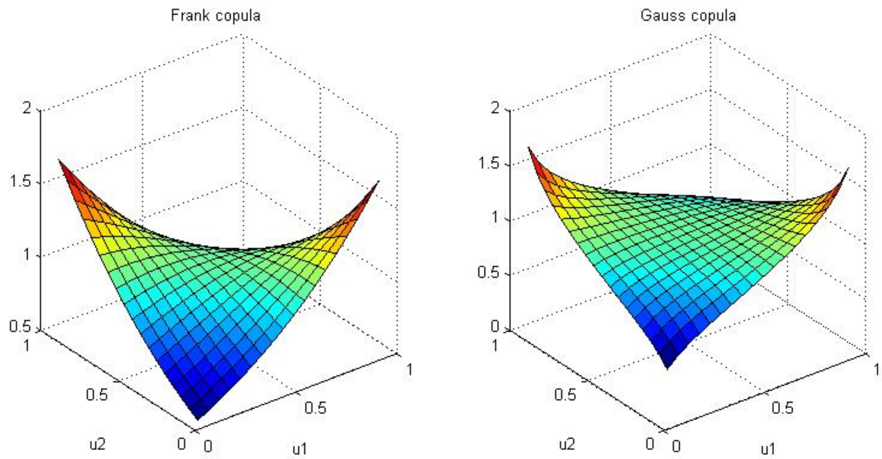

Figure 6.4: Frank and Gauss copulas in state 3.

7 Optimal asset allocation.

As mentioned in the introduction, the main task of the ALM department is to determine the optimal strategic asset allocation. Note that the optimality in this context is rather a subjective notion, given that it depends on return and risk criteria of each institutional investor. In practice, the optimal asset weights are obtained by an analysis of return and risk measures (expected return, standard deviation, VaR or TailVaR) obtained by Monte-Carlo simulations, after calibration of models describing the evolution of benchmarks. The same approach is adopted here to investigate the one year performance of static and dynamic policies of investment, in 1000 scenarios. The benchmarks tracked by stocks and bonds included in analyzed portfolios are respectively the CAC 40 and the SGIXBE10. As seen in the previous section, a good choice consists to model marginal returns by student t distributions and dependence by Gaussian or Frank copulas.

The dynamic strategy of investment detailed later in this paper takes advantage of the knowledge of state probabilities, computed by loglikelihood maximization. Our analysis will reveal in a first time that there is a strong link between the state of the hidden Markov chain and the performance of static portfolios. Next, our results tend to prove that the performance of the dynamic strategy is far better than the static policy.

But before any further developments of results, we briefly depict the procedure to simulate benchmark returns. For each scenario, we have generated the daily evolution of a 3 states Markov chain, whose the daily transition probabilities are those presented in tables 6.4 and 6.7. Three cases are considered: each one related to the value of α_t taken in (1, 2 or 3). Once that a trajectory is computed, we simulate the marginal return of benchmarks (Student t distributed) either with the Frank copula and Gaussian copula. The algorithm used in this work to simulate Archimedian and Gaussian copulas are detailed in appendix A.

7.1 Static strategy of investment.

This paragraph presents the statistics measuring both the performance (mean return) and the risk (standard deviation, VaR, TVaR) of static portfolios on a time horizon of 252 trading days. Portfolios contains bonds and stocks, with weights running from 10% to 90%, by step of 10%. Tables 7.1 to 7.6 display the statistics related to the return of static portfolios for different values of the Markov chain α_t at time $t = 0$ and for different structure of dependence.

If the hidden Markov chain is initially in state one ($\alpha_{t=0} = 1$), which corresponds to a bear market, the higher one year return and the smaller risk are obtained by a portfolio overweighted in bonds, whatsoever the copula chosen to model the dependence. The main issue faced by the investor is that $\alpha_{t=0}$ is not directly observable. However, a probability of being in this state may be inferred by applying the calibration procedure developed in section 5. If this probability is high (above a certain trigger chosen by the investor), it is a good indicator that the best static portfolio contains mainly bonds. This observation is at the origin of the dynamic strategy proposed in the next subsection.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	-1.4%	-0.9%	-0.3%	0.3%	0.8%	1,4%	1,9%	2,5%	3,1%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7,8%	5,3%	3.3%
VaR 5%	-39.0%	-34.3%	-29.2%	-24.1%	-19.6%	-14.7%	-10.1%	-5.5%	-2.0%
TVaR 5%	-46.6%	-40.8%	-35.0%	-29.3%	-23.6%	-18.1%	-12,7%	-7.6%	-3.6%

Table 7.1: Returns after 252 days, initial state: 1, Gaussian copula.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	-4.5%	-3.6%	-2.7%	-1.7%	-0.8%	0.1%	1.1%	2.0%	2.9%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-46%	-40.2%	-34.5%	-29.0%	-23.3%	-17.6%	-12.3%	-7.2%	-2.7%
TVaR 5%	-53.8%	-47.2%	-40.7%	-34.2%	-27.7%	-21.4%	-15.2%	-9.5%	-4.5%

Table 7.2: Returns after 252 days, initial state: 1. Frank copula.

When α_t is initially in state one, that corresponds to a bull market, the best average return is obtained by overweighting stocks. However, the risk increases proportionally with the fraction of stocks hold in portfolio. In this case, the choice of the optimal static allocation is driven by the risk aversion of the investor. If the one year acceptable VaR level is 10%, the best assets mix is 70% of bonds and 30% of stocks. We note again that statistics are not significantly different for Gaussian and Frank copulas.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	10.7%	9.8%	8.9%	8.0%	7.2%	6.3%	5.4%	4.5%	3.7%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-30.0%	-26.3%	-22.5%	-18.6%	-14.7%	-10.6%	-6.9%	-3.9%	-1.5%
TVaR 5%	-37.6%	-33.0%	-28.4%	-23.8%	-19.2%	-14.7%	-10.3%	-6.3%	-2.9%

Table 7.3: Returns after 252 days, initial state: 2. Gaussian copula.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	11.0%	10.1%	9.2%	8.3%	7.4%	6.5%	5.6%	4.6%	3.7%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-28.1%	-24.4%	-20.9%	-17.4%	-13.9%	-10.3%	-6.9%	-3.5%	-0.9%
TVaR 5%	-37.6%	-32.8%	-28.1%	-23.4%	-18.7%	-14.1%	-9.7%	-5.5%	-2.2%

Table 7.4: Returns after 252 days, initial state: 2. Frank copula

In time of crisis, which corresponds to $\alpha_{t=0} = 3$, the investor has again interest to overweight bonds to minimize risk and improve the one year expected return. Again this conclusion is valid for both Gaussian and Frank copulas.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	-4.3%	-3.4%	-2.4%	-1.4%	-0.4%	0.5%	1.5%	2.5%	3.5%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-43.5%	-37.7%	-32,4%	-26.9%	-21.7%	-16.1%	-10,8%	-6.1%	-2.1%
TVaR 5%	-51.5%	-45.0%	-38.6%	-32.2%	-25.8%	-19.6%	-13.6%	-8.1%	-3.9%

Table 7.5: Returns after 252 days, initial state: 3. Gaussian copula.

Bonds %	10%	20%	30%	40%	50%	60%	70%	80%	90%
Mean	-6.0%	-4.9%	-3.8%	-2.6%	-1.5%	-0.4%	0.7%	1.9%	3.0%
Deviation	24.3%	21.5%	18.7%	15.9%	13.2%	10.5%	7.8%	5.3%	3.3%
VaR 5%	-45.9%	-40.0%	-33.9%	-28.6%	-22.8%	-16.8%	-11.7%	-6.8%	-2.7%
TVaR 5%	-52.1%	-45.6%	-39.1%	-32.7%	-26.3%	-20.1%	-14.1%	-8.7%	-4.5%

Table 7.6: Returns after 252 days, initial state: 3. Frank copula.

7.2 Dynamic strategy of investment.

The state of the hidden Markov chain considerably influences the return and the risk. In state 1 and 3, the fund manager should adopt a defensive strategy and invest mainly in bonds. In state 2, the fund manager seeking a high return should rather invests massively in stocks. However, the Markov chain α_t is not directly observable. But, the calibration procedure provides, an efficient way to calculate the probability of being in a certain state based upon the observed achieved returns. At the end of each day, based upon achieved past returns $O_1, \dots O_{k-1}$, we can compute by equation (5.1) the probabilities:

$$p_i(t_{k-1}|\Theta, O_1, \dots O_{k-1}) \; i = 1, \, 2, \, 3 \; ,$$

of being in state i. If one of this probability is above a certain trigger κ chosen by the investor, we can assume that the economy is well in the detected state at time t_{k-1}. If the probability of remaining in this state is high (as in our application, see matrix 6.4) , the policy of investment can be adjusted for the next day. E.g. Based upon the observations done in the previous subsection, if we detect that the market is in state 2 (bull market) with a high probability, the part of the fund invested in stocks can be increased. On the contrary, if we detect that the chain is in state 1 or 3 with a high probability, it is interesting to increase the percentage of bonds hold in portfolio. This strategy should naturally leads to a better return than a static approach.

To test the performance of this strategy, we have performed a Monte-Carlo simulation similar to the one used to test static strategies. 1000 scenarios of 252 days of trading are simulated. For each scenario, the daily state probabilities are computed according to equation (5.1) where O_k and Θ are respectively the simulated daily returns of benchmarks, and the parameters estimated by MLE in the calibration phase. The trigger level κ is set to 60%. The rules determining the assets allocation are detailed in table 7.7. We have chosen a bipolar strategy. I.e. in a bull market, nearly the whole portfolio is invested in stocks while in the two other states, 90% of the portfolio is invested in bonds. If none of the $p_i(...)$ is above the trigger, the asset allocation remains unchanged. Note that, we didn't take into account any transaction costs.

	Bonds	Stocks
$p_1(t_{k-1}|\Theta, O_1, \dots) \geq \kappa$	90%	10%
$p_2(t_{k-1}|\Theta, O_1, \dots) \geq \kappa$	10%	90%
$p_3(t_{k-1}|\Theta, O_1, \dots) \geq \kappa$	90%	10%
$p_1, \, p_2, \, p_3 \leq \kappa$	No change	

Table 7.7: Rules for assets allocation.

Tables 7.8 and 7.9 provides statistics about the return and the risk of the proposed dynamic strategy when dependence is modeled by a Gaussian and a Frank copula. As the initial state, $\alpha_{t=0}$, of the Markov chain influences widely the one-year return, we have done Monte-Carlo simulations for each value of the Markov chain α_t at time $t = 0$. Those results reveal that in any case, the return achieved is high (around 8% if $\alpha_{t=0} = 1$, 2 and 16% if $\alpha_{t=0} = 3$) while the risk is relatively low (the tail VaR is around 5.5% whatsoever the initial state). The positive skewness indicates that the right tail of the 1y-return distribution is longer than the left tail. The kurtosis when $\alpha_{t=0} = 1$, 3 is above 3, the kurtosis of the normal. In this cases, the distribution of the 1y-return has fatter tails than a normal. If $\alpha_{t=0} = 2$, the kurtosis is close to the one of a normal distribution.

	Initial state		
	State 1	State 2	State 3
Mean	8.1%	15.7%	7.8%
Deviation	9.8%	9.8%	9.8%
VaR 5%	-4.2%	-2.4%	-3.5%
TVaR 5%	-5.8%	-5.3%	-5.6%
Skewness	1.1983	0.6633	1.2147
Kurtosis	4.9317	3.3311	5.1503

Table 7.8: Returns after 252 days, dynamic strategy. Gaussian copula.

	Initial state		
	State 1	State 2	State 3
Mean	8.02%	15.13%	7.43%
Deviation	9.95%	12.3%	9.03%
VaR 5%	-3.56%	-1.96%	-3.59%
TVaR 5%	-5.58%	-4.44%	-6.14%
Skewness	1.3257	0.6275	1.1257
Kurtosis	5.2291	3.0647	4.566

Table 7.9: Returns after 252 days, dynamic strategy. Frank copula.

As mentioned earlier, there are no transaction fees in our model. In reality, transaction fees can seriously decrease the return achieved by a dynamic strategy of investment, particularly if the number of portfolio reallocations is high. Tables 7.10 and 7.11 present some statistics about the number of assets reallocation of our dynamic strategy, for both type of dependence (Gaussian and Frank copula). Whatsoever the initial state, the portfolio is rebalanced, on average, seven times during the year of trading. Even the maximum number of assets reallocation (around 25) is still acceptable in practice compared with existing strategies such CPPI that usually requires at least a weekly rebalancing. Those observations tend to indicate that dynamic strategies similar to the one studied should perform relatively well even with transaction fees.

	Initial state		
	State 1	State 2	State 3
Mean	8	7	7
Deviation	14	13	17
Min	1	3	0
Max	25	24	25

Table 7.10: Number of assets reallocation after 252 days. Gaussian copula.

	Initial state		
	State 1	State 2	State 3
Mean	8	7	6
Deviation	12	10	14
Min	1	2	0
Max	21	19	19

Table 7.11: Number of assets reallocation after 252 days. Frank copula.

8 Conclusion

This work explores the strategic allocation of a fund of stocks and bonds tracking the CAC 40 and the SGI Bond 10 years. The novelty of our approach is to consider that marginal distributions of indices returns and their dependence structure both depend on the same unobservable Markov chain. The calibration of such kind of model is done by the Hamilton filter. This procedure, applied to a three states Markov chain, on daily data from 2003 to end 2009, allow us to identify each state and the past probabilities of sojourn in each state. State one corresponds to a bear market: the return is negative and the volatility is high. State two is coupled to a bull market: the return is high and risk is low. In the third state, the market is seriously depressed. An analysis of probabilities of sojourn in this state allows us to detect the financial crisis related to the Irak war in 2003 and the recent credit crunch. The loglikelihood is maximized when marginal returns are fitted by a Student t law and when the dependence is modeled either by a Gaussian or either by a Frank copula.

After discussion of the calibration procedure and results, we investigate the one year return of two kinds of investment strategies: static and dynamic policies. Static investment strategies are widely influenced by the initial state of the hidden Markov chain. Furthermore, their one year performance is poor excepted in a bull market. As alternative, we propose a dynamic investment strategy based upon the estimated state probabilities of the Markov chain. If the Markov chain is in a certain state with a probability above a given trigger, the investment policy is adapted to provide the highest yield in this state. The numerical tests tend to prove that a good average return may be achieved by investing mainly in stocks if we detect a bull market state and to reverse the position in bonds if we detect any other state. Furthermore, the one year VaR and tail VaR are relatively low and the number of reallocations needed remains is reasonable

Appendix A Simulation of copulas.

In this section, we briefly review the methods to simulate Archimedian, Gaussian and Student copulas.

Simulation of an Archimedian copula.

The algorithm to simulate Archimedian copulas is based upon the relation between the Laplace transform of a random variable and the generator function. Let us note $\Lambda(\alpha_t)$ a family of random variable indexed by the Markov chain α_t, defined such that their Laplace transform is equal to the inverse of the generator $\varphi^{-1}(.)$:

$$\varphi^{-1}(u, \alpha_t) \quad = \quad \mathbb{E}\left(e^{-\Lambda(\alpha_t)u}\right)$$

The distribution of $\Lambda(\alpha_t)$, for the three families of copulas introduced in section 4, are presented in table 8.1 where $\theta(\alpha_t)$ is the parameter of the Archimedian copula, indexed by the Markov chain.

Family	Generator $\varphi(.)$	Laplace Transform, $L_\Lambda(u,\alpha_t)$	Distribution of $\Lambda(\alpha_t)$
Clayton	$u^{-\theta(\alpha_t)}-1$	$(1+u)^{-\frac{1}{\theta(\alpha_t)}}$	Gamma law
Gumbel	$(-\ln u)^{\theta(\alpha_t)}$	$\exp\left(-u^{\frac{1}{\theta(\alpha_t)}}\right)$	Positive stable law
Frank	$\ln\frac{e^{\theta(\alpha_t)u}-1}{e^{(\alpha_t)}-1}$	$\frac{1}{\theta(\alpha_t)}\ln\left(1+e^{u}(e^{\theta(\alpha_t)}-1)\right)$	Law of logarithmic series distribution on the positive integers

Table 8.1: Generator and related Laplace transforms

By definition (4.2), the multivariate cumulative density function of returns, whose the dependence is modeled by a Archimedian copula, is related to marginal distributions as follows:

$$H(y^1,\, y^2\,|\,\alpha_t) \quad = \quad L_\Lambda\left(L_\Lambda^{-1}(F_1(t,y^1),\alpha_t)+L_\Lambda^{-1}(F_2(t,y^2),\alpha_t)\,,\,\alpha_t\right)$$

Based upon this observation, Marshall & Olkin (1988) have proposed the following algorithm to simulate a random hit of $\left(Y_t^1,Y_t^2\right)$:

1. Generate a random variable $\Lambda(\alpha_t)$ whose the Laplace transform is $L_\Lambda(u,\alpha_t)$.
2. Generate two U_1 , U_2 independent uniform (0,1) random numbers.
3. For $k=1,2$ calculate

$$u_k \quad = \quad \exp\left(-\left(-\frac{1}{\theta(\alpha_t)}\ln U_k\right)\right) \qquad k=1,2$$

4. Invert the marginal distributions of returns to retrieve a random hit of $\left(Y_t^1,Y_t^2\right)$:

$$y^k \quad = \quad F_1^{-1}(t,u_k) \qquad k=1,2$$

For details about the law of logarithmic series distribution, the reader can refer to Jeffrey (1988) for details.

Simulation of a Gaussian or Student copula.

Simulating Gaussian or Student copulas is easy because we can take advantage of the underlying multivariate statistical distributions:

1. Generate a random bivariate normal variable , (U_1,U_2), having a null mean and a correlation matrix $\Sigma(\alpha_{t_i})$.
2. To retrieve a random hit of $\left(Y_t^1,Y_t^2\right)$, invert the marginal distributions of returns :

$$y^k \quad = \quad F_1^{-1}(t,U_k) \qquad k=1,2$$

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