An interest rate tree driven by a Lévy process.

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**Abstract**

We propose an interest rate model driven by a particular Lévy Process, the normal inverse Gaussian (NIG) process, in which the SDE chosen to govern the evolution of the short term rate is directly inspired from the Hull-White model. The interest rate dynamics is still mean reverting but the Brownian motion is replaced by a NIG process. The principal motivation for this approach stems from the empirical evidence that a NIG process provides a better fit of bond returns than those driven by a Brownian motion. Above all, it captures the asymmetry and the leptokurticity of short term rates distribution. We show that derivatives may be priced numerically by setting up a pentanomial tree, fitting the first four moments of the NIG process. Finally, we have compared its performance with those of the Hull-White model. The estimated parameters exhibit stability and consistency over a variety of yield curves. Furthermore, our tests reveal that one parameter, $\beta$, plays a relatively strong role in distinguishing between the curve shapes.

**Keywords**: term structure of interest rates, Lévy process, Hull-White model.

**Introduction.**

In the bond market, the transactions which determine prices are made at discrete times rather than continuously. The assumption that interest rates are led by a pure diffusion model is therefore put into question. Furthermore, Lekkos [1999] has shown that the probability distribution of interest rates displays significant higher moments which are inconsistent with a pure normal or lognormal distribution and that the distribution of interest rates exhibits excess kurtosis and skewness. As illustrated by the empirical studies done by Raible [2000] in his Ph.D. dissertation, replacing the Brownian motion present in the diffusion model...
by a particular Lévy process, the normal inverse Gaussian (denoted NIG hereafter), provides a better fit of bond logreturns. Numerous papers generalizing the Heath-Jarrow-Morton term structure model originate from this analysis, see e.g. Eberlein and Raible [1999] and Eberlein and Kluge [2004].

The drawback for better fitting models of interest rates distribution is an increase in computation complexity when they are used to price derivatives. Contrary to Gaussian models such as the Hull-White Model [1990], there is to our knowledge no analytical expression of swaption prices for models driven by Lévy processes. One can either rely on the Fast Fourier Transform as done in Madan and Seneta [1990] to price options on stocks or try to build a discrete tree. This alternative has been explored by Kuan and Webber [2001] who propose a random trinomial lattice, for interest models driven by time changed Brownian motions, a class of Lévy processes that includes NIG. In this case the calibration is time consuming because it uses a Monte-Carlo approach through successive generations of trees.

Our work looks at an interest rate model in which the dynamics of rates is mean reverting, as in the Hull-White model, but it differs in that the Brownian motion is replaced by a NIG process. A mean reverting dynamics was studied by Cariboni and Schoutens [2007] for the pricing of CDS. Directly inspired from the results of Maller et al. [2006] who demonstrate the efficiency of multinomial approximation to price options on stocks driven by Lévy processes, we propose, in this paper, to discretize the NIG process by using a pentanomial tree so as to capture the asymmetry and leptokurticity of the underlying rates distribution. This type of tree has already been successfully implemented by Bollen [1998] to price options in regime switching models. Next, the calibration of this tree using the Arrow Debreu approach is briefly reviewed. Finally, the NIG is compared to the Hull-White model using several typical cases.

The NIG process.

By definition, a Lévy process \( Z_t \), has independent, stationary increments and is continuous in probability. The most interesting feature of such processes is that they can be reformulated as the sum of three components, a deterministic drift, a Brownian motion and a jump process. This property is called the Lévy Ito decomposition in the literature. Hence, the family of Lévy processes not only encompasses the majority of stochastic processes used in option pricing models but also other less common processes such as subordinated Brownian motions. These processes are Brownian motions \( W_t \), observed on a new time scale (sometimes called business time) given by \( S_t \), which is a positive random variable:

\[
Z_t = W_{S_t}.
\]
$S_t$ may be seen as the integrated rate of information arrival. In financial models based upon subordinated Brownian motions, each economic agent assumes that the instantaneous asset return is normal but that the times of trading are randomly distributed according to $S_t$, which is referred to as the subordinator. This point is illustrated in exhibit 1. More detailed information about subordinated Brownian motions can be found in Cont and Tankov [2004] or Applebaum [2004].

In the remainder of this paper, we focus on a particular subordinated Brownian: the Normal Inverse Gaussian process. For this category of processes, $S_t$, the inter-arrival time between two successive trades, is assumed to be Inverse Gaussian (IG). The density of the Inverse Gaussian depends on three parameters $\alpha, \beta, \delta$ and is given by the following expression:

$$f_{S_t}(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \sqrt{\alpha^2 - \beta^2} s^{-3/2}} \exp \left( -\frac{1}{2} \left( \frac{t^2 \delta^2}{s} + (\alpha^2 - \beta^2) s \right) \right)$$

under the constraint $\alpha^2 - \beta^2 \geq 0$. The name of “inverse” Gaussian can be misleading. It is inverse only in that while the Gaussian describes a Brownian motion’s level at a fixed time, the IG describes the distribution of the time a Brownian motion with positive drift takes to reach a fixed positive level. The process $Z_t$, as defined in equation (1) is called the Normal Inverse Gaussian (NIG). It exhibits important features such leptokurticity and asymmetry. We refer the interested reader to the paper by Barndorff-Nielsen [1998] for a detailed analysis of this process. In this paper, we intentionally restrict the scope of our presentation to useful properties for the interest rate model. As for most Lévy processes, the characteristic function of $Z_t$ is known analytically:

$$E(e^{iuZ_t}) = \exp \left( \delta t \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2} \right) \right)$$

We recall that all moments of $Z_t$ may be inferred by successive differentiations of the characteristic function with respect to $u$:

$$E(Z^n_t) = (-i)^n \frac{\partial^n E(e^{iuZ_t})}{\partial u^n}(0) \quad n = 1, 2, ...$$

Furthermore, the characteristic function also plays a crucial role in the pricing of options by the Fast Fourier Transform (e.g. see Carr and Madan [1999] for an application to option pricing). If we denote $\gamma = \sqrt{\alpha^2 - \beta^2}$, mean, variance, skewness and kurtosis of $Z_t$ can be inferred from (2) and (3) given by:

$\text{Note that in the literature there are different but equivalent formulations of the Inverse Gaussian. In this paper, we have adopted the formula used by Barndorff-Nielsen (1998).}$
Mean\( (Z_t) = \frac{\delta t \beta}{\gamma} \) \hspace{1cm} (4)

Variance\( (Z_t) = \frac{\delta t (\beta^2 + \gamma^2)}{\gamma^3} \) \hspace{1cm} (5)

Skewness\( (Z_t) = 3 \frac{\beta}{\alpha \sqrt{\delta t \gamma}} \) \hspace{1cm} (6)

Kurtosis\( (Z_t) = 3 \frac{\alpha^2 + 4 \beta^2}{\delta t \alpha^2 \gamma} \) \hspace{1cm} (7)

The probability density function of \( Z_t \), noted \( g(z, \alpha, \beta, \delta, t) \), is also known analytically:

\[
g(z, \alpha, \beta, \delta, t) = \pi^{-1} e^{\delta t \sqrt{\alpha^2 - \beta^2}} q\left(\frac{z}{\delta t}\right)^{-1} K_1(\delta t \alpha q\left(\frac{xz}{\delta t}\right)) e^{\beta z}
\]

where \( q(x) = \sqrt{1 + x^2} \) and \( K_1(x) \) is a Bessel function of the second kind\(^2\). In the remainder of this paper, we denote the probability space, the filtration and the risk neutral probability associated with \( Z_t \) by \((\Omega, F_t, Q)\).

Some empirical evidence.

To explain the motivation behind this research, we present some empirical evidence that a NIG process provides a relatively good representation of the evolution of short term rates. We retrieved the Eonia overnight rates from the 1st of July 2009 to 5th of May 2010 (852 values), and computed their daily variation. Next, we fitted these variations using a normal \( N(\mu, \sigma) \) and a NIG \( Z_1(\alpha, \beta, \delta) \) distribution, by log-likelihood maximization. Exhibit 2 compares the empirical distribution of interest rate movements with the normal and NIG distributions. This clearly shows that the adjusted NIG distribution is closer to the real distribution of interest rates compared to the normal distribution. Indeed, the log-likelihood of the fitted NIG distribution is 1495 and is greater than 1214, the log-likelihood of the fitted normal distribution. The mean and standard deviation of the fitted normal are respectively equal to \( \mu = -1.86 \times 10^{-7} \) and \( \sigma = 8.51 \times 10^{-4} \). The parameters of \( Z_1(\alpha, \beta, \delta) \) take the values \( \alpha = 100.14, \beta = 5.42 \) and \( \delta = 6.36 \times 10^{-5} \). On the basis of these observations, we propose in the next section an interest rate dynamics driven by a NIG process to explain the interest rate variation.

The interest rate model.

Hull-White [1990] proposed a generalisation of the Vasicek model by introducing to it a time-varying parameter so that the model could fit any given term.

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\(^2\text{We draw the reader’s attention to the fact that the literature is sometimes confusing. E.g. in Cont and Tankov [2004], } K_1(x) \text{ is referred as the Bessel function of the third kind, which is not correct.}\)
structure. This model implies a normal distribution for a short rate process and therefore has the useful property of analytical tractability for options pricing. The main drawback of their model is that it does not capture the leptokurticity and the asymmetry exhibited by real short interest rates, as shown in the previous section. This also explains in part why the Hull-White model does not always fit the volatility smile. To remedy this situation, we propose to replace the Brownian motion in the Hull-White model by a NIG process. More precisely, the instantaneous interest rate is assumed to be the sum of a deterministic function \( \varphi(t) \) and of a random process \( Y_t \) under \( Q \), the risk neutral measure:

\[
r_t = \varphi(t) + Y_t
\]

(8)

where \( Y_t \) is a mean reverting process driven by a NIG process, \( Z_t \), and of initial value \( Y_0 = 0 \). The function \( \varphi(t) \) is adjusted to fit the observed term structure of interest rates. We come back to this point at the end of this section. As in the Hull-White model, the process \( Y_t \) is mean reverting and a solution of the following equation:

\[
dY_t = -aY_t dt + \sigma dZ_t
\]

(9)

where \( a \) and \( \sigma \) are positive constants. In this setting, one can easily demonstrate (it is a direct consequence of the Lévy Ito formula applied to \( e^{a_s Y_s} \)) that

\[
Y_s = Y_t e^{-a(s-t)} + \sigma \int_t^s e^{-a(s-\theta)} dZ_\theta
\]

(10)

The statistical distribution of \( Y_s \) is unknown but we will see later that its moments can be obtained from its characteristic function. According to the standard financial theory, the price, noted \( P(t, s) \), of a zero coupon bond of maturity \( s \) at time \( t \), is the following expectation under \( Q \):

\[
P(t, s) = E \left( e^{-\int_t^s r_u du} | \mathcal{F}_t \right)
\]

\[
= e^{-\int_t^s \varphi(u) du} E \left( e^{-\int_t^s Y_u du} | \mathcal{F}_t \right)
\]

(11)

So as to value this last expectation, we need to calculate the integral of \( Y_u \) on an interval \([t, s]\). This integral, denoted \( \Lambda_{t,s} \), is given by:

\[
\Lambda_{t,s} = \int_t^s Y_u e^{-a(u-t)} du + \sigma \int_t^s \int_t^u e^{-a(u-\theta)} dZ_\theta \, d\theta \]

(12)

\[
= Y_t \frac{1}{a} \left( 1 - e^{-a(s-t)} \right) + \sigma \int_t^s 1 - e^{-a(s-\theta)} dZ_\theta
\]

(13)

The expectation of \( \exp(-\Lambda_{t,s}) \), involved in the value of \( P(t, s) \), can be valued by the following result, proposed by Eberlein and Raible [1999]:

**Proposition 1.** Let \( Z_t \) be a Lévy process having a cumulant transform defined as follows

\[
k(\theta) = \log E(\exp(\theta Z_1))
\]
and let $f: \mathbb{R}^+ \rightarrow \mathbb{C}$ be a complex valued left continuous function such that $|\text{Re}(f)| \leq M$ then

$$E \left( \exp \left( \int_0^t f(\theta) \, d\theta \right) \right) = \exp \left( \int_0^t k(\theta) \, d\theta \right)$$

(14)

In particular, if $Z_t$ is a NIG process, its cumulant transform is equal to:

$$k(u) = \delta \left( \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2} \right)$$

(15)

and the price of the zero coupon bond $P(t, s)$ is equal to:

$$P(t, s) = \exp \left( - \int_t^s \varphi(u) \, du \right) E(\exp(-\Lambda_{t,s})|\mathcal{F}_t)$$

$$= \exp \left( - \int_t^s \varphi(u) \, du - Y_t \frac{1}{a} \left( 1 - e^{-a(s-t)} \right) \right)$$

$$\exp \left( \int_t^s k \left( -\frac{\sigma}{a} \left( 1 - e^{-a(s-\theta)} \right) \right) d\theta \right)$$

(16)

The integral of the cumulant transform in eq. (16) has no simple analytical solution but can be easily computed numerically. The equation (16) also defines the link between the observed term structure of interest rates and the deterministic function $\varphi(u)$. Given that $Y_0 = 0$, then we have:

$$\exp \left( - \int_0^t \varphi(u) \, du \right) = \frac{P(0,t)}{\exp \left( \int_0^t k \left( -\frac{\sigma}{a} \left( 1 - e^{-a(t-\theta)} \right) \right) d\theta \right)}$$

(17)

However, as explained below, it is preferable to directly calibrate a discretized function $\varphi(t)$ during the tree construction, using the Arrow Debreu method. Hence, relation (17) is not be needed in the remaining developments.

We end this section with an illustration of the ability of our model to generate a wide range of slopes. As shown in exhibit 4, if the function $\varphi(t)$ is set to a constant and the parameter values in exhibit 3 for the NIG model are adopted, it is possible to generate three typical shapes of yield curves, decreasing, increasing and humped slopes.

The characteristic function of $Y_t$ and its moments.

In the previous section, it was established in eq. (10) that $Y_t$ is the sum of a deterministic function and of a stochastic integral with respect to $Z_t$. Whilst we have of the moments of $Y_t$, there is no known closed form expression for $Y_t$. Its characteristic function,

$$E( e^{i u Y_t + \Delta t} | \mathcal{F}_t ) = \exp \left( Y_t e^{-\alpha \Delta t} + \sigma \int_t^{t+\Delta t} e^{-a(t+\Delta t-\theta)} dZ_{\theta} \right)$$

6
may indeed be obtained by applying the proposition 1. This leads to the following result:

\[ E(e^{iuY_{t+\Delta t}}|\mathcal{F}_t}) = \exp\left(iue^{-a\Delta t}Y_t + \int_t^{t+\Delta t} k\left(iu\sigma e^{-a(t+\Delta t-\theta)}\right) d\theta\right) \]  

(18)

where

\[ k\left(iu\sigma e^{-a(t+\Delta t-\theta)}\right) = \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu\sigma e^{-a(t+\Delta t-\theta)})^2}\right) \]  

(19)

The moments, that provide information about the distribution, can be calculated by differentiating the characteristic function as follows:

\[ E(Y_{n+\Delta t}|\mathcal{F}_t) = (-)^n \frac{\partial^n E(e^{iuY_{t+\Delta t}}|\mathcal{F}_t})}{\partial a^n}(0) \quad n = 1, 2, \ldots \]  

(20)

We conducted these somewhat lengthy calculations for the first four moments. The expectation of \( Y_t \) is equal to

\[ E(Y_{t+\Delta t}|\mathcal{F}_t) = Y_te^{-a\Delta t} + \frac{\delta}{\sqrt{\alpha^2 - \beta^2}} \frac{\beta\sigma}{a} (1 - e^{-a\Delta t}) \]  

(21)

The moment of order 2 is given by:

\[ E(Y_{t+\Delta t}^2|\mathcal{F}_t) = (E(Y_{t+\Delta t}|\mathcal{F}_t))^2 + \frac{\delta^2\sigma^2}{2a} (1 - e^{-2a\Delta t}) \left(\frac{\beta^2}{\gamma^3} + \frac{1}{\gamma}\right) \]  

(22)

The third order moment is equal to:

\[ E(Y_{t+\Delta t}^3|\mathcal{F}_t) = \frac{\delta\sigma^3}{a} (1 - e^{-3a\Delta t}) \left(\frac{\beta^3}{\gamma^5} + \frac{\beta}{\gamma^3}\right) + (E(Y_{t+\Delta t}|\mathcal{F}_t))^3 + 3E(Y_{t+\Delta t}|\mathcal{F}_t) \left(E(Y_{t+\Delta t}^2|\mathcal{F}_t) - (E(Y_{t+\Delta t}|\mathcal{F}_t))^2\right) \]  

(23)

And finally, the fourth order moment is:

\[ E(Y_{t+\Delta t}^4|\mathcal{F}_t) = 6(E(Y_{t+\Delta t}|\mathcal{F}_t))^4 - 12(E(Y_{t+\Delta t}|\mathcal{F}_t))^2 E(Y_{t+\Delta t}^2|\mathcal{F}_t) + 3(E(Y_{t+\Delta t}^2|\mathcal{F}_t))^2 + 4E(Y_{t+\Delta t}|\mathcal{F}_t) E(Y_{t+\Delta t}^3|\mathcal{F}_t) + \left(\frac{15\beta^4}{\gamma^7} + 18\frac{\beta^2}{\gamma^5} + \frac{3}{\gamma^3}\right) \frac{\delta\sigma^4}{4a} (1 - e^{-4a\Delta t}) \]  

(24)

Those moments are used in the next section to build a numerical pricing procedure.
The pentanomial tree.

As mentioned in the introduction, the penalty for a better fit of the interest rates distribution is an increase in complexity, and the lack of analytical formulas for the pricing of basic derivatives such as caps, floors and swaptions. In practice, we can remedy this by adopting a numerical approach capturing the main features of the NIG model. Among the set of available methods, we have opted for the lattice approach, largely adopted by the quant community for its tractability. When the source of randomness is Brownian, the evolution of short term rates is commonly approximated by a recombining trinomial tree. At each node, the lattice approximates the normal law of rates using a trinomial distribution. This distribution is constructed by imposing that the conditional local mean and variance at each node are equal to those of the basic continuous-time process. The interested reader should refer to Brigo and Mercurio [2006] annex F for details. The geometry of the tree is then designed to ensure all branching probabilities remain positive.

If we calibrate a trinomial tree to an interest rate model driven by a NIG process, there are not enough degrees of freedom to replicate its skewness and kurtosis, which are important characteristics of our model. To overcome this drawback, we decided to build a recombining pentanomial tree. As the short term rate is the sum of one deterministic function $\varphi(t)$ and of the process $Y_t$, the first stage of our approach is the building of a tree for $Y_t$. The second step consists in calibrating a discretized version of $\varphi(t)$ and to then add it to all nodes of the tree.

To discretize the process (10) from time 0 to $T$, we first choose a finite set of times $0 = t_0 < t_1 < ... < t_n = T$. The length of time intervals are noted $\Delta t_i = t_{i+1} - t_i$. At each time $t_i$, we have a finite number of equispaced nodes, $y_{i,j}$. If we note $\Delta y_i$ the distance between nodes at time $t_i$, the value of nodes is defined as $y_{i,j} = j\Delta y_i$ where $j$ is an integer.

If the process is at the node $y_{i,j}$ at time $t_i$, it is assumed that it can move to the following nodes $y_{i+1,k-2}$, $y_{i+1,k-1}$, $y_{i+1,k}$, $y_{i+1,k+1}$, $y_{i+1,k+2}$, with the respective probabilities $p_{dd}$, $p_d$, $p_m$, $p_u$ and $p_{uu}$ (to simplify notation, we drop the indexes $i$ and $j$ hereafter). The node $y_{i+1,k}$ is the closest node to the theoretical expectation $E(Y_{t+\Delta t_i}|Y_t = y_{i,j})$. These transition probabilities are chosen so that the first four moments of the discretized process match the moments of the continuous process. We adopt the following notation for the out local moments of $Y_{t+\Delta t_i}$:

$$
Ey = E(Y_{t+\Delta t_i}|Y_t = y_{i,j})
$$
$$
Ey^2 = E(Y_{t+\Delta t_i}^2|Y_t = y_{i,j})
$$
$$
Ey^3 = E(Y_{t+\Delta t_i}^3|Y_t = y_{i,j})
$$
$$
Ey^4 = E(Y_{t+\Delta t_i}^4|Y_t = y_{i,j})
$$
To simplify the equations, we introduce the following terms:

\[
A = -4\, (Ey^3) + 6\, (Ey^2)\, y - 4\, (Ey)\, y^2 + y^3
\]

\[
B = (Ey^2) + y\, (y - 2\, (Ey))
\]

\[
C = (Ey^3) - y\, (3\, (Ey^2) - 3\, (Ey)\, y + y^2)
\]

The probabilities matching those moments are given by the following expressions, in which the indexes \(i, j\) of \(\Delta y_i\) and \(y_{ij}\) have been dropped:

\[
p_{dd} = \frac{1}{24\, \Delta y^4} \left[ Ey^4 + 2\, \Delta y^3 (Ey - y) + y\, A - \Delta y^2\, B - 2\, \Delta y\, C \right]
\]

\[
p_d = -\frac{1}{6\, \Delta y^4} \left[ Ey^4 + 4\, \Delta y^3 (Ey - y) + y\, A - 4\, \Delta y^2\, B - \Delta y\, C \right]
\]

\[
p_m = \frac{1}{4\, \Delta y^4} \left[ Ey^4 + 4\, \Delta y^4 + y\, A - 5\, \Delta y^2\, B \right]
\]

\[
p_u = \frac{1}{6\, \Delta y^4} \left[ -Ey^4 + 4\, \Delta y^3 (Ey - y) - y\, A + 4\, \Delta y^2\, B - \Delta y\, C \right]
\]

And \(p_{uu} = 1 - p_u - p_m - p_d - p_{dd}\).

The discretization parameters have to be chosen so that the probabilities remain positive. As in the Hull-White tree, it has been determined that a value of

\[
\Delta y_i = \sqrt{3\, \sqrt{\text{Variance}(\sigma Z\Delta t_i)}}
\]

\[
= \sigma \sqrt{3\, \delta \Delta t_i (\beta^2 + \gamma^2) / \gamma^3}
\]

generally provides a tree with positive transition probabilities.

Once the tree coupled to the process \(Y_t\) is built, the discretized version of the function \(\varphi(t)\), that fits the current yield curve still needs to be determined. This discretized function has the value \(\varphi_i\), on the interval \([t_i, t_{i+1}]\). The nodes in the tree are then shifted by \(\varphi_i\) to get the tree that discretizes the short term rate process, \(r_t\). The calculation of \(\varphi_{i=0...n}\) is based upon Arrow-Debreu state prices, which are fundamental securities that constitute the building blocks of all other securities. Let \(Q_{i,j}\) be the present value of the instrument paying 1 if the node \((i, j)\) is reached and zero otherwise. By definition \(Q_{0,0} = 1\) and once the tree is built, the other \(Q_{i,j}\) can be recursively inferred. Given that we set \(y_0 = 0\), \(\varphi_0 = r_{\Delta t_0}\), the first state prices are:
The value of $\phi_1$ is determined such that the price of the zero coupon of maturity $t_2$ is equal to the expectation of discount factors:

$$P(0, t_2) = \sum_{j=-2}^{2} Q_{1,j} e^{-\phi_1 \Delta y_1 \Delta t_1}$$

(30)

Once $\phi_1$ is known, one can compute the $Q_{2,j}$ and so on. More generally, the value of $\phi_{i=1...n}$ are obtained by solving the equations:

$$P(0, t_{i+1}) = \sum_{j=j_{i,min}}^{j_{i,max}} Q_{i,j} \exp (- (\phi_i + j \Delta y_i) \Delta t_i) \quad i = 1...n$$

(31)

Whose solution is

$$\phi_i = \frac{1}{\Delta t_i} \ln \frac{\sum_{j=j_{i,min}}^{j_{i,max}} Q_{i,j} \exp (-j \Delta y_j \Delta t_i)}{P(0, t_{i+1})}$$

(32)

The interest rate at node $(i, j)$ is therefore equal to $r_{i,j} = y_{i,j} + \phi_i$. This rate is used as a discount rate for all ascendant nodes when pricing derivatives with standard backward iterations. This procedure is common for all types of trees. We again refer the interested reader to Brigo and Mercurio’s book [2006], chapter 3, section 3.11 for details.

**Numerical tests.**

To evaluate the performance of the NIG Model implemented in the form of a pentanomial tree, we conducted a series of numerical simulations and compared the results to those obtained from the one factor Hull-White model. The pentanomial tree is fitted to reproduce both the interest rate curve provided in the appendix (exhibit 12) and the prices of a basket of ten caplets. The chosen caplets used for calibration are defined on the one year rate with maturities that range from one year to ten years. The strike prices are set to the forward one year rate (at the money caplets). Caplets are traditionally priced by Black’s formula and quoted in the market by their implied volatilities. We have chosen eight curves of implied volatilities, provided in appendix (exhibit 13), representative of different realistic market conditions. Four curves are decreasing with
various degrees of steepness (curves 1 to 4) and four are humped (curves 5 to 8). They are plotted in exhibits 5 and 6.

The calibration algorithm minimizes the mean relative error between caplet market prices and prices produced by the NIG and the Hull-White models. For a calibration basket which counts \( N \) caplets, the mean relative error is defined as:

\[
ME = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\text{Price}^\text{model}_i - \text{Price}^\text{market}_i}{\text{Price}^\text{market}_i} \right|
\]  

(33)

Given that the price of a caplet has an analytical expression in the Gaussian scheme of Hull-White, the calibration procedure of this model is quasi instantaneous. On the other hand, the calibration of our NIG pentanomial tree is completely numerical. However, it is relatively quick given that the lattice is recombining - less than 30 seconds in C++ for a time horizon of 10 years and 40 time steps.

The NIG tree consistently outperformed the Hull-White model as can be seen by the results presented in exhibit 7. The parameters obtained for the NIG model are displayed in exhibit 8. We note that the results are well behaved and consistent in so much that all the parameters exhibit stability and some form of erratic variations. For example, the parameter \( \sigma \) either increases or decreases in a monotonic fashion for each of the two types of curves. In the tests that were carried out, less than 1% of transition probabilities were negative and all transition probabilities, in absolute values, were in the range \([0,1]\). In less than 1% of cases, the transition probabilities were negative. It is interesting to note that the parameters \( \delta \) and \( \gamma \) hardly show any variation compared to the \( \beta \) parameter which must therefore play a discriminating role in obtaining the desired fits.

The parameters derived from the HW model are shown in exhibit 9. We note a strong resemblance with their counterparts in the NIG model. Not only are the magnitudes very similar but they also display the same variations in sign for particular curve types. This morphological resemblance is intuitively satisfying.

The role of the \( \beta \) parameter, which is particular to the NIG model, is of special interest as it plays a relatively important role in distinguishing between the curve shapes whilst at the same time displaying stable and consistent behaviour. This parameter is the principal source of added information provided by the NIG model. This insight is confirmed by exhibit 10 containing the centered moments of eight fitted NIG processes, over a period of one year, calculated from formulas (4) to (7). The \( \beta \) parameter mainly acts on the skewness of \( Z_1 \) and on the expectation. However, the influence on the expectation is immediately offset by an adjustment of \( \varphi(t) \) so as to fit the zero coupon curve. We clearly see that the skewness is higher for curve C1 than for curve C4. This tends to reveal that steeper slopes of implied volatilities lead to higher predicted asymmetry in the
To close this section, exhibit 11 emphasizes the influence of $\beta$ on the shape of the probability density function of $Z_1$. The $\beta$ parameter takes the values 0, 1.5 and 3, while the other parameters are those obtained from the calibration of the tree to the second curve of implied volatilities, $\delta = 5.409$, $\gamma = 5.443$, $\sigma = 1.39\%$. This graph shows that increasing beta accentuates the right asymmetry of the distribution and also shifts the mean to the right.

Conclusions.

Our work looks at an interest rate model in which the dynamics of rates is driven by a Lévy process rather than by a Brownian motion. The main motivation of the approach is to develop a model that captures the excess of kurtosis and skewness exhibited by interest rates distributions. As illustrated by empirical evidences, the Normal Inverse Gaussian (NIG) process provides a more realistic model of the behavior of interest rates. On the basis of this observation, we have chosen to model instantaneous interest rates as the sum of a mean reversion NIG process and of a deterministic function of time, which is used to fit the current yield curve. Our model is, at a first sight, identical to the Hull-White scheme, except that we have replaced the Brownian component by a NIG process. In this case we can infer the characteristic function of short term rates and an analytical formula of zero coupon bond prices. Furthermore, the first four moments of the NIG mean reverting process can be obtained by successive differentiations of its characteristic function.

Among the different alternatives to price interest rate derivatives, it was decided to explore the calibration and pricing by backward iterations via a pentanomial tree. We show that the first four moments of short term rates can be effectively reproduced by building a tree with five branches per time step and hence preserve the leptokurticity and skewness of the underlying process. Despite the somewhat lengthy expressions of moments and transition probabilities, the tree can easily be implemented in most programming languages.

As shown in numerical tests, the NIG model consistently outperformed the Hull-White model by a small but significant margin, based on a set of 8 typical curves of caplet volatilities. The parameters estimated via the pentanomial tree exhibited stability and consistency over a variety of term structures with typical shapes. Moreover, the NIG model produced a good match of the more restricted parameter set (speed of mean reversion and variance) of the Hull-White model, thus providing added intuitive justification to this approach. One parameter of the NIG model, the $\beta$ parameter, is of special interest as it plays a relatively strong role in distinguishing between the curve shapes. This parameter is indeed the principal driving factor behind the asymmetry of the process and it enables us to fit very steep implied volatility curves.
Appendix.

Exhibits 12 and 13 contain the zero coupon rates and implied volatilities used to price the caplets included in the calibration baskets of numerical tests.

Acknowledgment.

We gratefully acknowledge the support of TFM quantitative modeling department of Dexia Bank Brussels.

References


Exhibits.

Exhibit 1. Normal Inverse Gaussian Process

Exhibit 2. Comparison of fitted cdfs with empirical cdf.
<table>
<thead>
<tr>
<th></th>
<th>Curve 1</th>
<th>Curve 2</th>
<th>Curve 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>0.9538</td>
<td>0.1160</td>
<td>0.3911</td>
</tr>
<tr>
<td>(\varphi(t))</td>
<td>-0.1592</td>
<td>0.0948</td>
<td>0.0069</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.6766</td>
<td>0.0601</td>
<td>0.2312</td>
</tr>
<tr>
<td>(\delta)</td>
<td>0.9569</td>
<td>1.0246</td>
<td>1.0618</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>6.6777</td>
<td>2.4918</td>
<td>0.7521</td>
</tr>
<tr>
<td>(\beta)</td>
<td>-0.3006</td>
<td>0.3480</td>
<td>0.3386</td>
</tr>
</tbody>
</table>

Exhibit 3. Parameters.

Exhibit 4. Example of yield curves.

Exhibit 5. Decreasing volatilities.
Exhibit 6. Humped volatilities.

<table>
<thead>
<tr>
<th>Curve</th>
<th>H-W</th>
<th>NIG</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.80%</td>
<td>1.39%</td>
</tr>
<tr>
<td>2</td>
<td>0.55%</td>
<td>0.49%</td>
</tr>
<tr>
<td>3</td>
<td>1.51%</td>
<td>0.16%</td>
</tr>
<tr>
<td>4</td>
<td>4.05%</td>
<td>2.72%</td>
</tr>
<tr>
<td>5</td>
<td>17.84%</td>
<td>15.72%</td>
</tr>
<tr>
<td>6</td>
<td>9.76%</td>
<td>7.89%</td>
</tr>
<tr>
<td>7</td>
<td>5.39%</td>
<td>3.73%</td>
</tr>
<tr>
<td>8</td>
<td>4.32%</td>
<td>2.59%</td>
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</tbody>
</table>

Exhibit 7. Mean relative errors.

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<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
<th>C8</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ</td>
<td>1.57%</td>
<td>1.39%</td>
<td>1.32%</td>
<td>1.16%</td>
<td>0.83%</td>
<td>1.08%</td>
<td>1.41%</td>
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<tr>
<td>α</td>
<td>0.070</td>
<td>0.037</td>
<td>0.030</td>
<td>-0.002</td>
<td>-0.010</td>
<td>0.066</td>
<td>0.167</td>
</tr>
<tr>
<td>δ</td>
<td>5.434</td>
<td>5.409</td>
<td>5.398</td>
<td>5.421</td>
<td>5.224</td>
<td>5.429</td>
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<tr>
<td>β</td>
<td>0.248</td>
<td>0.196</td>
<td>0.0149</td>
<td>-0.023</td>
<td>-0.032</td>
<td>-0.073</td>
<td>-0.144</td>
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<tr>
<td>γ</td>
<td>5.427</td>
<td>5.443</td>
<td>5.455</td>
<td>5.432</td>
<td>5.709</td>
<td>5.4201</td>
<td>5.347</td>
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</table>

Exhibit 8. NIG parameters.

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<th>C4</th>
<th>C5</th>
<th>C6</th>
<th>C7</th>
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</thead>
<tbody>
<tr>
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<td>1.75%</td>
<td>1.52%</td>
<td>1.37%</td>
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<td>0.85%</td>
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<tr>
<td>α</td>
<td>0.080</td>
<td>0.042</td>
<td>0.018</td>
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<td>0.053</td>
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Exhibit 9. HW parameters.
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</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
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<td>0.195</td>
<td>-0.030</td>
<td>0.065</td>
<td>-0.029</td>
<td>-0.073</td>
<td>-0.021</td>
<td>-0.239</td>
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<tr>
<td>Var.</td>
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<td>0.998</td>
<td>1.003</td>
<td>1.001</td>
<td>1.045</td>
<td>0.999</td>
<td>1.002</td>
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</tr>
<tr>
<td>Skew.</td>
<td>0.025</td>
<td>0.020</td>
<td>-0.003</td>
<td>0.007</td>
<td>-0.003</td>
<td>-0.007</td>
<td>-0.002</td>
<td>-0.023</td>
</tr>
<tr>
<td>Kurt.</td>
<td>0.103</td>
<td>0.103</td>
<td>0.102</td>
<td>0.102</td>
<td>0.101</td>
<td>0.102</td>
<td>0.098</td>
<td>0.099</td>
</tr>
</tbody>
</table>

Exhibit 10: Variance, Skewness, Kurtosis of $Z_1$. 

Exhibit 11. pdf’s of $Z_1$, influence of $\beta$.

<table>
<thead>
<tr>
<th>rates (%)</th>
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<tbody>
<tr>
<td>1 y</td>
</tr>
<tr>
<td>2 y</td>
</tr>
<tr>
<td>3 y</td>
</tr>
<tr>
<td>4 y</td>
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<td>5 y</td>
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<tr>
<td>6 y</td>
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<td>7 y</td>
</tr>
<tr>
<td>8 y</td>
</tr>
<tr>
<td>9 y</td>
</tr>
<tr>
<td>10 y</td>
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</table>

<table>
<thead>
<tr>
<th>$\sigma_{\text{caplets}}$ (%)</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
<th>C5</th>
<th>C6</th>
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<tr>
<td>1 y</td>
<td>36.48</td>
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<td>27.83</td>
<td>25.00</td>
<td>22.17</td>
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<td>19.76</td>
<td>22.44</td>
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</tr>
<tr>
<td>3 y</td>
<td>26.98</td>
<td>25.37</td>
<td>23.76</td>
<td>22.15</td>
<td>19.52</td>
<td>20.15</td>
<td>20.78</td>
<td>21.41</td>
</tr>
<tr>
<td>4 y</td>
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<td>23.76</td>
<td>22.87</td>
<td>21.98</td>
<td>20.00</td>
<td>19.50</td>
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<td>18.52</td>
</tr>
<tr>
<td>5 y</td>
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<td>22.68</td>
<td>22.22</td>
<td>21.76</td>
<td>19.48</td>
<td>18.40</td>
<td>17.33</td>
<td>16.25</td>
</tr>
<tr>
<td>6 y</td>
<td>22.15</td>
<td>21.94</td>
<td>21.74</td>
<td>21.53</td>
<td>18.50</td>
<td>17.15</td>
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<tr>
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<td>15.91</td>
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<tr>
<td>9 y</td>
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<td>20.85</td>
<td>20.92</td>
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<td>12.29</td>
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<tr>
<td>10 y</td>
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<td>20.67</td>
<td>20.76</td>
<td>14.09</td>
<td>12.75</td>
<td>11.42</td>
<td>10.08</td>
</tr>
</tbody>
</table>

Exhibit 13. Caplets volatilities.