LIFE ANNUITIZATION: WHY AND HOW MUCH?

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ABSTRACT. This paper addresses the problem that a majority of retired individuals have to face: why and in what proportion should they invest in a life annuity to maximize the utility from their future consumption or legacy? The market considered in this work is composed of three assets: a life annuity, a risky asset and a cash account. As this problem doesn’t admit any suitable explicit solution, it is numerically solved by the Markov Chain approximation developed by Kushner and Dupuis. Without bequest motive, we observe that the optimal planning of consumption is subdivided in two periods and that the optimal asset allocation should include the risky asset. Next, the influence of a legacy on the consumption and investment pattern is developed. We show that even with a bequest motive, the pensioner should allocate a part of his wealth to the purchase of an annuity.

1. Introduction.

This paper addresses the problem of the optimal individual asset allocation, at the age of retirement, between classical financial investments and life annuities. Even if it is generally admitted that a life annuity is an efficient insurance against a decrease of the standard of living, the success of such product is still limited. However, during the last years, life annuitization has become a serious option for individuals in retirement. One argument pleading in favour of annuitization is the huge volatility of alternative investments such those available on financial markets and the low return of state bonds. In comparison, a life annuity is an interesting choice because of its longevity protection, coupled to an interest rate guarantee which is a long term protection against financial disappointments. This work presents a model providing rational arguments in favour of the integration of life annuity in the individual asset allocation of a pensioner who wishes to maximize the utility arising both from its consumption and from a legacy to relatives.

The issue of the optimal individual asset allocation at retirement, also known as the annuity puzzle, is widely studied in the actuarial and social literature. We mention here papers motivating this work. The starting point of our research is the seminal paper of Yaari (1965), who has proved that an agent, without bequest motive and under certain assumptions, should prefer a full annuitization in a market without risky asset. An intuitive proof of this result is that the rate of return of an annuity is always higher than the risk free rate, because it includes a mortality risk credit. The dynamic programming approach, applied by Merton to consumption/investment issues (1969, 1971) was then extended by Richard (1974) to describe the optimal behaviour of an individual throughout his lifetime, in the presence
of deterministic incomes. The paper of Kapur and Orzag (1999) focuses on the pensioner’s portfolio strategy with pure endowments. Milevsky (1998, 2001) and Young (2000) have developed a model in which the individual defers the purchase of a life annuity until it is not possible to beat the rate of return of an annuity. Davidoff et al. (2003) have proved that full annuitization is optimal in a more general setting than the one of Yaari. Purcal and Pigott (2004) have calibrated the Richard’s model to the Japanese market in order to explain the demand in life annuity. An extension of the Richard’s model to stochastic incomes is numerically solved by Purcal (2004). Another attempt of calibration of the Yaari’s model to the U.S. market is done by Petrova (2004). Finally, we refer to Devolder and Hainaut (2005) for the treatment of a similar problem by a Lagrangian approach, in a deterministic market.

This paper presents a numerical scheme solving the pensioner’s asset allocation problem. We assume that the agent can invest his capital in a life annuity and in a financial market, composed of cash and of one risky asset. The decision about the amount annuitized is made once for all at the age of retirement. Our aim is to determine both the optimal investment strategy and the optimal consumption, which maximize the expected utility drawn from the future consumption and eventually from a future legacy to relatives. After formulation of the problem, we have derived the Hamilton Jacobi Bellman equation coupled to this stochastic control problem and solved it numerically. We have adapted the method proposed by Kushner and Dupuis (2001) and developed an implicit iterative framework of resolution. The main technical difficulties consist in the choice of parameters of discretization to insure the convergence of the algorithm.

The main results of this paper are summarized as follows. For an individual without bequest motive, a small fraction of the capital should be dedicated to the purchase of risky assets. Her optimal consumption pattern can be subdivided in two periods. During the first one, she consumes the totality of the annuity and a part of her fund of equities. Crossed a certain age, the consumption is equal to the annuity and her savings are depleted. An interesting observation is that the optimal asset allocation still includes a life annuity if the retiree wishes to leave a legacy to his relatives. The optimal investment strategy is in agreement with the generally admitted principle, that the quantities of risky asset decrease with age.

The outline of this paper is as follows. In section 2, we have presented the general model of optimal consumption and the related Hamilton Jacobi Bellman equation. Section 3 develops a numerical method based on the implicit discretisation of the HJB equation. In section 4, we have applied this methodology to the optimal choice without bequest motive; similarly, section 5 is devoted to the case with bequest motive. Finally, section 6 concludes this paper.

2. The market and the individual’s maximization problem.

We consider the case of an individual who retires at age \( x \) and has to allocate his wealth, noted \( W_0 \), between a cash account, a risky asset (equities) and a life annuity. The amount annuitized is chosen once and for all at the age of retirement. Let \( \alpha \) be the fraction of the initial capital devoted to the purchase of a continuous annuity, \( B_x \) (the annuity rate is indexed by \( x \) because it depends on the pensioner’s age at issuance of the contract). The remaining capital is managed by the agent and invested in a fund of cash and equities. This
fund is noted $F_0 = (1 - \alpha)W_0$. The life annuity rate $B_x$ is constant and calculated by the classical actuarial formula:

$$B_x = \frac{\alpha W_0}{\bar{a}_{x,x+T}} \cdot \frac{1}{1 - \epsilon} \quad (2.1)$$

Where:

- $\epsilon$: commercial loading.
- $\bar{a}_{x,x+T} = \int_x^{x+T} e^{-rtf(s-x)} s p^t_x ds$ is a continuous life annuity from age $x$ to age $x + T$. $\tau^t_x$ is the interest rate granted to the costumer and $s p^t_x$ is the survival probability from age $x$ to age $x + s$. $T$ is the maximum age that a human being could reach.
- $s p^t_x = e^{-\int_t^s \mu^t(z) dz}$ where $\mu^t(z)$ is the mortality rate of the tariff. The real survival probability and the associated mortality rate are respectively noted $s p_x$ and $\mu(t)$. They will be used later in the formulation of the problem.

The individual has access to a financial market composed of two assets, the cash and equities, respectively noted $S_0$ and $S_1$, with the following dynamics:

$$dS_0 = S_0.r.dt \quad (2.2)$$
$$dS_1 = S_1.m.dt + S_1.\sigma.dW_t \quad (2.3)$$

The risk free rate $r$, the expected return of equities $m$ and the volatility $\sigma$ are constant. $(W_t)_{0 \leq t}$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $(\mathcal{F}_t)_{0 \leq t}$ is the natural filtration of $W_t$. The fund at time $t$ is noted $F_t$ and the fraction of $F_t$ invested in equities is written $\pi_t$. The consumption of the individual at time $t$ is noted $c_t$. The dynamic of the savings managed by the individual is the sum of the financial return on investments and of the spread between the annuity and the consumption:

$$dF_t = (1 - \pi_t).F_t.dS_0 + \pi_t.F_t.dS_1 + (B_x - c_t).dt \quad (2.4)$$

Combining this last equation with (2.2) and (2.3), we obtain the following SDE for the individual’s savings:

$$dF_t = ((r + \pi_t.(m - r)).F_t - c_t + B_x).dt + \pi_t.F_t.\sigma.dW_t$$

The dynamic of the wealth being established, we now define the objective of the retired agent. Let $U_1(c_t)$ and $U_2(F_t)$ be respectively the utility provided by the consumption and the utility of the legacy of $F_t$, if the individual deceases at time $t$. We assume that those utilities are C.R.R.A, with risk aversion parameters $\gamma_i = 1, 2 < 1$:

$$U_1(c_t) = \frac{c_t^{\gamma_1}}{\gamma_1} \quad U_2(F_t) = u_2.\frac{F_t^{\gamma_2}}{\gamma_2}$$

$u_2$ is a parameter that worths 0 if the individual has no bequest motive, 1 otherwise. The objective pursued by the pensioner at time $t$, is to determine the consumption and investment patterns maximizing his total expected utility. If $\tau$ is the random instant of decease, and $\rho$ is a discount factor corresponding to the rush to consume, the value function at time $t$, is
defined by:

\[(2.5) \quad v(F_t, t) = \max_{c_t, \pi_t \in U} \mathbb{E}\left( \int_t^\tau e^{-\rho(s-t)} U_1(c_s) ds + e^{-\rho(\tau-t)} U_2(F_\tau) \big| \mathcal{F}_t \right)\]

Where \(U\) is the set of admissible strategies:

\[U = \{c_t, \pi_t : F_t \geq 0 \quad \pi_t \leq \pi_b \quad \forall t \in [0, \tau] \}\]

The set \(U\) is delimited by a constraint on the wealth, stipulating that the wealth may not be negative and by a constraint on investments, specifying that the position in risky asset has an upper bound. This bound is usually set to 100% because a short position in cash is not allowable for an individual saver. We refer to Richard (1974) to prove that the equation \((2.5)\) is equivalent to:

\[(2.6) \quad v(F_t, t) = \max_{c_t, \pi_t \in U} \mathbb{E}\left( \int_t^\tau e^{-\rho(s-t)} s-tp_{x,t} \cdot (U_1(c_s) + \mu(x+s)U_2(F_s)) ds \big| \mathcal{F}_t \right)\]

From the theory of stochastic control (see for e.g. Fleming and Rishel 1975), we know that the value function is solution of the Hamilton Jacobi Bellman equation:

\[(2.7) \quad 0 = \frac{\partial v(F_t, t)}{\partial t} - (\rho + \mu(x + t)) v(F_t, t) + \sup_{c_t, \pi_t \in U} \left( L^{c_t, \pi_t} v(F_t, t) + \frac{c_t}{\gamma_1} + \mu(x + t) u_2, \frac{F_t^{\gamma_2}}{\gamma_2} \right)\]

Where \(L^{c_t, \pi_t} v(F_t, t)\) is the infinitesimal generator of \(v(F_t, t)\):

\[L^{c_t, \pi_t} v(F_t, t) = ((r + \pi_t(m-r)) \cdot F_t + B_x - c_t) \cdot \frac{\partial v(F_t, t)}{\partial F_t} + \frac{1}{2} F_t^2 \cdot \pi_t^2 \cdot \sigma^2 \cdot \frac{\partial^2 v(F_t, t)}{\partial F_t^2}\]

And with the terminal condition:

\[v(F_T, T) = u_2, \frac{F_T^{\gamma_2}}{\gamma_2}\]

The classical method used to solve the HJB equation consists in proposing a form for the value function and next to inject it in the SDE \((2.7)\) to reduce it to one or two ODE. The belonging of the optimal controls to the set of admissible solutions is generally checked a posteriori. This approach doesn’t lead to an analytic solution for the problem treated in this paper.

3. The numerical method.

3.1. The Markov chains approximation. As the problem of the individual asset allocation doesn’t admit an analytic solution, we have opted for a numerical approach, namely the Markov chain approximation of Kushner and Dupuis (2001). The first step consists in delimiting and fractioning a domain of resolution:

\[(3.1) \quad \mathbb{D} = \{(t, f) : t \in (0, \Delta t, \ldots, T) \quad f \in (f_{ib}, f_{ib} + \Delta f, \ldots, f_{ub})\}\]

The choice of \(f_{ib}\) and \(f_{ub}\) and the method of discretization at those bounds are discussed in paragraphs 3.2 and 3.3. Developments done in this paragraph concern interior points of \(\mathbb{D}\).
The Bellman’s equation (2.7) is discretized by replacing the partial derivatives by their finite difference equivalents.

\[ 0 = v_t(f, t) - (\rho + \mu(x + t)) \cdot v(f, t) + \sup_{c_t, \pi_t \in U} \left( \left( (r + \pi_t.(m - r)) \cdot f + B_x \right) \cdot v_t^+(f, t) \right) \]

\[ (3.2) \]

\[-c_t v_F^-(f, t) + 1/2. \pi_t^2 \cdot f^2 \cdot \sigma^2 \cdot v_{FF}(f, t) + U_1(c_t) + \mu(x + t) \cdot U_2(f) \]

With

\[ (3.3) \]

\[ v_t(f, t) = \frac{v(f, t + \Delta t) - v(f, t)}{\Delta t} \]

\[ (3.4) \]

\[ v_F^+(f, t) = \theta \cdot \frac{v(f + \Delta f, t) - v(f, t)}{\Delta f} + (1 - \theta) \cdot \frac{v(f + \Delta f, t + \Delta t) - v(f, t + \Delta t)}{\Delta f} \]

\[ (3.5) \]

\[ v_F^-(f, t) = \theta \cdot \frac{v(f, t) - v(f - \Delta f, t)}{\Delta f} + (1 - \theta) \cdot \frac{v(f, t + \Delta t) - v(f - \Delta f, t + \Delta t)}{\Delta f} \]

\[ v_{FF}(f, t) = \theta \cdot \frac{v(f + \Delta f, t) + v(f - \Delta f, t) - 2v(f, t)}{\Delta f^2} + (1 - \theta) \cdot \frac{v(f + \Delta f, t + \Delta t) + v(f - \Delta f, t + \Delta t) - 2v(f, t + \Delta t)}{\Delta f^2} \]

\[ \theta \] is a parameter of overrelaxation. As the finite difference equivalents of the derivatives depend on \( v(., t) \) and \( v(., t + \Delta t) \), the algorithm developed in subsection 3.4, is implicit and iterative. The optimal consumption and investment policies are respectively noted \( c_t^* \) and \( \pi_t^* \). In order to simplify further calculations, the following notations are adopted:

\[ (3.7) \]

\[ b^+ = (\left( (r + \pi_t^*. (m - r)) \cdot f + B_x \right) \]

\[ (3.8) \]

\[ b^- = c_t^* \]

\[ (3.9) \]

\[ a = \pi_t^2 \cdot f^2 \cdot \sigma^2 \]

Combining equations (3.2) to (3.9), and regrouping terms, allow us to rewrite (3.2) as :

\[ (3.10) \]

\[ v(f, t) \cdot (1 + (\rho + \mu(x + t)) \cdot \Delta \tilde{t}) = \sum_{i=0,1} \sum_{j=-1,0,1} p(t + i \cdot \Delta t, f + j \cdot \Delta f | \pi_t^*, c_t^*) \cdot v(f + j \cdot \Delta f, t + i \Delta t) \]

\[ + (U_1(c_t^*) + \mu(x + t) \cdot U_2(f)) \cdot \Delta \tilde{t} \]

Where

\[ p(t, f | \pi_t^*, c_t^*) = 0 \]

\[ p(t + i \cdot \Delta t, f + j \cdot \Delta f | \pi_t^*, c_t^*) = \frac{b^+ \cdot \theta \cdot \Delta f + \frac{1}{2} \cdot a \cdot \theta \cdot \Delta f^2}{(1 + b^+ \cdot \theta \cdot \Delta f + b^- \cdot \theta \cdot \Delta f + a \cdot \theta \cdot \Delta f^2)} \]

\[ p(t + i \cdot \Delta t, f + j \cdot \Delta f | \pi_t^*, c_t^*) = \frac{b^- \cdot \theta \cdot \Delta f + \frac{1}{2} \cdot a \cdot \theta \cdot \Delta f^2}{(1 + b^+ \cdot \theta \cdot \Delta f + b^- \cdot \theta \cdot \Delta f + a \cdot \theta \cdot \Delta f^2)} \]
\[ p(t + \Delta t, f \mid \pi_t^*, c_t^*) = \frac{1 - (b^- + b^+).((1 - \theta) \cdot \frac{\Delta t}{\Delta f} - a.(1 - \theta) \cdot \frac{\Delta t}{\Delta f^2})}{(1 + b^+ \cdot \theta \cdot \frac{\Delta t}{\Delta f} + b^- \cdot \theta \cdot \frac{\Delta t}{\Delta f^2} + a \cdot \theta \cdot \frac{\Delta t}{\Delta f^2})} \]

\[ p(t + \Delta t, f + \Delta f \mid \pi_t^*, c_t^*) = \frac{b^+.(1 - \theta) \cdot \frac{\Delta t}{\Delta f} + \frac{1}{2} a.(1 - \theta) \cdot \frac{\Delta t}{\Delta f^2}}{(1 + b^+ \cdot \theta \cdot \frac{\Delta t}{\Delta f} + b^- \cdot \theta \cdot \frac{\Delta t}{\Delta f^2} + a \cdot \theta \cdot \frac{\Delta t}{\Delta f^2})} \]

\[ p(t + \Delta t, f - \Delta f \mid \pi_t^*, c_t^*) = \frac{b^-.(1 - \theta) \cdot \frac{\Delta t}{\Delta f} + \frac{1}{2} a.(1 - \theta) \cdot \frac{\Delta t}{\Delta f^2}}{(1 + b^+ \cdot \theta \cdot \frac{\Delta t}{\Delta f} + b^- \cdot \theta \cdot \frac{\Delta t}{\Delta f^2} + a \cdot \theta \cdot \frac{\Delta t}{\Delta f^2})} \]

\[ \Delta \tilde{t} = \frac{\Delta t}{(1 + b^+ \cdot \theta \cdot \frac{\Delta t}{\Delta f} + b^- \cdot \theta \cdot \frac{\Delta t}{\Delta f^2} + a \cdot \theta \cdot \frac{\Delta t}{\Delta f^2})} \]

If all \( p(\cdot \mid \pi_t^*, c_t^*) \) may be interpreted as probabilities of transition of a discrete random variable \( Z \in \{v(t + i \Delta t, f + j \Delta f) \mid i \in \{0, 1\} \land j \in \{-1, 0, 1\}\} \):

\[ \sum_{i=0,1}^{1} \sum_{j=-1,0,1}^{1} p(t + i \Delta t, f + j \Delta f \mid \pi_t^*, c_t^*) = 1 \]

\[ p(t + i \Delta t, f + j \Delta f \mid \pi_t^*, c_t^*) \geq 0 \quad \forall i \in \{0, 1\} \land j \in \{-1, 0, 1\} \]

then, the equation (3.10) may be seen as a discrete version of the dynamic programming principle:

\[ (3.13) \quad Z \cdot (1 + (\rho + \mu(x + t)) \cdot \Delta \tilde{t}) = \mathbb{E}(Z \mid c_t^*, \pi_t^*) + (U_1(c_t^*) + \mu(x + t) \cdot U_2(f)) \cdot \Delta \tilde{t} \]

Or, if \( \Delta \tilde{t} \) is sufficiently small,

\[ (3.14) \quad Z = e^{-(\rho + \mu(x+t)) \cdot \Delta \tilde{t} \cdot \max_{c_t, \pi_t \in \mathbb{U}}} \mathbb{E}(Z \mid c_t, \pi_t) + (U_1(c_t) + \mu(x + t) \cdot U_2(f)) \cdot \Delta \tilde{t} \]

Equation (3.14) means that the current value of \( Z \) is equal to the maximum, amongst all admissible controls, of the expected discounted value of \( Z \), plus the gain of utility, realized during the small time interval \( \Delta \tilde{t} \). In this framework, the convergence of the approximation of \( v(f, t) \) when \( \Delta t \to 0 \) \( \Delta f \to 0 \), has been proved both by a viscosity approach (see Fleming and Soner, chapter 9) and by a probabilistic approach (see Dupuis and Kushner 2001, chapter 14). We insist on the fact that parameters of discretization \( \Delta t, \Delta f \), and of overrelaxation \( \theta \) must be chosen such that conditions (3.11) and (3.12) are satisfied. In general a value of \( \theta \) close to one insures that all \( p(\cdot \mid c_t, \pi_t) \) may be interpreted as probabilities of transition.

We end this paragraph by calculating the optimal consumption and the optimal investment policy. If we derive the discretized Bellman’s equation with respect to \( c_t \) and \( \pi_t \), we get that:

\[ (3.15) \quad c_t^* = \min \left( c_b, \left( v_F^{-1}(f, t) \right)^{1 \over 2} \right) \]

\[ (3.16) \quad \pi_t^* = \min \left( \pi_b, \frac{m - r}{\sigma^2} \frac{1}{f} v_{FF}^{-1}(f, t) \right) \]
$c_b$ is a bound on the consumption related to the constraint on the wealth ($F_t \geq 0$). More details on $c_b$ are presented in the next subsection. The bound on the investment policy $\pi_t \leq \pi_b \leq 100\%$ is justified by the fact that the optimal strategy can be a short position in cash and a long one in equities. This kind of speculative portfolio cannot be carried out by an individual saver.

3.2. **The constraint on the wealth and the domain.** In the continuous formulation (2.7), the set of admissible controls, $\mathbb{U}$, is such that the wealth remains positive and that the fraction of risky assets in portfolio is limited to a certain level.

$$\mathbb{U} = \{ c_t, \pi_t : F_t \geq 0 \quad \pi_t \leq \pi_b \quad \forall t \in [0, \tau] \}$$

The constraint on the wealth is inserted in the framework of numerical resolution as a constraint on the consumption. During the small step of time $\Delta t$, the consumption may not exceed the sum of the market value of the fund and of the annuity paid in during this period:

$$c_t \leq c_b = \frac{F_t}{\Delta t} + B_x$$

In view of the previous expression, the bounds $f_{ib}$ of domain of resolution $\mathbb{D}$ defined by (3.1), is therefore equal to:

$$f_{ib} = -B_x \Delta t$$

The upper bound, $f_{ub}$ is chosen in function of the desired range of results.

3.3. **Approximations at the boundary.** We briefly describe the approximations adopted on the boundary of $\mathbb{D}$, and more precisely when $f \in \{f_{ib}, f_{ub}\}$.

When the wealth reaches the upper bound $f_{ub}$, the probabilities of remaining in the state $(f_{ub}, t)$ and the transition probabilities toward $(f_{ub}, t + \Delta t)$ are modified in order to take into account that the states $(f_{ub} + \Delta f, \cdot)$ aren’t defined:

$$p(t, f_{ub} | \pi^*_t, c^*_t) = 1 - p(t, f_{ub} - \Delta f | \pi^*_t, c^*_t)$$

$$p(t + \Delta t, f_{ub} | \pi^*_t, c^*_t) = 1 - p(t + \Delta t, f_{ub} - \Delta f | \pi^*_t, c^*_t)$$

Whereas missing approximated derivatives are replaced by:

$$v^+_F(f_{ub}, t) = v^+_F(f_{ub} - \Delta f, t)$$

$$v_{FF}(f_{ub}, t) = v_{FF}(f_{ub} - \Delta f, t)$$

Similarly, when the wealth attains the lower bound $f_{lb}$, the probabilities of remaining in the state $(f_{lb}, t)$ and the transition probabilities toward $(f_{lb}, t + \Delta t)$ are:

$$p(t, f_{lb} | \pi^*_t, c^*_t) = 1 - p(t, f_{lb} + \Delta f | \pi^*_t, c^*_t)$$

$$p(t + \Delta t, f_{lb} | \pi^*_t, c^*_t) = 1 - p(t + \Delta t, f_{lb} + \Delta f | \pi^*_t, c^*_t)$$

And missing derivatives are replaced by:

$$v^+_F(f_{lb}, t) = v^+_F(f_{lb} + \Delta f, t)$$

$$v_{FF}(f_{lb}, t) = v_{FF}(f_{lb} + \Delta f, t)$$
3.4. Approximation in value space. This paragraph presents the recursive approach used to compute the value function at each point of the domain \( \mathbb{D} \). First, we introduce some matricial notations. Let \( V(t) \) be the vector of \( v(f_j,t) \) and \( U(c_t,t) \) be the vector of utility gains \( (U_1(c_t) + \mu(x+t).U_2(f_j)) \). \( M \) and \( P \) contain respectively the factors multiplying \( V(t) \) and \( V(t+\Delta t) \) in the expression (3.10). The dynamic programming equation (3.10) becomes therefore:

\[
M(c^*_t, \pi^*_t).V(t) = P(c^*_t, \pi^*_t).V(t+\Delta t) + U(c^*_t, t) \quad \forall t \in \{0, \Delta t, \ldots, T-\Delta t\}
\]

At time \( t = T \), the value function is equal to the utility of bequest.

\[
V(T) = u_2 \cdot \frac{(f_j)^{\gamma_2}}{\gamma_2} \quad \forall f_j \in \{f_{lb}, f_{lb} + \Delta f, \ldots, f_{ub}\}
\]

The algorithm of approximation in value space (also known as the method of successive approximations) performs backward iterations from \( t = T - \Delta t \) to \( t = 0 \), by step of time \( -\Delta t \). At each instant \( t \), the optimal consumption and investment policies \( c^*_t, \pi^*_t \), are calculated by the formulae (3.15) and (3.16). Matrices \( M(c^*_t, \pi^*_t) \) and \( P(c^*_t, \pi^*_t) \) are then computed and the vector of value function \( V(t) \) is obtained by the resolution of the linear system of equations (3.18). This system is easily solved by Gaussian elimination. As the finite difference method for approximating the derivatives of \( v(f, t) \) is implicit, the backward iterations from \( T - \Delta t \) to \( 0 \) are repeated till convergence. Please note that the accuracy \( \epsilon_{f,t} \) at the point \( (f, t) \in \mathbb{D} \) can be estimated by the Bellman’s equation:

\[
\epsilon_{f,t} = v_t(f,t) - \left( \rho + \mu(x+t) \right) . v(f,t) + \left( \left( r + \pi^*_t \cdot (m - r) \right) . f + B_x \right) . v_F^2(f,t)
\]

\[
- c_t^* . v_F(f,t) + 1/2 . \pi^*_t^2 . f^2 . \sigma^2 . v_{FF}(f,t) + U_1(c^*_t) + \mu(x+t).U_2(f)
\]

(3.19)

4. Optimal choice without bequest motive.

In this section, the situation of a retired agent without bequest motive \( (u_2 = 0) \) is analysed and some general conclusions are drawn from numerical results. In the next subsection, the optimal pattern of consumption and the evolution of the fund managed by the individual, are developed for three scenarios, when 70\% of the initial wealth \( (W_0 = 100) \) is devoted to the purchase of an annuity. In paragraph 4.2, the same exercise is done for a full annuitization. The section is then finished by the valuation of the optimal allocation of the initial wealth between an annuity and a self managed fund.

4.1. Partial annuitization. We consider a 60 years old pensioner, who decides to invest 70\% of his capital in a life annuity, providing a continuous income of \( B_{60} = 4.68 \). Table 1 presents the parameters of the financial market.

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<th>Table 1. market parameters.</th>
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<tbody>
<tr>
<td>( r )</td>
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<td>( m )</td>
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The mortality rates (real and of the tariff) are given by a Gompertz-Makeham distribution (see appendix for details). There isn’t any commercial loading \( \epsilon = 0 \). The risk aversion parameter related to the consumption, \( \gamma_1 \), is set to 60\% and the psychological discount factor, \( \rho \), is assumed equal to 2\%. The discretization parameters are: \( \Delta t = 0.5 \), \( \Delta f = 0.1 \)
and $\theta = 0.9999$. Three scenarios, in which the return is constant (by return, we mean the financial return of the fund, $F_t$, see equation (2.4)), are studied: a low return 0.75%, a return equal to $r$ and a high return scenario of 5.25%. Figures 4.1 and 4.2 respectively depict the consumption and the evolution of the fund.

**Figure 4.1.** Evolution of the consumption, $\alpha=70\%$.

In each scenario, it clearly emerges that the pensioner will deplete his savings before the age of 83 years. Crossed this age, the individual consumes the pension $B_{x=60}$, in its totality. In a market with only one risk free rate asset, similar patterns of consumption were already established in a previous paper (Devolder and Hainaut 2005), with a method combining Lagrangian multipliers and the HJB equation.

**Figure 4.2.** Evolution of the fund, $\alpha=70\%$.
In this example, the optimal investment policy consists in investing the whole wealth in the risky asset, $\pi_t = 100\%$, which is the chosen upper bound on $\pi_t$. Relaxing this bound leads to an optimal solution with a huge long position in equities and a short position in cash. In this situation, it seems that the optimal proportion of stocks is partially proportional to the sum of the market value of the fund and of the mathematical reserve. But as mentioned previously, this solution is quite unrealistic and for this reason, we didn’t develop it.

4.2. **full annuitization.** Under the same assumptions as previously, the optimal pattern of consumption/investment is computed when the agent decides to invest 100% of his initial wealth in a life annuity, with a rate $B_{x=60}$ of 6.70.

![Figure 4.3. Evolution of the consumption, $\alpha=100\%$.](image)

The optimal pattern of consumption is decomposable into three periods. The first one, extending from age 60 to 68, is a phase of capitalization. A major part of the annuity is consumed whereas the residue is totally invested in the risky asset. During the second period, from 68 to 83 years old, the individual consumes more than the annuity and depletes the fund. Finally, in the third period, the consumption is equal to the annuity. The existence of the first phase of capitalization clearly reveals that in a market including a risky asset, the agent without bequest motive has interest to invest a part of his patrimony in it. The amount placed in the financial market, is evidently function of the retiree’s risk appetite, $\gamma_1$ and of the psychological discount factor $\rho$. This observation motives developments done in the next paragraph.

4.3. **The annuity puzzle.** We have seen that in the case of full annuitization, a part of the annuity is reinvested in the fund. It is therefore interesting to seek the optimal fraction $\alpha^*$ of the initial wealth that the pensioner should dedicate to the purchase of an annuity, in order to maximize the discounted value of the expected utility arising from the consumption.

$$\alpha^* = \arg \sup_{\alpha} v((1 - \alpha).W_0, 0)$$
A first answer is given by the graph 4.4 which presents the value function for different levels $\alpha$ of annuitization, and for different ages of annuity purchase. All other assumptions (volatility, mortality, ...) are identical to those used in previous examples. We observe that, even if the curvature is not important, a maximum emerges at each age of purchase. This maximum goes from $\alpha^* = 80\%$ when the agent is 50 years old, to $\alpha^* = 100\%$ at 65 years old. This is an interesting observation: a young retired individual, without bequest motive, should not neglect the opportunities of the financial market. We mitigate the results of Yaari in the sense that a small diversification between market risk and mortality risk is the optimal personal strategy of investment.

**Figure 4.4.** Value function by age and $\alpha$.

**Figure 4.5.** Value function by volatility and $\alpha$. 
As showed in figure 4.5, the optimal asset allocation for a 60 years old individual, is however very sensitive to the volatility of the risky asset (when $m=6\%$). Without surprise, there is a negative correlation between the volatility of equities and the optimal fraction of capital dedicated to the self managed fund. When the volatility reaches 35\%, it is nearly no more interesting to invest in the financial market.

We close this section by presenting the influence of the mean stocks return (with $\sigma =30\%$) on the asset allocation for a 60 years old retiree (figure 4.6). The higher is the risk premium of risky assets, the smaller is the fraction of the initial endowment devoted to purchase an annuity. For stocks return inferior to 5.5\%, the optimal allocation is a full annuitization.

**Figure 4.6.** Value function by return of stocks $m$ and $\alpha$.

5. **Optimal choice with bequest motive.**

We now consider the case of an individual who wishes both to leave a legacy ($u_2 = 1$) to his relatives and to consume partly his capital. The chosen risk aversion parameters $\gamma_1, \gamma_2$ are: $\gamma_1 = 20\%$, $\gamma_2 = 60\%$. The first subsection presents the pattern of consumption/investment in case of partial annuitization. Next, the optimal allocation of the wealth between an annuity and a fund is analysed for different ages, volatilities-returns of the risky asset and different bequest motives.

5.1. **Partial annuitization.** Again, the analysis is done for a 60 years old pensioner who dedicates 70\% of his wealth to purchase an annuity, providing an income of $B_{x=60} = 4.68$ . Others parameters are identical to those used in previous examples and the same three scenarios of return (0.75\%, 3.25\% and 5.75\%) are developed.
Figures 5.1 and 5.2 respectively depict the evolution of the optimal consumption and of the retiree’s savings. For each of the three scenarios, the consumption rate is inferior to the annuity during the first years. As we will see on graph 5.3, the fraction of the annuity capitalized is totally invested in risky asset during this phase of accumulation. After 4, 7 or 14 years, in function of the asset return, the consumption becomes higher than the annuity. However, when the financial return is high enough (scenario 5.75%), the individual’s fund doesn’t decrease. In this case, the spread between the consumption and the annuity is financed by the financial performance of assets. We close this section by a comment on the evolution of the investment policy, figure 5.3. As mentioned early, the totality of the fund
is placed in risky asset, during the first years. We observe that the quicker is the growth of the fund, the quicker is the decrease of the position in equities. When the fund performs well (scenario 5.75%), the position in risky asset is reduced after 7 years. Whereas in case of poor performance (scenario 0.75%), the part of risky asset is reduced only after 27 years. Such investment strategies are coherent with the well admitted principle that the position in equities should always decrease with age.

**Figure 5.3.** Investment policy.

5.2. **The annuity puzzle.** As in paragraph 4.3, we seek the optimal fraction $\alpha^{*}$ of the initial capital that should be devoted to the purchase of an annuity, in order to maximize the expected utility.

**Figure 5.4.** Value function by age and $\alpha$. 
The graph 5.4, presents the value function for different levels of annuitization and for different ages of annuity purchase. The set of assumptions is identical to the one of the previous examples. The curvature of the value functions is not important but it is possible to draw a maximum at each age of purchase. For the chosen preference parameters ($\gamma_1 = 20\%$ $\gamma_2 = 60\%$), the optimal level of annuitization, 75\%, is independent of the age of the pensioner.

**Figure 5.5.** Value function by volatility and $\alpha$.

![Value function by volatility and $\alpha$.](image)

**Figure 5.6.** Value function by return of stocks $m$ and $\alpha$.

![Value function by return of stocks $m$ and $\alpha$.](image)

Figures 5.5 and 5.6 respectively show the influence of the volatility (with $m=6\%$) and of the stocks mean return (with $\sigma=30\%$) on the optimal level of annuitization, for a 60 years
old retired individual. There is a positive correlation between the annuitized capital and the volatility. If the stocks volatility falls to 20%, 60% of the wealth should be annuitized. On the contrary, the level of annuization is negatively correlated to the mean stocks return. For an average return of 7%, the amount annuitized decrease to 70%. We also observe that, despite the bequest motive, the level of annuitization climbs to 95% if the mean stocks return is only of 4%.

Finally, we analyze the influence of the bequest motive on the optimal asset allocation. For a given consumption utility, $\gamma_1 = 20\%$, Figure 5.7 depicts the impact of the bequest parameter, $\gamma_2$, on the value function, for a 60 years old retiree. In agreement with our intuition, the smaller is the bequest motive, the higher is the proportion of the initial wealth devoted to the purchase of a life annuity.

**Figure 5.7.** Value function by $\gamma_2$ and $\alpha$.

6. Conclusions.

This paper solves the individual asset allocation problem for a pensioner who tries to maximize the expected utility of his future consumption, with or without bequest motive. After a brief presentation of the market, one establishes the HJB equation coupled to this maximization problem. As an analytic solution is currently unavailable, we have opted for the numerical method developed by Kushner and Dupuis (2001), which mainly consists in the implicit discretization of the partial derivatives included in the HJB equation. If parameters of discretization are well chosen, the HJB equation can be rewritten as a discrete dynamic programming equation and the convergence toward the unique viscosity solution is guaranteed.

An important part of this work is devoted to the analysis of examples from which we draw some general remarks. At first, we address the case of an individual without bequest motive. In such situation, we observe that the pensioner should invest a fraction of his wealth in a life annuity. The annuitized amount is proportional to the age of the pensioner and to the
volatility of the risky asset and inversely proportional to the stocks return. The consumption pattern can also be decomposed in two periods: during the first one, the wealth constraint is inactive ($F_t > 0$) and the individual’s savings decrease. In the second period, the constraint is active ($F_t = 0$) and the optimal consumption is equal to the annuity.

Finally, we show that, even with a legacy motive, the agent should annuitize an important part of his patrimony to maximize his expected utility. Again, the capital dedicated to the purchase of equities is function of the pensioner’s age, of the volatility and expected return of the risky asset. In many scenarios, the optimal consumption is lower than the annuity during the first years and the saving grows up (it is a capitalization phase). We also observe that the optimal part of the fund invested in the risky asset decreases with the age.

In conclusion, this paper presents rational arguments in favour of the integration of an annuity in the individual asset allocation of an individual in retirement. Intuitively, our results may be explained by the fact that the mortality risk credit is in many case more attractive than the risk premium of other risky assets.

7. Appendix.

In the examples presented in this paper, we assume that the real mortality rates and the mortality rates used for pricing, $\mu(x + t)$ are given by a Gompertz-Makeham distribution. The parameters are those defined by the Belgian regulator for the pricing of a life insurance purchased by a man. For an individual of age $x$, the mortality rate is:

$$\mu(x) = \mu^{tf}(x) = a_\mu + b_\mu \cdot e^{x} \quad a_\mu = -\ln(s_\mu) \quad b_\mu = \ln(g_\mu) \cdot \ln(c_\mu)$$

Where the parameters $s_\mu, g_\mu, c_\mu$ take the values showed in the table 2. Table 3 presents the evolution of mortality rates in function of the age of the individual.

**Table 2.** Belgian legal mortality, for life insurance products, and for a male population.

<table>
<thead>
<tr>
<th>$s_\mu$</th>
<th>0.999441703848</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_\mu$</td>
<td>0.99973441115</td>
</tr>
<tr>
<td>$c_\mu$</td>
<td>1.116792453830</td>
</tr>
</tbody>
</table>

**Table 3.** Mortality rates.

<table>
<thead>
<tr>
<th>Age x</th>
<th>$\mu(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.10%</td>
</tr>
<tr>
<td>40</td>
<td>0.18%</td>
</tr>
<tr>
<td>50</td>
<td>0.37%</td>
</tr>
<tr>
<td>60</td>
<td>0.88%</td>
</tr>
<tr>
<td>70</td>
<td>2.23%</td>
</tr>
<tr>
<td>80</td>
<td>5.74%</td>
</tr>
</tbody>
</table>
REFERENCES


Acknowledgement. We gratefully acknowledge the financial support of the “Communauté française de Belgique” under the “Projet d’Action de Recherches Concertées”.