Multi Dimensional Lee-Carter model with switching mortality processes.

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Abstract
This paper proposes a multi dimensional Lee Carter model, in which the time dependent components are ruled by switching regime processes. The main feature of this model is its ability to replicate the changes of regimes observed in the mortality evolution. Changes of measure, preserving the dynamics of the mortality process under a pricing measure, are also studied. After a review of the calibration method, a 2-D, 2-regimes model is fitted to the male and female French populations, on the period 1946-2007. Our analysis reveals that one regime corresponds to longevity conditions observed during the decade following the second world war, while the second regime is related to longevity improvements observed during the last 30 years. To conclude, we analyze, in a numerical application, the influence of changes of measure affecting transition probabilities, on prices of life and death insurances.

KEYWORDS: Longevity, mortality, Lee-Carter.

1 Introduction.
The continuous improvement of longevity is a matter of concerns for the insurance and pension fund industry. This evolution is mainly related to the reduction of mortality linked to infectious diseases, and to the decrease of mortality from chronic diseases affecting older ages. The interested readers can refer to the work of McDonald et al. (1998) for a detailed presentation of recent mortality trends. This evolution has also been at the origin of the development of numerous methods to forecast future mortality rates, and to model the uncertainty around these forecasts. Lee and Carter (1992) were the first to propose a simple and efficient method to forecast mortality. In their approach the logarithm of mortality rates is the sum of a fixed age component and of a age specific rate multiplied by a time component. Lee (2000) has written a review of recent applications of the Lee Carter methodology, which has been widely adopted by the practitioners community. There exist many alternatives or variants to the Lee-Carter model. For a general survey of other projection methods, we refer to Pitacco (2004), Wong-Fupuy and Haberman (2004) or Cairns (2008).

As underlined by Sweeting (2009), the mortality process exhibits clear changes of trends during the last century. This paper proposes a multifactor model, in which the time dependent components are driven by switching regime random walks. This work is an extension of the multifactor model developed by Renshaw and Haberman (2003) and of the work ofMilidonis et al. (2011), who studied a one dimension switching Lee Carter model. Regime switching models were introduced in the eighties and used by Hamilton (1989) in econometric applications. They are particularly well adapted to tackle changes of behaviors exhibited by financial series. Our work reveals that this technique is also efficient to build a model that capture changes of trends observed in mortality rates.

The outline of the paper is the following. We start by a presentation of the multi dimensions Lee Carter in continuous time. Next, the hidden Markov process driving the time components of the model is developed. After a brief presentation of calibration procedures, we present the family of changes of measure available for pricing. As illustration, a model with 2 dimensions and 2 states is fitted to the French male and female populations, on the period 1946-2007. This section is followed by a discussion of results and a comparison with the one dimension model ofMilidonis et al. (2011). The paper ends up with an analysis of the influence of transition probabilities changes on prices of annuities and death insurances.

2 Continuous Multi dimensions Lee Carter.
For a given time horizon $T$, we consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P\right)$ on which the death time is modeled as a stopping time $\tau$ with respect to $\mathcal{F}_t$. Furthermore, we assume that $\mathcal{F}$ is the enlarged
filtration $\mathcal{H} \vee \mathcal{G}$ where $\mathcal{H}$ is the filtration related to risk factors and $\mathcal{G}$ is the complement such that $\tau$ is a stopping time on $\mathcal{F}$. As mentioned by Biass (2009), $\mathcal{H}$ may be seen as the filtration carrying information based on medical/demographic data at the population level, while $\mathcal{G}$ captures the occurrence of death. The time of death of an individual of age $x$ is the first jump of a Poisson process, noted $N^x_t$, whose intensity is a non-negative predictable process, the mortality rate, $\mu^x_t$ is defined on $\mathcal{H}_t$. Conditionally to the path followed by the mortality rates, the survival probability is given by:

$$P(\tau \geq t | \mathcal{H}_t) = e^{-\int_0^t \mu^x_s \, ds} \quad 0 \leq t \leq T.$$ 

and the survival probability is

$$P(\tau \geq t | \mathcal{F}_u) = \mathbb{1}_{\tau > u} E\left(e^{-\int_u^T \mu^x_s \, ds} | \mathcal{H}_u\right).$$

Furthermore, the process

$$M^x_t = N^x_t - \int_0^{t \wedge \tau} \mu^x_s \, ds$$

is a martingale with respect to $\mathcal{F}$. In the multidimensional Lee-Carter model with $r$ factors, such as proposed by Renshaw and Haberman (2003), the mortality rates obey to the following dynamics:

$$\mu^x_t = \exp\left(\alpha_{(x+t)} + \sum_{i=1}^r \beta^x_{(x+t),i} \kappa^x_i\right)$$

(2.1)

where $\beta^{i=1...r}$ are age effects, while $\kappa^x_i = (\kappa^x_i)^{i=1...r}$ are processes capturing the evolution of mortality over time. The calibration of parameters in discrete time is discussed in a next section. Renshaw and Haberman have assumed that the vector $\kappa^x_i$ is ruled by independent ARIMA processes. In this way, so as to take into account the change of trends observed in the dynamics of the vector $\kappa^x_i$, we rather assume that their variations are distributed as a multivariate normal, whose mean and covariance depend on the state of an unobservable Markov process $\eta_t$ that takes its value in the set $\mathcal{N} = \{1, 2, ..., N\}$. If the instantaneous mean and covariance matrix of $\kappa^x_i$ are noted $a(\eta_t)$ and $C(\eta_t) = \Sigma(\eta_t)\Sigma(\eta_t)$ with

$$a(\eta_t) = \begin{pmatrix} a_1(\eta_t) \\ \vdots \\ a_r(\eta_t) \end{pmatrix} \quad \Sigma(\eta_t) = \begin{pmatrix} \sigma_{11}(\eta_t) & \cdots & \sigma_{1r}(\eta_t) \\ \vdots & \ddots & \vdots \\ \sigma_{r1}(\eta_t) & \cdots & \sigma_{rr}(\eta_t) \end{pmatrix} \quad \eta_t = 1, \ldots, N.$$ 

The dynamics of the vector $\kappa^x_i$ is given by:

$$d\kappa^x_t = a(\eta_t) dt + \Sigma(\eta_t) dW_t,$$

(2.2)

where $W_t$ is a vector of $r$ independent Brownian motions, defined on the filtration $\mathcal{H}$.

3 The Markov process.

The indicator of the state at time $t$, is a hidden Markov process $\eta_t$. As mentioned previously, $\eta_t$ takes its value in the set $\mathcal{N} = \{1, 2, ..., N\}$ and has an intensity matrix $H$ whose elements, noted $h_{i,j}$, satisfy the following conditions:

$$h_{i,j} \geq 0 \quad \forall i \neq j \quad \sum_{j=1}^N h_{i,j} = 0 \quad \forall i \in \mathcal{N}.$$ 

(3.1)

The probabilities of transitions (under the real measure) between times $t$ and $s \geq t$ are computed as the (matrix) exponential of $H$:

$$P(t,s) = \exp \left( H(s-t) \right).$$

(3.2)

The elements of the matrix $P(t,s)$ are noted $p_{i,j}(t,s), i, j \in \mathcal{N}$. And $p_{i,j}(t,s)$ is the probability of jumping from state $i$ at time $t$ to state $j$ at time $s$:

$$p_{i,j}(t,s) = P(\eta_s = j | \eta_t = i) \quad i, j \in \mathcal{N}.$$ 

(3.3)

The probability of being in state $i$ at time $t$, noted $p_i(t)$ depends upon the initial probabilities $p_{k=1...N}(0)$ at time $t = 0$, as follows:

$$p_i(t) = P(\eta_t = i) = \sum_{k=1}^N p_k(0)p_{k,i}(0,t) \quad \forall i \in \mathcal{N}.$$ 

(3.4)
Whether the Markov process has been running for a sufficiently long enough period of time, we can show that this probability is independent from the initial state:  
\[ p_i = \lim_{t \to +\infty} p_i(t), \quad \forall i \in \mathcal{N}. \]
In this framework, we denote by \( \tau_i \) the random time at which the Markov chain \( \eta \) changes of state, for the \( i^{th} \) times. Among the approach chosen to model the Markov chain, we adopt the marked point process approach for its simplicity. We define a mark space \( E \) which includes all possible regime switching as:
\[
E = \{ z = (i, j) : i \in \{1, \ldots, N\}, j \in \{1, \ldots, N\}, i \neq j \}.
\]
The \( \sigma \)-algebra generated by \( E \) is noted \( \mathcal{E} \). As the Markov process is not directly observable \( \mathcal{E} \) is not included in the filtration \( F \). On \( \mathcal{E} \) we define a mark point process \( \nu(t, \cdot) \), see Bremaud (1981) for an introduction. If \( A \) is a subset of \( E \), \( \nu(t, A) \) counts the cumulative number of regime shifts that belong to \( A \) during \((0; t]\). The compensator of \( \nu(t, \cdot) \) is given by
\[
\gamma(dt, dz) = \sum_{i \neq j} h_{i,j} I(\eta_\tau = i)\epsilon_{(i,j)}(dz)dt,
\]
where \( I(.) \) is the indicator function and \( \epsilon_{(i,j)} \) denotes the Dirac measure at point \( z = \{i, j\} \). The filtration coupled to the market point process \( \nu(t, \cdot) \) at time \( t \) is noted \( (\mathcal{E}_t)_{t \geq 0} \). The Markov process \( \eta \) is equal to an integral on \( E \) of the function \( \varphi(z) = \varphi(i, j) = j - i \) with respect to time and to the marked point process:
\[
\eta_t = \int_0^t \int_E \varphi(z) \nu(ds, dz).
\]
By definition, \( \eta_t \) is \( \mathcal{E}_t \)-adapted. Furthermore,
\[
M^n_t = \eta_t - \int_0^t \int_E \varphi(z) \gamma(ds, dz)
\]
is a local martingale under the real measure \( P \).

4 Moments of \( \mu^x_t \).

In the 1D Lee Carter model, the mortality rates are lognormally distributed. In our setting, the density has no closed form expression. However, all moments of \( \mu^x_t \) can be calculated as shown in the next proposition.

**Proposition 4.1.** Let us denote \( \{e_1, e_2 \ldots e_N\} \) the set of unit vectors and \( 1 \), a vector of \( N \) ones. The moment of order \( m \) of \( \mu^x_t \) is given by the following expression:
\[
E((\mu^x_t)^m) = e^{m(\alpha_m + \sum_{i=1}^N \beta_x^m i\kappa^i)} \sum_{l=1}^N p_l(0) \left( (\exp(B_m l) e_i ; 1) \right)
\]
where
\[
B_m = H + \text{diag} \left( \begin{array}{c} w_1 \\ \vdots \\ w_N \end{array} \right),
\]
with
\[
w_l = \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \left( m \beta_x^m a_i(l) + \frac{1}{2} m^2 (\beta_x^m l)^2 \sigma_i(l) \right),
\]
and \( \sigma_i(l) \) is the \( i^{th} \) line of the matrix \( \Sigma(l) \).

**Proof.** In view of equations (2.1) and (2.2), we have that
\[
E((\mu^x_t)^m) = e^{m(\alpha_m + \sum_{i=1}^N \beta_x^m i\kappa^i)}
\]
and if we denote by \( \delta(i, \alpha) \) an indicator variable that is worth 1 if we are in the state of the world \( i \), 0 else, the processes \( \kappa^i \) are equal to:
\[
\kappa^i = \kappa_0^i + \sum_{l=1}^N \int_0^t a_i(l) \delta(l, \eta_u) du + \sum_{l=1}^N \int_0^t \delta(l, \eta_u) \left( \sigma_{i1}(l) \ldots \sigma_{iN}(l) \right) dW_u \quad i = 1 \ldots r.
\]
According to the result of Bungton and Elliott (2002), the following equality is satisfied:

$$\text{rewritten as follows:}$$

$$T$$

Let us denote $$T_i$$ the total time spent into state $$l$$, during the interval of time $$[0, t]$$. As the sum of $$T_i$$ is equal to the interval of time, we have the constraint $$T_N = t - \sum_{i=1}^{N-1} T_i$$. The expectation eq. (4.7) may then be rewritten as follows:

$$\mathbb{E} \left( (\mu_t^X)^n \right) = e^{m(\alpha + \sum_{i=1}^{\infty} \beta_i^e \xi_i^0)} \mathbb{E} \left( e^{\left( \sum_{i=1}^{N-1} w_i - w_N \right) T_i} \right) e^{w_N t}. \tag{4.8}$$

According to the result of Buffington and Elliott (2002), the following equality is satisfied:

$$\mathbb{E} \left( e^{\sum_{i=1}^{N-1} (w_i - w_N) T_i} \right) = \left\langle \exp \left( (H' + \text{diag} (w_1 - w_N, \ldots, w_{N-1} - w_N, 0)) t \right) \delta(0) ; 1 \right\rangle, \tag{4.9}$$

where $$\delta(t) = (\delta(i, \alpha(t)) i \in N \rangle$$ is a vector taking its values in the set of units vectors $$\{e_1, e_2 \ldots e_N\}$$. Equation (4.8) becomes then

$$\mathbb{E} \left( (\mu_t^X)^n \right) = e^{m(\alpha + \sum_{i=1}^{\infty} \beta_i^e \xi_i^0)} \mathbb{E} \left( \langle \exp (B_{m} t) \delta(0) \rangle ; 1 \right)$$

$$= e^{m(\alpha + \sum_{i=1}^{\infty} \beta_i^e \xi_i^0)} \sum_{i=1}^{N} p_i(0) \langle \exp (B_{m} t) \rangle \delta(0) ; 1 \right\rangle. \tag{4.8}$$

This result is an alternative to Monte Carlo simulations to infer the mean of future $$\mu_t^X$$. If the moments of order $$m$$ are noted $$\mu_m$$, the cumulants of order $$m$$, $$\kappa_m$$, are given by the following relations (found in Gardiner 1993):

$$\kappa_1 = e_1,$$

$$\kappa_2 = e_2 - (e_1)^2,$$

$$\kappa_3 = e_3 - 3e_1e_2 + 2(e_1)^3,$$

$$\kappa_4 = e_4 - 4e_1e_3 - 3(e_2)^2 + 12(e_1)^2 e_2 - 6e_1^2.$$

And based on the cumulants, the statistical distribution of $$\mu_t^X$$ can in theory be approached by the Edgeworth expansion:

$$f_{\mu_t^X}(\mu) = \frac{1}{\sqrt{2\pi}c_2} \exp \left[ -\frac{(x - c_1)^2}{2c_2} \right] \left[ 1 + \frac{c_3}{3c_2^{3/2}}H_3 \left( \frac{\mu - c_1}{c_2^{1/2}} \right) + \frac{c_4}{4!c_2^2}H_4 \left( \frac{\mu - c_1}{c_2^{1/2}} \right) \right].$$

where $$H_3(x) = x^3 - 3x$$ and $$H_4(x) = x^4 - 6x^2 + 3$$ are Hermite polynomials.

### 5 Changes of measure.

In this section we discuss changes of measure for pricing purposes. We note $$\mathcal{I}$$ the enlarged filtration $$\mathcal{F} \vee \mathcal{E}$$. An equivalent measure $$\bar{P}$$ to $$P$$ on the $$\sigma$$-algebra $$\mathcal{I}$$ is a measure of probability that grants a null probability to all events having a null probability under $$\bar{P}$$. Given two equivalent measures $$\bar{P}$$ and $$P$$, there exists a $$\mathcal{F}$$-measurable random variable, $$\frac{d\bar{P}}{dP}$$, called Radon-Nykodym derivatives which is strictly positive and such that $$\mathbb{E} \left( \frac{d\bar{P}}{dP} \mid \mathcal{F}_0 \right) = 1$$. The Radon Nikodym density process is defined as follows:

$$\xi_t := \mathbb{E} \left( \frac{d\bar{P}}{dP} \mid \mathcal{F}_t \right).$$
If \( V(t) \) is a mortality derivative that delivers a payoff \( V(T) \) at time \( T \), its price is equal to the discounted payoff at the risk free rate \( r \), under the chosen pricing measure \( \tilde{P} \):

\[
V(t) = \mathbb{E}^\tilde{P} \left( e^{-r(T-t)} V(T) \mid \mathcal{F}_t \right) = \frac{\mathbb{E}^P \left( \xi_T e^{-r(T-t)} V(T) \mid \mathcal{F}_t \right)}{\mathbb{E}^P (\xi_T \mid \mathcal{F}_t)} .
\]

**Proposition 5.1.** An equivalent measure \( \tilde{P} \) to \( P \) is defined by the following Radon Nikodym derivatives:

\[
\xi_t = \xi_t^W \xi_t^\tau \xi_t^\eta , \tag{5.1}
\]

where

\[
\xi_t^W = \exp \left( - \int_0^t \phi_s^W dW_s - \frac{1}{2} \int_0^t \langle \phi_s^W, \phi_s^W \rangle ds \right),
\]

\[
\xi_t^\tau = \exp \left( \int_0^{t \wedge \tau} \ln (1 + \phi_s^\tau) dN_s - \int_0^{t \wedge \tau} \phi_s^\tau \mu_s^\tau ds \right),
\]

\[
\xi_t^\eta = \exp \left( \int_0^t \ln (1 + \phi_s^\eta (z)) \varphi(z) \nu(ds, dz) + \int_0^t \phi_s^\eta (z) \varphi(z) \gamma(ds, dz) \right).
\]

**Proof.** According standard results on decomposition of martingale (see Kusuoka 1999), this \( \mathcal{I} \) martingale can be written as a sum of four terms:

\[
\xi_t = 1 - \int_0^t \xi_{s-} \phi_s^W dW_s + \int_0^t \xi_{s-} \phi_s^\tau dM_s^\tau + \int_0^t \xi_{s-} \phi_s^\eta ds , \tag{5.2}
\]

where \( \phi_s^W = (\phi_s^{W1}, \ldots, \phi_s^{Wn}) \), \( \phi_s^\tau \), \( \phi_s^\eta \) are \( \mathcal{I} \)-adapted predictable processes. According to the result of Protter (2004, p84) the solution of equation (5.2) is well equation (5.1) \( \Box \)

Note that if we refer to the Girsanov theorem (see Protter 2004, p134), the following processes

\[
\tilde{W}_t = W_t + \int_0^t \phi_s^W ds ,
\]

\[
\tilde{M}_s^\tau = N_s^\tau - \int_0^{t \wedge \tau} (1 + \phi_s^\tau) \mu_s^\tau ds ,
\]

\[
\tilde{M}_s^\eta = \eta_t - \int_0^t \int_E (1 + \phi_s^\eta (z)) \varphi(z) \gamma(ds, dz) .
\]

are martingales under the probability measure \( \tilde{P} \). Note that the changes of measure of Markov processes are also discussed in Bremaud (1981), p241 theorems T10. In our setting, there is no warranty that the mortality process under \( P \) keeps the same structure under the equivalent measure. Given that \( \phi_s^\eta \) depends on the enlarged filtration \( \mathcal{I} \) and not only of \( \mathcal{H} \), the transition probabilities of the chain \( \eta_t \) may depend on the evolution of the Brownian motion and hence loses its Markov property. In the remainder of this paper, we focus on changes of measure preserving the structure of the process. For this reason, the constraint \( \phi_s^\tau > -1 \) is required to keep a positive mortality rate. On another side, the first necessary condition to guarantee that the process \( \eta_t \) is still Markov under \( \tilde{P} \), is that the process \( \phi_s^\eta (z) = \phi_t^\eta (z) \) when \( z = (i,j) \) , in view of (3.1) the second required condition is:

\[
\phi_{i,j} > -1 \quad \forall \ i \neq j \quad \sum_{j=1}^N h_{i,j} (1 + \phi_{i,j}) = 0 \quad \forall \ i \in N . \tag{5.3}
\]

Under \( \tilde{P} \), the mortality process \( N_t^\tau \) has the following intensity:

\[
\tilde{\mu}_t^\tau = (1 + \phi_t^\tau) \exp \left( \alpha_{(x+t)} + \sum_{i=1}^r \beta_{(x+t)}^i \kappa_t^i \right) , \tag{5.4}
\]

while the dynamics of the vector \( \kappa_t \) is given by:

\[
d\kappa_t = \left( \begin{array}{c}
a_{1}(\eta_t) \\
\vdots \\
a_r(\eta_t) \\
\end{array} \right) + \Sigma(\eta_t) \left( \begin{array}{c}
\phi_t^{W1} \\
\vdots \\
\phi_t^{W2} \\
\end{array} \right) dt + \Sigma(\eta_t) d\tilde{W}_t . \tag{5.5}
\]
In this section, we explain how to calibrate the multi dimensions Lee Carter. The probability that an individual of age \( x \) dies before time \( t \) is noted \( q(t,x) \). The intensity of jumps, \( \mu(t,x) \), for an individual of age \( x \) at time \( t \) is the instantaneous mortality rate. The relation between the probability of death and this rate is the following:

\[
q(t,x) = 1 - \exp\left(-\int_t^{t+1} \mu(s,x+s-t) \, ds\right).
\]

In practice, the mortality rate is assumed to be constant on \( [t, t+1[ \times [x, x+1[ \), and is hence equal to

\[
\mu(s,y) = -\ln (1-q(t,x)) \quad \forall s \in [t, t+1[ \quad y \in [x, x+1[.
\]

In the multidimensional Lee-Carter model with \( r \) factors, the mortality rates obey to the following dynamics:

\[
\mu(t,x) = \exp(\alpha_x + \sum_{i=1}^r \beta^i_x \kappa^i_x),
\]

where \( \beta^i_x \) are age effects, while \( \kappa^i_t \) are processes capturing the evolution of mortality over time. The following constraints are added

\[
\sum_x \beta^i_x = 1 \quad \sum_t \kappa^i_t = 0 \quad \forall i,
\]

to ensure the model identification. If we have at disposal data from year \( t_{\text{min}} \) to \( t_{\text{max}} \), for ages \( x_{\text{min}} \) to \( x_{\text{max}} \), the estimator \( \hat{\alpha}_x \) of \( \alpha_x \) is the mean of the observed log mortality rates:

\[
\hat{\alpha}_x = \frac{1}{t_{\text{max}} - t_{\text{min}} + 1} \sum_{t=t_{\text{min}}}^{t_{\text{max}}} \ln \mu(t,x) \quad x = x_{\text{min}}, \ldots, x_{\text{max}}.
\]

The parameters \( \beta^i_x \) and \( \kappa^i_t \) are endogenous and hence not directly observable. Their estimators are inferred from the singular value decomposition of the matrix \( Z \) defined as:

\[
Z = (\ln \mu(t,x) - \hat{\alpha}(t,x))_{t=t_{\text{min}} \ldots t_{\text{max}}, x=x_{\text{min}} \ldots x_{\text{max}}} = \sum_{i \geq 1} \sqrt{\lambda_i} v_i^t w_i
\]

where \( \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \) are the eigenvalue of \( Z'Z \), \( v_i \) and \( u_i \) are respectively the eigenvectors and the normed eigenvectors of \( Z'Z \). The estimators of \( \beta^i_x \) and \( \kappa^i_t \) are then

\[
\left(\hat{\beta}^i_x\right)_{x=x_{\text{min}} \ldots x_{\text{max}}} = \frac{1}{\sum_j v_{i,j}} v_{i,x_{\text{max}}}, \quad i = 1, \ldots, r
\]

and

\[
\left(\hat{\kappa}^i_t\right)_{t=t_{\text{min}} \ldots t_{\text{max}}} = \sqrt{\lambda_i} \sum_j v_{i,j} u_{t_{\text{min}}}, \quad i = 1, \ldots, r.
\]
The model fits the crude mortality rates \( \mu(t, x) = d_{x,t}/e_{x,t} \) where \( d_{x,t} \) and \( e_{x,t} \) are respectively the number of deaths and the exposure (size of the population) of age \( x \) at time \( t \). Usually, the factor \( \kappa_i^t \) is adjusted to ensure that the actual total deaths are identical to the total expected deaths for each \( t \):

\[
\sum_{x=x_{\text{min}}}^{x_{\text{max}}} d_{x,t} - \sum_{x=x_{\text{min}}}^{x_{\text{max}}} e_{x,t} \exp \left( \alpha_x + \sum_{i=1}^{r} \beta_x^j \kappa_i^t \right) = 0.
\]

Finally, processes \( \hat{\kappa}_i^{t-1} \) and \( \hat{\alpha}_x \) are adjusted to satisfy the constraints (6.2). Once that the estimator of \( \kappa_i^t \) are obtained, we can fit a discrete version of (8.1) to them.

7 Calibration of the switching regime model.

The calibration method used in this paper is directly inspired from the Hamilton filter (1989), which estimates parameters by maximum likelihood estimation. We note \( O_1, O_2, \ldots, O_n \) be the \( n \) observed realized values of \( (\kappa_i^1, \ldots, \kappa_i^r) \), at equidistant times \( t = t_1, t_2, \ldots, t_n \), spaced by \( \Delta t \). We assume that \( \eta_t \) changes of state only at discrete times. Hence, if we are in the \( j^{th} \) state at time \( t_{i-1} \), according to eq. (8.1), the variation of the process \( \Delta \kappa_{ti} = \kappa_{ti} - \kappa_{ti-1} \) on \( [t_{i-1}, t_i] \) is a multivariate random variable \( O_i \sim N(a(\eta_t) \Delta t, \Sigma(\eta_t) \sqrt{\Delta t}) \). We note \( \Theta \) the set of parameters of our model

\[
\Theta = \{a_1 \ldots r(\eta_t), \sigma_1 \ldots r(\eta_t), h_{j=1 \ldots N, j=1 \ldots N}\}.
\]

The log-likelihood function, \( \log L \), is defined as follows:

\[
\log L = \log f(O_1|\Theta) + \log f(O_2|\Theta, O_1) + \log f(O_3|\Theta, O_1, O_2) + \ldots + \log f(O_n|\Theta, O_1, \ldots, O_{n-1}),
\]

where \( f(O_k|\Theta, O_1, \ldots, O_{k-1}) \) is the density function of variation \( \Delta \kappa_{ti} \) on the \( k^{th} \) period, conditionally to a set of given model parameters \( \Theta \) and to previous observations \( O_1, \ldots, O_{k-1} \). The parameters calibrating the switching regime model are the one maximizing the log-likelihood function. Hamilton has shown that the conditional density, involved in the calculation of \( f(O_k|\Theta, O_1, \ldots, O_{k-1}) \), may be recursively calculated:

\[
f(O_k|\Theta, O_1, \ldots, O_{k-1}) = \sum_{i=1}^{N} \sum_{j=1}^{N} p_i(t_{k-1}|\Theta, O_1, \ldots, O_{k-1}) p_{ij}(t_{k-1}, t_k|\Theta) f(O_k|\Theta, \eta_t = j).
\]

where

- \( f(O_k|\Theta, \eta_t = j) \) is the Gaussian multivariate density of \( \Delta \kappa_{ki} \) in state \( j \), \( N(a(j) \Delta t, \Sigma(j) \sqrt{\Delta t}) \),
- \( p_{ij}(t_{k-1}, t_k|\Theta) \) is the probability of transition, as defined by eq. (3.3), from state \( i \) at time \( t_{k-1} \) to state \( j \) at time \( t_k \) for the set of parameters \( \Theta \),
- \( p_i(t_{k-1}|\Theta, O_1, \ldots, O_{k-1}) \) is the probability of being in state \( i \) at time \( t_{k-1} \), conditionally to previous observations.

The probability \( p_i(t_{k-1}|\Theta, O_1, \ldots, O_{k-1}) \) may be inferred recursively from \( f(O_{k-1}|\Theta, O_1, \ldots, O_{k-2}) \) as follows:

\[
p_i(t_{k-1}|\Theta, O_1, \ldots, O_{k-1}) = \frac{\sum_{j=1}^{N} p_j(t_{k-2}|\Theta, O_1, \ldots, O_{k-2}) p_{ji}(t_{k-2}, t_{k-1}|\Theta) f(O_{k-1}|\Theta, \eta_{t_{k-1}} = i)}{f(O_{k-1}|\Theta, O_1, \ldots, O_{k-2})}. \tag{7.1}
\]

In order to initiate the recursion, we need to determine \( f(O_1|\Theta) \). Hamilton assumes that the Markov chain has been running for a sufficiently long enough period of time, so as to apply the stationary property of Markov chains, mentioned in section 3. In particular, we get that:

\[
f(O_1|\Theta) = \sum_{i=1}^{N} p_i(\Theta) f(O_1|\Theta, \eta_t = i),
\]

where \( p_i(\Theta) \) are the stationary probabilities of the Markov process \( \eta_t \), for the set of parameters \( \Theta \) (\( p_i(\Theta) = \lim_{t \to \infty} p_i(t) \)).
8 An application to the French population.

The male and female French mortality rates have been retrieved from the human mortality database (www.mortality.org), for the period 1946 to 2007. The period before 1946 has been ignored given the shocks on mortality rates related to first and second world wars. In this paragraph, we have explored the features of a two dimensions Lee-Carter model, led by a two states hidden Markov chain. The ages considered range from 20 to 100 years. Figure 8.1 presents the components $\alpha_x$ for the male and female population. $\alpha_x$ may be interpreted as the average trend of the logarithms of $\mu_x$. Its increasing shape is directly related to the increase of death probabilities with age.

![Figure 8.1: $\alpha_x$ for the male and female French population](image1)

Figures 8.2 present the values of $\beta_1^x$ and $\kappa_1^t$ for the French population. The process $\kappa_1^t$ is clearly decreasing and displays a low volatility. The steepness of this curve seems however to change slightly between the periods 1946-1963 and 1964-2007. The $\beta_1^x$ curves have two peaks: one around thirty years and one around seventy years. As $\kappa_1^t$ decreases, this reveals that the mortality improvements embedded in the first component have been more significant during the last half century, for the ages ranges 20 to 40 and 60 to 80 years.

![Figure 8.2: $\beta_1^x$ and $\kappa_1^t$ for the male and female French population](image2)

Figures 8.3 present $\beta_2^x$ and $\kappa_2^t$. The curves $\beta_2^x$ display a peak around the age of 27 years, for male and female populations. Note that the curve of female $\beta_2^x$ has been rescaled by a factor 1/100 to make the figures comparable with the male $\beta_2^x$. The main difference is the amplitude. The oscillation of the $\beta_2^x$ for the female population is more significant than for the male population. The processes $\kappa_2^t$ have a similar behavior but the
volatility seems higher for men than for women. During the period of increase of \( \kappa_2^t \), as \( \beta_2^t \) is maximum around 27 years, the mortality of generation 20 to 30 years tends to rise.

\[
\begin{align*}
\left( \frac{d\kappa_1^t(\eta_t)}{d\kappa_2^t(\eta_t)} \right) &= \left( \begin{array}{c} a_1(\eta_t) \\ a_2(\eta_t) \end{array} \right) \ dt + \left( \begin{array}{c} \sigma_1(\eta_t)dW_1^1(\eta_t) \\ \sigma_2(\eta_t)dW_2^2(\eta_t) \end{array} \right) \quad \eta_t = 1, 2
\end{align*}
\]

where \( \sigma_1(\eta_t) = \sqrt{(\sigma_{11}(\eta_t) + \sigma_{21}(\eta_t))} \), \( \sigma_2(\eta_t) = \sqrt{(\sigma_{22}(\eta_t) + \sigma_{12}(\eta_t))} \) and where \( W_1^1(\eta_t), W_2^2(\eta_t) \) are correlated Brownian motions whose correlation is noted \( \rho(\eta_t) \). The correlations are presented in table 8.1. In state 1, the correlation between components \( \kappa_1^t \) and \( \kappa_2^t \) are negative and significant, for both women and men. In state 2, the correlation are close to zero.

<table>
<thead>
<tr>
<th>( \rho(1) )</th>
<th>( \rho(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4596</td>
<td>-0.7462</td>
</tr>
<tr>
<td>-0.0015</td>
<td>-0.1365</td>
</tr>
</tbody>
</table>

Table 8.1: Correlations

Table 8.2 contains the parameters of the switching processes \( \kappa_1^{1,2} \). In state 1, the average trend \( a_1(1) \), of \( \kappa_1^t \) is negative, but twice bigger for women than for men. The volatilities are comparable and around 0.80. In state 2, the average trend \( a_1(2) \) is negative and identical for the male and female populations. The means of \( \kappa_1^t \) in state 1 and 2 are close to zero, for men and women. Only the volatilities differ between the two populations.

<table>
<thead>
<tr>
<th>( a_1(1) )</th>
<th>( \sigma_1(1) )</th>
<th>( \sigma_2(1) )</th>
<th>( \kappa_1^{1} )</th>
<th>( \kappa_1^{2} )</th>
<th>( a_1(2) )</th>
<th>( \sigma_1(2) )</th>
<th>( \sigma_2(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2648</td>
<td>0.7343</td>
<td>0.0054</td>
<td>0.0054</td>
<td>0.0054</td>
<td>-0.6208</td>
<td>0.8817</td>
<td>0.0001</td>
</tr>
<tr>
<td>-0.0015</td>
<td>0.0001</td>
<td>0.2576</td>
<td>0.2576</td>
<td>0.2576</td>
<td>-0.0014</td>
<td>0.0015</td>
<td>0.0015</td>
</tr>
<tr>
<td>-0.3001</td>
<td>0.3001</td>
<td>0.0032</td>
<td>0.0032</td>
<td>0.0032</td>
<td>-0.2999</td>
<td>0.3903</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

Table 8.2: Means and standard deviations

The transition probabilities on a time horizon of one year are given in table 8.3. To lighten notations, the probabilities \( p_{i,j}(t, t+1) \) are noted \( p_{i,j} \). At a first sight the probabilities to remain in the same state are high. If we’re in state one, there is a 95% probability to still be in this state, one year later. We note that the probabilities to switch of states are higher for the male than for the female mortality process.
Table 8.3: 1 year transition probabilities

<table>
<thead>
<tr>
<th></th>
<th>Men</th>
<th></th>
<th>Women</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p_{1,1}$</td>
<td>0.9577</td>
<td>$p_{1,1}$</td>
<td>0.9713</td>
</tr>
<tr>
<td></td>
<td>$p_{1,2}$</td>
<td>0.0423</td>
<td>$p_{1,2}$</td>
<td>0.0287</td>
</tr>
<tr>
<td></td>
<td>$p_{2,1}$</td>
<td>0.0486</td>
<td>$p_{2,1}$</td>
<td>0.0128</td>
</tr>
<tr>
<td></td>
<td>$p_{2,2}$</td>
<td>0.9514</td>
<td>$p_{2,2}$</td>
<td>0.9872</td>
</tr>
</tbody>
</table>

Figures 8.4 display the probabilities of presence in states 1 and 2 (defined by equation 7.1). The male mortality process was in state 1, with a high probability, from 1946 to 1973, while the female mortality process switches from state 1 to state 2 around 1960. A sudden change of state is observed in 2003. During a short period, the male mortality process and to a lesser extent the female mortality process transit in state 1. This change of behavior is in fact explained by the overmortality of elderly population, caused by the wave of heat observed in France, during the summer 2003. Based on this observation, we can infer that state 1 corresponds to a deterioration of longevity conditions, or at least to a slow down of longevity improvements.

To end this section, we compare in Table 8.4, the logarithms of expected moments obtained by proposition 4.1 and by Monte-Carlo simulations. This comparison is done for mortality rates of a woman, 50 years old in 2007, after 1, 5 and 10 years. This confirms that empirical moments are close to moments obtained by Monte-Carlo simulations, but both are very small. The smallness of moments seriously harms to the efficiency of the Edgeworth expansion, as illustrated in figure 8.5 that compares the empirical distribution of $\log \mu_{x=50}^{t=10}$ with the Edgeworth expansion.

<table>
<thead>
<tr>
<th></th>
<th>log $E\left(\mu_{x=50}^{t=1}\right)$</th>
<th>log $E\left(\mu_{x=50}^{t=2}\right)$</th>
<th>log $E\left(\mu_{x=50}^{t=3}\right)$</th>
<th>log $E\left(\mu_{x=50}^{t=4}\right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=1$, theo.</td>
<td>-6.1299</td>
<td>-12.2597</td>
<td>-18.3896</td>
<td>-24.5195</td>
</tr>
<tr>
<td>$t=1$, MC</td>
<td>-6.1299</td>
<td>-12.2597</td>
<td>-18.3896</td>
<td>-24.5195</td>
</tr>
<tr>
<td>$t=5$, theo.</td>
<td>-5.8623</td>
<td>-11.7237</td>
<td>-17.5842</td>
<td>-23.4438</td>
</tr>
<tr>
<td>$t=5$, MC</td>
<td>-5.8658</td>
<td>-11.7312</td>
<td>-17.5962</td>
<td>-23.4607</td>
</tr>
<tr>
<td>$t=10$, theo.</td>
<td>-5.5143</td>
<td>-11.0259</td>
<td>-16.5350</td>
<td>-22.0414</td>
</tr>
<tr>
<td>$t=10$, MC</td>
<td>-5.5644</td>
<td>-11.1270</td>
<td>-16.6876</td>
<td>-22.2461</td>
</tr>
</tbody>
</table>

Table 8.4: Comparison of theoretical versus empirical log of moments
9 Comparison with existing models.

In this section, we study the new insights that we obtain from our model (noted 2D 2S) with the Lee-Carter model (LC) and with the one dimension, 2 states, model (1D 2S) proposed by Milidonis et al. (2011). Table 9.1 presents the loglikelihoods obtained after calibration of models to series $\kappa_1^1$ and $\kappa_2^2$. These figures reveals that the improvements of loglikelihoods related to the introduction of switching regimes are significant both for one and two dimensions models.

<table>
<thead>
<tr>
<th></th>
<th>Men</th>
<th>Women</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log L (2D 2S)$</td>
<td>212.38</td>
<td>305.68</td>
</tr>
<tr>
<td>$\log L (1D 2S)$</td>
<td>-40.67</td>
<td>-35.35</td>
</tr>
<tr>
<td>$\log L (LC)$</td>
<td>-61.35</td>
<td>-61.69</td>
</tr>
</tbody>
</table>

Table 9.1: Log likelihoods

So as to estimate the influence of a second dimensions on mortality forecasts, we have first computed with the 2D 2S model, the expected log mortality rates of the cohort, 20 years old in 2007. Figure 9.1 compares the expected log-mortality rates with those observed during the year 2007 and those computed with the LC and 1D 2S models. The log-mortality rates are obtained by Monte Carlo simulations (10 000 scenarios). We have assumed that the hidden Markov process is in state 2 in 2007, which is coherent with the observation of figure 8.4. At a first glance, the LC, 1D 2S and 2D 2S models forecast an improvement of longevity. The 1D 2S model mortality forecasts nearly coincide with those of the Lee Carter model, while the 2D 2S yields a less significant decrease of longevity rates at older ages.

To understand the behavior of the 2D 2S model, we have also projected the expected $\log(q(t,x))$, for different ages. The results are plotted in graphs 9.2 and 9.3 which compare them with the expected log mortality rates from the LC and 1D 2S models. For ages 20, 30 and 40, the 2D 2S model forecasts a sharper decrease of log mortality rates than the LC and 1D 2S models. This trend inverts between 40 and 50 years: the 2D 2S model forecasts for higher ages, a smaller improvement of mortality rates than the LC and 1D 2S models. This confirms that the second dimension of the 2D 2S model introduces an age specific enhancement of mortality rates. The 2D 2S model foresees a most important downturn in crude mortality rates, roughly in the age range 20-40 years than for age above 50. This result is partly explained by the negative correlation between $\kappa_1^1$ and $\kappa_2^2$. Furthermore, our model does not exclude to switch back to state 1 (state observed at least during a decade after the second world war), in which longevity improvements at higher ages are less significant than in state 2. The model of Milidonis yields future mortality rates closer to those obtained with the Lee Carter model. The downturn depicted by forecast mortality rates is furthermore similar whatever the age.
Figure 9.1: $\mathbb{E}(\log(\mu^x_t))$ for the male and female French population

Figure 9.2: Evolution of log mortality rates, male.
10 Pricing under a change of measure.

Our model allows the mortality process to jump from state 2, corresponding to human mortality observed during the last decades, to state 1, which is related to mortality observed after the second world war. The insurer can adapt the transition probabilities between states of \( \eta \), in function of his own perception of the longevity evolution. An insurer selling life annuities, fears an improvement of longevity and his tariff will probably be based on the assumption that the mortality process has little chance to meet again post war conditions. On another side, a company selling death insurances fears a deterioration of longevity. Its tariff will hence rather rely on the assumption that the longevity improvement will slow down and may again know post war conditions. Those adaptations of tariff are theoretically justified by choosing a change of measure affecting the transition probabilities of the hidden Markov process \( \eta \). If we remember section 5, transition probabilities are adjusted under the pricing measure \( \tilde{P} \) by the component \( \xi_\eta^t \) in equation (5.1). Even if it is also possible to adapt values of \( \alpha_x \), \( \beta^{1,2}_x \) and dynamics of processes \( \kappa^{1,2}_t \) (involved directly in the dynamics of \( \mu^x_t \)) under the pricing measure \( \tilde{P} \), we limit the scope of our study to changes of measure that modify transition probabilities of \( \eta \) under \( \tilde{P} \).

The interested reader may refer to the work of BiFFis et al. (2009) for a detailed presentation of changes of measure adjusting other parameters of the Lee Carter model. Note that, It does not mean that the mortality or longevity risk is basically just a regime change risk. But we want to isolate the eventual impact of adapting the probabilities of regime switches on the tariff of insurance products. We believe that adapting transition probabilities in function of the insurance type (life or death) is an easy way to include a safety margin in the tariff. Furthermore, each regime being clearly identified to a period of time (post war versus current mortality evolutions), we think that it would be easier to accept by regulators than a modification of the whole dynamics of mortality rates. The reader interested by pricing approaches unifying both actuarial and financial valuations techniques can refer to the works of Bauer et al. (2010 a, b), to the paper of BiFFis and Blake (2009) or to the paper of Cairns et al (2006) for such attempt of reconciliation. The surveys of Blake et al. (2009) and of Cairns et al. (2008) present the most recent developments in longevity pricing. However, at our knowledge, none of these papers investigate the cost of regime switches as proposed in this work.

The model chosen for our investigation is the 2D Lee Carter, driven by a 2 states Markov chain, fitted on the French population. In this setting, the pricing measure \( \tilde{P} \) is defined by the following change of measure:

\[
\xi_\eta^t := \mathbb{E} \left( \frac{d\tilde{P}}{dP} | \mathcal{F}_t \right),
\]

where

\[
\xi_\eta^t = \exp \left( \int_0^t \int_E \ln (1 + \phi(z)) \varphi(z) \nu(ds,dz) + \int_0^t \int_E \phi(z) \varphi(z) \gamma(ds,dz) \right). \tag{10.1}
\]
If we note $\phi_{i,j} = \phi_i^j(z)$ when $z = (i,j)$, the necessary conditions (5.3) to guarantee that the process $\eta_t$ remains Markov under $\tilde{P}$ are:

\[
\begin{align*}
    h_{1,1}(1 + \phi_{1,1}) + h_{1,2}(1 + \phi_{1,2}) &= 0 \quad \phi_{1,2} > -1 \\
    h_{2,1}(1 + \phi_{2,1}) + h_{2,2}(1 + \phi_{2,2}) &= 0 \quad \phi_{2,1} > -1
\end{align*}
\]

Under $\tilde{P}$, the intensity matrix of $\eta_t$ is given hence by:

\[
\tilde{H} = \begin{pmatrix}
    h_{1,1}(1 + \phi_{1,1}) & -h_{1,1}(1 + \phi_{1,1}) \\
    -h_{2,2}(1 + \phi_{2,2}) & h_{2,2}(1 + \phi_{2,2})
\end{pmatrix}.
\]

The value of $\phi_{i,j}$ directly influences the one year probabilities of transition $p_{i,j}(t, t + 1)$. As illustrated in figure (10.1), the higher is $\phi_{1,1}$, the lower is the probability $p_{1,1}(t, t + 1)$ of remaining in state 1. In the same way, when $\phi_{2,2}$ rises, the probability $p_{2,2}(t, t + 1)$ declines. In fact, to each level of transition probabilities under $\tilde{P}$, corresponds a couple of values $(\phi_{1,1}, \phi_{2,2})$ which distorts the real transition probabilities. For this reason, the analysis of the impact of changes of measure (10.1) on prices, is performed in terms of probabilities $(p_{1,1}, p_{2,2})$ under $\tilde{P}$ rather then in function of $(\phi_{1,1}, \phi_{2,2})$.

![Figure 10.1: $p_{1,1}(t, t + 1), p_{2,2}(t, t + 1)$, in function of $\phi_1$.](image)

The influence of transition probabilities on prices is analyzed on two categories of product: a life annuity and a death insurance. The single premium of an annuity purchased at age $x$ which delivers one monetary unit per year is equal to the expected value of discounted future cash-flows:

\[
an_x = \sum_{t=1}^{\omega-x} \mathbb{E}^\tilde{P}\left(e^{-rt} \int_0^x \mu(s,x+s)ds\right) = \sum_{t=1}^{\omega-x} e^{-rt} \tilde{p}_x.
\]

where $\omega$ is the maximum age of the life table (here set to 100) and $\tilde{p}_x$ is the survival probability from age $x$ to age $x + t$, under $\tilde{P}$. This premium has been computed by Monte Carlo simulations (10 000 scenarios), for a 60 years old woman and for different level different levels of probabilities $(p_{1,1}, p_{2,2})$ ranging from 0.05 to 0.95, by step of 0.15. The interest rate is $r = 1\%$. Table 10.1 presents the results of those computations. They reveal that the highest annuity price is obtained with a low probability of sojourn in state 1 and a high probability of sojourn in state 2. This confirms our intuition: a life annuities seller has interest to build his tariff based on the assumption that the mortality process has little chance to meet again post war conditions.

<table>
<thead>
<tr>
<th>$p_{1,1} / p_{2,2}$</th>
<th>0.95</th>
<th>0.80</th>
<th>0.65</th>
<th>0.50</th>
<th>0.35</th>
<th>0.20</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>22.77</td>
<td>22.41</td>
<td>22.27</td>
<td>22.23</td>
<td>22.19</td>
<td>22.16</td>
<td><strong>22.15</strong></td>
</tr>
<tr>
<td>0.80</td>
<td>22.91</td>
<td>22.63</td>
<td>22.49</td>
<td>22.38</td>
<td>22.30</td>
<td>22.28</td>
<td><strong>22.29</strong></td>
</tr>
<tr>
<td>0.65</td>
<td>22.97</td>
<td>22.73</td>
<td>22.58</td>
<td>22.51</td>
<td>22.45</td>
<td>22.40</td>
<td><strong>22.36</strong></td>
</tr>
<tr>
<td>0.50</td>
<td>23.00</td>
<td>22.80</td>
<td>22.67</td>
<td>22.59</td>
<td>22.52</td>
<td>22.47</td>
<td><strong>22.43</strong></td>
</tr>
<tr>
<td>0.35</td>
<td>23.02</td>
<td>22.85</td>
<td>22.72</td>
<td>22.64</td>
<td>22.58</td>
<td>22.51</td>
<td><strong>22.47</strong></td>
</tr>
<tr>
<td>0.20</td>
<td>23.02</td>
<td>22.89</td>
<td>22.78</td>
<td>22.70</td>
<td>22.63</td>
<td>22.59</td>
<td><strong>22.53</strong></td>
</tr>
<tr>
<td>0.05</td>
<td><strong>23.04</strong></td>
<td>22.90</td>
<td>22.81</td>
<td>22.74</td>
<td>22.65</td>
<td>22.61</td>
<td><strong>22.58</strong></td>
</tr>
</tbody>
</table>

Table 10.1: Annuities, women, 60 years old
The single premium of a death insurance purchased at age \( x \) and delivering one monetary unit at the time of death, is given by the following formula:

\[
d^x = \sum_{t=0}^{\omega-x} E^P \left( e^{-rt} \int_0^{t} \mu(s, x+s)ds \left( 1 - e^{-\int_t^{t+1} \mu(s, x+s-t)ds} \right) \right).
\]

The price of this contract has been computed by Monte Carlo simulations for a 60 years old woman. As emphasized by table 10.2, the highest price is obtained with a low probability of sojourn in state 2 and a high probability of sojourn in state 1. The insurer has hence interest to assume that the mortality process can know similar conditions to the post war period, with a high probability. This again confirms our insight.

<table>
<thead>
<tr>
<th>( p_{11} ) / ( p_{22} )</th>
<th>0.95</th>
<th>0.80</th>
<th>0.65</th>
<th>0.50</th>
<th>0.35</th>
<th>0.20</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>0.7713</td>
<td>0.775</td>
<td>0.776</td>
<td>0.7766</td>
<td>0.777</td>
<td>0.7773</td>
<td><strong>0.7775</strong></td>
</tr>
<tr>
<td>0.80</td>
<td>0.7699</td>
<td>0.7727</td>
<td>0.7742</td>
<td>0.7757</td>
<td>0.7758</td>
<td>0.7765</td>
<td></td>
</tr>
<tr>
<td>0.65</td>
<td>0.7693</td>
<td>0.7717</td>
<td>0.773</td>
<td>0.7739</td>
<td>0.7745</td>
<td>0.775</td>
<td><strong>0.7752</strong></td>
</tr>
<tr>
<td>0.50</td>
<td>0.7689</td>
<td>0.7708</td>
<td>0.7722</td>
<td>0.7729</td>
<td>0.7739</td>
<td>0.7741</td>
<td><strong>0.7748</strong></td>
</tr>
<tr>
<td>0.35</td>
<td>0.7687</td>
<td>0.7705</td>
<td>0.7715</td>
<td>0.7725</td>
<td>0.7731</td>
<td>0.7736</td>
<td><strong>0.7741</strong></td>
</tr>
<tr>
<td>0.20</td>
<td>0.7686</td>
<td>0.7699</td>
<td>0.7711</td>
<td>0.772</td>
<td>0.7725</td>
<td><strong>0.7732</strong></td>
<td><strong>0.7733</strong></td>
</tr>
<tr>
<td>0.05</td>
<td><strong>0.7686</strong></td>
<td>0.7698</td>
<td>0.7706</td>
<td>0.7716</td>
<td><strong>0.7722</strong></td>
<td><strong>0.7727</strong></td>
<td><strong>0.7732</strong></td>
</tr>
</tbody>
</table>

Table 10.2: Death insurances, women, 60 years old

However, the transition probabilities have also an impact on the capital that the insurer has to bring to guarantee his solvency. According to Solvency II, this capital is the difference between the expected value of future discounted payments and their percentile, at a given level of confidence. Table 10.3 presents the capital computed at 99.5% percent, for the annuity and death insurance previously priced, while figure 10.2 shows the empirical distribution of annuity values (10,000 scenarios).

If we set probabilities of sojourn to \( p_{11} = 0.95 \) and \( p_{22} = 0.05 \), the capital at 99.5%, required for the sale of annuity or death insurances, is lower than the one obtained with \( p_{11} = 0.05 \) and \( p_{22} = 0.95 \). The probabilities of sojourn chosen for pricing purposes have a non negligible impact on the capital requirements: it can be twice higher. We note that the probabilities of sojourn leading to the highest annuity price (\( p_{11} = 0.05 \) and \( p_{22} = 0.95 \)), leads also to the higher capital need. We observe the opposite for the death insurance: the highest price is obtained with probabilities of sojourn that minimizes the capital need.

<table>
<thead>
<tr>
<th></th>
<th>( p_{11} = 0.05 ) / ( p_{22} = 0.95 )</th>
<th>( p_{11} = 0.95 ) / ( p_{22} = 0.05 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital at 99.5% ( an_{x=60} )</td>
<td>1.1156</td>
<td>0.5192</td>
</tr>
<tr>
<td>Capital at 99.5% ( dn_{x=60} )</td>
<td>0.0112</td>
<td>0.0058</td>
</tr>
</tbody>
</table>

Table 10.3: Cost capital

![Figure 10.2: Distributions of \( an_{x=60} \).](image-url)
Our conclusions are valid for 60 years old individuals. Do they remain valid for younger/older people? To answer this question, we have computed the expected log of female mortality rates after changes of transition probabilities and compared them with the log mortality rates observed during year 2007. In figure 10.3, the curve “Life operations” is the curve of log mortality rates leading to the highest annuity value for older individuals \((p_{11} = 0.05, p_{22} = 0.95)\), while the curve “Death insurances” is obtained with \(p_{11} = 0.95, p_{22} = 0.05\). The curve of log mortality rates “Life operations” is always below the curve of mortality rates observed in 2007. However, the spread between curves narrows for age smaller than 50. The curve “death insurance” is well above the log mortality rates observed in 2007 for older individuals but falls below it for age inferior to 50. This observation mitigates our previous result: post war conditions (state 1) were well less favorable for an improvement of longevity for person older than 50 years but not for the younger population.

![Figure 10.3: ln(qx) after change of measures, female French population.](image)

11 Conclusions.

This work studies the features of a multifactor Lee-Carter model, in which the time dependent components are driven by switching regime random walks. The main motivation to undertake this research is the observation of changes of trends in the mortality process, as underlined by the work of Sweeting (2009), who proposed a methodology to detect the times of switches. Despite the apparent complexity of this model, the procedure of calibration is rather simple to implement. Furthermore, the model presents a certain degree of analytical tractability: the moments of mortality rates have a closed form expression, which can eventually be used in an Edgeworth expansion to approach their statistical distribution. We also present the family of changes of measure, that can be used for pricing purposes.

As illustration, a 2D-2 states model is fitted to the French male and female populations, on the period 1946-2007. Our analysis reveals that the first state of mortality process was mainly observed during the decade after the second world war, while the second state is related to much recent mortality evolutions. A sudden switch to state 1 is observed in 2003 and is explained by the overmortality of elderly population, caused by the wave of heat observed in France, during the summer 2003. An analysis of forecast log mortality rates reveals that the 2D switching model introduces an age specific enhancement of mortality rates, not present in the 1D switching model of Milidonis (2011). In particular, the 2D 2S model projects a most important downturn in crude mortality rates, roughly in the age range 20-40 years than for age above 50.

The insurer can adapt the transition probabilities between states, in function of his own perception of the longevity evolution. An insurer selling life annuities, fears an improvement of longevity and designs his tariff with the assumption that the mortality process has little chance to meet again post war conditions. On the opposite, the tariff of death insurance will rather rely on the assumption that the longevity improvement can slow down and be similar to the post war period. Those adaptations of tariff can theoretically be justified by a suitable change of measure affecting only transition probabilities. This approach can then also serve to justify eventual tariff assumptions to the regulator. Finally, we have also shown that the considered changes of measure are not neutral on capital requirements.
References


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