A fractal version of the Hull-White interest rate model.

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Abstract

This paper develops a new version of the Hull-White’s model of interest rates, in which the volatility of the short term rate is driven by a Markov switching multifractal model. The interest rate dynamics is still mean reverting but the constant volatility of the Brownian motion is replaced by a multifractal process so as to capture persistent volatility shocks. In this setting, we infer properties of the short term rate distribution, a semi closed form expression for bond prices and their dynamics under a forward measure. Finally, our work is illustrated by a numerical application in which we assess the exposure of a bonds portfolio to the interest risk.

Keywords. Hidden Markov process, switching Brownian motion, Interest rates, Hull-White model, Switching volatility, Markov modulated volatility.

1 Introduction.

Hull-White (1990) proposed a generalization of the Vasicek model (1977) by introducing to it a time-varying parameter so that the model could fit any given term structure. This mean reverting model implies a normal distribution for a short rate process and therefore has the useful property of analytical tractability for options pricing. The main drawback of this model is that it does not capture the leptokurticity and the asymmetry exhibited by real short interest rates, as shown in Lekkos (1999). This also explains in part why the Hull-White model does not always fit the smile of implied volatilities.

To remedy this situation, we propose to replace the constant volatility of the Brownian motion in the Hull-White model by a Markov Switching Multifractal (MSM) process such as developed by Calvet and Fisher (2001) and Calvet (2004). This type of process presents many interesting features. The volatility specification is highly parsimonious and requires only a few parameters. The MSM model is also consistent with the slowly declining autocovariograms and fat tails of financial series. In this setting, the dynamics of interest rates can be reformulated as a mean reverting Brownian process with a Markov modulated volatility. We can then infer a closed form likelihood and use the Hamilton filter (1989) to fit the model to time series of interest rates. Existing models of interest rates based on multiple regimes, such developed by Kalimipalli and Susmel (2004), Mills and Wang (2006), Elliott and Siu (2009), Seungmoon (2009), Elliott et al. (2011) or by Zhou and Mamon (2012), consider that the volatility takes at most two or three values. The main reasons motivating this choice are the analytical tractability and the over parametrization of models with more than two regimes, which prevents to fit them to real time series. We don’t face these drawbacks with the MSM model: more than hundred regimes can easily be defined with six parameters. Furthermore, the econometric calibration reveals that the MSM model realistically captures the changes of economic regimes. As illustrated by numerical applications presented in this work, fractal models seem well suited for risk management purposes, and in particular to assess the capital required to cover the interest rate risk.

Using fractal models for pricing of bonds and derivatives is less evident. Hansen and Poulsen
(2001) extended the Vasicek model by including jumps in the local mean. Landen (2000) was among the first to study this issue and built the system of SDE driving prices. But solving this system remains a tricky exercise, particularly with a high number of states. In this work, we explore an alternative approach which consists to calculate bond options by a Monte Carlo method, but under a forward measure. This approach, well known from quantitative analysts, allows us to remove inaccuracies related to the approximation of the discount factor multiplying the derivative payoff. One contribution of this paper is to establish the dynamics of bond prices under this forward measure.

The paper is organized as follows. In section 2, we introduce the interest rate model and define the fractal process ruling the volatility. In the next section, models with various number of regimes are fitted to the time serie of 1-year Euribor rates, daily observed on a period of one year and a half. Loglikelihoods are compared with a GARCH model, with a 2 states switching and basic Hull-White models. In section 4, we infer the formula of bond prices, their dynamics and an expression for the deterministic trend fitting a yield curve. The next paragraph brings some new elements about the pricing of bond derivatives by Monte Carlo under a forward measure. We end this work with numerical applications.

2 The Interest rate model.

Hull-White (1990) proposed a generalization of the Vasicek model by introducing a time-varying parameter so that the model could fit any given term structure. The success of their approach has been guaranteed by its analytical tractability and by existence of closed form expressions for caps, floors and swaptions. However, most of the time, this model is insufficient to capture the erratic volatility exhibited by short term rates. So as remedy to this problem, the constant volatility is replaced in this work by a Markov Switching Multifractal processes (noted MSM in the remainder of the paper). Before providing more details on this, we first introduce the Hull-White model, in a similar way to Brigo-Mercurio (2007).

We consider a probability space \((\Omega, \mathcal{F}, P)\) endowed with some filtration \(\mathcal{F}_t\). On this filtration is defined the interest rate process, \(r_t\). This instantaneous interest rate is assumed to be the sum of a deterministic function \(\varphi(t)\) and of a random process \(Y_t\) under \(P\):

\[
r_t = \varphi(t) + Y_t
\]

where \(Y_t\) is a mean reverting process of initial value \(Y_0 = 0\). The function \(\varphi(t)\) is adjusted to fit the observed term structure of interest rates. We come back to this point later. The process \(Y_t\) is solution of the following equation:

\[
dY_t = -aY_t dt + \sigma_t dW_t
\]

where \(a, \sigma_t\) and \(W_t\) are respectively a positive constant, the volatility process and a Brownian motion. The volatility process is not directly observable. It is then defined on a filtration \(\mathcal{G}_t\) different from the filtration \(\mathcal{F}_t\) of \(r_t\). \(\sigma_t\) is the product of a constant \(\sigma_0\) and of the \(n\) elements of a Markov state vector, \(S_t\):

\[
S_t = (S_{1,t}, S_{2,t} \ldots S_{n,t}) \in \mathbb{R}_+^n,
\]

and

\[
\sigma_t = \sigma_0 \left( \prod_{k=1}^{n} S_{k,t} \right)^{1/2}.
\]

The components of \(S_t\) are mutually independent. For each \(k = \{1, \ldots, n\}\), the multiplier \(S_{k,t}\) is drawn from a fixed distribution \(S\) with probability \(\gamma_k dt\), and is otherwise equal to its previous
value \( S_{k,t+dt} = S_{k,t} \). Calvet and Fisher (2001) recommend the following distribution for \( S \):

\[
S = \begin{cases} 
  s_0 & p_0 = \frac{1}{2} \\
  2 - s_0 & 1 - p_0 = \frac{1}{2} 
\end{cases}
\]  

(2.3)

that is fully determined by the parameter \( s_0 \in [0,1] \) and whose expectation is equal to one. A component \( S_i,t \), that is equal to \( 2 - s_0 \) (resp. \( s_0 \)), increases (resp. decreases) the volatility. The probabilities \( \gamma_{k=1...n} \) depend on two parameters \( \gamma_1 \in (0,1) \) and \( c \in (1,\infty) \) as follows:

\[
\gamma_{k,dt} = \gamma^{k-1}_{1} \quad k = 1, \ldots, n
\]  

(2.4)

This rule of construction guarantees that \( \gamma_1 \leq \ldots \leq \gamma_n < 1 \). This means that the last factor \( S_{n,t} \) changes of value more frequently than the first component factor \( S_1,t \). The main advantage of this model is its ability to capture low-frequency regime shifts of the volatility process (See Calvet and Fisher, 2002 for a discussion of this feature of MSM models). Furthermore, it allows a parsimonious representation (only four parameters, \( \sigma_0, s_0, \gamma_1, c \)) of a high dimensional state space.

In this setting, \( \sigma_t \) takes a finite number of values, \( 2^n \). We can reformulate it as a continuous Markov process having \( d = 2^n \) states. More precisely, if we note the driving Markov process \( \delta_t \) and \( \sigma = (\sigma_1, \ldots, \sigma_d) \) the vector of possible realizations of \( \sigma_t \), the volatility of short term rate is the following scalar product:

\[
\sigma_t = \langle \sigma', \delta_t \rangle
\]  

(2.5)

where \( \delta_t \) is defined on \( (\Omega, \mathcal{G}) \) and taking its value in a finite state space \( (E, \mathcal{E}) \). This state space is identified by the set of unit vectors in \( \mathbb{R}^d \): \( E = \{e_1, \ldots, e_d\} \). Where \( e_j = (0, \ldots, 0, 1, \ldots, 0) \) is a standard unit vector with one in the coordinate \( j \) and zeros elsewhere. Each element of the state space \( E \) corresponds to the occurrence of the state vector \( S_t \), that is noted \( s^1, \ldots, s^d \in \mathbb{R}_+^n \). For a given realization \( s^j \), the volatility of short term rates in state \( j \) is given by

\[
\sigma_j = \sigma_0 \sqrt{\prod_{k=1}^{n} s^j(k)}
\]

where \( s^j(k) \) is the \( k^{th} \) element of the vector \( s^j \). The \( d \times d \) matrix of transition probabilities between \( t \) and \( t + \Delta t \) is noted

\[
P(t, t + \Delta t) = (p_{i,j}(t, t + \Delta t))_{1 \leq i,j \leq d}
\]

whose elements are fully determined by the \( \gamma_{k=1...n} \) : 

\[
p_{i,j}(t, t + \Delta t) = P(S_{t+\Delta t} = s^j | S_t = s^i) = \prod_{k=1}^{n} \left( \gamma_k \Delta t \frac{1}{2} + (1 - \gamma_k \Delta t) I_{\{s^j(k) = s^i(k)\}} \right)
\]  

(2.6)

Elliott et al. (1995) proved that the process

\[
M_t = \delta_t - \delta_0 - \int_0^t Q^\prime \delta_s ds,
\]

(2.7)

where \( Q \) is the \( d \times d \) intensity matrix of \( \delta_t \), is a \( \mathcal{G} \)-martingale. \( Q \) is related to the matrix of transition probabilities as follows:

\[
P(t, t + \Delta t) = \exp(Q \Delta t)
\]

and its elements, noted \( q_{i,j} \), satisfy the following conditions:

\[
q_{i,j} \geq 0 \quad \forall i \neq j \quad \sum_{j=1}^{d} q_{i,j} = 0 \quad i = 1, \ldots, d.
\]  

(2.8)
3 Calibration.

So as to justify the choice of a MSM volatility in the short term rate dynamics, we fitted a mean reverting process (2.1) to daily observations of the one week Euribor (EURo Inter Bank Offered Rate), from the 2/06/2010 to 16/12/2011 (400 occurrences). Interest rates observed during this period, and their variations, are plotted in figure 3.1.

![Daily variations of 1-week Euribor, from the 2/06/2010 to 16/12/2011.](image)

The dynamics of short term rates has been discretized in steps of \( \Delta = 1/256 \). The function \( \varphi(t) \) is introduced in the dynamics to fit the observed term structure of interest rates and cannot be retrieved directly from the time serie of interest rates. For this reason, we assume that \( \varphi(t) \) is defined as follows

\[
\varphi(t) = b \left(1 - e^{-at}\right)
\]

where \( b \) is constant over time and can be seen as a mean interest rate to which the one week Euribor reverts. In section 4, we will explain how to build \( \varphi(t) \) so as to fit an observed yield curve. Under the assumption (3.1), the series of \( Y_t \) can be retrieved by deducting \( \varphi(t) \) from the serie of \( r_t \) and its dynamics can be approached as follows:

\[
\Delta Y_t = Y_t + \Delta t - Y_t \\
\approx -aY_t \Delta t + (\sigma, \delta_t) \Delta W_t
\]

where \( \Delta W_t \) is a normal random variable \( N(0, \sqrt{\Delta t}) \). As mentioned in the previous section, the volatility can be seen as a Markov process taking \( d = 2^n \) values. Each of its values, \( \sigma_j \), is built as the product of \( \sigma_0 \) and of an occurrence of the state vector \( S_t \). For a given occurrence of the state vector \( S_t = s^j \), the variation of \( Y_t \), on \([t, t + \Delta t]\\\),
is then normally distributed:

$$\Delta Y_t = N\left(-aY_t \Delta t, \sigma_0 \prod_{k=1}^{n} s^j(k) \Delta t\right), \quad (3.2)$$

and we note its density, $f(\Delta Y_t)$. The state vector $S_t$ is not directly observable, but the filtering technique developed by Hamilton (1989) and inspired from the Kalman’s filter (1960) allows us to retrieve the probabilities of being in a state given previous observations. We briefly summarize this filter. Let us note $\Delta Y_{t=0,...,t}$ the observed variation of short term rates on the past periods. Let us define the probabilities of presence in a certain state as:

$$\pi^j_t = P(S_t = s^j | \Delta r_1, \ldots, \Delta r_t).$$

Hamilton has proved that the vector $\pi_t = E(\delta_t | F_t) = (\pi^j_t)_{j=1,...,d}$ can be calculated as a function of the probabilities of presence during previous periods:

$$\pi_t = f(\Delta Y_t) \ast \left(\pi^j_{t-1} P(t, t + \Delta t)\right) \langle f(\Delta Y_t) \ast \left(\pi^j_{t-1} P(t, t + \Delta t)\right), 1\rangle, \quad (3.3)$$

where $1 = (1, \ldots, 1) \in \mathbb{R}^d$ and $x \ast y$ is the Hadamard product $(x_1 y_1, \ldots, x_d y_d)$. To start the recursion, we assume that the Markov process $\delta_t$ has reached its stable distribution, $\pi_0$ is then set to the ergodic distribution of $\delta_t$, which is the eigenvector of the matrix $P(t, t + \Delta t)$, coupled to the eigenvalue equal to 1. If we observed the interest rate process on $t$ periods, the loglikelihood function is:

$$\ln L(\Delta Y_1, \ldots, \Delta Y_T) = \sum_{t=0}^{T} \ln \langle f(\Delta Y_t), (\pi^j_{t-1} P(t, t + \Delta t))\rangle. \quad (3.4)$$

The most likely parameters, $(a, b, \gamma_1, c, s_0, \sigma_0)$, are obtained by numerical maximization of (3.4). The variance of an estimator of a parameter $\theta \in (a, b, \gamma_1, c, s_0, \sigma_0)$ is computed numerically from the asymptotic Fisher information:

$$Var(\theta) = -\left(\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}\right)^{-1}.$$

Table 3.1 presents these parameters for different sizes of the state vector. At our knowledge, there does not exist any parametric statistics to test the goodness of fit of a regime switching model. However, if we define $\tilde{\sigma}^2$ as the mean expected variance over the last $k$ observations

$$\tilde{\sigma}^2 = \frac{1}{k} \sum_{t=T-k}^{T} \langle \sigma^2, \pi_t \rangle$$

from relation (3.2), empirical tests reveal that the statistics

$$Z = \sum_{t=T-k}^{T} \frac{(\Delta Y_t - aY_t \Delta t)^2}{\tilde{\sigma}^2} \quad (3.5)$$

is approximately $\chi^2_{d-6}$ distributed, where $6$ is the number of parameters. $\tilde{\sigma}^2$ depends on estimated probabilities of presence, $\pi_t$ that are initially set to the ergodic distribution of $\delta_t$. To ensure that this assumption does not influence the statistics, it must then be calculated in priority with the last observations of the sample. In table 3.1, we report
p-values coupled to this statistics, computed for the last 200 hundreds observations.

The loglikelihood is optimized for \( n = 3 \) but the \( \chi^2 \) test is less good than for \( n = 4 \). For this reason, we will work with \( n = 4 \) parameters in numerical applications of section 6. Standard errors are all acceptable. The mean reversion level \( b \) is around 0.5% whatever the number of fractal components. The speed of reversion, \( a \), increases from 0.62 to 1.12 with the number of fractal components. We have compared the MSM model with a 2D switching regime model of interest rates. This model has exactly the same number of degrees of freedom as the MSM model. Excepted that the volatility takes only two values and is function of a 2 states hidden Markov process \( \alpha_t \).

\[
\Delta Y_t = -aY_t \Delta t + \sigma(\alpha_t)\Delta W_t
\]

The daily probabilities of transition from states 1 or 2 to states 2 or 1 are respectively noted \( p_{1,2}^{\alpha} \) and \( p_{2,1}^{\alpha} \). As the MSM model, the 2D switching model is fitted by the Hamilton filter. The estimates are provided in table 3.2. The loglikelihood of this model is clearly smaller than the one of multifractal models with \( n \geq 2 \). But it is identical to the one of an univariate fractal model (\( n = 1 \)), that also counts two distinct states. The \( \chi^2 \) statistics is even better for the 2D Hull-White model than for the univariate fractal model. The two states in the 2D Hull-White model respectively correspond to periods during which the volatility of short term rates is high (0.57%) and low (0.07%).

Table 3.3 presents the results of the calibration of a simple Hull-White model. We note that the volatility obtained for this model (0.41%) is not far from the volatility of fractal models (with one, two and three components) and that the level of mean reversion is higher than these of fractal models. The loglikelihood is however lower than these of other models and the \( \chi^2 \) test (which is here an exact statistic) rejects this approach.

Figure 3.2 compares the expected volatility \( \langle \sigma', \pi_t \rangle \) with the daily variations of the Euribor. It reveals a correspondence between periods during which we observe peaks of volatility and the highest variations of interest rates. Figure 3.3 shows two sample paths of of short term rates variations, on a 1 year period, simulated daily with 4 fractal components and parameters presented in table 3.1. This shows the ability of our model to generate picks of volatility followed by periods of low activity.

To conclude this section, we fit a GARCH(1,0) model to variations of interest rates. GARCH models, first introduced by Engle (1982) and extended by Bollerslev (1986)

![Figure 3.2: Expected volatility, MSM model n = 4.](image-url)
Table 3.2: Parameters of a 2D switching dynamics for $r_t$

can capture important volatility clustering. In this framework, the dynamics of short term rates is as follows:

$$\Delta r_t = \mu + \sigma_t \epsilon_t$$

where $\epsilon \sim N(0, 1)$ and where the variance $\sigma^2_t$ depends both on $\Delta r_{t-1}$ and on $\sigma^2_{t-1}$:

$$\sigma^2_t = \alpha_0 + \beta_1 \sigma^2_{t-1}$$

The results of the fit are presented in table 3.4. The GARCH(1,0) performs better than the simple Hull-White model, but its loglikelihood is lower than these obtained with switching regime models. This conclusion is in line with the results obtained by Smith (2002), who compares Garch, stochastic volatility and switching models, to fit treasury bills rates. We also tested the Garch(1,1) in which $\sigma^2_t = \alpha_0 + \beta_0 \Delta r^2_{t-1} + \beta_1 \sigma^2_{t-1}$, but the parameter $\beta_0$ was not significant. Note that contrary to the Hull-White model, the Garch does not model the whole term structure of interest rates, but only the short term rate.

Table 3.1: Parameters of MSM models for $r_t$. 

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Estimates</th>
<th>Std. Error</th>
<th>Loglik.</th>
<th>p val. $\chi^2$</th>
</tr>
</thead>
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<td>0.0173</td>
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<tr>
<td>$b$</td>
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<td>0.7899</td>
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<td>0.0001</td>
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<tr>
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<td>$p_{12}$</td>
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<td>$p_{21}$</td>
<td>0.1119</td>
<td>0.0287</td>
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Table 3.1: Parameters of MSM models for $r_t$. 

<table>
<thead>
<tr>
<th>Parameters</th>
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<th>Std Err.</th>
<th>$n = 2$</th>
<th>Std Err.</th>
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<th>Std Err.</th>
<th>$n = 4$</th>
<th>Std Err.</th>
<th>$n = 5$</th>
<th>Std Err.</th>
<th>$n = 6$</th>
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<td>0.0135</td>
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<td>0.0109</td>
<td>0.1775</td>
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<tr>
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<td>0.3401</td>
<td>1.6412</td>
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<td>0.0212</td>
<td>0.0016</td>
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<tr>
<td>$b$</td>
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<td>0.0046</td>
<td>0.0006</td>
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4 Zero coupon prices.

In this section, we consider that the dynamics of short term rates is still defined by equations (2.1) and (2.2) but on a different probability space \((\Omega, \mathcal{F}, Q)\), where \(Q\) is the risk neutral measure. We remind to the reader who is not familiar with this concept, that the pricing of all financial securities is done under this measure to avoid arbitrage. The absence of arbitrage entails that all financial assets growth on average at the risk free rate. On this filtration, is defined the interest rate process, \(r_t\). The price of a zero coupon bond of maturity \(T\) is the expectation of the discount factor. Given that the filtration \(\mathcal{F}_t \subset \mathcal{F}_t \lor \mathcal{G}_t\), this expectation may be rewritten as follows:

\[
P(t, T) = \mathbb{E} \left( e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right) = \mathbb{E} \left( \mathbb{E} \left( e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \lor \mathcal{G}_t \right) \mid \mathcal{F}_t \right) = \mathbb{E} \left( P(t, T, \delta_t) \mid \mathcal{F}_t \right)
\]

where \(P(t, T, \delta_t)\) is the price of a zero coupon bond, when the Markov process that defines the volatility is visible. This price can be calculated by the following proposition:

**Proposition 4.1.** Let us denote \(Q\) the matrix of transition probabilities of the Markov process \(\delta_t\) and \(F\) the diagonal matrix of variances

\[
F = \text{diag} \left( (\sigma_j^2)_{j=1\ldots d} \right).
\]
If we define $B(t, T)$ as follows
\[ B(t, T) = \frac{1}{a} \left( 1 - e^{-a(T-t)} \right). \] (4.2)

The value of $P(t, T, \delta_t)$ is given by the following expression,
\[ P(t, T, \delta_t) = e^{-\int_t^T \varphi(s)ds - Y_tB(t, T)} A(t, T, \delta_t) \] (4.3)

where the vector of $(A(t, T, e_j))_{j=1...N}$, noted $\hat{A}(t, T)$, is solution of the ODE system:
\[ \frac{\partial}{\partial t} \hat{A}(t, T) + \left( \frac{1}{2} B(t, T)^2 F + Q \right) \hat{A}(t, T) = 0, \] (4.4)

with the terminal boundary condition
\[ \hat{A}(t, T, j) = 1 \quad j = 1...d \]

A proof of this result in a regime switching jump augmented Vasicek model can be found in Siu (2010). We sketch an adapted version of this proof to the MSM model in appendix A. In most cases, differential equations of the type $\frac{dA}{dt} = C(t)A$, where $C$ is a matrix function of time, don't admit any analytical solution. In particular, $\hat{A} \neq e^{\int_t^0 C(s)ds}$ where $e^\cdot$ is the matrix exponential. Here, despite the particular form of the system (4.4), we cannot find any analytical solution for $\hat{A}$. However, the system (4.4) can easily be solved numerically by Euler’s method. The price of the zero coupon bond, when the fractal process is not observable, is the weighted sum of $P(t, T, e_j)$ by probabilities of being in state $j$ at time $t$. If we note these probabilities $\pi_{j,t} = E(\delta_j | \mathcal{F}_t)$, the price is equal to
\[ P(t, T) = \mathbb{E}(P(t, T, \delta_t) | \mathcal{F}_t) \]
\[ = \sum_{j=1}^d \mathbb{E}(\delta_j | \mathcal{F}_t) P(t, T, e_j) \]
\[ = \sum_{j=1}^d \pi_{j,t} P(t, T, e_j) \]

**Corollary 4.2.** The function $\varphi(.)$ fitting at time $0$, the observed curve of zero coupon prices is such that
\[ \varphi(t) = -\frac{\partial}{\partial t} \ln P(0, t) - Y_0 e^{-at} + \frac{1}{\sum_{j=1}^d \pi_{j,t} A(0, t, e_j)} \sum_{j=1}^d \left[ (Q \pi_0)_j A(0, t, e_j) + \pi_{j,0} \frac{\partial}{\partial t} A(0, t, e_j) \right] \] (4.5)

**Proof.** From the relation (4.3), we obtain the equation defining the integral of $\varphi(t)$
\[ \int_0^t \varphi(s)ds = -\ln P(0, t) - Y_0 B(0, t) + \ln \left( \sum_{j=1}^d \pi_{j,0} A(0, t, e_j) \right) \] (4.6)

Given that $\pi_t = E(\delta_t | \mathcal{F}_t)$ is the vector of probabilities of presence and that,
\[ \delta_t = \delta_0 + \int_0^t Q^t \delta_s ds + \int_0^t dM_s \]
we can infer that
\[
\frac{\partial}{\partial t} \pi_t = \frac{\partial}{\partial t} \mathbb{E} \left( \delta_0 + \int_0^t Q' \delta_s ds + \int_0^t dM_s | \mathcal{F}_t \right) \\
= \mathbb{E} \left( \frac{\partial}{\partial t} \mathbb{E} \left( \delta_0 + \int_0^t Q' \delta_s ds + \int_0^t dM_s | \mathcal{F}_t \cup \mathcal{G}_t \right) | \mathcal{F}_t \right) \\
= Q' \pi_t
\]

The relation (4.5) is then obtained by differentiation of equation (4.6).

In practice, the calculation of \( \varphi(.) \) by the relation (4.5), can be done by fitting a function (such the Nelson Siegel curve) to the initial yield curve \( P(0,t) \) and to \( A(0,T,e_j) \) so as to calculate the analytical differentials of these quantities. However, in most applications, it is sufficient to start from equation (4.6), to infer a staircase proxy of \( \varphi(.) \). In the next corollary, we develop the dynamics of bond prices. This result will be used later to propose a pricing method for options on zero coupon bonds.

**Corollary 4.3.** The zero coupon bond \( P(t,T,\delta_t) \) is driven by the following stochastic differential equation on the enlarged filtration \( \mathcal{F}_t \cup \mathcal{G}_t \):

\[
dP(t,T,\delta_t) = r_t P(t,T,\delta_t) \, dt - \langle \sigma, \delta_t \rangle B(t,T) P(t,T,\delta_t) \, dW_t \\
+ \int_E (P(t,T,z) - P(t,T,\delta_t)) \langle dM_t, dz \rangle.
\]

where \( M_t \) is defined by equation (2.7).

**Proof.** From the Itô’s lemma, we know that the dynamics of the bond price obeys to the following SDE under the enlarged filtration \( \mathcal{F}_t \cup \mathcal{G}_t \):

\[
dP = \left[ \frac{\partial P}{\partial t} - aY_t \frac{\partial P}{\partial Y} + \frac{1}{2} \left( \Sigma, \delta_t \right) \frac{\partial^2 P}{\partial Y^2} \right] dt + \langle \sigma, \delta_t \rangle \frac{\partial P}{\partial Y} dW_t \\
+ \int_E (P(t,T,\delta_t) - P(t,T,\delta_{t-})) \, d\delta_t
\]

From equations (4.3) (7.4), we infer the following relations:

\[
\frac{\partial P}{\partial Y} = -B(t,T)P(t,T,\delta_t)
\]

and for \( \delta_t = e_j \),

\[
\frac{\partial P}{\partial Y^2} = B(t,T)^2 P(t,T,\delta_t)
\]

and for \( \delta_t = e_j \),

\[
\frac{\partial P}{\partial t} = \left( \varphi(t) + Y_t e^{-a(T-t)} \right) P(t,T,\delta_t) \\
- \sum_{k \neq j} q_{j,k} \left( P(t,T,e_k) - P(t,T,e_j) \right)
\]

Substituting (4.9), (4.10) and (4.11) in (4.7) yields the result.

Equation (4.7) reveals that when the short term rate changes of regime, prices of zero coupon bonds (if the state \( \delta_t \) is observed) jump. Contrary to the short term rate, bond prices have then discontinuous trajectories.
Corollary 4.4. The price $P(t, T, \delta_t)$ on the enlarged filtration $\mathcal{F}_t \vee \mathcal{G}_t$ satisfies the following relation

$$P(t, T, \delta_t) =$$

$$P(s, T, \delta_s) \exp \left( \int_s^t r_u du - \frac{1}{2} \int_s^t \langle \Sigma, \delta_u \rangle B(u, T)^2 du - \int_s^t \langle \sigma, \delta_u \rangle B(u, T) dW_u \right) \times$$

$$\exp \left( \int_s^t \int_E \left( 1 - \frac{P(u, T, z)}{P(u, T, \delta_u)} \right) \langle Q' \delta_u, dz \rangle du + \int_s^t \int_E \left( \ln \frac{P(u, T, z)}{P(u, T, \delta_u)} \right) \langle d\delta_u, dz \rangle \right).$$

(4.12)

This result is a direct consequence of the Itô’s lemma applied to $\ln P(t, T, \delta_t)$ and of equation (4.7) and will be useful to simulate the evolution of a portfolio of zero coupon bond prices, as detailed in the the next section.

5 Some results about the pricing of derivatives.

In this section, we infer the dynamics of interest rates in the MSM model, under a forward measure. This result is at our knowledge new in the literature and as illustrated in following paragraphs is useful to implement an accurate pricing of derivatives.

Let us denote by $V(T, r_T)$, the payoff paid at time $T$ by an European option written on interest rates. The price of this option is the expectation of this discounted payoff under the risk neutral measure. Again, using the relation $\mathcal{F}_t \subset \mathcal{F}_t \vee \mathcal{G}_t$, the option price can be rewritten as the expectation of the expected discounted payoff on the enlarged filtration:

$$\mathbb{E} \left( e^{-\int_t^s r_u ds} V(T, r_T) \mid \mathcal{F}_t \right) = \mathbb{E} \left( \mathbb{E} \left( e^{-\int_t^s r_u ds} V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t \right) \mid \mathcal{F}_t \right).$$

(5.1)

If we denote by

$$F(t, r_t, \delta_t) = \mathbb{E} \left( e^{-\int_t^s r_u ds} V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t \right)$$

(5.2)

the price of the derivative under the enlarged filtration, can be obtained, as shown in Landen (2000), by solving the following system of Feynman-Kac equations:

$$(\varphi(t) + Y_t) F = \frac{\partial F}{\partial t} - a Y \frac{\partial F}{\partial Y} + \frac{1}{2} \langle \Sigma, e_j \rangle \frac{\partial^2 F}{\partial Y^2} + \sum_{k \neq j} q_{j,k} (F(t, r_t, e_k) - F(t, r_t, e_j)) \quad j = 1, \ldots, d$$

(5.3)

We don’t discuss in this work the numerical solving of this equation, which is in practice a difficult exercise, particularly when the number of states, $d$, is important. Once that the system (5.3) is solved, the price of the derivative is calculated as the sum of $F(t, T, e_j)$ weighted by probabilities of sojourn:

$$\mathbb{E} \left( e^{-\int_t^s r_u ds} V(T, r_T) \mid \mathcal{F}_t \right) = \sum_{j=1}^d \pi_{j,t} F(t, T, e_j).$$

Instead of solving the system (5.3), the prices $F(t, T, \delta_t)$ can be computed by a Monte-Carlo simulation, on the enlarged filtration. However, practitioners usually avoid to simulate payoffs of derivatives directly under the risk neutral measure. Because, typically, this requires to approximate the discounting term $e^{-\int_t^s r_u ds}$ with $e^{-\Sigma_i \Delta s_i}$ and the $\Delta s_i$ have to be small in order for the approximation to be accurate. It is there advisable to work under a forward measure. Under this measure, we can draw out the discount factor of the equation (5.2). If the market admits at least one risk neutral measure $Q$ on the enlarged filtration, we can define equivalent probability measures to $Q$ by the technique of changes of numéraires. The $T$-forward measure has as numéraire, the zero coupon bond of maturity $T$. Under this measure, the price of any financial assets, divided by the numéraire $P(t, T)$, is a martingale. We have the following interesting result:
**Proposition 5.1.** Under the $T$-forward measure, the price of a derivative delivering a payoff $V(T, r_T)$ at time $T$, is provided by the following expectation:

$$
E \left( e^{-\int_0^T r_s ds} V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t \right) = P(t, T, \delta_t) E^{Q^T} \left( V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t \right),
$$

Where $Q^T$ points out the forward measure under which the dynamics of short term rate is given by $r_t = \varphi(t) + Y_t$ with

$$
dY_t = (-aY_t - \langle \Sigma, \delta_t \rangle B(t, T)) \, dt + \langle \sigma, \delta_t \rangle dW_t^T
$$

and where $\delta_t$ is a non homogeneous Markov process with an intensity matrix $Q^T(t, T)$ whose elements are the following

$$
\begin{align*}
q_{i,j}^F(t) &= q_{i,j} A(t, T, j) A(t, T, e_i) \\
q_{i,i}^F(t) &= - \sum_{j \neq i} q_{ij} A(t, T, j) A(t, T, e_i)
\end{align*}
$$

(5.4)

**Proof.** Let us denote $B(t)$, the market value of a cash account

$$
B_t = e^{\int_0^t r_s ds}
$$

The Radon-Nykodym derivative defining the measure having $P(t, T)$ as numeraire, is equal to:

$$
\frac{dQ^T}{dQ} = \frac{1}{B_T P(0, T, \delta_0)}.
$$

Its conditional expectation (under $Q$), according to equality (4.12), is then given by:

$$
E \left( \frac{dQ^T}{dQ} \mid \mathcal{F}_t \vee \mathcal{G}_t \right) = \frac{B_0}{B_t P(0, T, \delta_0)} P(t, T, \delta_t)
$$

(5.5)

By the Bayes rule, the expected payoff (under the $T$ forward measure) of an European option of maturity $T$, on a zero coupon bond of maturity $S$, is equivalent to:

$$
E^{Q^T} \left( V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t \right) = \frac{E \left( \frac{dQ^T}{dQ} \right) V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t}{E \left( \frac{dQ^T}{dQ} \mid \mathcal{F}_t \vee \mathcal{G}_t \right)} = \frac{1}{P(t, T, \delta_t)} E \left( \frac{B_t}{B_T} V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t \right)
$$

(5.6)

The expectation of the discounted payoff, under the risk neutral measure is then equal to the product of a bond price and of the expected payoff under forward measure:

$$
E \left( e^{-\int_0^T r_s ds} V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t \right) = P(t, T, \delta_t) E^{Q^T} \left( V(T, r_T) \mid \mathcal{F}_t \vee \mathcal{G}_t \right)
$$

(5.6)

Replacing in equation (5.5) the bond price by its expression (4.12) leads to the following result:

$$
E \left( \frac{dQ^T}{dQ} \mid \mathcal{F}_t \vee \mathcal{G}_t \right) = \exp \left( -\frac{1}{2} \int_0^t \langle \Sigma, \delta_u \rangle B(u, T)^2 du - \int_0^t \langle \sigma, \delta_u \rangle B(u, T) dW_u \right) \times \\
\exp \left( \int_0^t \int_E \left( 1 - \frac{A(u, T, z)}{A(u, T, \delta_u)} \right) Q' \delta_u, dz \right) + \int_0^t \left( \ln \frac{A(u, T, z)}{A(u, T, \delta_u)} \right) \langle d\delta_u, dz \rangle du
$$

(5.7)
We recognize in the first term of the right hand side of this last equation, a change of measure affecting the Brownian motion. The second term modifies the frequency of jumps of \( \delta_t \). According to equation (5.7), we have then that

\[
dW_t^F = dW_t + \langle \sigma, \delta_t \rangle B(t, T) \, dt
\]

is a Brownian motion under the \( T \) forward measure. While the process \( \delta_t \) has a transition matrix defined by (5.4) under the forward measure.

This result is particularly interesting and at our knowledge is new in the literature. It reveals that under a forward measure, the transition probabilities of the Markov process \( \delta_t \), become time dependent.

6 Numerical Applications.

As seen in section 3, using the fractal interest rates model is justified from an econometric point of view. By adjusting the deterministic trend, we can also replicate any yield curves. To illustrate this point, we fit the fractal model to the curve of zero coupon rates, bootstrapped from the swaps yield curve on the 16th of December 2011 (see appendix, table 7.1). We consider a MSM model with 4 fractal components that can be reformulated as a regime switching model with 16 states. The parameters defining the process \( Y_t \) are assumed to be equal to those obtained by the econometric calibration on historical data. This assumption is realistic if the model is used as interest rates generator in a risk management system. For this type of applications, users rather seek a model able to generate realistic yield curves than a model fitting perfectly prices of derivatives. If we intend to use the model for derivatives pricing, we should instead choose the parameters of \( Y_t \) so as to minimize the spread between market and modeled prices. The probabilities of presence in each of these 16 states (noted \( \pi_{j,t} \) in previous sections) are retrieved from the Hamilton filter, on the 16th of December 2011. These probabilities and the related volatilities, are reported in appendix table 7.2 and a summary of these results is provided in table 6.1. Note that even if the volatilities can be the same in two different states, they are well the result of different occurrences of the state vector \( S_t \). We observe that the process is in states 4,6,7,10,11,13 with a cumulative probability of 61.04%. And in these states, the volatility is 0.2815%. The probability of being in states 2,3,5 or 9 is also high (33.56%). And the corresponding volatility is low: 0.0869%. We also remark that the interest rate process can have an extremely low or high volatility (resp. 0.0268% and 2.9514%), both with very low probabilities (less than 1%).

Given that only the integral of \( \phi(t) \) is involved in the calculation of zero coupon bonds (formula (4.3)), we use the relation (4.6) to retrieve it, instead of fitting the function \( \phi(t) \) by formula (4.5). Figure 6.1 exhibits the logarithm of functions \( A(t, 10, \delta_t) \) and \( A(t, 20, \delta_t) \) computed numerically by an Euler’s discretization of equations ((4.4)). Whatever the state, they are decreasing convex functions.
After calibration of the model, we have implemented a Monte Carlo simulation to forecast 10,000 curves of zero coupon rates, in one year. The time step chosen to discretize the dynamics of $Y_t$ is a trading day ($\Delta t = 1/250$). Given that the initial state of the MSM process is not known with certainty on the 16th of December 2011, the simulation draw it from the discrete distribution of presences yield by the filter and reported in table 7.2 (see appendix). So as to compare our result, we have also forecast interest rates curves with the 2D and normal Hull-White models (fitted to the yield curve on the 16th December and with historical volatility and mean reversion of table 3.3).

Statistics about forecast interest rates are detailed in tables 6.2, 6.3 and 6.4. The same statistics for bonds prices are provided in tables 7.3 7.4 and 7.5, in appendix. If the zero coupon rate of maturity $(T-t)$ is noted $R(t,T) = -\ln (P(t,T))/(T-t)$, the VaR is defined here as the percentile of $R(1,T)$ and TVaR are here defined as follows:

$$TVaR_{1\%} = E(R(1,T) \mid R(1,T) \leq VaR_{1\%})$$

$$TVaR_{99\%} = E(R(1,T) \mid R(1,T) \geq VaR_{99\%})$$

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>15</th>
<th>17</th>
<th>20</th>
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<tr>
<td>Average (%)</td>
<td>1.0809</td>
<td>1.4760</td>
<td>1.8872</td>
<td>2.1949</td>
<td>2.5080</td>
<td>2.6800</td>
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<tr>
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<td>0.1013</td>
<td>0.0627</td>
<td>0.0449</td>
<td>0.0315</td>
<td>0.0242</td>
<td>0.0210</td>
<td>0.0185</td>
<td>0.0157</td>
</tr>
<tr>
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<td>1.1719</td>
<td>1.6982</td>
<td>2.0592</td>
<td>2.4129</td>
<td>2.6068</td>
<td>2.6563</td>
<td>2.6506</td>
<td>2.6437</td>
</tr>
<tr>
<td>99% VaR</td>
<td>1.7396</td>
<td>1.7891</td>
<td>2.0805</td>
<td>2.3333</td>
<td>2.6049</td>
<td>2.7545</td>
<td>2.7843</td>
<td>2.7636</td>
<td>2.7397</td>
</tr>
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<td>99% Tail VaR</td>
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<td>2.7813</td>
<td>2.8075</td>
<td>2.7841</td>
<td>2.7571</td>
</tr>
</tbody>
</table>

Table 6.2: MSM model: statistics about simulated ZC rates, $R(1,T)$, in one year. Initial State distributed according to $\pi_0$. 

Figure 6.1: Plot of $\log(A(t,10,\delta_t))$ and $\log(A(t,20,\delta_t))$ functions.
A first interesting observation is about volatilities of interest rates: they are clearly bigger in the 2D Hull-White model than in MSM and 1D Hull-White models. We also remark that volatilities decrease with the maturity whatever the model. But this trend is more significant in the Hull-White model than in fractal and 2D Hull-White models.

If we look at percentiles, the 1% VaR of short term yields (1 and 2 years) is slightly lower in the MSM model then in the 2D Hull-White model, but clearly below the 1D Hull-White model. The 1% and 99% tailVaRs of short term yields (1 to 4 years) in the fractal model are significantly lower and higher than their equivalents in Hull-White models. For longest maturities, tailVaRs in the fractal and 2D Hull-White models are comparable. This observation suggests that on a short-term time horizon, distributions of interest rates in the fractal model are slightly more leptokurtic than those obtained with Hull-White models. To confirm this, we compares in figure 6.2 the densities of 1y and 5y rates (in one year). The distribution of 1y rates in the MSM model seems well to have fatter tails than the one of the 2D Hull-White model.

Using the MSM model instead of a standard Hull-White model, in a risk management system, can have a huge impact on the calculation of the capital requirement to hedge the interest rate risk. Consider the case of an insurance company holding only one zero coupon bond of maturity 8 years at time zero (principal 100 Euros). Let us assume that the firm compute its required capital as the difference between the expected value of its asset in one year, minus the 99% VaR. Based on the Hull-White model (and on figures of tables 7.3 and 7.5), the capital required to cover the interest rate risk is hence equal to 85.75-85.55= 0.20 Euros. The same calculation, based on simulations done with the fractal model, leads to a capital of 85.76-84.93= 0.83 Euros, which is significantly higher (about 4 times more in our example).
As emphasized in the previous section, the calculation of interest rates derivatives by direct solving of the system of ODE (5.3) is a tricky exercise, particularly when the number of states is important. Another approach consists to simulate under the forward measure, the evolution of the underlying. We did this exercise to price caplet on 1y rates, for different maturities and strikes, on the 16th of December 2012. The MSM model uses 4 fractal components. Parameters defining the process $Y_t$ are assumed to be equal to those obtained by the econometric calibration on historical data. Maturities and strikes respectively range from a half year to five years and from 0.4% to 2.0%. If the maturity and strike are noted $t$ and $K$, the caplet price is calculated as follows:

$$\text{Caplet}(t, K) = P(0, t) \mathbb{E}^{Q_t} \left( (R(t, t+1y) - K)_+ \right).$$

These prices are plotted in figure 6.4 and a summary of prices computed is reported in table 7.6 (see appendix). In practice, caplet are quoted by their implied volatilities, which are retrieved by inverting the Black’s formula. These volatilities are reported in table 7.7 and presented in figure 6.4. We observe that the fractal model is able to generate humped curves of implied volatilities. Implied volatilities are minimal in an area in which the strike is around the forward rate (“At The Money” caplets), which is a realistic feature of volatilities of implied volatilities. We think that the MSM model can fit a large range of surfaces. However, using a Monte Carlo method to fit an existing surface of implied volatilities is time consuming and require a huge calculation power.
7 Conclusions.

Our work looks at an extension of the Hull-White interest rate model in which the constant volatility is replaced by a fractal process. The main motivation of this approach is to develop a model that captures the excess of kurtosis and skewness exhibited by interest rates distributions. As illustrated by the econometric calibration, the fractal volatility provides a more realistic model of the behavior of short term rates. Furthermore, the specification of the volatility is highly parsimonious and requires only a few parameters compared to classic switching regimes models. Even if we lose the analytical tractability of the Hull-White model, there still exists a formula for zero coupon bond prices, and their dynamics under the risk and forward measure are well identified. In particular, we have shown that The Markov process coupled to fractal components has an intensity which is time dependent under any forward measure.

As exhibited in numerical applications, the fractal extension of the Hull-White model can be
used for risk management purposes but will probably lead to an increase of capital required to cover the interest rate risk. This is mainly explained by the fact that interest rates yield by the fractal model have strongly leptokurtic distributions, compared to a traditional Gaussian model. Among the different alternatives to price interest rate derivatives, it was decided to explore the pricing by Monte Carlo simulations under a forward measure. This procedure can be used in practice but fitting the model to a given surface of implied volatility is time consuming.

Appendix A.

Proof of proposition 4.1.

\[ P(t, T, \delta_t) = \mathbb{E} \left( e^{-\int_t^T \varphi(s) ds} | \mathcal{F}_t \cup \mathcal{G}_t \right) \]

\[ = e^{-\int_t^T \varphi(s) ds} \mathbb{E} \left( e^{-\int_t^T Y(s) ds} | \mathcal{F}_t \cup \mathcal{G}_t \right) \]

given that the dynamics of interest rates such as defined in equation (2.2) can be rewritten as

\[ dY_t = -aY_t dt + \left\langle \sigma', \delta_t \right\rangle dW_t \]

In this setting, one can easily demonstrate (it is a direct consequence of the Ito formula applied to \( e^{a.s} Y_s \)) that

\[ Y_t = Y_s e^{-a(t-s)} + \int_s^t \left\langle \sigma', \delta_s \right\rangle e^{-a(t-s)} dW_s \]  \hspace{1cm} (7.1)

In view of equation (7.1), the integral of \( Y_t \) is given by:

\[ \int_t^T Y_s ds = Y_t \int_t^T e^{-a(s-t)} ds + \int_t^T \int_t^s \left\langle \sigma', \delta_s \right\rangle e^{-a(t-s)} dW_s ds \]

\[ = Y_t \frac{1}{a} \left( 1 - e^{-a(T-t)} \right) + \int_t^T \frac{1}{a} \left( 1 - e^{-a(T-s)} \right) \left\langle \sigma', \delta_s \right\rangle dW_s \]  \hspace{1cm} (7.2)

If we define the function \( B(t, T) \) as (4.2), the price of the zero coupon bond is rewritten as follows:

\[ P(t, T, \delta_t) = e^{-\int_t^T \varphi(s) ds - Y_s B(t,T)} \mathbb{E} \left( \exp \left( -\int_t^T B(\theta, T) \left\langle \sigma', \delta_\theta \right\rangle dW_\theta \right) | \mathcal{F}_t \cup \mathcal{G}_t \right) \]

Given that \( \mathcal{F}_t \cup \mathcal{G}_t \subset \mathcal{F}_T \cup \mathcal{G}_T \), by the principle of conditional expectations, we get that:

\[ \mathbb{E} \left( \exp \left( -\int_t^T B(\theta, T) \left\langle \sigma', \delta_\theta \right\rangle dW_\theta \right) | \mathcal{F}_t \cup \mathcal{G}_t \right) \]

\[ = \mathbb{E} \left( \mathbb{E} \left( \exp \left( -\int_t^T B(\theta, T) \left\langle \sigma', \delta_\theta \right\rangle dW_\theta \right) | \mathcal{F}_t \cup \mathcal{G}_T \right) | \mathcal{F}_t \cup \mathcal{G}_t \right) \]

\[ = \mathbb{E} \left( \exp \left( \int_t^T \frac{1}{2} B(\theta, T)^2 \Sigma, \delta_\theta \right) d\theta \right) | \mathcal{F}_t \cup \mathcal{G}_t \]  \hspace{1cm} (7.3)

where \( \Sigma = (\sigma_1^2, \ldots, \sigma_d^2) \) is the vector of possible short term rate variances. The price of the zero coupon bond becomes then

\[ P(t, T, \delta_t) = e^{-\int_t^T \varphi(s) ds - Y_s B(t,T)} A(t, T, \delta_t) \]

where

\[ A(t, T, \delta_t) = \mathbb{E} \left( \exp \left( \int_t^T \frac{1}{2} B(\theta, T)^2 \Sigma, \delta_\theta \right) d\theta \right) | \mathcal{F}_t \cup \mathcal{G}_t \). \]
We have that for all \( u \geq t \):

\[
A(t, T, \delta_t) = \mathbb{E} \left( \mathbb{E} \left( \exp \left( \int_t^T \left( \frac{1}{2} B(\theta, T)^2 \Sigma, \delta_\theta \right) d\theta \right) \big| \mathcal{F}_u \lor \mathcal{G}_u \right) \big| \mathcal{F}_t \lor \mathcal{G}_t \right),
\]

yielding, thanks to the definition of the quantity \( A \):

\[
A(t, T, \delta_t) = \mathbb{E} \left( e^{\int_t^T \left( \frac{1}{2} B(\theta, T)^2 \Sigma, \delta_\theta \right) ds} A(u, T, \delta_u) \big| \mathcal{F}_t \lor \mathcal{G}_t \right).
\]

Then, by assuming enough regularity to allow one to take the limit within the expectation, the following limit converges to zero:

\[
\lim_{u \to t} \mathbb{E} \left( e^{\int_t^T \left( \frac{1}{2} B(\theta, T)^2 \Sigma, \delta_\theta \right) ds} A(u, T, \delta_u) \big| \mathcal{F}_t \lor \mathcal{G}_t \right) - A(t, T, \delta_t) = 0.
\]

If we develop the exponential by its Taylor approximation of first order, we can rewrite this limit as:

\[
\lim_{u \to t} \frac{\mathbb{E} \left( A(u, T, \delta_u) \big| \mathcal{F}_t \lor \mathcal{G}_t \right) - A(t, T, \delta_t)}{u - t} = -\frac{1}{2} B(t, T)^2 \langle \Sigma, \delta_t \rangle A(t, T, \delta_t).
\]

The right hand term being calculable by the Itô formula for switching processes, we infer that \( A(u, T, \delta_u) \) is the solution of a system of partial integro-differential equations:

\[
\frac{\partial}{\partial t} A(t, T, e_j) + LA(t, T, e_j) = -\frac{1}{2} B(t, T)^2 \langle \Sigma, e_j \rangle A(t, T, e_j) \quad j = 1 \ldots d,
\]

where \( LA(t, T, e_j) \) is the generator:

\[
LA(t, T, e_j) = \sum_{k \neq j} q_{j,k} (A(t, T, e_k) - A(t, T, e_j))
\]

As \( \sum_{k \neq j} q_{j,k} = -q_{j,j} \), this last expression is also equivalent to:

\[
\frac{\partial}{\partial t} A(t, T, e_j) + \sum_k q_{j,k} A(t, T, e_k) = -\frac{1}{2} B(t, T)^2 \langle \Sigma, e_j \rangle A(t, T, e_j) \quad j = 1 \ldots d
\]

and if we define \( \hat{A}(t, T) \) a the vector of \( (A(t, T, e_j))_{j=1 \ldots N} \), this last system can be rewritten as follows:

\[
\frac{\partial}{\partial t} \hat{A}(t, T) + \left( \frac{1}{2} B(t, T)^2 \text{diag} \left( \langle (\Sigma, e_j) \rangle_{j=1 \ldots N} \right) + Q \right) \hat{A}(t, T) = 0.
\]

The boundary condition \( \hat{A}(T, T) = 1 \) \( j = 1 \ldots d \) is a direct consequence of the bond constraint \( P(T, T, \delta_t) = 1 \).

### Appendix B.

<table>
<thead>
<tr>
<th>State</th>
<th>( \pi_0 )</th>
<th>( \sigma_{t=0} ) (%)</th>
<th>State</th>
<th>( \pi_0 )</th>
<th>( \sigma_{t=0} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0073</td>
<td>0.0268</td>
<td>9</td>
<td>0.0539</td>
<td>0.0869</td>
</tr>
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<td>0.0869</td>
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<td>0.0869</td>
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<td>0.0869</td>
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<td>0.2146</td>
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</tr>
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<td>0.2815</td>
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<td>0.0022</td>
<td>0.9114</td>
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<td>0.9114</td>
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<tr>
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<td>0.9114</td>
<td>16</td>
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Table 7.2: Probabilities of presence and related volatilities, on the 16/12/11.
<table>
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<th>Maturity</th>
<th>Zero coupon rate</th>
</tr>
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<td>1y</td>
<td>1.36%</td>
</tr>
<tr>
<td>2y</td>
<td>1.22%</td>
</tr>
<tr>
<td>3y</td>
<td>1.27%</td>
</tr>
<tr>
<td>4y</td>
<td>1.45%</td>
</tr>
<tr>
<td>5y</td>
<td>1.62%</td>
</tr>
<tr>
<td>7y</td>
<td>1.98%</td>
</tr>
<tr>
<td>10y</td>
<td>2.32%</td>
</tr>
<tr>
<td>15y</td>
<td>2.64%</td>
</tr>
<tr>
<td>20y</td>
<td>2.63%</td>
</tr>
</tbody>
</table>

Table 7.1: Zero coupon rates bootstrapped from the swap curve, 16/12/11.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Average Price</th>
<th>Volatility</th>
<th>1% VaR</th>
<th>99% VaR</th>
<th>1% Tail VaR</th>
<th>99% Tail VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>98.93</td>
<td>0.21</td>
<td>98.28</td>
<td>99.56</td>
<td>98.04</td>
<td>99.78</td>
</tr>
<tr>
<td>3</td>
<td>95.67</td>
<td>0.29</td>
<td>94.77</td>
<td>96.55</td>
<td>94.45</td>
<td>96.86</td>
</tr>
<tr>
<td>5</td>
<td>91.00</td>
<td>0.29</td>
<td>90.12</td>
<td>91.86</td>
<td>89.81</td>
<td>92.16</td>
</tr>
<tr>
<td>7</td>
<td>85.76</td>
<td>0.27</td>
<td>84.93</td>
<td>86.58</td>
<td>84.64</td>
<td>86.86</td>
</tr>
<tr>
<td>10</td>
<td>77.82</td>
<td>0.24</td>
<td>77.07</td>
<td>78.56</td>
<td>76.80</td>
<td>78.82</td>
</tr>
<tr>
<td>13</td>
<td>70.58</td>
<td>0.22</td>
<td>69.90</td>
<td>71.26</td>
<td>66.96</td>
<td>71.49</td>
</tr>
<tr>
<td>15</td>
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<td>0.21</td>
<td>65.86</td>
<td>67.14</td>
<td>65.63</td>
<td>67.36</td>
</tr>
<tr>
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<td>63.72</td>
<td>62.30</td>
<td>63.94</td>
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<tr>
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<td>57.81</td>
<td>59.03</td>
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</tr>
</tbody>
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Table 7.3: MSM model with $n = 4$. Statistics about simulated ZC prices, in one year. Initial State distributed according to $\pi_0$.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Average Price</th>
<th>Volatility</th>
<th>1% VaR</th>
<th>99% VaR</th>
<th>1% Tail VaR</th>
<th>99% Tail VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>98.92</td>
<td>0.23</td>
<td>98.38</td>
<td>99.46</td>
<td>98.30</td>
<td>99.53</td>
</tr>
<tr>
<td>3</td>
<td>95.66</td>
<td>0.40</td>
<td>94.72</td>
<td>96.60</td>
<td>94.57</td>
<td>96.72</td>
</tr>
<tr>
<td>5</td>
<td>90.99</td>
<td>0.43</td>
<td>89.98</td>
<td>91.99</td>
<td>89.82</td>
<td>92.13</td>
</tr>
<tr>
<td>7</td>
<td>85.75</td>
<td>0.42</td>
<td>84.77</td>
<td>86.73</td>
<td>84.62</td>
<td>86.86</td>
</tr>
<tr>
<td>10</td>
<td>77.81</td>
<td>0.38</td>
<td>76.91</td>
<td>78.71</td>
<td>76.77</td>
<td>78.83</td>
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<td>13</td>
<td>70.58</td>
<td>0.35</td>
<td>69.76</td>
<td>71.39</td>
<td>69.63</td>
<td>71.50</td>
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<tr>
<td>15</td>
<td>66.50</td>
<td>0.33</td>
<td>65.73</td>
<td>67.26</td>
<td>65.60</td>
<td>67.36</td>
</tr>
<tr>
<td>17</td>
<td>63.12</td>
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<td>63.84</td>
<td>62.27</td>
<td>63.94</td>
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<td>20</td>
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<td>0.29</td>
<td>57.70</td>
<td>59.05</td>
<td>57.59</td>
<td>59.13</td>
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</table>

Table 7.4: 2D Hull White model. Statistics about simulated ZC prices, in one year. Initial State distributed according to $\pi_0$.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Average Price</th>
<th>Volatility</th>
<th>1% VaR</th>
<th>99% VaR</th>
<th>1% Tail VaR</th>
<th>99% Tail VaR</th>
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</thead>
<tbody>
<tr>
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<td>98.92</td>
<td>0.09</td>
<td>98.72</td>
<td>99.13</td>
<td>98.68</td>
<td>99.16</td>
</tr>
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<td>0.10</td>
<td>95.44</td>
<td>95.90</td>
<td>95.40</td>
<td>95.93</td>
</tr>
<tr>
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<td>90.99</td>
<td>0.10</td>
<td>90.77</td>
<td>91.22</td>
<td>90.74</td>
<td>91.25</td>
</tr>
<tr>
<td>7</td>
<td>85.75</td>
<td>0.09</td>
<td>85.55</td>
<td>85.97</td>
<td>85.51</td>
<td>86.00</td>
</tr>
<tr>
<td>10</td>
<td>77.81</td>
<td>0.08</td>
<td>77.63</td>
<td>78.01</td>
<td>77.60</td>
<td>78.03</td>
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<td>70.41</td>
<td>70.75</td>
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<td>66.50</td>
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<td>66.66</td>
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<td>66.69</td>
</tr>
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<td>63.12</td>
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<td>63.28</td>
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<tr>
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<td>58.23</td>
<td>58.52</td>
<td>58.21</td>
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</table>

Table 7.5: Hull White model. Statistics about simulated ZC prices, in one year. Initial State distributed according to $\pi_0$. 
### Table 7.6: Caplet Prices on 1y interest rates. Fractal model.

<table>
<thead>
<tr>
<th>Mat.</th>
<th>Strike price (in %)</th>
<th>0.40%</th>
<th>0.60%</th>
<th>0.80%</th>
<th>1.00%</th>
<th>1.20%</th>
<th>1.40%</th>
<th>1.60%</th>
<th>1.80%</th>
<th>2.00%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1y</td>
<td></td>
<td>0.6841</td>
<td>0.4912</td>
<td>0.3035</td>
<td>0.1344</td>
<td>0.0436</td>
<td>0.0194</td>
<td>0.0100</td>
<td>0.0054</td>
<td>0.0029</td>
</tr>
<tr>
<td>2y</td>
<td></td>
<td>0.9363</td>
<td>0.7629</td>
<td>0.5710</td>
<td>0.3823</td>
<td>0.2086</td>
<td>0.0693</td>
<td>0.0132</td>
<td>0.0071</td>
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</tr>
<tr>
<td>3y</td>
<td></td>
<td>1.5232</td>
<td>1.3310</td>
<td>1.1391</td>
<td>0.9475</td>
<td>0.7567</td>
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<td>0.3812</td>
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<tr>
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<td>0.8937</td>
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</table>

### Table 7.7: Caplet Prices on 1y interest rates. Implied Volatilities.

<table>
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<tr>
<th>Mat.</th>
<th>Vol. (in %)</th>
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<th>0.80%</th>
<th>1.00%</th>
<th>1.20%</th>
<th>1.40%</th>
<th>1.60%</th>
<th>1.80%</th>
<th>2.00%</th>
</tr>
</thead>
<tbody>
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<td>18.30</td>
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<td>44.46</td>
<td>36.34</td>
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<td>23.88</td>
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<tr>
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<td>59.76</td>
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<td>34.30</td>
<td>28.49</td>
<td>23.42</td>
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### References


