Assessing horizontal mergers under uncertain efficiency gains*

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Abstract

The analysis of horizontal mergers hinges on a tradeoff between unilateral effects and efficiency gains. The article examines the role of uncertainty (on the efficiency gains) in this tradeoff. Common wisdom is that the antitrust authorities should be very cautious about random gains. Our results show that dismissing efficiency gains on the sole ground that they are uncertain would not be theoretically founded. Indeed, the attitude towards uncertainty depends on the curvature of the social objective function. We exhibit a number of situations where the objective is convex in the efficiency gains, implying that antitrust enforcers should be prepared to take risks in approving mergers with uncertain efficiency gains. Implications for empirical merger analysis are exposed.

Keywords: Merger analysis, Antitrust, Efficiency gains, Uncertainty.

JEL Classification: K210, L120, L410

1 Introduction

This article reexamines the welfare tradeoffs put forward by Williamson (1968) when efficiency gains are uncertain. To this end, we consider horizontal mergers that create market power, but at the same time yield cost economies (or losses) of random magnitude.

In the modern antitrust langage, the variation of Nash equilibrium prices following the alteration of market structure is often referred to as “unilateral effects”. These effects have been recognized for a long time by U.S competition authorities. On the other side of the Atlantic ocean, the 1989 European Community Merger Regulation had been criticized by economists and practitioners because it was not clear that the prohibition criterion (creation or strengthening of a dominant position) encompassed unilateral effects. In January 2004, the substantive test has been reworded, so as to unambiguously fill the gap.

As regards efficiencies, it is fair to say that they are now widely taken into account by competition authorities, even though differences in the standards of proof may subsist.

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Section 4 of the 1997 U.S. Merger Guidelines, which is devoted to efficiencies, recognizes that “the primary benefit of mergers to the economy is their potential to generate such efficiencies” and acknowledges their potentially pro-competitive impact: “Efficiencies generated through merger can enhance the merged firm’s ability and incentive to compete, which may result in lower prices, improved quality, enhanced service, or new products.” In Europe, a rather skeptical view about efficiency gains seems to prevail. The 2004 EC Regulation imposes high standards regarding efficiency gains which are referred to as “development of technical and economic progress”: they are taken into account only “provided that it is to consumers’ advantage and does not form an obstacle to competition.” Moreover, under both jurisdictions, efficiency gains have to be merger specific, verifiable, and assessed net of costs. Having opened the scope for an efficiency defence, competition authorities emphasize the issue of uncertainty:

Efficiencies are difficult to verify and quantify, in part because much of the information relating to efficiencies is uniquely in the possession of the merging firms. Moreover, efficiencies projected reasonably and in good faith by the merging firms may not be realized.

(U.S. Merger Guidelines sec. 4)

This quote highlights two distinct informational features: asymmetry and imperfection. First, merging parties, arguably, know more about potential efficiency gains than competition authorities. Obviously, firms have strong incentives to dissemble about efficiencies; antitrust enforcers correctly anticipate that the merging firms often provide self-serving information so as to increase the likelihood of approval. In practice, they require substantiation of promised efficiency gains. A strand of literature addresses this issue in an asymmetric information framework (see for instance Besanko and Spulber (1993)).

Second, even under symmetric information, there would remain uncertainty about the realization and the magnitude of the efficiency gains. During the merger implementation, firms try to achieve cost reductions but this process is more or less successful. As put by Judge Thomas F. Hogan in his opinion about the Staples and Office Depot merger case:

The Court agrees with the defendants that where, as here, the merger has not yet been consummated, it is impossible to quantify precisely the efficiencies that it will generate. In addition, the Court recognizes a difference between efficiencies which are merely speculative and those which are based on a prediction backed by sound business judgment. Nor does the Court believe that the defendants must prove their efficiencies by “clear and convincing evidence” in order for those efficiencies to be considered by the Court. That would saddle Section 7 defendants with the nearly impossible task of rebutting a possibility with a certainty, a burden which was rejected in United States v. Baker Hughes, Inc.

(U.S. District Court for the District of Columbia, Civ. No. 97-701)

In other words, courts do not impose on merging firms to remove all uncertainty from their efficiency claims. They acknowledge that uncertainty about future costs is inevitable.
This paper investigates the next step: how should the decision rule account for this uncertainty? At the time of deciding the case, the antitrust authority faces a distribution of uncertainty that summarizes all available information, including the merger proposal itself. We assume that the objectives of society and of the competition authority are perfectly aligned and that the expectation of a weighted sum of consumers’ surplus and firms’ profits is maximized. This objective being in general nonlinear in the efficiency gains, society should not be indifferent to the uncertainty affecting efficiencies. To understand the social attitude towards this risk, the article endeavors to characterize the curvature of the social objective.

First, we identify, in a general competition model, a number of forces that tend to make consumers’ surplus and firms’ profits convex in efficiency gains. Second, we show that these forces dominate in several competitive environments. In those cases, antitrust enforcers should be prepared to take risks in approving mergers with uncertain efficiency gains. This finding contrasts with the common wisdom that uncertainty about efficiencies should weaken a merger case.

In the tradition of Deneckere and Davidson (1985), most of the analysis considers mergers in a multi-product industry under price differentiation and constant marginal costs of production. To check the robustness of our results, we examine some cases with homogenous goods, in particular Cournot competition, as in Farrell and Shapiro (1990). Consumers’ surplus and total welfare are convex as soon as demand functions are linear in prices, for any merger and any pre- and post-merger market structure, with price competition as well as with Cournot competition. No (e.g. symmetry) assumption on top of the linearity is required for this result to hold.1

Turning to nonlinear demand functions, we specialize to two polar cases: mergers to monopoly (strong unilateral effects) and Bertrand competition with homogenous products (no unilateral effect). In the former case, the monopoly profit is always convex; symmetry and an additional restriction (met, for instance, by CES demand systems) are needed for consumers’ surplus to be convex. For a Logit demand system, total welfare is convex in efficiency gains while consumers’ surplus can be either concave or convex depending on the value of the underlying parameters. Under Bertrand competition (and any demand function), consumers’ surplus cannot go up after a merger. Yet, uncertainty might still be welcomed if the antitrust authorities put enough weight on firms’ profits. Finally, when firms compete in quantity and costs are asymmetric, the consumers’ surplus is convex in efficiency gains for a large class of demand functions (and for an arbitrary number of outsiders).

These examples certainly do not imply that the convexity property is generic. The curvature of the social objective depends on the specifics of each case, in particular on the functional form of the demand system. Yet, our analysis shows that the specification chosen in econometric studies may entail an implicit attitude towards uncertain efficiency gains. We elaborate on the implications of our results for empirical merger analysis in the last

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1After completing the first version of our paper, we became aware of Barros and Cabral (2001) which addresses the same convexity issue. These authors examine the merger of two firms among n in three models: Cournot and price competition with linear demand, plus a “pyramid” model. Their analysis is, however, limited to symmetric and mono-product firms as well as to linear demand functions.
The remainder of the article is organized as follows. Section 2 presents the competition framework in which the merger takes place and discusses the merger assessment under uncertainty about efficiency gains. Section 3 solves the linear case. Section 4 analyzes mergers to monopoly under few restrictions on demands. Section 5 deals with Bertrand competition with homogeneous products. Section 6 extends the analysis to Cournot competition. Section 7 presents some implications of the results for merger control in practice, with a particular emphasis on the use of econometric models.

2 Framework

In this section, we introduce various notations and assumptions to describe a merger in a multi-products firms industry under constant marginal costs of production.

**Competitive environment:** Let \( N = \{1, \ldots, n\} \) denote the set of all brands. We use the index \( I \) for the merging parties and \( J \) for the outsiders. The structure of the industry before the merger is described by a partition of \( N \) into \( l + r \) subsets: \( \{I_1, \ldots, I_l, J_1, \ldots, J_r\} \).

The structure of the industry after the merger is described by a partition of \( N \) in \( 1 + r \) firms: \( \{I, J_1, \ldots, J_r\} \), where \( I = \bigcup_{i=1}^l I_i \) denotes the set of brands owned by the merged entity. Thus, \( l \) is the number of merging parties, \( r \) the number of outsiders.

**Costs and efficiency gains:** Before and after the merger, all goods are produced at constant marginal cost. Let \( c_k \) (\( k = 1 \) to \( n \)) denote the marginal cost of brand \( k \) before the merger. Without any loss of generality, the post-merger marginal costs of the merging firms can be noted: \((1 - \gamma_i) c_i\), with \( \gamma_i \leq 1 \).

In most of the paper, we suppose, for simplicity, that efficiency gains are one-dimensional,\(^2\) that is, they can be represented by a real-valued random variable \( \gamma \). We do not want, however, to assume that the gains are proportional to the pre-merger marginal cost \( c_i \) of each brand \( i \). To avoid this restriction, we introduce nonnegative fixed numbers \( \lambda_i \) with \( \max I \lambda_i = 1 \), and assume that the gains are proportional to \( \lambda_i c_i \). In other words, the post-merger marginal costs of the new entity are

\[
(1) \quad c_i (\gamma) = (1 - \lambda_i \gamma) c_i, \quad i \in I.
\]

If each variety benefits similarly from the efficiency gains, then \( \lambda_i = 1 \) for all \( i \). But marginal costs might vary differently: for instance, only one product could benefit from the efficiencies, that is, only one \( \lambda_i \) could be positive.

Since the costs of each variant must remain nonnegative, we have: \( \gamma \leq 1 \). The merger allows to reduce marginal costs when \( 0 < \gamma \leq 1 \). Efficiency losses, \( \gamma < 0 \), (caused, for instance, by clashes between corporate cultures) are not ruled out. Efficiency gains being merger specific do not affect the outsiders’ costs.

Efficiency gains may also reduce fixed costs of production and such cost savings can be crucial in the merger decision. This paper, however, focuses on merger analysis under

\(^2\)Some results hold with multidimensional gains. See Remark 2 in Section 3.
uncertainty; the results will be driven by the curvature of the objective function. Fixed costs do not directly affect consumers’ surplus and enter into firms’ profits in a linear way. Therefore, they only intervene through their expectation, and we do not need to model them explicitly.

**Demand:** We follow the standard assumption that all consumers have the same constant marginal utility for money, which allows to aggregate their preferences and postulate the existence of a representative consumer.\(^3\) For \(k = 1\) to \(n\), let \(p_k\) denote the price of variant \(k\), \(p = (p_1, \ldots, p_n)'\) the column vector of all prices, \(x_k \geq 0\) the quantity chosen by the consumers, and \(x\) the vector of all quantities. Consumers’ surplus is:

\[
V(p) = \max_x [U(x) - p'x]
\]  

(2)

The utility function \(U\) does not have to be symmetric in \(x_1, \ldots, x_n\). The functions \(U\) and \(V\) are assumed to be twice differentiable.

**Remark 1.** Consumer’s surplus \(V\) is convex in the price vector \(p\).

In economic terms, consumers benefit more from a price reduction than they suffer from an increase in price. This effect comes from their reaction to price changes. Facing a price fall (resp. rise), they increase (resp. reduce) the quantity purchased, which exacerbates (resp. mitigates) the gain (resp. loss).

Formally, Remark 1 follows from \(V\) being a maximum of affine functions. Moreover, standard duality results in convex analysis (see Rockafellar (1972) theorem 5.5, page 35) show that, given a convex indirect utility function \(V(p)\), there exists a (unique) concave direct utility \(U(x)\) satisfying (2). It follows that the convexity of \(V\) is necessary and sufficient for consistency with a well-posed consumer’s problem. In the remainder of the article, we work with \(V\) rather than with \(U\).

The demand functions follow from Roy’s identity:

\[
D^k(p) = -\frac{\partial V}{\partial p_k}(p) = -V_{p_k}, \quad k = 1, \ldots, n
\]

or in matrix notation: \(D = -V_p\), where \(D = (D^1, D^2, \ldots, D^n)'\). The \(n \times n\) matrix \(D_p = -V_{pp}\) is symmetric semi-definite negative. Throughout, we will use a subscript to denote differentiation with respect to prices.

After the merger, the profit function of the merged entity is

\[
\Pi^I(p) = \sum_{i \in I} (p_i - c_i(\gamma)) \cdot D^i(p) = (p^I - c^I(\gamma))' \cdot D^I = (m^I)' \cdot D^I,
\]

where \(m^I\) denotes the vector of price-cost margins for the products owned by the merged firm. Using similar notations, the profit function of an outsider \(J\) writes:

\[
\Pi^J(p) = \sum_{j \in J} (p_j - c_j) \cdot D^j(p) = (p^J - c^J)' \cdot D^J = (m^J)' \cdot D^J, \quad J = J_1, \ldots, J_r.
\]

\(^3\)On the aggregation result, see Blackorby, Primont, and Russel (1978). Also note that our quasi-linear specification assumes away income effect and risk aversion considerations. On these issues, see Turnovsky, Shalit, and Schmitz (1980) and Stennek (1999).
**Oligopoly game:** After the merger, the value of the random variable $\gamma$ is realized and becomes common knowledge: competition takes place under complete information.\(^4\) Except in Section 6, firms compete in prices with differentiated products. They choose simultaneously the prices of their brands. A similar situation has been analyzed in Deneckere and Davidson (1985), who, however, mainly focused on the profitability of the merger.

Throughout the paper, we assume that, before and after the merger, the Nash equilibrium exists and is unique. As our focus is not on entry/exit decisions,\(^5\) we restrict our attention to interior Nash equilibria: demand for each good is positive in equilibrium. In all specific examples under consideration, we provide sufficient conditions for these properties to hold.

**Objective function:** We use $p(\gamma) = (p_1(\gamma), \ldots, p_n(\gamma))^t$ and $S(\gamma)$ to denote the prices and the consumers’ surplus, viewed in (post-merger) equilibrium as functions of $\gamma$:

$$S(\gamma) = V(p(\gamma)).$$

For any function $u$ of $\gamma$, we use $\dot{u}$ and $\ddot{u}$ to note its first and second derivatives with respect to $\gamma$. For instance, the effect of a marginal increase in $\gamma$ on the price of good $k$ is denoted $\dot{p}_k(\gamma)$. Following Froeb, Tschantz, and Werden (2005), we interpret the vector $-\dot{p}$ as the (marginal) pass-through rate of efficiency gains.

Society’s objective, also seen as a function of the magnitude of efficiency gains, is a weighted sum of consumers’ surplus and firms’ profits:

$$W_\alpha(\gamma) = \alpha S + (1 - \alpha) \sum_K \Pi^K,$$

with $K = I, J_1, \ldots, J_r$ and $0 \leq \alpha \leq 1$. It is often assumed that antitrust authorities put more weight on consumers’ surplus than on firms’ profits. (Theoretical foundations can be found in Besanko and Spulber (1993) and Neven and Röller (2005).) Here, however, we do not a priori restrict to $\alpha \geq 1/2$.

**Existence of a critical threshold:** Given a weight $\alpha$, it is natural to ask whether there exists a critical threshold $0 < \hat{\gamma}_\alpha \leq 1$ such that

$$W_\alpha(\hat{\gamma}_\alpha) = W_\alpha^{\text{pre}},$$

where $W_\alpha^{\text{pre}}$ denotes the value of the weighted welfare at the pre-merger Nash equilibrium. Empirical studies often contain estimates of such thresholds. For example, Nevo (2000), Table 7 page 414, estimates $\hat{\gamma}_{1/2}$ for a number of actual and hypothetical mergers. See also Werden (1996), who expresses $\hat{\gamma}_1$ as a function of diversion ratios. In Section 7, we elaborate on the use of econometric studies for merger assessment when efficiency gains are uncertain.

\(^4\)On the contrary, Banal-Estañol (2005) and Amir, Diamantoudi, and Xue (2003) consider games where the outsiders do not know $\gamma$ at the time of production, which gives an informational advantage to the merging parties.

\(^5\)Werden and Froeb (1998) and Spector (2003) analyze the effects of mergers taking into account entry considerations.
In theory, the existence and the uniqueness of a threshold $\hat{\gamma}_\alpha > 0$ do not follow from simple general assumptions. The following conditions are sufficient for existence and uniqueness: (i) without efficiency gains, the merger reduces the objective: $W_\alpha(0) < W_\alpha^{\text{pre}}$; (ii) the objective $W_\alpha$ increases in $\gamma$ and (iii) 100% gains are sufficient to raise the objective: $W_\alpha(1) > W_\alpha^{\text{pre}}$. Unfortunately, these properties do not hold in general.

First, when firms compete in price and varieties are strategic complements, it is indeed true that a merger without efficiency gains raises prices, reduces consumers' surplus and increases the profits of the merged entity and of each outsider (unilateral effects). In our notations: $W_1(0) < W_1^{\text{pre}}$. This property does not hold, however, for total welfare ($\alpha = 1/2$). Even when goods are strategic complements, a merger without efficiency gains can raise total welfare (see Appendix A.1). Second, the monotonicity of the objective function $W_\alpha$ is not guaranteed: Appendix A.3 shows that the total welfare can even decrease with efficiency gains. Third, 100% gains are sometimes not sufficient to offset the unilateral effects; this happens, for instance, with zero (or very small) marginal costs. The Logit case studied in Section 4 yields a less obvious example.

If a threshold $\hat{\gamma}_\alpha$ does not exist, the tradeoff is trivial: irrespective of the magnitude of the efficiency gains, the merger is either beneficial or detrimental. We concentrate on the only cases of interest in which $\hat{\gamma}_\alpha$ exists. For deterministic efficiency gains, say $\gamma = \bar{\gamma}$, the assessment of the merger is straightforward: if $\hat{\gamma}_\alpha > \bar{\gamma}$, the merger is blocked, whereas if $\bar{\gamma} > \hat{\gamma}_\alpha$, the merger is welcomed.

**Decision under uncertainty:** The article investigates the role of uncertainty of efficiency gains in the tradeoff between unilateral effect and efficiency gains. We assume that society is risk neutral with respect to consumers' surplus and firms' profit, i.e. it maximizes $E_{\gamma}(W_\alpha)$.

What happens if antitrust authorities make their decision by comparing the expected gain $E_{\gamma}$ with the threshold $\hat{\gamma}_\alpha$? When the objective is globally convex or concave, such a decision rule may lead to the rejection (resp. acceptation) of welfare enhancing (resp. decreasing) mergers, i.e. to type I (resp. II) errors, as shown in Table 1.

<table>
<thead>
<tr>
<th>$W_\alpha$ is globally convex in $\gamma$</th>
<th>concave in $\gamma$</th>
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<tbody>
<tr>
<td>Merger is blocked ($\hat{\gamma}<em>\alpha &gt; E</em>{\gamma}$)</td>
<td>Type I errors may occur</td>
</tr>
<tr>
<td>Merger is cleared ($\hat{\gamma}<em>\alpha &lt; E</em>{\gamma}$)</td>
<td>No error</td>
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Of course, the relevant comparison is between $\hat{\gamma}_\alpha$ and $\gamma_{\alpha}^{\text{CE}}$, where $\gamma_{\alpha}^{\text{CE}}$ denotes the certainty equivalent of the lottery $\gamma$ for $W_\alpha(.)$, that is, $W_\alpha(\gamma_{\alpha}^{\text{CE}}) = E_{\gamma}(W_\alpha)$. If $\hat{\gamma}_\alpha < \gamma_{\alpha}^{\text{CE}}$, the merger enhances welfare, while if $\hat{\gamma}_\alpha > \gamma_{\alpha}^{\text{CE}}$, welfare is reduced. The computation of
\( \gamma^\text{CE} \) would, however, require the knowledge of the entire distribution of \( \gamma \).\(^6\)

**General results on the consumers’ surplus:**

The first and second derivatives of the consumers’ surplus write respectively

\[
\dot{S}(\gamma) = - \sum p_k(\gamma) D^k = -D' \hat{p} = (V_p)' \hat{p}
\]

and

\[
\ddot{S}(\gamma) = \sum_{k \in N} \sum_{h \in N} \hat{p}_k \hat{p}_h \frac{\partial D^k}{\partial p_h} - \sum_{k \in N} \hat{p}_k(\gamma) D^k = \hat{p}' V_{pp} \hat{p} - D' \hat{p}.
\] (3)

Consumers’ surplus \( S \) depends on \( \gamma \) through the price vector \( p \). The uncertainty on \( \gamma \) translates into uncertainty on \( p \), which tends to make \( S \) convex in \( \gamma \), since the indirect utility is convex in price \((\hat{p}' V_{pp} \hat{p} \geq 0)\). This channel also involves the curvature of \( p(\gamma) \), though. If the pass-through rate increases with \( \gamma \) \((\hat{p} \leq 0)\), the distribution of prices is distorted downwards compared to a constant pass-through rate,\(^7\) which reinforces the direct effect, implying that \( S \) is unambiguously convex. With a decreasing pass-through rate, the price-shifting effect is negative and the sign of \( \ddot{S} \) is indeterminate. In the remainder of the paper, however, we exhibit a number of cases where the first effect dominates.

**General results on firms’ profits:**

For any firm \( K \) (insider or outsider), the first order condition for profit maximization writes \( D^K_m^K + D^K = 0 \), therefore the equilibrium profit can be rewritten as

\[
\Pi^K(\gamma) = (m^K)' D^K = (m^K)' (-D^K_K) m^K.
\]

For a given \( \gamma \), the right-hand side is a quadratic form in \( m^K \) as the matrix \((-D^K_K)\) is symmetric semi-definite positive. This tends to make the profit convex in the efficiency gains. Yet, in general, the mapping of \( \gamma \) into prices can take any form and profits are not generically convex. We detail below the various effects at work.

In Appendix B, we express the second derivative of an outsider’s profit as functions of \( \hat{p}^H \) and \( \hat{p}^H \), where \( H = N \setminus J \) is the set of brands controlled by the other firms:

\[
\ddot{\Pi}^J = (\hat{p}^H)' \Pi^J_{HH} (-\Pi^J_{JJ})^{-1} \Pi^J_{JH} \hat{p}^H + (\hat{p}^H)' \Pi^J_{IH} \hat{p}^H + \Pi^H_{HH} \hat{p}^H.
\] (4)

Due to the concavity of \( \Pi^J \), the first term in (4) is positive. As for consumers’ surplus, it is an heterogeneity effect: other things being equal, the expected profit of an outsider,

\(^6\)In general, the curvature of the objective varies with \( \gamma \). Local properties can, however, be derived. Noting with \( \overline{\gamma} \) the mean value and \( \sigma^2 \) the variance of \( \gamma \), it is readily confirmed that

\[
E(W_\alpha(\gamma) - W_\alpha(\tau)) \sim \frac{1}{2} \overline{W}_\alpha(\tau) \sigma^2
\]

as \( \sigma^2 \) goes to zero. In particular, consider a merger such that \( W_\alpha(\tau) = W_\alpha^{\text{pre}} \) and \( \sigma^2 \) is small. The merger should be accepted if \( \overline{W}_\alpha(\tau) > 0 \), rejected if \( \overline{W}_\alpha(\tau) < 0 \).

\(^7\)Indeed, the change of variables formula \( dp = |\hat{p}(\gamma(p))| d\gamma = -\hat{p}(\gamma(p)) d\gamma \) shows that the distribution of \( p \) is more concentrated at the bottom when \( \hat{p}(\cdot) \) is increasing as \( \gamma(p) \) is decreasing in \( p \).
who can choose prices along its best reply map, increases when its competitors’ prices vary randomly around a given mean. Yet, when the prices of J’s competitors fluctuate, there is, also, a direct effect of J’s profit $\Pi_J$ through the curvature of J’s demand function w.r.t. its competitors’ prices. Finally, a price-shifting effect is also present: intuitively, $\Pi_J > 0$ and the sign of this effect is positive when the distributions of the competitors’ prices are distorted upwards ($\hat{p}_H > 0$).

As regards the curvature of the insider’s profit function, the same terms are at play (see the first line of (19) in Appendix B). There are, however, two additional terms coming from the direct effect of $\gamma$ through $c_I$ (second line of (19)). The first one of the last two writes $(c^I)'V_H (-\Pi_H)^{-1} V_Hc^I$, which is always nonnegative. It is a monopoly effect: a monopoly’s profit is convex w.r.t its constant marginal cost. That is, other things being equal, the insider would like its marginal cost $c_I$ to fluctuate rather than being constant. Under oligopolistic competition, however, these fluctuations bring about reactions in outsiders’ prices, which generate the other effects.

3 Linear demand system

The linear demand system is widely used to model price competition. Theorists often refer to it when they need a closed form equilibrium. Empiricists also use it to estimate demand (see, for instance, Pinkse and Slade (2004)). Here we use the most general version of the linear demand model. We impose the minimal restrictions necessary to ensure strategic complementarity and the existence and uniqueness of a Nash equilibrium. Unlike earlier works (Barros and Cabral (2001) and Section 5.4 of Motta (2004)), we do not limit our analysis to symmetric costs or demand systems. In particular, we do not restrict to Bertrand-Shubik demands (see Shubik and Levitan (1980)).

The consumers’ surplus rewrites as

$$V(p) = -A'p - \frac{1}{2}p'Bp,$$

where $A$ is a positive (constant) vector of size $n$ and $B = (b_{hk})_{h,k \in N}$ is a (constant) matrix of size $n$.

According to Remark 1, the matrix $B$ must be symmetric negative semi-definite. In what follows, we make the additional assumption that $B$ is nonsingular. These conditions are satisfied when $B$ is a symmetric diagonally dominant matrix with: $b_{hh} < 0$ and $\sum_{k \neq h} |b_{hk}| < |b_{hh}|$ for all $h = 1$ to $n$ (see Jacob (1990), pp. 240-241). These inequalities imply that if the prices of all varieties increase by the same amount, then all demands decrease. We do not, however, restrict the analysis to this particular case. Throughout, we shall rely only on $B$ being symmetric negative definite. The demand function, given by Roy’s identity, is

$$D(p) = -V_p = (D^I, D^{J_1}, \ldots, D^{J_r})' = A + Bp.$$

Let $D^K_I$ denote the submatrix $(b_{hk})_{h,k \in K}$. After the merger, the first order conditions for firm $K = I, J_1, \ldots, J_r$ write

$$D^K + D^K_I m^K = 0.$$

\[ (5) \]
Since the matrices $D^K_K$ are definite negative, the second-order conditions of the firms’ maximization problems are satisfied. In Appendix C.1, we solve for the best reply maps, which are implicitly given by (5), and for the equilibrium prices. Next, we turn to the main result of this section.

**Proposition 1.** When the demand system is linear in prices, all equilibrium prices are linear functions of $\gamma$ (that is, $\bar{p} = 0$) and, therefore, consumers’ surplus is strictly convex in $\gamma$. In addition, all profit functions (that is, of the merged entity as well as of each outsider) are convex in $\gamma$. As a result, $W_\alpha$ is strictly convex in $\gamma$.

**Proof.** If (for all $h, k$) $D^h$ is linear in $p_k$, then the first order condition system is linear in prices and in $\gamma$. Therefore the equilibrium price is linear in $\gamma$. (Its expression is given by (21) in Appendix C.1.) Equation (3) and $\ddot{p} = 0$ yield: $\ddot{S} = (p')' V_{pp} p$. Since $\dot{p}$ is not zero\(^8\) and $V_{pp} = -B$ is symmetric positive definite, we have: $\ddot{S} > 0$.

Next, we prove the convexity of the profit functions. Using the first order conditions (5), the equilibrium value of the profit of firm $K$ writes:

$$\Pi^K = (m^K)'D^K = (m^K)'(-D^K_K) m^K. \quad (6)$$

Therefore, using the symmetry of the constant matrix $D^K_K$ and the linearity of margins in $\gamma$, $\Pi^K = 2 (m^K)'(-D^K_K) \dot{m}^K$ and

$$\ddot{\Pi}^K = 2 (\dot{m}^K)'(-D^K_K) \ddot{m}^K. \quad (7)$$

Since the matrix $(-D^K_K)$ is symmetric positive, we have that $\ddot{\Pi}^K \geq 0$ for all $K$. \(\Box\)

Proposition 1 emphasizes that in the linear case the objective function of the antitrust authorities (which can be any combination of consumers’ surplus and profits) is convex. Therefore, whenever the linearity assumption is realistic, the authorities should not be afraid of uncertainty. On the contrary, uncertainty should be welcomed and firms’ should be encouraged to provide verifiable evidence about the variance of the efficiency gains and not only about its mean value. This result is obtained for any structure of the industry. That is, the merged entity as well as any outsider might control several brands. No symmetry assumption is required.

Proposition 1 does not rely on the assumption that efficiency gains are proportional to a scalar $\gamma$. Indeed, it generalizes to multidimensional efficiency gains, as shown in the following remark (proved in Appendix C.3).

**Remark 2.** Let $\gamma = (\gamma_1, \ldots, \gamma_l)'$, and assume that after the merger $c_i(\gamma) = (1 - \gamma_i)c_i$, $i \in I$. The consumers’ surplus and the profit of each firm are convex functions of the efficiency gain vector $\gamma$.

To end this section, we discuss the monotonicity of $W_\alpha(\gamma)$. The linearity of demand does not ensure that $W_\alpha$ is an increasing function of $\gamma$. Yet, provided that a simple technical condition is satisfied, Lemma 1 (see Appendix C.2 for a proof) shows that $W_1$ in increasing. That is, consumers unambiguously benefit from an increase in $\gamma$.

\(^8\)See (22) in Appendix C.1.
Lemma 1. Suppose that goods are strategic complements and the best reply map is a contraction. Then all equilibrium prices are decreasing with $\gamma$ ($\dot{p} \leq 0$). Consumers’ surplus is an increasing function of $\gamma$. Outsiders’ profits decrease with $\gamma$.

Yet, even under the assumptions of the lemma, the insider’s profit may decrease with $\gamma$, as shown by the example detailed in Appendix A.2. Therefore, if a weight large enough is put on firms, $W_\alpha$ can be decreasing. Looking more closely at total welfare ($\alpha = 1/2$), we obtain:

$$2\dot{W}_{1/2} = \dot{S} + \sum_{K} \dot{\Pi}^K = -D'\dot{p} + 2\sum_{K} (D^K)'\dot{m}^K$$

$$= -(D^I)'\dot{p}^I + 2(D^I)'\dot{m}^I + \sum_{J \neq I} (D^J)'\dot{p}^J.$$

(8)

Under the assumptions of the Lemma 1, the last term in (8) is negative: as far as the goods controlled by the outsiders are concerned, outsiders lose more than consumers gain from a higher efficiency of the new entity. The first term being always positive, the sign of $\dot{W}_{1/2}$ is indeterminate. Intuitively, total welfare should always increase with efficiency gains. Appendix A.3 shows that this is not true, even when the products are strategic complements and the best reply map is a contraction.

The linearity of demand is, therefore, not sufficient to rule out counter-intuitive situations. In contrast to the surprising properties of the first derivatives, Proposition 1 yields clear-cut results for the curvature of consumers’ surplus and profits. The linearity of demand implies that competition authorities should love the risk related to the magnitude of efficiency gains (irrespective of the relative weights assigned to consumers and firms). Next, we turn to more general demand functions.

4 Mergers to monopoly

In this section, we examine the attitude towards risk for mergers that lead to a monopoly ($I = N$), that is, the case where unilateral effects are expected to be the strongest. On the contrary, Section 5 posits Bertrand competition with homogenous products before and after the merger, that is, the weakest unilateral effects. Both sections can thus be thought of as complementary. These restrictions allow to carry out the analysis for general demand functions.

Lemma 2 (Convexity of monopoly profits). The equilibrium monopoly profit is a convex function of the magnitude of the efficiency gains.

Proof. The monopoly’s profit is $\Pi^I(\gamma) = \max_p \left[ (p - c(\gamma))' D(p) \right]$. As $c(\gamma)$ is an affine function of $\gamma$, $\Pi^I(\gamma)$ is convex in $\gamma$ as a maximum of affine functions. \end{proof}

9The middle term, which is the derivative of the insider’s profit, should intuitively be positive. Yet, it can be negative, as shown in Appendix A.2.
Lemma 2 shows that for any demand functions, the monopoly’s profit is convex in \( \gamma \). That is, firms are willing to take risks to create a monopoly. For a given mean, say zero, they are ready to accept a large variance.

To derive specific results on consumers’ surplus for general demand functions, we postulate a symmetric environment.

**Assumption 1 (Symmetry).** Before and after the merger, consumers perceive brands symmetrically. All marginal costs of production are identical.

Given the symmetry of demand functions in \( p_1, \ldots, p_n \), it is readily confirmed that at the optimum: \( p_1 = \ldots = p_n \). Therefore there is no loss of generality in assuming that the merged entity maximizes its profit under the constraint that \( p_1 = \ldots = p_n = p \). Let \( \Phi \) be the Chamberlin’s DD curve, i.e. the demand function for one product when all prices are equal (see e.g. Anderson, de Palma, and Kreider (2001))

\[
\Phi(p) = D^i(p_1, \ldots, p_n).
\]

Then the post-merger profit function is:

\[
\Pi^f(p) = n[p - (1 - \gamma)c] \Phi(p).
\]

Let \( p(\gamma) \) be the post-merger monopoly price. Hereafter, we continue to note with a dot derivatives with respect to \( \gamma \) (e.g. \( \dot{p} \)) and with a prime derivatives w.r.t. the price (e.g. \( \Phi' \)).

**Definition 1.** Let \( \varepsilon(p) = -\frac{p \Phi'}{\Phi} \) denote the price elasticity of the demand function \( \Phi \) and let \( E(p) = \frac{p \Phi''}{\Phi'} \) denote the price elasticity of \( \Phi' \) the derivative of the demand function. Finally let \( \Psi(p) = E/\varepsilon \) be the ratio of both elasticities.

Appendix D.1 shows that \( \Psi(p) \) is nothing but the elasticity of the slope of the inverse demand curve, evaluated at \( \Phi(p) \). Appendix D.2 presents the first-order condition of the monopoly problem and checks that the second-order condition is satisfied if and only if \( \Psi(p(\gamma)) + 2 > 0 \). Appendix D.3 computes the second-order derivatives of consumers’ surplus and firms profits, and establishes the following result.

**Lemma 3.** The welfare function \( W_\alpha \) is strictly convex at \( \gamma \) if and only if

\[
\frac{\alpha \Psi'}{2 + \Psi} + \left[ \alpha + (1 - \alpha)(2 + \Psi) \right] \frac{\varepsilon}{p(\gamma)} > 0,
\]

(9)

where \( \Psi, \Psi' \) and \( \varepsilon \) are evaluated at \( p(\gamma) \).

Because of the second-order condition, the bracketed term in (9) is strictly positive, which yields Proposition 2.

**Proposition 2.** If \( \Psi \) is nondecreasing with \( p \), then, for all \( 0 \leq \alpha \leq 1 \), the objective \( W_\alpha \) is globally convex in \( \gamma \).
Proposition 2 characterizes a large class of demand functions for which \( W_\alpha \) is globally convex. In consequence, for this class, for a given mean, uncertainty about efficiency gains raises the expected value of \( W_\alpha \).

As a first illustration of Proposition 2, suppose goods are homogenous and demand is given by \( (a - bp)^\beta \), with \( \beta b > 0 \). In this case, the function \( \Psi \) is constant and equal to \( (1 - \beta)b/\beta \). When \( \beta / \in [-1, 0] \), the second-order condition \( \Psi + 2 > 0 \) is satisfied and Proposition 2 applies. Similarly, if \( \Phi = A \exp[a - bp] \) \( (A, b > 0) \), then \( \Psi = -1 \).

Next, we present examples with imperfect substitutes. First, we consider constant elasticity of substitution (CES) demand system (see Vives (1999), pages 147-148)

\[
D_i(p_1, \ldots, p_n) = \left( \beta \theta \right)^{1/(1-\beta\theta)} \frac{p_i^{1/(\beta-1)}}{\left(p_1^{\beta/(\beta-1)} + \ldots + p_n^{\beta/(\beta-1)}\right)^{(1-\theta)/(1-\beta\theta)}},
\]

with \( 0 < \beta\theta < 1, \beta \leq 1 \) and \( \theta < 1 \). The elasticity of substitution, defined as the derivative of \( \ln(D_i/D_j) \) with respect to \( \ln(p_i/p_j) \), is \( 1/(1 - \beta) \). For \( \beta = 1 \), goods are homogenous.

Corollary 1 (Constant elasticity of substitution). If the demands \( D_i \) exhibit a constant-elasticity of substitution, then \( \Psi' = 0 \), and therefore \( W_\alpha \) is globally convex in \( \gamma \), for all \( 0 \leq \alpha \leq 1 \).

Proof. In that case,

\[
\Phi(p) = n^{(1-\theta)} \left( \beta \theta \right)^{1/(1-\beta\theta)} p^{-1/(1-\beta\theta)},
\]

whence we deduce that: \( \Psi = -2 + \beta\theta > -2 \). The function \( \Psi \) is constant and the second-order conditions are satisfied. Proposition 2 applies.

Second, we turn to the symmetric Logit demand system with an outside option\(^{10}\)

\[
D_i(p_1, \ldots, p_n) = \frac{\exp(-p_i/\sigma)}{\exp(-p_0/\sigma) + \ldots + \exp(-p_n/\sigma)}
\]

where \( \sigma > 0 \) and \( p_0 \) is the price of the outside good. The Chamberlin’s demand is

\[
\Phi(p) = \frac{\exp(-p/\sigma)}{\exp(-p_0/\sigma) + n \exp(-p/\sigma)}.
\]

A simple change of variables shows that the monopoly price \( p(\gamma) \) is such that \( p(\gamma)/\sigma \) only depends on \( c/\sigma \) and \( p_0/\sigma \). After the merger, the market share covered by the monopolist is \( n\Phi(p(\gamma)) \).

Lemma 4. With a Logit symmetric demand, the objective \( W_\alpha \) is a convex function of \( \gamma \) if and only if the market coverage by the monopolist is lower than \( 1/(2\alpha) \).

\(^{10}\)See Werden and Froeb (1994) for a thorough study of mergers relying on the Logit model.
Proof. It is readily confirmed that

\[ \Psi = n \exp[(p_0 - p)/\sigma] - 1, \quad \Psi' = (-n/\sigma) \exp[(p_0 - p)/\sigma], \quad \varepsilon = \frac{1/\sigma}{n \exp[(p_0 - p)/\sigma] + 1}. \]

Straightforward computations show that (9) writes:

\[ \exp[(p - p_0)/\sigma] > n(2\alpha - 1), \]

which is equivalent to \( n\Phi(p(\gamma)) < 1/(2\alpha) \).

Since market coverage is lower than 100 %, the following corollary obtains.

Corollary 2. Suppose \( \alpha \leq 1/2 \). With a Logit symmetric demand, the objective \( W_\alpha \) is globally (strictly) convex in \( \gamma \).

Corollary 2 implies in particular that the total welfare is globally convex in \( \gamma \) (case \( \alpha = 1/2 \)). Next, we analyze the curvature of the consumers’ surplus \( S \) (case \( \alpha = 1 \)). Here, we get only local results around \( \gamma_1 \). Recall that \( \gamma_1 \) is the critical threshold for efficiency gains associated with the consumers’ surplus. Appendix D.4 shows that

Corollary 3. With a Logit symmetric demand, the consumers’ surplus is locally convex at \( \gamma_1 \) if and only if

\[ \frac{p_0}{\sigma} - \frac{c}{\sigma} \leq \frac{2n}{2n - 1} - \ln n. \quad (10) \]

The set of parameters defined by condition (10) must be compared to the admissible region, namely the set of \((c/\sigma, p_0/\sigma)\) for which efficiency gains are potentially able to offset unilateral effects. The admissible region, depicted on Figure 1, is located below the curve \( \gamma_1(c/\sigma, p_0/\sigma) = 1 \) corresponding to 100 % gains. For efficiency gains to be able to compensate the unilateral effect, the outside good must be sufficiently attractive \((p_0/\sigma\) not too high) and the potential for cost reduction must be sufficiently high \((c/\sigma\) not too small). The straight line \( \tilde{S}(\gamma_1) = 0 \) splits the admissible region in two areas where the consumers’ surplus is locally concave (resp. convex) at \( \gamma_1 \). This example shows that no general conclusion as to the curvature of the consumers’ surplus can be drawn.

Lemma 5. Efficiency gains can offset the unilateral effect if only if

\[ p_0 \leq p^+(c) + \sigma \ln \left[ \frac{p^+(c) - (\sigma + c)}{(\sigma n + (n - 1)c) - (n - 1)p^+(c)} \right] \quad (11) \]

where

\[ p^+(c) = \frac{\sigma + c}{2} + \sqrt{\left(\frac{\sigma + c}{2}\right)^2 + \frac{\sigma c}{n - 1}} \quad (12) \]

Proof. See Appendix D.4

\[ \gamma_1 = 1 \text{ and } \tilde{S}(\gamma_1) = 0 \text{ intersect at } \sigma/\sigma = 2/3 \text{ and } p_0/\sigma = 2 - \ln 2. \] As a consequence, for any admissible set of parameter \((c, p_0, \sigma)\) satisfying \( p_0 \leq 1.3 \sigma \), the consumers’ surplus is locally convex at \( \gamma_1 \).
5 Mergers without any unilateral effect

After having investigated mergers leading to monopolies, we turn to situations where, on the contrary, the post-merger environment is highly competitive. To this end, we assume Bertrand competition with homogeneous goods, the fiercest kind of competition. In particular, if firms share the same technology, prices are driven down to the common marginal cost of production. Yet, thanks to a merger, a firm can obtain some market power if its marginal cost becomes lower than that of its competitors. Since we are interested in post-merger competitive environments, we impose that, after the merger, each firm exerts competitive pressure on the others. That is, we rule out drastic efficiency gains and losses.

Formally, let $c$ be the common pre-merger constant marginal cost. After the merger, the marginal cost of the merged entity amounts to $(1 - \gamma)c$ with $\gamma \leq 1$. The marginal cost of the outsider(s) remains at $c$. We note $p^m(\cdot)$ the (unconstrained) monopoly price as a function of the marginal cost: $p^m(c) = \arg \max_p (p - c) D(p)$ and we adopt the following assumption:

**Assumption 2.** Efficiency gains and losses are (almost) never drastic, i.e. the inequalities

\[ c \leq p^m((1 - \gamma)c) \quad \text{and} \quad (1 - \gamma)c \leq p^m(c) \]

hold (almost) everywhere on the support of the random variable $\gamma$.

Suppose that $n$ mono-product and symmetric firms initially compete à la Bertrand and consider a merger involving $l$ insiders, with $l \leq n - 1$. Absent efficiency gain or
loss, the consumers’ surplus remains unchanged, as well as the industry’s profit (still at zero). There is no unilateral effect. In our notations: \( \tilde{\gamma}_1 = 0 \). Moreover, when there are two or more outsiders \( (l \leq n - 2) \) and as long as Assumption 2 holds, the merger leaves the price unchanged, \textit{irrespective of the sign and magnitude of efficiency gains}. In particular, efficiency losses, if any, play no role. Society benefits from efficiency gains (which are, however, entirely kept by the merged entity), but cannot suffer from efficiency losses. Things are different when the merger creates an asymmetric duopoly \( (l = n - 1) \). Here, consumers and society as a whole suffer from efficiency losses. We now concentrate on this case.

Under Assumption 2, all sales are made by the firm with the lowest marginal cost and the post merger equilibrium price is the largest one: \( p(\gamma) = \max \{c, (1 - \gamma)c\} \). Figure 2 depicts the post-merger firms’ profits, consumers’ surplus and total welfare \( 2W_{1/2} \) as functions of \( \gamma \). From the expression of \( p(\gamma) \), it is transparent that the price cannot decrease and therefore consumers’ surplus cannot increase. As soon as there is a positive probability that \( \gamma < 0 \), it is even worse: consumers’ surplus decreases in expectation. That is, in this set-up, efficiency gains are never passed on to consumers, but efficiency losses are: the pass-through rate is asymmetric. Such a merger would not be allowed by antitrust authorities, should they care only about consumers’ surplus \( (\alpha = 1) \). On the other hand, as soon as authorities recognize that firms benefit from the merger \( (\alpha < 1) \), they face a tradeoff between increased efficiency and creation of market power.

![Figure 2: Profits, consumers’ surplus and total welfare for a Bertrand duopoly](image)

**Proposition 3.** \textit{Under Bertrand competition and Assumption 2, there exists a threshold value } \( \hat{\alpha} \in (1/2, 2/3) \text{ such that, for all } \alpha \leq \hat{\alpha}, \text{ any merger creating a duopoly and satisfying } \mathbb{E}\gamma = 0 \text{ raises the expected value of the objective function } W_{\alpha}. \text{ Formally}

\[
\mathbb{E}_{\gamma} W_{\alpha} \geq W_{\alpha}(\mathbb{E}\gamma) = W_{\alpha}(0) = W_{\alpha}^{\text{pre}}.
\]
Proposition 3 establishes that if a weight large enough is put on firms and if efficiency gains are zero in expectation ($\mathbb{E} \gamma = 0 = \hat{\gamma}_1$), then uncertainty of efficiency gains is welcomed. The proof, in Appendix E, shows that $\hat{\alpha} = [D(c) + D(p^m(c))]/[2D(c) + D(p^m(c))]$.

The inequality (13) does not follow from Jensen’s lemma since the objective function $W_\alpha$ is not globally convex in $\gamma$ (see Figure 2). It may, however, be locally convex around $\hat{\gamma}_1 = 0$. More precisely, Figure 2 shows that the industry’s profit $\Pi^{\text{Insider}} + \Pi^{\text{Outsider}}$ has a convex kink at 0, while the consumers’ surplus has a concave kink at 0. It is easy to check that the right and left derivatives of $W_\alpha$ at 0 are $(1 - \alpha)cD(c)$ and $(2\alpha - 1)cD(c)$ respectively. Therefore, $W_\alpha$ has a convex kink at 0 if and only if $\alpha \leq 2/3$. It follows that, for any $\alpha \leq 2/3$, it exists $\eta > 0$ such that, if the random variable $\gamma$ has zero mean and takes its values in $[-\eta, \eta]$, then the merger raises $\mathbb{E} \gamma (W_\alpha)$. Proposition 3 introduces a threshold $\hat{\alpha} \leq 2/3$ such that this “convexity-like” property is valid without any restriction on the support of $\gamma$ (besides Assumption 2).

Finally, note that $\hat{\alpha}$ is always (strictly) larger than 1/2: from the ex ante point of view, a merger with zero efficiency gains in expectation raises the standard welfare criterion, which puts equal weights to firms and consumers.

6 Mergers under Cournot competition

In this section, we examine the effects of uncertainty of efficiency gains for mergers when firms compete à la Cournot. Costs are asymmetric and mergers under consideration involve an arbitrary number of parties, $2 \leq l \leq n$. All notations used below have been introduced in Section 2.

After the merger, the $r$ outsiders use the same technology as before, while the merged entity enjoys a marginal cost of $(1 - \gamma)c^I$, where $c^I$ denotes the marginal cost of the merged entity absent efficiency gains ($\gamma = 0$). In the spirit of Farrell and Shapiro (1990), we distinguish cost reductions due to the reallocation of production, which lead to $c^I$, from the ones linked to efficiency gains, which lead to $(1 - \gamma)c^I$. In the Cournot model with constant return to scale, a merger of two (or more) asymmetric firms should, in principle, lead to the closing of the less efficient insider(s). In that case, $c^I$ would be equal to the smallest marginal cost among insiders’. Yet, such a rationalization of the production might not be achievable by the merging firms, so we assume $c^I \geq \min_{i \in I} c_i$ but we do not a priori impose equality.

The inverse demand curve is denoted $P(Q)$, where $Q$ is the sum of the quantities produced by all firms. Let $Q(\gamma)$ denote the equilibrium quantity after the merger. From now on, a prime is used for derivatives with respect to quantity (e.g. $P'$), while the dot still indicates a derivative w.r.t. $\gamma$ (e.g. $\dot{Q}$). The elasticity $\Theta(Q) = QP''(Q)/P'(Q)$ of the slope of the inverse demand function plays a crucial role in the analysis. When evaluated at $Q = \Phi(p)$, it corresponds to the function $\Psi(p)$ used in Section 4. The assumption $\Theta + 2 > 0$ ensures that the maximization problem of each firm is concave (for any configuration of the industry). By adding the f.o.c. of each firm, we check that $Q(\gamma)$ is the unique solution to

$$(r + 1)P(Q) + QP'(Q) = (1 - \gamma)c^I + \sum_{j \in J} c_j, \quad (14)$$

17
where $J$ denotes the set of outsiders. Lemma 6 provides a necessary and sufficient condition for efficiency gains to be able to compensate unilateral effects. This condition only depends on the pre-merger price and on the sum of the marginal costs of the insiders before the merger.

**Lemma 6** (Farrell and Shapiro (1990), Proposition 1). Let $p^\text{pre}$ denote the pre-merger equilibrium price. It is possible that efficiency gains offset the unilateral effects of the merger (from consumers’ point of view) if and only if

$$\sum_{i \in I} \frac{p^\text{pre} - c_i}{p^\text{pre}} \leq 1.$$  \hfill (15)

**Proof.** The pre-merger equilibrium quantity $Q^\text{pre}$ is solution to $nP(Q) + QP'(Q) = \sum_{k \in N} c_k$. Noting $p^\text{pre} = P(Q^\text{pre})$ the pre-merger price, we get: $Q^\text{pre}P'(Q^\text{pre}) = \sum_{k \in N} c_k - np^\text{pre}$. Using (14), it is straightforward that the price after the merger is exactly equal to $p^\text{pre}$ if and only if $c^I\gamma = (l - 1)p^\text{pre} - \sum_{i \in I} c_i + c^I$. The condition $\gamma \leq 1$ leads to (15).

For the inequality (15) to hold, insiders should not be too efficient\textsuperscript{12} before the merger. Absent efficiency gains, the merger harms consumers (See Farrell and Shapiro (1990), Proposition 2). That is, the merger is harmful (due to unilateral effects) but if (15) holds, amendable (thanks to efficiency gains). The critical threshold $\hat{\gamma}_1$ is, then, given by $c^I\hat{\gamma}_1 = -(p^\text{pre} - c^I) + \sum_{i \in I} p^\text{pre} - c_i$.

The effect of the uncertainty about $\gamma$ on the expected value of the consumers’ surplus remains linked to the curvature of $S$ with respect to $\gamma$.

**Lemma 7.** The consumers’ surplus is locally convex at $\gamma$ if and only if

$$1 - \frac{Q\Theta'}{r + 2 + \Theta} > 0.$$  \hfill (16)

Lemma 7, proved in Appendix F, parallels Lemma 3 of Section 4. Both results coincide exactly when we look at the consumers’ surplus for mergers that create a monopoly (replace $r$ by 0 in Equation (16) and $a$ by 1 in Equation (9)). This similarity is not coincidental, as a monopoly selling an homogeneous good can equivalently maximize in price or quantity. Recall, however, that in this section (contrary to Section 4), firms do not share the same technology and the mergers under consideration do not necessarily create a monopoly.

**Proposition 4.** If $\Theta$ is non increasing in $Q$ and larger than -2, then the consumers’ surplus is globally convex in $\gamma$ ($\hat{S} > 0$).

Proposition 4, which follow directly from condition (16), applies to a large class of demand functions. A subclass of interest consists of demand functions such that the function

\textsuperscript{12}In particular, if all marginal costs equal $c$, then (15) writes $\frac{p^\text{pre} - c}{p^\text{pre}} \leq 1/l$. If, in addition, demand is linear ($P = a - bQ$), then (15) becomes $c \geq (l - 1)a/(n + l)$, which defines a lower bound for $c$.
Θ is constant and larger than -2 (to ensure that the second-order conditions hold). This family is described by three parameters:

\[ P(Q) = \max\left\{ a + \frac{b}{1+\theta} Q^{1+\theta}; 0 \right\}, \]

with \(-2 < \theta \neq -1\) and \(b < 0\). If \(\theta > -1\), the intercept \(a\) must be positive to get a positive demand. If \(\theta < -1\), \(a\) must be negative to guarantee that the demand tends to zero as the price goes to infinity. For \(\theta = -1\), we define

\[ P(Q) = \max\{a + b \log(Q); 0\}, \]

with \(a > 0\) and \(b < 0\). This family contains demands with constant elasticity, \(P(Q) = a - b \varepsilon Q^{-1/\varepsilon}\), with \(\varepsilon > 1\) (take \(\theta = -(1 + 1/\varepsilon)\)). For all demands in this family, the function \(\Theta\) is equal to the constant \(\theta\) and Proposition 4 applies. Another particular case encompassed in this family is the linear demand (\(\theta = 0\)) for which the curvature of the profit functions can also be determined.

**Lemma 8.** If demand is linear, \(P = a - bQ\), with \(a, b > 0\), then, for all \(0 \leq \alpha \leq 1\), the objective function \(W_\alpha\) is globally convex in \(\gamma\).

**Proof.** The equilibrium profit of any firm writes \(\Pi^k = b (q^k)^2\) where \(q^k\) is linear w.r.t. marginal costs. Therefore, all profit functions are convex in \(\gamma\). Proposition 4 shows that \(S\) is convex, therefore \(W_\alpha\) is convex as a combination of convex functions. \(\square\)

To summarize, in the Cournot environment with asymmetric costs and an arbitrary number of outsiders, the post-merger consumers' surplus appears to be a convex function of the efficiency gains for a large family of demand functions. The curvature of the profits function is harder to characterize. For linear demand, profits are convex.

### 7 Lessons for merger control

The analysis of horizontal mergers hinges on a tradeoff between unilateral effects and efficiency gains. This article examines the role of uncertainty (on efficiency gains) in such a tradeoff. Common wisdom is that antitrust authorities should be very cautious about random gains. Our results show that dismissing efficiency gains on the sole ground that they are uncertain would not be theoretically founded. Indeed, the attitude towards uncertainty depends on the curvature of the social objective function. We have exhibited a number of situations where the objective is convex in the efficiency gains, implying that competition authorities should welcome the risk for a given expectation of efficiency gains.

In particular, we have shown that the linearity of demand ensures the convexity of profits and consumers' surplus in an otherwise general price competition setup. When mergers create a monopoly, the profit is always convex in the efficiency gains; the consumers surplus is convex for CES demand systems; total welfare is convex for a Logit demand. When the merger creates a duopoly and goods are homogenous, a convexity-like result holds for total
welfare (for any demand function). When firms compete in quantity with asymmetric costs, the consumers’ surplus is convex in efficiency gains for a large class of demand functions. This is not to say that the convexity property is general. The Logit example shows that, for a merger to monopoly, the consumers’ surplus can be locally convex or concave, depending on the precise value of the underlying structural parameters.

Our results should not be misunderstood; they do not imply that antitrust enforcers should keep themselves in the dark. They, of course, should use all available information to reduce *ex ante* uncertainty. In particular, they should require that the parties substantiate their efficiency claims. Information about the distribution of efficiency gains is valuable because it can help to reduce the probability of type I or II errors. However, at the end of the merger assessment, authorities are left with inevitable uncertainty about future costs; at this point, under the conditions identified in the paper, they should be ready to take risks in accepting mergers with unknown efficiencies.

The article provides some guidance for empirical merger analysis. Our results imply that functional specifications may entail implicit restrictions on the attitude towards risk regarding the magnitude of efficiency gains. When two different demand systems fit the data equally well, welfare could be convex in one case but concave in the other. In the former case, it is more likely that efficiency gains are found to compensate the unilateral effect than in the latter case.

Firms and antitrust authorities commonly produce econometric studies for judges ruling on merger cases. Baker and Rubinfeld (1999) offers a thorough survey of empirical methods used in antitrust and merger litigation. They compare reduced forms and structural approaches for the estimation of pass-through rates and for the simulation of post-merger prices. In principle, structural studies make it possible to identify post-merger conduct. Such studies estimate oligopolistic models and use these estimations to simulate the post-merger market equilibrium under various assumptions. A huge body of empirical literature has grown, estimating unilateral effects for mergers in various industries: see, among others, Nevo (2000) in the U.S. ready-to-eat cereals industry, Pinkse and Slade (2004) in the U.K. beer industry, and Ivaldi and Verboven (2005) for the European truck industry. These studies allow to estimate critical thresholds for the magnitude of efficiency gains, i.e. lower bounds for gains to offset unilateral effects.

Demand and cost equations in empirical studies, of course, include error terms that account for unobserved variables and misspecifications. These terms, however, do not account for the uncertainty about efficiency gains which is of a different nature: efficiency gains affect future costs and are therefore impossible or extremely difficult to estimate *ex ante*. Empirical studies usually provide *point* estimates for the threshold above which efficiency gains offset unilateral effect under the implicit assumption that gains are certain. Under uncertainty, when the social objective is concave, the use of such estimates may lead to authorize welfare-deteriorating mergers (type II errors); when the social objective

13 In particular, they discuss Ashenfelter, Ashmore, Baker, and McKernan (1998) which relies on a linear regression model to estimate the pre-merger pass-through rates in the Staples and Office Depot case. Focarelli and Panetta (2003) also uses linear regression models to analyze, in the Italian bank sector, the tradeoff between unilateral effects and efficiency gains.
is convex it might lead to block welfare-enhancing mergers (type I errors). For instance, Pinkse and Slade (2004) use a linear demand system to evaluate the impact of mergers in the U.K. brewing industry. They write page 641 that “the costs of the merging firms would have to fall by about 20% to just offset the increase in market power.” They conclude that “a reduction of the required magnitude would not have been possible”. Their calculation is done under the implicit assumption that $\gamma$ is perfectly known. Under the chosen specification, we have seen that consumers’ surplus is convex. If they had allowed $\gamma$ to be distributed over an interval, the convexity of $S$ would have implied a requirement for the expectation of efficiency gains lower than 20%. Our results suggest that presenting the objective as a function of future costs might yield a more accurate idea of the market after the merger.

It would be of interest to extend our analysis to situations where the merging firms do not know the exact value of the post-merger cost, but have a better information about its distribution than the authorities (in the spirit of Besanko and Spulber (1993) or Lagerlöf and Heidhues (2005)). Our results might help to better understand the information disclosure issue in such cases. Although this article focuses on horizontal mergers, the argument applies to any situations with a market power - efficiency tradeoff. Williamson (1968) points dissolution, vertical and conglomerate mergers. Many other practices under the scrutiny of antitrust authorities (e.g. joint-ventures, collusion, bundling) give rise to such tradeoffs. In all these cases, the curvature of the social objective plays a crucial role whenever efficiencies are uncertain.

References


Appendix

A Some properties of mergers under price competition

In this Appendix, we present three counter-intuitive properties of mergers when firms compete in prices. The examples below rely on linear demand systems, which are analyzed systematically in Section 3 and Appendix C. For all examples, we have checked that (i) the demand is consistent with a well-posed consumer’s problem (see Remark 1), (ii) there exists a unique Nash equilibrium, (iii) all varieties are produced at the pre- and post-merger equilibria, (iv) the varieties are strategic complements and (v) the best reply map is a contraction. (See Lemma 1, Appendix C and Footnote 17 for more details.)

A.1 A merger without efficiency gains can raise total welfare

When firms compete in price and varieties are strategic substitutes, it is well-known that a merger can reduce price and raise total welfare.\(^{14}\) We show in this section that this may also happen for substitute goods under strategic complementarity. The intuition goes as follows. A merger might involve relatively inefficient firms whose goods are (relatively) not valued much by consumers and (relatively) distinct from outsiders’ goods. As a result of the merger, all prices increase but the outsiders’ prices increase less than insiders’ ones. Therefore, while the quantities produced by the insiders decrease, some of the quantities produced by the outsiders might increase. If one of the outsider is very efficient, the merger

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\(^{14}\)For instance, two firms selling complementary goods exert a negative price externality on each other. If they merge, they reduce prices and at the same time earn more profit. This property is known as the Cournot effect.
shifts production from inefficient firms to an efficient one, which reduces total costs of production. This gain might be greater than consumers’ loss due to higher prices.

To illustrate, take \( n = 3 \) monoproduct firms and consider the following demand system:

\[
\begin{align*}
D^1 &= 10 - p_1 + \frac{1}{2}p_2 + \frac{1}{20}p_3 \\
D^2 &= 10 + \frac{1}{2}p_1 - p_2 + \frac{1}{20}p_3 \\
D^3 &= 10,000 + \frac{1}{20}p_1 + \frac{1}{20}p_2 - p_3.
\end{align*}
\]

Assume that the marginal costs of production are

\[ c_1 = c_2 = 9 \quad \text{and} \quad c_3 = 0. \]

The pre-merger Nash equilibrium is:

\[
p^{\text{pre}} = \begin{pmatrix} 179.633 \\ 179.633 \\ 5008.98 \end{pmatrix}, \quad D^{\text{pre}} = \begin{pmatrix} 170.633 \\ 170.633 \\ 5008.98 \end{pmatrix}, \quad \Pi^{\text{pre}} = \begin{pmatrix} 29,115.5 \\ 29,115.5 \\ 25,089,897.0 \end{pmatrix}.
\]

Consumers’ surplus is \( S^{\text{pre}} = -37,622,304.2 \), while total welfare is \( W^{\text{pre}}_{1/2} = -12,474,176.1 \). Suppose that firms 1 and 2 merge without generating any efficiency gains. The post-merger Nash equilibrium is:

\[
p = \begin{pmatrix} 265.163 \\ 265.163 \\ 5013.26 \end{pmatrix}, \quad D = \begin{pmatrix} 128.081 \\ 128.081 \\ 5013.26 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 65,619.4 \\ 25,132,757.2 \end{pmatrix}.
\]

After the merger, consumers’ surplus is \( S = -37,669,283.4 \), while total welfare is \( W_{1/2} = -12,470,906.7 \). The merger increases welfare by 3,269.4. Moreover, it is privately profitable and, as expected, consumers’ surplus is reduced.

### A.2 The profit of the merged entity can decrease with efficiency gains

The intuition behind this result is fairly simple. Efficiency gains are not identical for all the varieties produced by the merged entity. Some marginal costs might (almost) not be affected by the merger while others might be significantly reduced. The prices of all products, however, decline after the merger as soon as some efficiency gains exist. That is, the profit margin of the merged entity might decline for some of its products and these profit losses might not be compensated by gains on other brands. To illustrate, take \( n = 3 \) monoproduct firms and consider the following demand system:

\[
\begin{align*}
D^1 &= 800 - \frac{3}{5}p_1 + \frac{1}{5,000}p_2 + \frac{9}{20}p_3 \\
D^2 &= 10 + \frac{1}{5,000}p_1 - p_2 + \frac{9}{20}p_3 \\
D^3 &= 10 + \frac{9}{20}p_1 + \frac{9}{20}p_2 - p_3.
\end{align*}
\]

Assume that the marginal costs of production are

\[ c_1 = 0, \quad \text{and} \quad c_2 = c_3 = 5. \]
The pre-merger Nash equilibrium is:

\[
p_{\text{pre}} = \begin{pmatrix} 735.70 \\ 48.99 \\ 184.05 \end{pmatrix}, \quad D_{\text{pre}} = \begin{pmatrix} 441.42 \\ 43.99 \\ 179.05 \end{pmatrix}, \quad \Pi_{\text{pre}} = \begin{pmatrix} 324,748 \\ 1,934.72 \\ 32,060 \end{pmatrix}
\]

Consumers’ surplus is \( S_{\text{pre}} = -475,372 \); total welfare is \( W_{1/2}^{\text{pre}} = -116,629 \).

Suppose that firms 1 and 2 merge and generate efficiency gains: \( c_2(\gamma) = 5(1 - \gamma) \). For \( \gamma = 0 \), the post-merger Nash equilibrium is:

\[
p = \begin{pmatrix} 735.71 \\ 49.06 \\ 184.07 \end{pmatrix}, \quad D = \begin{pmatrix} 441.42 \\ 43.92 \\ 179.07 \end{pmatrix}, \quad \Pi = \begin{pmatrix} 326,690 \\ 32,067.5 \end{pmatrix}, \quad S = -475,386,
\]

and \( W_{1/2} = -116,628 \). Therefore total welfare decreased by 13.94 after the merger, which is privately profitable.

It is readily confirmed\(^{15}\) that, for \( \gamma = \hat{\gamma}_1 \approx 4.1\% \), consumers’ surplus is unaffected by the merger, \( \Pi^{1+2} = 326,690, \Pi^3 = 32,058 \) and \( W = -116,624 > -116,629 \). That is, the merged entity benefits society as a whole, consumers are indifferent (they might be made slightly better of), the merged entity profit increases and the outsider profit decreases. Yet, this merger is problematic because the profit of the merged entity \( \Pi^{1+2} \approx 326,690.15 - 8.61\gamma + 5.58\gamma^2 \) is a decreasing function of \( \gamma \), for \( \gamma \in [0,0.77] \). Once the merger is authorized, the merged entity would have no incentive to implement the efficiency gains (it seems reasonable to assume that while the merged entity cannot achieve gains greater than \( \gamma \), it could nevertheless achieve gains lower than \( \gamma \)). But, absent efficiency gains, consumers’ surplus and total welfare would fall (relatively to the pre-merger situation). If the merging firms cannot credibly commit to reduce their costs, antitrust authorities should block a would be profitable merger. All this has a prisoner dilemma flavor.

A.3 Total welfare can decrease with efficiency gains

Take \( n = 4 \) monoproduct firms, and consider the following demand system:

\[
\begin{align*}
D^1 &= 5.10^6 - p_1 + \frac{99}{100}p_3 + \frac{1}{10,000}p_4 \\
D^2 &= 1 - p_2 + \frac{1}{1,000}p_3 + \frac{4}{5}p_4 \\
D^3 &= 10 + \frac{99}{100}p_1 + \frac{1}{1,000}p_2 - p_3 \\
D^4 &= 10 + \frac{1}{10,000}p_1 + \frac{4}{5}p_2 - p_4
\end{align*}
\]

Assume that the marginal costs of production are

\[ c_1 = c_3 = c_4 = 0 \quad \text{and} \quad c_2 = 1/2. \]

\(^{15}\)\( S \approx -475,386.00 + 340.34\gamma + 2.88\gamma^2. \)
As \( D^1 \) (resp. \( D^2 \)) is independent of \( p_2 \) (resp. \( p_1 \)), for \( \gamma = \hat{\gamma}_1 = 0 \), the post-merger Nash equilibrium is identical to the pre-merger Nash equilibrium (no unilateral effects):

\[
\begin{align*}
p^{\text{pre}} &= p = \begin{pmatrix} 3,311,371.59 \\ 1,057.79 \\ 1,639,134.46 \\ 593.68 \end{pmatrix}, \\
D^{\text{pre}} &= D = \begin{pmatrix} 3,311,371.59 \\ 1,057.29 \\ 1,639,134.46 \\ 593.68 \end{pmatrix}.
\end{align*}
\]

Firms 1 and 2 merge and generate efficiency gains: \( c_2(\gamma) = (1 - \gamma)/2 \). The post-merger total welfare is given by

\[
W_{1/2}(\gamma) \approx -1.45 \times 10^{12} - 6.72\gamma + 0.078\gamma^2
\]

which is a non increasing function of \( \gamma \) for \( \gamma \in [-\infty, 1] \) (if \( \gamma \) is too negative, brand 2 is no longer produced). Thus, total welfare is decreasing in the magnitude of efficiency gains on the whole relevant range, that is, up to 100% gains. Note that in this example, the profit \( \Pi^{1+2} \) of the merged entity is also a non increasing function of \( \gamma \). The merger is, however, privately beneficial from an \textit{ex ante} perspective if the firms’ expectations put enough weight on \( \gamma < 0 \).

**B Preliminary results on the profit functions**

**For an outsider:** First note that from the f.o.c. \( \Pi^J_2 = 0 \) it follows that

\[
\Pi^J_2 \hat{p}^J + \Pi^J_{2H} \hat{p}^H = 0 \quad \text{that is} \quad \hat{p}^J = (\Pi^J_{2J})^{-1} (\Pi^J_{2H}) \hat{p}^H \quad \text{where} \quad H = N \setminus J. \quad (17)
\]

Now, differentiating \( \Pi^J \) w.r.t. \( \gamma \) (using the envelope theorem) yields: \( \hat{\Pi}^J = \Pi^J_H \hat{p}^H \). Differentiating again leads to

\[
\hat{\Pi}^J = (\hat{p}^H)' \Pi^J_{JH} \hat{p}^J + (\hat{p}^H)' \Pi^J_{HH} \hat{p}^H + \Pi^J_H \hat{p}^H
\]

which, combined to (17), gives (4).

**For the insider:** from the f.o.c. \( \Pi^I_i = 0 \) and using \( \Pi^I_i = -D^I_i = V^I_i \) it follows that

\[
\Pi^I_{II} \hat{c}^I + \Pi^I_{IK} \hat{p}^K + V^I_i \hat{c}^I = 0
\]

and therefore

\[
\hat{p}^I = (-\Pi^I_{II})^{-1} V^I_i \hat{c}^I + (-\Pi^I_{II})^{-1} (\Pi^I_{IK}) \hat{p}^K,
\]

where \( K = N \setminus I \). Now, differentiating \( \Pi^I \) w.r.t. \( \gamma \) and using the envelope theorem, it comes \( \hat{\Pi}^I = (\hat{c}^I)' (-D^I) + \Pi^I_K \hat{p}^K \). Differentiating once more, using (18) and rearranging terms it follows that

\[
\hat{\Pi}^I = (\hat{p}^K)' \Pi^I_{IK} (-\Pi^I_{II})^{-1} V^I_i \hat{c}^I + 2 (\hat{c}^I)' (V^I_K + V^I_i (-\Pi^I_{II})^{-1} \Pi^I_{IK}) \hat{p}^K
\]

where \( K = N \setminus I \).
C Linear demand system

In this section, we derive the best reply maps for linear demand systems and solve for the Nash equilibrium prices. Then, we examine the variations of prices and consumers’ surplus with the magnitude of efficiency gains.

C.1 Best reply map and Nash equilibrium prices

Let \( \Delta \) denote the block diagonal matrix, whose blocks are \( D^K_K \), \( K = I, J_1, \ldots, J_r \) and let \( I_n \) denote the \( n \times n \) identity matrix. The matrix \( \Delta \) has the same properties as the matrix \( B \): it is symmetric negative definite.

For all \( K, K = I, J_1, \ldots, J_r \), equation (5) rewrites \( m^K = - (D^K_K)^{-1} D^K \) that is:

\[
m + \Delta^{-1}D = 0 \quad \text{or} \quad p - c + \Delta^{-1}(Bp + A) = 0.
\]

It follows that the best reply map, which gives prices of each firm as functions of the prices of the other firms, writes

\[
R(p) = \frac{1}{2}c - \frac{1}{2}\Delta^{-1}A + Rp,
\]

where \( R \) is the constant \( n \times n \) matrix given by

\[
R = \frac{1}{2}(I_n - \Delta^{-1}B).
\]

Note that the diagonal blocks of the matrix \( R \) are zero, so that each price given by (20) only depends on the prices of goods controlled by other firms.\(^{16}\) Strategic complementarity is equivalent to each term of \( R \) being non negative.

The Nash equilibrium vector price is defined by: \( p = R(p) \). So there exists a unique interior equilibrium if and only if the matrix \( I_n - R \) is nonsingular and the demand is positive for each good at the solution to \( p = R(p) \). The Nash equilibrium vector price is given by

\[
p = (I_n + \Delta^{-1}B)^{-1}(c - \Delta^{-1}A) = \frac{1}{2} (I_n - R)^{-1} (c - \Delta^{-1}A).
\]

Differentiating equation (21) yields:

\[
\dot{p} = \frac{1}{2}(I_n - R)^{-1}\dot{c}.
\]

It follows that \( \dot{p} \neq 0 \) unless if \( \dot{c} = 0 \).

\(^{16}\)The off-diagonal blocks of \( R \) are \( R_{K,H} = \frac{1}{2} (-D^K_K)^{-1} D^K_H \) for \( K \neq H, K, H = I, J_1, \ldots, J_r \).
C.2 First derivative properties (proof of Lemma 1)

In this subsection, we assume that the best reply map \( R(\cdot) \) is a contraction. This is equivalent to \( \| R \| < 1 \), where \( \| R \| \) is any matrix norm. Under this assumption, we can write

\[
(I_n - R)^{-1} = \sum_{t=0}^{+\infty} R^t.
\]

Now, using equation (22) yields:

\[
\dot{p} = \frac{1}{2} \left( \sum_{t=0}^{+\infty} R^t \right) \dot{c} \leq 0
\]

since all elements of \( R \) are nonnegative (strategic complementarity) and all components of \( \dot{c} \) are nonpositive. It follows that the consumers’ surplus (the profit of each outsider \( J \) respectively) is nondecreasing (resp. nonincreasing) in \( \gamma \):

\[
\dot{S} = -D \dot{p} \geq 0 \quad \text{and} \quad \dot{\Pi}^J = 2 \left( D^J \right)' \left( \dot{p}^J \right)' \leq 0,
\]

which completes the proof of Lemma 1. Note that the contraction assumption and the strategic complementarity are only sufficient assumptions for these results to hold.

C.3 Multidimensional gains

In this subsection, we note \( \frac{d}{d\gamma} \) and \( \frac{d^2}{d\gamma^2} \) the first and second derivatives of a scalar function with respect to the vector \( \gamma \) of efficiency gains. For instance, \( \frac{dS}{d\gamma} \) is a column vector of size \( l \) and \( \frac{d^2S}{d\gamma^2} \) is a symmetric \( l \times l \) matrix, where \( l \) is the number of goods controlled by the new entity. Since prices are linear in \( \gamma \), we have

\[
\frac{dS}{d\gamma} = -\frac{dp'}{d\gamma} D \quad \text{and} \quad \frac{d^2S}{d\gamma^2} = -\frac{dp'}{d\gamma} D \frac{dp}{d\gamma'},
\]

where the generic term of the matrix \( \frac{dp'}{d\gamma} \) is \( \frac{dp_i}{d\gamma_j} \), \( 1 \leq i \leq l, 1 \leq j \leq n \). (We use \( \frac{dp}{d\gamma} \) for the transposed matrix of \( \frac{dp'}{d\gamma} \).) Since \( D \) is symmetric negative definite, the \( l \times l \) matrix \( \frac{d^2S}{d\gamma^2} \) is symmetric positive semi-definite. Similarly, Equation (7) extends to multidimensional gains

\[
\frac{d^2\Pi^J}{d\gamma^2} = 2 \frac{d(m^J)'}{d\gamma} \left( -D^J \right) \frac{dm^J}{d\gamma'}.
\]

As in the one-dimensional case, the convexity results follow from the concavity of demand.

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\(^{17}\)A matrix norm is defined as \( \| R \| = \sup_{\| x \| \leq 1} \| Rx \| \), where \( \| \cdot \| \) stands for any norm of \( \mathbb{R}^n \). For instance, consider the norm: \( \| x \|_\infty = \max_{i} |x_i| \). The associated matrix norm is \( \| R \|_\infty = \max \sum_j |r_{ij}| \), where \( r_{ij} \) is the generic element of \( R \). Using this norm, we checked that the best reply maps in all three examples of Appendix A are contractions.
D Merger to monopoly

D.1 The $\Psi$ function

Let $P(\cdot)$ denote the inverse demand function. From $P(\Phi(p)) = p$, we deduce that

$$\Phi'(p) = \frac{1}{P'(\Phi(p))} \text{ and } \Phi''(p) = \frac{-(\Phi'(p))^2 P''(\Phi(p))}{P'(\Phi(p))}.$$  

It follows that

$$\Psi(p) = \frac{E}{\varepsilon} = \frac{-\Phi(p)\Phi''(p)}{(\Phi'(p))^2} = \frac{\Phi(p)P''(\Phi(p))}{P'(\Phi(p))} = \Theta(\Phi(p)), $$

where $\Theta(Q) = Q P''(Q)/P'(Q)$. In words, $\Psi(p)$ is the elasticity of the slope $P'(Q)$ of the inverse demand function, evaluated at $Q = \Phi(p)$.

D.2 First and second order conditions

The first order condition of the monopolist’s problem $\frac{\partial \Pi_I}{\partial p} = 0$ writes:

$$\Phi(p) + [p - (1 - \gamma)c] \Phi'(p) = 0 \text{ or } \frac{p - (1 - \gamma)c}{p} = \frac{1}{\varepsilon}. \quad (24)$$

The second order condition $\frac{\partial^2 \Pi_I}{\partial p^2} < 0$ writes:

$$2\Phi'(p) + (p - (1 - \gamma)c) \Phi''(p) < 0 \text{ or } \left(\frac{p - (1 - \gamma)c}{p}\right) \frac{p \Phi''(p)}{\Phi'(p)} > -2. \quad (25)$$

It follows that the second-order condition is satisfied at the solution to (24) if and only if

$$E > -2\varepsilon \text{ or } \Psi > -2.$$

D.3 Convexity of $S$ and $W_\alpha$

The first derivatives of the consumers’ surplus and the monopolist’s profit are:

$$\dot{S}(\gamma) = -n \dot{p} \Phi(p) \text{ and } \dot{\Pi}_I(\gamma) = nc \Phi(p).$$

The second derivatives are:

$$\ddot{S}(\gamma) = -n \dot{p} \Phi(p) - n \dot{p}^2 \Phi'(p) \text{ and } \ddot{\Pi}_I(\gamma) = nc \dot{p} \Phi'(p).$$

From the first order condition (24), if follows that

$$\dot{p} = \frac{-c}{2 + E/\varepsilon} = \frac{-c}{2 + \Psi} \quad (26)$$

and therefore (using that $d\Psi/d\gamma = \dot{p} \Psi'$)

$$\ddot{p} = \frac{(\dot{p})^3}{c} \Psi'. \quad (27)$$

29
Using (27), $\tilde{S}$ becomes:

$$\tilde{S} = np^2 \Phi \left[ \frac{\Psi'}{2 + \Psi} + \frac{\varepsilon}{p} \right] > 0$$

Now, using

$$\ddot{\Pi}(\gamma) = ncp\Phi'(p) = np^2 \Phi (2 + \Psi) \frac{\varepsilon}{p}$$

The second derivative of the welfare function is obtained by combining $\tilde{S}$ and $\ddot{\Pi}$.

### D.4 Proof of Corollary 3 and Lemma 5

First we prove Lemma 5. Note that for a Logit demand, $\partial D_i / \partial p_i = -D_i (1 - D_i) / \sigma$ which simplifies the f.o.c. Let $p^{\text{pre}}$ denote the pre-merger Nash equilibrium price, namely the solution to

$$(p - c)(1 - \Phi(p)) = \sigma.$$ 

Since $(p - c)(1 - \Phi(p))$ increases in $p$, we know that $p^{\text{pre}} / \sigma$ increases in $c / \sigma$ and $p_0 / \sigma$. The post merger price $p(\gamma)$ is solution to

$$[p - (1 - \gamma)c](1 - n\Phi(p)) = \sigma.$$ 

Therefore, efficiency gains can offset the unilateral effect if and only if there exists $\gamma \leq 1$ such that

$$(p^{\text{pre}} - (1 - \gamma)c)(1 - n\Phi(p^{\text{pre}})) = \sigma,$$

which is equivalent to (taking the value of the left hand side for $\gamma = 1$)

$$p^{\text{pre}}(1 - n\Phi(p^{\text{pre}})) \geq \sigma,$$

and, using the pre-merger f.o.c. $(p^{\text{pre}} - c)(1 - \Phi(p^{\text{pre}})) = \sigma$ to eliminate $\Phi$, it comes

$$- (p^{\text{pre}})^2 + (\sigma + c)p^{\text{pre}} + \frac{\sigma c}{n - 1} \geq 0.$$  

(28)

Let $p^+(c)$ denote the largest solution of the equation $-p^2 + (\sigma + c)p + \frac{\sigma c}{n - 1} = 0$ (the lowest solution is negative). The expression of $p^+(c)$ is given by (12). Condition (28) is equivalent to $p^{\text{pre}} \leq p^+$ which (using the fact that $(p - c)(1 - \Phi)$ is an increasing function of $p$) is equivalent to

$$(p^+(c) - c)[1 - \Phi(p^+(c))] \geq \sigma,$$

which simplifies to prove (11) and completes the proof of Lemma 5. We turn now to Corollary 3.

By Lemma 4, the consumers’ surplus $S = W_1$ is convex at $\tilde{\gamma}_1$ if and only if $\Phi(p(\tilde{\gamma}_1)) \leq 1/(2n)$. By definition of $\tilde{\gamma}_1$: $p(\tilde{\gamma}_1) = p^{\text{pre}}$. Now observe that the function $\varphi(.)$ defined by $\varphi(c) = \Phi(p^{\text{pre}}(c))$ is decreasing. Therefore the condition $\Phi(p^{\text{pre}}) < 1/(2n)$ is equivalent to $c > \tau = \varphi^{-1}(1/(2n))$. The equation $\varphi(c) = 1/(2n)$ combined with $(p^{\text{pre}} - c)(1 - \Phi(p^{\text{pre}})) = \sigma$ gives the value of $\tau$ and proves (10).
E  Bertrand competition with homogenous goods

This section presents a proof of Proposition 3. Before the merger, the objective is $W^\text{pre}_\alpha = \alpha S(c)$. After the merger, the objective is given by

$$W_\alpha(\gamma) = \begin{cases} \alpha S(c) + (1 - \alpha) \gamma cD(c) & \text{if } \gamma \geq 0 \\ \alpha S((1 - \gamma)c) - (1 - \alpha) \gamma cD((1 - \gamma)c) & \text{if } \gamma \leq 0. \end{cases}$$

The convexity of the consumers’ surplus and the identity $S'(p) = -D(p)$ yield, for $\gamma \leq 0$:

$$S((1 - \gamma)c) - S(c) \geq D(c)\gamma c.$$  \(29\)

Let $F$ be the cumulative distribution function of the random variable $\gamma$. Using successively (29), the assumption that $(1 - \gamma)c \leq p^m(c)$ for $\gamma \leq 0$ (no drastic efficiency loss) and $E\gamma = \int_{\gamma \geq 0} \gamma dF(\gamma) + \int_{\gamma \leq 0} \gamma dF(\gamma) = 0$, we get

$$E\gamma(W_\alpha) - W^\text{pre}_\alpha = \begin{align*} &\int_{\gamma \geq 0} [W_\alpha(\gamma) - \alpha S(c)] dF(\gamma) + \int_{\gamma \leq 0} [W_\alpha(\gamma) - \alpha S(c)] dF(\gamma) \\ &\geq (1 - \alpha)cD(c) \int_{\gamma \geq 0} \gamma dF(\gamma) + \int_{\gamma \leq 0} \{\alpha[S((1 - \gamma)c) - S(c)] - (1 - \alpha)\gamma cD((1 - \gamma)c)\} dF(\gamma) \\ &\geq (1 - \alpha)cD(c) \int_{\gamma \geq 0} \gamma dF(\gamma) + \int_{\gamma \leq 0} \{\alpha \gamma cD(c) - (1 - \alpha)\gamma cD(p^m(c))\} dF(\gamma) \\ &= c [(1 - \alpha)D(c) - \alpha D(c) + (1 - \alpha)D(p^m(c))] \int_{\gamma \geq 0} \gamma dF(\gamma) \\ &= c \left[ (1 - \alpha)D(c) - \alpha D(c) + (1 - \alpha)D(p^m(c)) \right] \int_{\gamma \geq 0} \gamma dF(\gamma) \\ &= \frac{D(c) + D(p^m(c))}{2D(c) + D(p^m(c))}. \end{align*}$$

which is non negative for any $\alpha \geq \hat{\alpha} = \frac{D(c) + D(p^m(c))}{2D(c) + D(p^m(c))}$.

F  Convexity of $S$ under Cournot competition

Consumers’ surplus is

$$S(\gamma) = \int_0^{Q(\gamma)} P(u) du - P(Q(\gamma)) Q(\gamma)$$

whence

$$\dot{S}(\gamma) = -\dot{Q}QP'(Q).$$

From

$$(r + 1)P(Q(\gamma)) + Q(\gamma) P'(Q(\gamma)) = (1 - \gamma)c' + \sum_{j \in J} c_j,$$
it follows that (by differentiating with respect to $\gamma$)

$$-\dot{Q}P'(Q) = \frac{c_I}{r + 2 + \Theta(Q)}$$

therefore, $\dot{S}$ becomes

$$\dot{S} = \frac{c_I Q}{r + 2 + \Theta(Q)}$$

and

$$\ddot{S}(\gamma) = \frac{c_I \dot{Q}}{r + 2 + \Theta(Q)} + \frac{-c_I Q \dot{Q} \Theta'}{(r + 2 + \Theta(Q))^2}$$

$$= \frac{c_I \dot{Q}}{r + 2 + \Theta(Q)} \left[1 - \frac{Q \Theta'}{r + 2 + \Theta(Q)}\right].$$