Intertemporal pricing with unobserved consumer arrival times*

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Abstract

We examine optimal selling mechanisms with ex-ante commitment for a non-durable good when the seller does not observe the times at which strategic consumers arrive on the market and how much they are willing to pay for the good. Assuming consumer risk neutrality, we demonstrate in this two-dimensional screening problem that stochastic mechanisms are suboptimal. In practice, this means that quantity rationing and behavior-based price discrimination do not improve the profit compared to a simple time-dependent price schedule. We explain how the optimal profit may be achieved with a first-come first-served policy.

Keywords: Intertemporal pricing, strategic consumers, arrival date, heterogeneous cohorts, two-dimensional screening.

JEL Classification: D11, D42, D82.

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1 Introduction

We derive the optimal pricing strategy with ex ante commitment of a monopolist who sells an indivisible non-durable product (e.g. an air, train or event tickets, a hotel room) to a population of strategic buyers who arrive in the market at different times and differ in their valuation for the good. Both the arrival times and the valuations are unobserved to the seller. Buyers are strategic in the sense that they decide to purchase immediately upon arrival or later in the selling period.

First, we examine the most general selling mechanisms in this two-dimensional screening problem. We show that any mechanism can be approximated by a behavior-based pricing policy combined with quantity rationing, whereby the seller lets the price offered and the probability of serving a given customer depend on the history of his purchase attempts. Yet we find that when consumers are risk-neutral such instruments do not improve the profit: the optimal strategy of the seller is a simple time-dependent price schedule.

Next, we characterize the optimal price schedule. We explain how the seller optimally exploits the statistical link between valuations and arrival times, given the buyers' ability to postpone their purchase. We find that optimal prices are nondecreasing over time. For instance, airline and railway companies commonly sell tickets in advance and change their prices as the departure date approaches. If the buyers become less price sensitive as time passes, the standard pricing formula holds at each date, relating the optimal mark-up to the elasticity of demand. Otherwise, if the price elasticity is non monotonic over time, it may be optimal to offer the same price to consumers arriving at different times.

Finally, we show that when the consumers can delay their purchase at no cost, a first-come first-served policy makes it possible to achieve the optimal profit. Under such a policy, the seller offers units at different prices and allocates them according to the order of arrival: consumers who arrive earlier get better deals. First-come first served policies are easier to implement than full-fledged time-dependent price schedules because they involve only the order of arrival of consumers and do not require to let the price explicitly depend on precise purchase times.

Our framework is primarily related to Akan et al. (2011). Akan et al. (2011) assume that all consumers are present in the market at the beginning of the selling period, each of them privately knowing the date at which he will learn his valuation and the distribution from which it will be drawn. In contrast, we assume here that the consumers simultaneously recognize their need for the good and discover their willingness to pay for it, but that the arrival times differ across consumers. These two frameworks can thus be seen as polar cases in a general environment where the consumers would discover their need at different times and thereafter learn their valuation progressively. Although the relevant instruments differ in the two frameworks (e.g. refund policies are irrelevant in the context we consider), in both cases the purchase time is a valuable screening device when less elastic consumers arrive later.

The article is organized as follows. Section 2 relates our results to the existing literature. Section 3 presents the framework. Section 4 considers a general mechanism design approach of the problem and shows that stochastic mechanisms are sub optimal. Section 5 characterizes the optimal time-dependent price schedule and Section 6 introduces first-come first-served policies as a convenient way to implement the optimal mechanism under zero waiting costs.

2 Literature review

To clarify the connection of the present article to the existing literature, it is important to highlight the contrast between consumer preferences for durable and non-durable goods:

- In most of the durable good literature, buyers are assumed to be impatient as delaying purchase entails a waiting cost due to the privation of consumption. Decreasing prices are obtained if sellers are more patient than consumers (Stokey (1979), Landsberger and Meilijson (1985)). High-valuation consumers buy first to enjoy consumption of the good sooner; as time goes by, the firm has an incentive to lower prices to induce purchase by low-valuation consumers. Consumers face an intertemporal tradeoff between buying now at a high price or later at a lower price, which crucially depends on their degree of patience.[1]

- The economic setting is strikingly different for non-durable goods. There is no opportunity cost due to the privation of consumption when purchasing later. For instance, buying an airplane ticket sooner does not yield any additional utility - in the absence of uncertainty - though a longer period of consumption. In addition, due to the relatively short period of time between purchase and consumption,

[1] When demand varies over time and when the incoming demand is made up of identical consumer cohorts, Conlisk et al. (1984) show that decreasing price cycles emerge at equilibrium: sooner or later, the firm must charge a low price to make the cumulated stock of low-valuation consumers purchase. In a more general setting, Board (2008) explains how the firm can time discriminate between cohorts when incoming demand is allowed to vary over time.

consumer discounting is unlikely to play a key role. Accordingly, unlike Board (2008) and Board and Skrzypacz (2010) who adopt a durable good perspective,[2] we assume that buyers do not suffer any utility loss per se by delaying their purchase and are infinitely patient (the discount factor is one).[3] We show that time discrimination across cohorts may still arise in this setting because of heterogeneous price-sensitivity.

Second, while the literature on intertemporal price discrimination emphasizes the role of demand uncertainty and capacity constraints,[4] we argue here that there is no need for demand uncertainty or capacity constraints to obtain nondecreasing price patterns.

Closely related to our analysis is the revenue management literature (see Talluri and van Ryzin (2004)). Many articles in this field, however, postulate a particular form for the price scheme (e.g. a number of fare classes) and look for the corresponding optimal prices. In contrast, we impose no a priori restriction on the shape of the price schedule.

The present article is also related to the recent literature on behavior-based pricing with customer recognition (see Fudenberg and Villas-Boas (2006) for a survey). Indeed, we allow the seller to condition the price offered to a customer on the history of his purchase attempts and thus to discriminate between first-time visitors and returning, previously rationed customers (see Section 4). The sales techniques studied by Armstrong and Zhou (2011) entail a somewhat similar sort of discrimination, implemented in a different context (consumer search and oligopolistic competition). However, our model differs from the general approach of the literature on behavior-based pricing that considers environments with repeated purchases that may at least partially revealed consumers' preferences over time.

Wilson (1988) suggests a rationale for first-come first-served policies, assuming that consumers can buy many units and that they all have the same individual demand function. He finds that the seller optimally charges no more than two different prices and rations sales at the lower price. In contrast, we find that the optimal scheme generally involves more than two prices, because consumer preferences change over time.

[2] See Gershkov and Moldovanu (forthcoming) for a review of models with intertemporal pricing in environments wherein consumers arrive over time.

[3] Introducing a common discount factor for the monopolist and consumers in our model does not alter the analysis

[4] A number of articles consider environments with scarce capacity and uncertain aggregate demand (e.g. Gale and Holmes (1993), and Dana (2001)). These papers often resort to two-periods models with two consumer segments (price-sensitive and price-insensitive consumers) and show the optimality of increasing price schedules. Some articles (e.g. Courty and Li (2000), Möller and Watanabe (2010) and Nocke et al. (2010)) combine individual uncertainty, consumer heterogeneity, and possibly capacity constraints. They explain how a seller can take advantage of the uncertainty to screen consumers.

3 Model

A monopolistic seller operates over a finite time horizon (selling period), the length of which is normalized to one. At the start of the time horizon, the seller is endowed with an inventory of $\bar{Q}$ units. In the railroad example, $\bar{Q}$ accounts for the (exogenous) capacity of the train. At the end of the selling period, units that are left over have zero value. We denote by $c_d \geq 0$ the constant distribution cost per unit sold, which corresponds to the fee of emitting one ticket.

We assume that the monopolist announces the price schedule at the start of the selling period and is able to credibly stick to the schedule for the entire period. Price commitment is widely observed in practice, and this assumption is common in the advance-selling literature (see Möller and Watanabe (2010), Nocke et al. (2010), and Akan et al. (2011)).

The size of the population of potential buyers is denoted by N. Each consumer simultaneously recognizes his need for the product and discovers his willingness to pay for it. The time at which these two events occur is called the "consumer arrival time". The consumers differ in their valuations and arrival times. Both characteristics are unobserved to the seller, and hence constitute the private information of a consumer. In particular, the monopolist cannot condition the price on the consumer's arrival date. Consumers are thus represented by two characteristics, their arrival time, denoted by t, and their valuation, denoted by v. Each consumer buys at most one unit, which he cannot resell.

We allow the consumers to purchase immediately upon arrival or later with no waiting cost if they choose to delay their purchase.

We denote by $G(.)$ the distribution of arrival times, which is assumed to admit a positive density $g(.)$ on the selling period $[0, 1]$. We denote by $F(v|t)$ the distribution of valuations conditional on arrival times. We assume that this distribution admits a continuous and positive density, $f(v|t)$ on its support $[\underline{v}, \bar{v}]$. The aggregate demand function is defined by

$$D(p) = N \int_0^1 [1 - F(p|t)]\, g(t)\ \mathrm{d}t. \tag{1}$$

We introduce the price elasticity in cohort t (set of consumers arriving at time t):

$$\varepsilon(p|t) = \frac{pf(p|t)}{1 - F(p|t)}.$$

To ensure the existence of instantaneous monopoly prices, we assume hereafter that, for any arrival time t and any value of $c \geq c_d$, the instantaneous profit function, $\pi(p|t) = (p-c)[1-F(p|t)]$, is single-peaked, i.e. first increases then decreases as the price p rises.[5] (When the capacity constraint is binding, the relevant value for c is higher than c_d, see Section 5.1.)

Assumption 1 (Nondecreasing price elasticity)**.** *For all t, the elasticity of demand, $\varepsilon(p|t)$, is nondecreasing in p.*

Assumption 1 implies that the instantaneous profit is single-peaked. Indeed the derivative of $\pi(p|t)$ with respect to p can be written as $fp[1/\varepsilon - (p-c)/p)]$. Under Assumption 1, the bracketed term is decreasing in p, and hence the derivative is first positive then negative as p rises (see Maskin and Riley (1984a)). The assumption holds in particular in the iso-elastic case where consumer valuations in cohort t are Pareto-distributed: $1-F(p|t) = (p/\underline{v})^{-\varepsilon(t)}$, for some $\varepsilon(t) > 0$; the elasticity of demand in cohort t, $\varepsilon(t)$, does not depend on the level of price.

The variation of the elasticity $\varepsilon(p|t)$ with respect to the arrival time t depends on the statistical link between the random variables t and v, as stated in Lemma 1 below.

Lemma 1. *The elasticity of demand, $\varepsilon(p|t)$, does not depend on the arrival time t if and only if the random variables v and t are statistically independent.*

If the elasticity increases (decreases) over time, the valuation v first-order stochastically decreases (increases) with the arrival time t.

Proof: Since $\varepsilon(p|t) = p\,h(p|t)$, the elasticity of demand varies with t in the same way as the hazard rate h given by

$$h(v|t) = \frac{f(v|t)}{1-F(v|t)}.$$

Integrating between $\underline{v}$ and v yields

$$\int_{\underline{v}}^{v} h(x|t)\,\mathrm{d}x = -\ln[1-F(v|t)].$$

If the elasticity does not depend on (increases with, decreases with) t, the same is true for the hazard rate, and hence also for the cumulative distribution function $F(v|t)$, which yields the results.[6] $\square$

[5]To simplify the presentation, we assume that for all cohorts t, the profit function, $\pi(p|t)$, attains its maximum in the interior of the interval $(\underline{v}, \bar{v})$.

[6]The variable v first-order stochastically decreases (increases) with t if and only if $F(v|t)$ increases (decreases) with t.

4 General analysis

In Section 4.1, we describe and characterize the general class of direct mechanisms. In Section 4.2, we interpret these mechanisms in terms of sales strategies seen in practice, namely quantity rationing and behavior-based price discrimination. Assuming consumer risk neutrality, we show in Section 4.3 that these extra instruments do not enhance the seller's profit. Risk-aversion[7] is thus needed for such instruments to be optimal. In a static framework, Gauthier and Laroque (2012) provide necessary and sufficient conditions under which stochastic mechanisms shall be used; in particular, they notice that local randomization is always useless under risk neutrality.

4.1 Direct mechanisms

From the revelation principle, we know that there is no loss of generality restricting attention to direct mechanisms whereby consumers announce their valuations, their arrival times and the seller offers a price p as well as a probability of getting the good, x, depending on the announcements. Under such a mechanism, the consumer of type (v, t) enjoys the indirect utility

$$U(v|t) = \sup_{v', t' \geq t} x(v', t')[v - p(v', t')]. \tag{2}$$

The mechanism is incentive-compatible if and only if it induces truth-telling, i.e. (v, t) is solution to the maximization problem (2). We denote by $U'(v|t)$ the derivative of U with respect to v.

Lemma 2. *A mechanism $(x(v, t), p(v, t))$ is incentive-compatible if and only if the indirect utility U is non-increasing in t and convex in v, with $U'(v|t) = x(v, t)$.*

Proof. The envelope theorem yields $U'(v|t) = x(v, t) \in [0, 1]$. For any given arrival time, the function U is the upper bound of a family of affine functions of v, and hence is convex in v. Finally, the inspection of (2) shows that U is non-increasing in t.

Conversely, suppose that U is convex in v, with $U' = x$, and non-increasing in t.

[7] Liu and van Ryzin (2008) develop a two-period model with risk aversion in which all consumers are present from the start. They study strategic rationing, but do not consider behavior-based pricing.

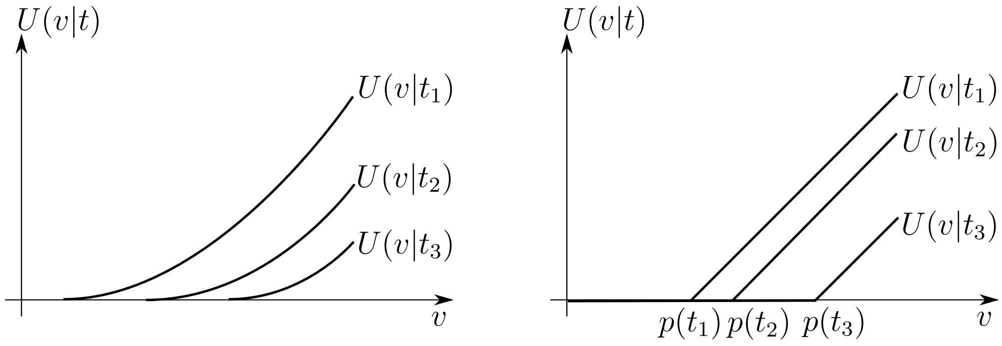

Figure 1: Indirect utility functions in a general mechanism (left panel) and in a price mechanism (right panel), with $t_1 < t_2 < t_3$

Then for any v, v', t, t' with $t' \geq t$, we have

$$\begin{aligned} U(v|t) = x(v,t)[v - p(v,t)] &\geq U(v|t') \geq U(v'|t') + x(v',t')(v - v') \\ &= x(v',t')[v' - p(v',t')] + x(v',t')(v - v') \\ &= x(v',t')[v - p(v',t')], \end{aligned}$$

which yields incentive-compatibility. □

Under a general mechanism, the indirect utility functions form a non-increasing family of convex curves, as represented on the left panel of Figure 1. Under a price mechanism, the indirect utility is given by $U(v|t) = \max(0, v - p(t))$, so the slope U' is first 0, then 1, with a convex kink at $v = p(t)$, as represented on the right panel of the Figure 1.

4.2 Quantity rationing under customer recognition

Stochastic mechanisms are not an abstract issue in the present problem. They can be implemented in practice if we allow the seller to commit on quantities as well as on prices and to let her policy depend on the observed visiting history of each consumer. Observing and conditioning upon consumer past behavior is made possible by technological evolutions in online retailing. For instance, an airline company can easily place cookies in the web browser of its online visitors and, in principle, might condition the price displayed on their screen upon their browsing behavior.

Rationing under customer recognition takes the following form. Suppose a buyer shows up at some time t_0 and in case he is denied the good returns at ulterior dates

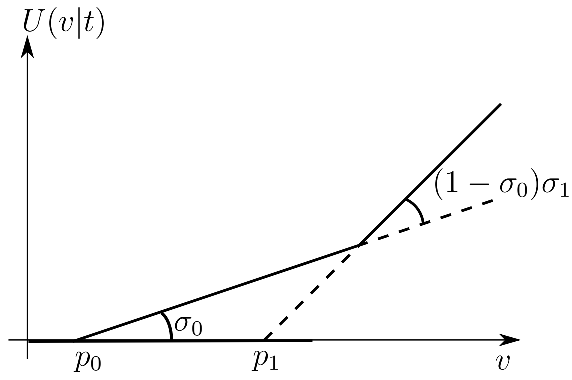

Figure 2: Rationing and behavior-based price discrimination for a sequence $(\sigma_0, p_0), (\sigma_1, p_1)$

$t_1, \ldots, t_n, \ldots$ The scheme consists of a sequence $(p_i, \sigma_i)_i$, where p_i and σ_i are the price and the probability of getting the good at the i-th attempt:

$$\mathcal{H}(t_0, t_1, \ldots, t_n, \ldots) \to \{(\sigma_0, p_0), (\sigma_1, p_1), \ldots, (\sigma_n, p_n), \ldots\}.$$

In the above expression, the time t_0 does not refer to a consumer arrival time, which is unobserved, but to the time the consumer first attempts to purchase the good. The seller serves such a customer with probability σ_0 at price p_0, serves him with probability σ_1 at price p_1 at a later time t_1 should he be rationed at time t_0 and return at t_1, with probability σ_2 at time $t_2 > t_1$ should he again be rationed at t_1 and return at t_2, etc.

When the values of $(\sigma_{t_i}, p_{t_i}), i > 0,$ depend only on the t_i's, the seller uses mere quantity rationing: all customers trying to purchase at a given date t_i are treated similarly; they all have the same probability of getting the good, at the same price. The pricing is behavior-based when these values vary across histories (which requires customer recognition).

A consumer who has unsuccessfully attempted to purchase at time t_0 keeps trying as long as his valuation v remains above p_i. Assuming that the sequence of prices, (p_i), is increasing, the stopping rule is given by the last index $j(v, t_0)$ such that $p_j \leq v < p_{j+1}$. The consumer derives the following expected indirect utility from the sequence $\mathcal{H}(t_0)$:

$$U(v; \mathcal{H}(t_0)) = \sigma_0(v - p_0) + \sigma_1(1 - \sigma_0)(v - p_1) + \cdots + \sigma_j(v - p_j) \prod_{i=0}^{j-1}(1 - \sigma_i), \quad (3)$$

where the index $j = j(v, t_0)$ represents the last purchase attempt. Since there is no cost of delaying purchase, the exact dates of the successive purchase opportunities do not affect consumer utility. The consumer gets the good with probability

$$\sigma_0 + \sigma_1(1 - \sigma_0) + \cdots + \sigma_j \prod_{i=0}^{j-1}(1 - \sigma_i),$$

which is non-increasing in v because the stopping rule $j(v, t_0)$ is non-increasing. Figure 2 shows the indirect utility functions of consumers arriving at time t_0 when these consumers are offered a sequence of two purchase opportunities $(\sigma_0, p_0), (\sigma_1, p_1)$. Increasing the size of the sequence allows to approximate any convex curve $U(v|t_0)$ arbitrarily closely, for a given t_0.

We have seen that the most general incentive-compatible mechanism is characterized by a non-increasing family of convex indirect utility functions $U(v|t)$. For each arrival date t, we can replicate the indirect utility function with a behavior-based pricing policy, i.e. with a sequence (σ_i, p_i) starting at $t_0 = t$. Because $U(v|t)$ is non-increasing in t, the consumer arriving at time t prefers these sequence to any other sequence starting later. Once engaged in a sequence, the consumer has no access to purchase opportunities included in other sequences because of customer recognition.

4.3 Stochastic mechanisms are suboptimal

The mechanism design literature has given sufficient conditions for stochastic mechanisms to be suboptimal (Maskin and Riley (1984b), Fudenberg and Tirole (1991) p. 306, Laffont and Martimort (2002) p. 65). To the best of our knowledge, however, the existing conditions apply only to one-dimensional models, and even for those models, the sufficient conditions hold only when the optimal allocation does not exhibit bunching (see e.g. Strausz (2006), Schotmüller and Boone (2012)). These results, therefore, do not apply in the present environment where consumers differ along two unobserved dimensions (their arrival date and their valuation for the good) and bunching may very well occur at the deterministic optimum (see Section 5).

Proposition 1. *Suppose that consumers are risk-neutral and that the instantaneous profit function, $\pi(p|t) = (p - c)[1 - F(p|t)]$, is concave in p.*

Then the optimal profit under quantity rationing and behavior-based pricing is the same as under a simple time-dependent price schedule.

The monopolist cannot improve upon the profits of the complete problem by introducing rationing and behavior-based pricing. This result extends Nocke and Peitz (2007)'s Proposition 1 to a framework with elastic demands, new inflows of consumers at each time and behavior-based pricing.

Proof. We assume that the consumer with the lowest valuation above the marginal cost never purchases the good, hence $\tilde{U}(\min(\underline{v}, c)|t) = 0$ and $\tilde{U}'(\min(\underline{v}, c)|t) = 0$.

The profit earned on a consumer with valuation v arriving at time t is the total surplus he generates, $(v - c)\tilde{U}'(v|t)$, minus his expected rent. The profit earned on the cohort arriving at time t is therefore

$$\tilde{\Pi}(\tilde{U}|t) = \int_{\underline{v}}^{\bar{v}} \left[(v-c)\tilde{U}'(v|t) - \tilde{U}(v|t) \right] f(v|t)\, \mathrm{d}v.$$

Using $\tilde{U}(\underline{v}|t) = \tilde{U}'(\underline{v}|t) = 0$ and integrating by part, we get

$$\tilde{\Pi}(\tilde{U}|t) = \int_{\underline{v}}^{\bar{v}} \tilde{U}''(v|t)\, \pi(v|t)\, \mathrm{d}v,$$

A second integration by part, together with $\tilde{U}'(\underline{v}|t) = \pi(\bar{v}|t) = 0$, yields

$$\tilde{\Pi}(\tilde{U}|t) = -\int_{\underline{v}}^{\bar{v}} \tilde{U}'(v|t)\, \frac{\mathrm{d}\pi(v|t)}{\mathrm{d}v}\, \mathrm{d}v.$$

A last integration by parts yields:

$$\tilde{\Pi}(\tilde{U}|t) = \int_{\underline{v}}^{\bar{v}} \left[\tilde{U}(v|t) - \tilde{U}(\bar{v}|t)\right] \frac{\mathrm{d}^2\pi(v|t)}{\mathrm{d}v^2}\, \mathrm{d}v - \tilde{U}(\bar{v}|t)\frac{\mathrm{d}\pi(\underline{v}|t)}{\mathrm{d}v}.$$

It follows that the total profit earned by the seller is given by

$$\Pi(\tilde{U}) = \int_0^1 \int_{\underline{v}}^{\bar{v}} \left[\tilde{U}(v|t) - \tilde{U}(\bar{v}|t)\right] \frac{\mathrm{d}^2\pi(v|t)}{\mathrm{d}v^2}\, \mathrm{d}v\, g(t)\, \mathrm{d}t - \tilde{U}(\bar{v}|t)\frac{\mathrm{d}\pi(\underline{v}|t)}{\mathrm{d}v}.$$

We have seen above that the indirect utility function $\tilde{U}(v|t)$ satisfies the following properties: $\tilde{U}(\underline{v}|t) = \tilde{U}'(\underline{v}|t) = 0$ for all t; $\tilde{U}(v|t)$ is convex in v, with slope between zero and one; $\tilde{U}(v|t)$ is non-increasing in t.

Let $\hat{U}(v|t) = \max(0, \tilde{U}(\bar{v}|t) + v - \bar{v})$. It is easy to check that $\hat{U}$ satisfies all the above properties. Actually $\hat{U}$ is the lowest function satisfying these properties and coinciding with $\tilde{U}$ for $v = \bar{v}$. The function $\hat{U}$ is attained when the seller does not use rationing nor behavior-based price discrimination, and simply offers a single price at each time. The time-dependent price schedule corresponding to indirect utility $\hat{U}$ is given by $\tilde{U}(\bar{v}|t) + p(t) - \bar{v} = 0$. We have: $\hat{U} \leq \tilde{U}$ with equality at $\bar{v}$. Using the concavity of the instantaneous profit π, we find that

$$\Pi(\hat{U}) \geq \Pi(\tilde{U}).$$

The derivative of $\hat{U}(v|t)$ with respect to v is either zero or one: at each time, the seller offers a single price (that is nondecreasing in time), and all consumers whose valuation exceeds this price level are served. We conclude that any scheme that maximizes the seller's profit is such that only one single price is offered at each date, with no rationing.

□

5 The optimal time-dependent price schedule

In this section, we study the optimal mechanism which is according to the previous section a price as a function of the purchase time. At the start of the selling period ($\tau = 0$), she commits to a price schedule, which we denote by $p(\tau)$, where τ is the purchase time, $0 \leq \tau \leq 1$.

In Section 5.1, we show that the consumer's ability to delay his purchase translates into a monotonicity constraint on the price schedule. In Section 5.2, we ignore the monotonicity constraint and solve the "relaxed problem". In Section 5.3, we solve the complete problem.

5.1 The monotonicity constraint

A consumer arriving at time t, should he decide to buy the product, purchases when the price is minimum. We denote by $\hat{p}(t)$ the minimum of prices offered after date t:

$$\hat{p}(t) = \min_{\tau \geq t} p(\tau). \tag{4}$$

The consumer decides to purchase the good if and only if his valuation for the good exceeds the minimum price, i.e. if and only if $\hat{p}(t) \leq v$, thus getting indirect utility $v - \hat{p}(t)$. The purchase time, τ, depends on the arrival time, t, but not on the valuation v, so we denote it by $\tau(t)$. The monopolist maximizes her intertemporal profit

$$\Pi = N \int_0^1 [p(\tau(t)) - c_d]\,[1 - F(p(\tau(t))|t)]\,g(t)\,\mathrm{d}t \tag{5}$$

subject to the capacity constraint

$$N \int_0^1 [1 - F(p(\tau(t))|t)]\, g(t)\, \mathrm{d}t \leq \bar{Q}.$$

The above inequality expresses the fact that the monopolist cannot sell more units than her initial inventory. Denoting by μ the Lagrange multiplier associated to the capacity constraint, we can rewrite the monopolist's objective function as

$$\mathcal{L} = N \int_0^1 [p(\tau(t)) - c]\, [1 - F(p(\tau(t))|t)]\, g(t)\, \mathrm{d}t + \mu\bar{Q}, \tag{6}$$

with $c = c_d + \mu$. The total cost c is the sum of the distribution cost, c_d, and the shadow cost of an extra unit, μ.

Clearly, no consumer purchases during periods where $p(t)$ is decreasing as it is then profitable to delay purchase. The next lemma introduces a transformation that removes such regions.

Lemma 3. *Let $p(t)$ be any intertemporal price schedule. Changing from schedule p to schedule $\hat{p}$, where $\hat{p}$ is given by (4), does not alter the seller's profit. The price schedule $\hat{p}(t)$ is nondecreasing over time.*

Proof: The monotonicity of $\hat{p}$ is obvious from (4). Under the schedule $\hat{p}$, the consumers should purchase upon arrival as they will not get a lower price later.[8] Hence, the seller's profit under $\hat{p}$ is given by

$$N \int_0^1 [\hat{p}(t) - c_d]\, [1 - F(\hat{p}(t)|t)]\, g(t)\, \mathrm{d}t,$$

which is the same as expression (5) for the profit Π because $\hat{p}(t) = p(\tau(t))$. □

It follows from Lemma 3 that the seller's optimal profit, denoted hereafter by Π^*, is obtained by maximizing

$$\Pi = N \int_0^1 [p(t) - c_d]\, [1 - F(p(t)|t)]\, g(t)\, \mathrm{d}t \tag{7}$$

under the capacity constraint

$$N \int_0^1 [1 - F(p(t)|t)]\, g(t)\, \mathrm{d}t \leq \bar{Q} \tag{8}$$

[8]In periods where the original schedule p is decreasing, the new schedule $\hat{p}$ is constant. The consumers arriving at such times indifferently purchase upon arrival or later, which does not affect the price paid nor the seller's profit.

and the monotonicity constraint:

$$p'(t) \geq 0 \tag{9}$$

for all $t \in [0,1]$. Hereafter, we refer to this problem as to the "complete problem".

5.2 The relaxed problem

In this section, we first ignore the monotonicity constraint, maximizing the profit (7) under the sole capacity constraint (8). Then we give a sufficient condition for the monotonicity constraint to be satisfied at the relaxed solution.

The Lagrangian of the relaxed problem

$$\mathcal{L} = N \int_0^1 [p(t) - c]\,[1 - F(p(t)|t)]\, g(t)\, \mathrm{d}t + \mu \bar{Q} \tag{10}$$

can be maximized separately at each time t. As the instantaneous profit is single-peaked (see Section 3), the solution of the relaxed problem, denoted $p^r(.)$, is characterized by the first-order condition

$$\frac{p^r(t) - c}{p^r(t)} = \frac{1}{\varepsilon(p^r(t)|t)}. \tag{11}$$

The relaxed solution is solution to the complete problem if and only if

$$\frac{\mathrm{d}p^r}{\mathrm{d}t} \geq 0, \tag{12}$$

for all $t \in [0,1]$. A necessary condition for the relaxed solution to be nondecreasing over time is that the price elasticity of the instantaneous demand, evaluated at the relaxed solution, $\varepsilon(p^r(t)|t)$, be non-increasing in t. Proposition 2 provides a sufficient condition.

Proposition 2. *Suppose that the elasticity of demand, $\varepsilon(p|t)$, is nondecreasing in price (Assumption 1) and non-increasing in time. Then the relaxed solution, given by (11), is also solution to the complete problem.*

Proof: Assume that the demand becomes less elastic as time passes, i.e. $\varepsilon(p|t)$ is non-increasing in the arrival time, t. This implies that

$$\frac{p - c}{p} - \frac{1}{\varepsilon(p|t)}$$

is non-increasing in t. Under Assumption 1, this term increases with p. We conclude that $p^r(.)$ is nondecreasing over time, and hence is solution to the complete problem. □

Under Assumption 1, the intertemporal price schedule increases over time when the consumers who arrive later are less price sensitive. Proposition 2 holds in particular when the elasticity of demand is constant over time, i.e. when the arrival time and the valuation are independent (see Lemma 1). In this case, the optimal price is constant over time: the monopolist uses a uniform price. In contrast, when demand becomes strictly less elastic as time passes, the optimal price increases over time: using a uniform price would be suboptimal.

We now provide two examples to illustrate the effects of price discrimination. We denote by p^u the best uniform price, which maximizes $(p-c)\,D(p)$, where the aggregate demand function $D(.)$ is given by (1). We compare the performances of p^u and p^r given by (11) under the assumptions of Proposition 2. Consumer surplus is given by

$$S = \int_0^1 \left(\int_{p(t)}^{\overline{v}} v f(v|t)\,\mathrm{d}v - p(t)[1 - F(p(t)|t)] \right) g(t)\,\mathrm{d}t, \tag{13}$$

and the welfare is

$$W = \int_0^1 \left(\int_{p(t)}^{\overline{v}} v f(v|t)\,\mathrm{d}v - c[1 - F(p(t)|t)] \right) g(t)\,\mathrm{d}t, \tag{14}$$

where $p(t) = p^r(t)$ or $p(t) = p^u$.

In both examples, the elasticity is constant in price and non-increasing over time, the minimum valuation is $\underline{v} = 1$, consumers arrive uniformly during the selling period, i.e. $g = 1$ on $[0, 1]$, the market size is $N = 1000$, and the cost is $c = 1$.

- Suppose that the demand elasticity decreases linearly from three to two over the selling period: $\varepsilon(p|t) = 3 - 2t$, for all p and $t \in [0, 1]$. Then price discrimination raises profits by 14.7% and reduces consumer surplus by 2.5% and welfare by 1.8%; 199 units (resp. 225 units) are sold under uniform pricing (resp. at the optimum).

- Suppose that $\varepsilon(t) = 5$ for consumers arriving before $t = .5$ and $\varepsilon(t) = 1.5$ for consumers arriving after $t = .5$. Then price discrimination raises profits, consumer surplus and welfare by respectively 18.1%, 1.4% and 5.4%; 122 units (resp. 260 units) are sold under uniform pricing (resp. at the optimum).

Intertemporal price discrimination always increases profits. It may reduce (first example) or raise (second example) aggregate consumer surplus and total welfare.

5.3 The complete problem

We now investigate the situation where the monotonicity constraint (9) is binding, i.e. the relaxed solution violates (12). In this case, the optimal price schedule remains constant on some time intervals, which we call bunching periods.

To derive the first-order conditions in this context, we employ the method of Mussa and Rosen (1978), considering infinitesimal variations of the intertemporal price schedule that are admissible, i.e. that respect the monotonicity constraint. Translating slightly the schedule on a bunching period (t_0, t_1), we get

$$\int_{t_0}^{t_1} \left[\frac{p-c}{p} - \frac{1}{\varepsilon(p|t)} \right] f(p|t) g(t) \, \mathrm{d}t = 0, \tag{15}$$

where p is the constant price during the bunching period. In other words, the first-order condition (11) does not hold pointwise as is the case under separation, but only in expectation on bunching intervals. The optimal margin rate, $(p-c)/p$, is equal to the average inverse elasticity during the bunching period.

Increasing slightly the price schedule on subinterval (s, t_1) yields

$$\int_{s}^{t_1} \left[\frac{p-c}{p} - \frac{1}{\varepsilon(p|t)} \right] f(p|t) g(t) \, \mathrm{d}t \geq 0, \tag{16}$$

for all s in (t_0, t_1). Hence, the optimal markup is below (above) the inverse elasticity at the start (end) of bunching periods. As the bracketed term increases in p, the solution to the complete problem is below (above) the relaxed solution p^r at the start (end) of bunching periods, as represented on Figure 3.

Finally, increasing (decreasing) slightly the price at the left (right) of t_0 (t_1), we get

$$\frac{1}{\varepsilon(p_0^-|t_0)} \leq \frac{p_0^- - c}{p_0^-} \quad \text{and} \quad \frac{p_1^+ - c}{p_1^+} \leq \frac{1}{\varepsilon(p_1^+|t_1)}, \tag{17}$$

where p_0^- and p_1^+ are the limits of $p(t)$ as t tends to t_0 from below and t tends to t_1 from above. Assuming that the densities $f(p|t)$ and $g(t)$ are continuous in the arrival time, t, we conclude from (16) and (17) that the price schedule is continuous at the extremities of the bunching periods, where it coincides with the relaxed solution.[9] The first-order

[9]Letting s tend to t_1 in (16), we get

$$\frac{p-c}{p} \geq \frac{1}{\varepsilon(p|t_1)},$$

which, combined with the right inequality of (17), shows the continuity of the price schedule at t_1.

conditions at the extremities, together with (15), jointly determine t_0, t_1 and the level p of the price during the bunching period.

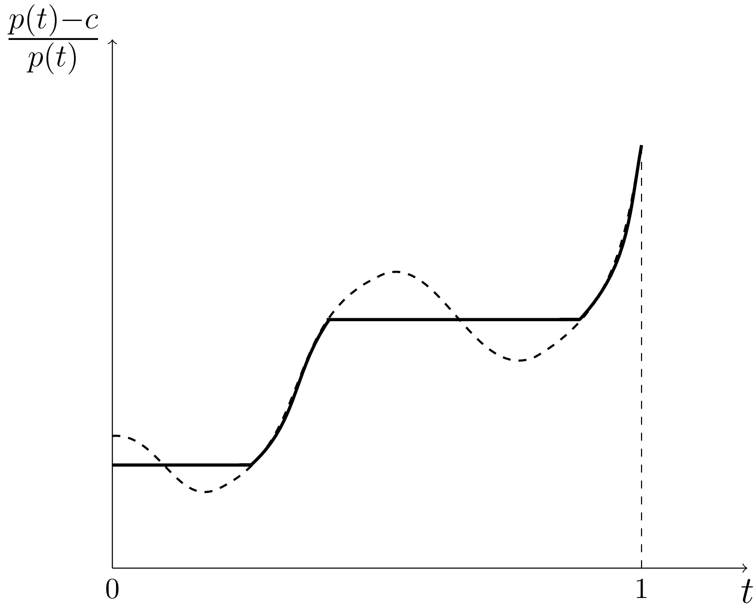

Figure 3: Mark-up for the relaxed solution (dashed line) and for the complete solution (solid line), with $c > 0$.

Outside bunching intervals, the seller is able to discriminate consumers on the basis of their purchase times, and the price scheme is solution to the relaxed problem, given by (11): the earlier the consumers arrive, the more price sensitive they are (the corresponding price elasticity being evaluated at the offered price).

In contrast, during bunching periods, consumers buying at different times are offered the same price. Using a uniform price is optimal if and only if the solution of the complete problem exhibits *full* bunching. This occurs in particular when the elasticity $\varepsilon(p|t)$ monotonically increases over time, or equivalently, when valuations first-order stochastically decrease with arrival times (see Lemma 1), in which case the relaxed solution, $p^r(.)$, is monotonically decreasing over time.

A last remark, before considering implementation, is that our analysis of the optimal price schedule would remain unchanged with we introduce a common discount factor r for consumers and the monopolist. All computations would remain the same replacing $p(t)$ by $p(t)e^{[r(T-t)]}$, where t is the purchase date, T the consumption date and consumers who pay upon purchasing.

6 First-come first-served policy

Since the seller cannot earn more than the optimal profit of the complete problem, even when equipped with sophisticated instruments, it is important to find selling mechanisms that yield this profit level and are easy to implement in practice. In this section, we show that first-come first-served policies meet these requirements.

A first-come first-served policy can be described by a pair $(Q, r(q))$, where $Q \leq \bar{Q}$ is the total number of units on sale and $r(q)$ is the price of the q-th cheapest unit, q being expressed as a percentage of the total number of units on sale. All units are made available at time zero and are allocated on a first-come first-served basis: consumers who arrive earlier get better price offers.

By construction, the function $r(q)$, which represents the q-th percentile of the distribution of offered fares, is nondecreasing on $[0, 1]$. Its inverse function, denoted by $q(r)$, is the fraction of units on sale that are offered at a price lower than or equal to r. For a uniform pricing policy, the support of the distribution of offered prices is a single point $\{p^u\}$ and hence $q(.)$ is a step function that jumps from zero to one at p^u.

Proposition 3. *When the consumers can delay their purchase at no cost, the optimal first-come first-served policy yields the same profit as the optimal time-dependent price schedule,* Π^**, defined as the maximum of (7).*

Proof: Under a first-come first-served policy, consumers who arrive earlier get better deals, so the induced intertemporal price schedule is nondecreasing: the monotonicity constraint (9) is satisfied.

Conversely, let $p(.)$ be the optimal intertemporal price profile, i.e. the solution to the complete problem derived in Section 5. The total number of units sold is

$$Q = N \int_0^1 [1 - F(p(t)|t)]\, g(t)\, \mathrm{d}t. \tag{18}$$

Let $q(t)$ be the fraction of units purchased before time t:

$$q(t) = \frac{\displaystyle\int_0^t [1 - F(p(s)|s)]\, g(s)\, \mathrm{d}s}{\displaystyle\int_0^1 [1 - F(p(s)|s)]\, g(s)\, \mathrm{d}s}. \tag{19}$$

The function $q(.)$ is continuous and increasing, with $q(0) = 0$ and $q(1) = 1$. The nondecreasing functions $q(t)$ and $p(t)$ parameterize a curve $(q(t), p(t))$ in the (q, r) space, implicitly defining a first-come first-served scheme $r(q)$ that satisfies $r(q(t)) = p(t)$ for all t.

The cheapest units are sold at price $r(0) = p(0)$ and the most expensive ones at price $r(1) = p(1)$. Using (18) and differentiating (19), we get

$$Q\,\mathrm{d}q = N\,[1 - F(p(t)|t)]\,g(t)\,\mathrm{d}t.$$

Changing variables q and t in (7) and using $r(q(t)) = p(t)$ shows that between times t and $t + dt$, $Q\,\mathrm{d}q$ units are sold at the optimal price $p(t)$, which yields

$$\Pi = Q \int_0^1 [r(q) - c_d]\,\mathrm{d}q.$$

The seller's profit is equal to the average price of a sold unit minus the unit cost, $\int_0^1 r(q)\,\mathrm{d}q - c_d$, multiplied by the number of units sold. $\square$

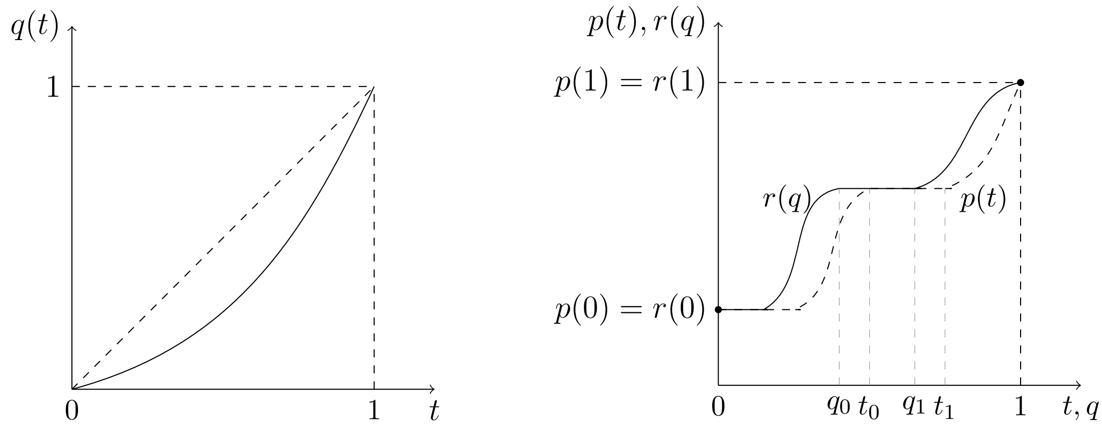

Figure 4: Left panel: Fraction of units purchased by early consumers (assuming that early consumers buy relatively less units). Right panel: Optimal intertemporal price schedule $p(t)$ (dashed); first-come first-served scheme $r(q)$ (solid).

As shown on the right panel of Figure 4, bunching at the optimum of the complete problem translates into flat parts of the first-come first-served scheme $r(q)$. If the optimal price is constant and equal to p between times t_0 and t_1, the schedule $r(q)$ is constant and equal to p on (q_0, q_1), where $q_0 = q(t_0)$ and $q_1 = q(t_1)$ are given by equation (19). The number of units sold at price p is therefore

$$(q_1 - q_0)Q = N \int_{t_0}^{t_1} [1 - F(p|t)]\,g(t)\,\mathrm{d}t.$$

Among consumers who arrive at time t, the fraction $\sigma(t) = 1 - F(p(t)|t)$ gets the good. If early consumers buy relatively less units, i.e. $\sigma(t)g(t)$ increases over time, then the function q is convex in t and hence the speed at which units are sold, $Q \,\mathrm{d}q/\,\mathrm{d}t$, increases over time. In this case, shown on Figure 4, we have: $q(t) \leq t$ and $p(t) = r(q(t)) \leq r(t)$ for all t. If early consumers buy relatively more units, i.e. $\sigma(t)g(t)$ decreases over time, then q is concave and the speed at which units are sold decreases over time. In this case: $q(t) \geq t$ and $p(t) = r(q(t)) \geq r(t)$. Finally, if $\sigma(t)g(t)$ remains constant, the flow of units sold is constant over time, $q(t) = t$ and $p(t) = r(t)$.

7 Discussion

We have examined optimal selling mechanisms when the seller does not observe the times at which her customers arrive on the market and how much they are willing to pay for the good. Assuming that the consumers are risk neutral and can delay their purchase at no cost, and that they simultaneously recognize their need for the good and their willingness to pay for it, we have demonstrated that quantity rationing and behavior-based price discrimination (or more generally any stochastic mechanism) do not improve the seller's profit compared to a simple time-dependent price schedule. We have also shown that the optimal profit may be achieved with a first-come first-served policy, whereby the price depends only on the orders of arrival, and not on the exact purchase times.

The above result does not mean that the seller cannot do better that the simple price mechanism if other, more powerful instruments are available. Suppose for instance that the seller announces that she will charge the static monopoly price at each time and that she will once-for-all stop selling in the event that the observed sales are not in line with what she expected. If the consumers believe the seller's commitment, they cannot do better than purchase upon arrival. Indeed, absent uncertainty, the seller is able to perfectly anticipate demand and hence would immediately detect any deviation, should some customers delay their purchase during high price periods. It follows that this policy makes it possible for the seller to earn the relaxed profit, which is higher than the complete profit if the monotonicity constraint is binding. We argue, however, that such a commitment is extreme because it involves giving up all sales after a deviation is observed –a very costly policy ex post, given that leftover units have zero values at the end of the selling period. For example, this commitment could involve letting a train depart with very few passengers. It seems unrealistic because it is based on aggregate data and it is unclear how the seller would let individual consumers know about her policy in practice.

Three directions for further research is worth mentioning. First, we have assumed that the consumers arrive in the market at exogenously given times. Yet one purpose of intertemporal pricing might be to encourage buyers to actively monitor the seller's offers. Further research is needed to understand optimal pricing strategies under endogenous arrival times. Second, introducing risk aversion in the model would emphasize the importance of resorting to stochastic mechanisms and thus of having a behavior-based price policy. Finally, relaxing the commitment assumption would also be a challenging extension of this work, allowing for the firm to revise its announced schedule.

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