Intertemporal pricing with unobserved consumer arrival times*

Philippe Choné
ENSAE-CREST
15, boulevard Gabriel Péri
92245 Malakoff cedex, France
Email: Philippe.Chone@ensae.fr
Phone: +33(0)141175390

Romain de Nijs
PSE (Ecole des Ponts ParisTech)-CREST
48, boulevard Jourdan
75014 Paris, France
Email: denijs@pse.ens.fr
Phone: +33(0)141176086

Lionel Wilner
INSEE-CREST
15, boulevard Gabriel Péri
92245 Malakoff cedex, France
Email: lionel.wilner@ensae.fr
Phone: +33(0)141176018

*We thank seminar participants at CREST-LEI and especially Philippe Février, Laurent Linnemer and Michael Visser for useful suggestions.
Intertemporal pricing with unobserved consumer arrival times

Abstract

We examine optimal selling mechanisms when the seller does not observe the times at which consumers arrive on the market and how much they are willing to pay for the good. Assuming consumer risk neutrality, we demonstrate that quantity rationing and behavior-based price discrimination do not improve the profit compared to a simple time-dependent price schedule. We explain how the level of the waiting costs and the variations of the instantaneous price elasticity affect the shape of the optimal price schedule. Under zero waiting costs, the optimal profit may be achieved with a first-come first-served policy.

Keywords: Intertemporal pricing, strategic consumers, arrival date, heterogeneous cohorts.

JEL Classification: D11, D42, D82.
1 Introduction

Airline and railway companies commonly sell tickets in advance and change their prices as the departure date approaches. For instance, the French national railway company offers tickets for sale three months before the train’s departure. The pricing relies on “fare classes”. Once a class is closed to the reservation, the next one opens, with a higher price, resulting in prices that increase over time. For each fare class, the company defines a quota before tickets are even opened to reservation. Such quantity instruments, which enable firms to ration consumer demand at a given price, are also observed in the airline and the hotel industries. They might be combined with various advances brought about by new information technologies, in particular the possibility to recognize customers based on their online fingerprint.

Online sellers can indeed track prospective buyers, for instance by placing cookies on their computer. They are thus able to let their policy depend on detailed information about their customers’ online visiting behavior. The use of such an instrument is not theoretical, as evidenced by current online firm behaviors. A study ‘Open to Exploitation’ (Turow et al. (2005)) conducted in 2005 by the Anneberg Center at the University of Pennsylvania discusses the case of a retail photography Web site that charges different prices for the same digital cameras and related equipment, depending on whether the shoppers had previously visited popular comparison sites. In addition, as noted by Zhang (2009), “targeted pricing is an evolving practice, and new ways to implement targeted pricing may emerge all the time”.

In this article, we model the above described situations in a simple framework wherein a monopolist sells an indivisible product to a population of strategic buyers who arrive in the market at different times and differ in their valuation for the good. Both the arrival times and the valuations are unobserved to the seller. The distribution of valuations within a cohort of consumers may vary with the arrival time. Buyers are strategic in the sense that they decide to
purchase immediately upon arrival or later in the selling period.

As a benchmark, we start with the simplest mechanism, namely a price that varies with the purchase time. We explain how the seller optimally exploits the statistical link between valuations and arrival times, given the buyers’ ability to postpone their purchase. We find that optimal prices are nondecreasing over time. If the buyers become less price sensitive as time passes, the standard pricing formula holds at each date, relating the optimal mark-up to the elasticity of demand. Otherwise, if the price elasticity is non monotonic over time, it is optimal to offer the same price to consumers arriving at different times, a phenomenon called “bunching” in the nonlinear pricing literature. We explain how the pricing formula is modified during bunching periods.

Next, we consider more sophisticated sales strategies, namely quantity rationing and behavior-based pricing. We suppose that the seller may credibly ration quantities at any time. Moreover, we assume that she is able to tag rationed customers, to recognize them when they come back, and to condition the available price and quantity on past behavior. As rationing creates uncertainty for consumers, we need to specify their attitude towards risk. Assuming risk neutrality, we demonstrate that strategic rationing and behavior-based pricing do not improve the seller’s profit compared to a simple time-dependent price schedule. Given that our result is negative, we allow, for theoretical completeness, the monopolist to condition her policy on the whole history of each consumer’s purchase attempts. Regardless of its complexity, such a selling mechanism cannot improve upon the optimal time-dependent price schedule.

Finally, we look for simple ways to implement the optimal mechanism. We show that when the consumers can delay their purchase at no cost, a first-come first-served policy makes it possible to achieve the optimal profit. Under such a policy, the seller offers units at different prices and allocates them according to the order of arrival: consumers who arrive earlier get
better deals. First-come first served policies are easier to implement than full-fledged time-
dependent price schedules because they involve only the order of arrival of consumers and do
not require to let the price explicitly depend on precise purchase times.

It is worth comparing our framework with that of Akan et al. (2011). These authors assume
that all consumers are present in the market at the beginning of the selling period, each of them
privately knowing the date at which he will learn his valuation and the distribution from which
it will be drawn. In contrast, we assume here that the consumers simultaneously recognize their
need for the good and discover their willingness to pay for it, but that the arrival times differ
across consumers. These two frameworks can thus be seen as polar cases in a general envi-
ronment where the consumers would discover their need at different times and thereafter learn
their valuation progressively. Although the relevant instruments differ in the two frameworks
(e.g. refund policies are irrelevant in the context we consider), in both cases the purchase time
is a valuable screening device when less elastic consumers arrive later.

The article is organized as follows. In Section 2, we provide a short review of the litera-
ture. Section 3 presents the framework and Section 4 characterizes the optimal time-dependent
price schedule. In Section 5, we consider strategic rationing and behavior-based pricing under
customer recognition and discuss the role of the seller’s commitment power. In Section 6, we
introduce first-come first-served policies as a convenient way to implement the optimal mech-
anism under zero waiting costs.

2 Literature review

The literature on intertemporal price discrimination emphasizes the role of demand uncertainty.
A number of articles consider environments with scarce capacity and uncertain aggregate de-
mand (e.g. Desiraju and Shugan (1999), Gale and Holmes (1993), and Dana (2001)). These
papers often resort to two-periods models with two consumer segments (price-sensitive and price-insensitive consumers) and show the optimality of increasing price schedules. Environments where buyers face uncertainty about their individual demand are studied in the advance-selling literature. Some papers (e.g. Xie and Shugan (2001), Shugan and Xie (2005)) point out that buyers are less heterogenous before they learn their valuation than after they get this information, which gives an incentive for advance-selling. Finally, some articles (e.g. Courty and Li (2000), Möller and Watanabe (2010) and Nocke et al. (2010)) combine individual uncertainty, consumer heterogeneity, and possibly capacity constraints. They explain how a seller can take advantage of the uncertainty to screen consumers.

Our approach is different because we assume away demand uncertainty. Moreover, the presence of capacity constraints does not change our qualitative results. We argue that there is no need for demand uncertainty or capacity constraints to obtain nondecreasing price patterns.

Closely related to our analysis is the revenue management literature (see Talluri and van Ryzin (2004)). Many articles in this field, however, postulate a particular form for the price scheme (e.g. a number of fare classes) and look for the corresponding optimal prices. In contrast, we impose no a priori restriction on the shape of the price schedule.

Our article is also related to the recent literature on behavior-based pricing with customer recognition (e.g. Villas-Boas (1999), Acquisti and Varian (2005), and Fudenberg and Villas-Boas (2006) for a survey). Indeed, we allow the seller to condition the price offered to a customer on the history of his purchase attempts and thus to discriminate between first-time visitors and returning, previously rationed customers (see Section 5.1). The sales techniques studied by Armstrong and Zhou (2011) entail a somewhat similar sort of discrimination, implemented in a different context (consumer search and oligopolistic competition).

Wilson (1988) suggests a rationale for first-come first-served policies. Contrary to us, he
assumes that consumers can buy many units and that they all have the same individual demand function, which makes the seller’s problem linear in the pricing policy. Wilson finds that the seller optimally charges (no more than) two different prices and rations sales at the lower price. Unlike our article, the seller’s problem is nonlinear and the optimal scheme generally involves more than two prices.

Finally, Busch and Curry (2011) show that a monopolist can use a lineup to screen consumers with different valuations and “qualities”. In contrast with the present article, the screening mechanism does not depend on the statistical link between the consumer characteristics, but on a social externality (the average quality of attendees is assumed to affect positively the surplus of each individual consumer).

3 Model

A monopolistic seller operates over a finite time horizon (selling period), the length of which is normalized to one. At the start of the time horizon, the seller is endowed with an inventory of $\bar{Q}$ units. In the railroad example, $\bar{Q}$ accounts for the (exogenous) capacity of the train. At the end of the selling period, units that are left over have zero value. We denote by $c_d \geq 0$ the constant distribution cost per unit sold, which corresponds to the fee of emitting one ticket.

We assume that the monopolist announces the price schedule at the start of the selling period and is able to credibly stick to the schedule for the entire period. There are many examples where price commitment is used in practice. For instance, conference or sport event organizers often commit to prices by announcing participation fees as a function of the registration date (Möller and Watanabe (2010)). In other instances, the commitment is implicit and made credible by repeated interactions. For example, although airlines usually do not commit to prices in
advance, consumers expect that prices increase as departure dates approach.¹ The commitment assumption is common in the advance-selling literature (see Xie and Shugan (2001), Shugan and Xie (2005), Möller and Watanabe (2010), Fay and Xie (2010), Nocke et al. (2010), and Akan et al. (2011)).

The size of the population of potential buyers is denoted by $N$. Each consumer simultaneously recognizes his need for the product and discovers his willingness to pay for it. The time at which these two events occur is called the “consumer arrival time”. The consumers differ in their valuations and arrival times. Both characteristics are unobserved to the seller, and hence constitute the private information of a consumer. In particular, the monopolist cannot condition the price on the consumer’s arrival date. Consumers are thus represented by two characteristics, their arrival time, denoted by $t$, and their valuation, denoted by $v$. Each consumer buys at most one unit, which he cannot resell.

We allow the consumers to purchase immediately upon arrival or later. Typically, when railroad passengers first recognize they need a travel ticket, they visit the railway company’s web site to check prices and decide to purchase a ticket immediately or later. If they do not purchase immediately, they must think about it later and monitor prices, which may entail an opportunity cost. Accordingly, we introduce a waiting cost that increases with the time elapsed between need recognition and purchase. For expositional convenience, we start by assuming zero waiting cost. In Section 4.4, we show that the results straightforwardly extend to positive waiting costs.

We denote by $G(.)$ the distribution of arrival times, which is assumed to admit a positive density $g(.)$ on the selling period $[0, 1]$. We denote by $F(v|t)$ the distribution of valuations.

¹In the railroad example mentioned in the introduction, the company does not publicly commit to a detailed price scheme, but much of the pricing is done before tickets are opened for booking and the pricing is roughly known by regular customers.
conditional on arrival times. We assume that this distribution admits a continuous and positive
density, \( f(v|t) \) on its support \([\underline{v}, \bar{v}]\). We define the instantaneous demand from consumers
arriving at time \( t \) as \( D(p|t) = N(1 - F(p|t))g(t) \). The aggregate demand function is defined
by
\[
D(p) = N \int_0^1 [1 - F(p|t)] g(t) \, dt.
\]
(1)
Given the demand function \( D(p|t) \), the elasticity of the instantaneous demand at time \( t \) is given
by
\[
\varepsilon(p|t) = \frac{pf(p|t)}{1 - F(p|t)}.
\]
To ensure the existence of instantaneous monopoly prices, we assume hereafter that, for any
arrival time \( t \) and any value of \( c \geq c_d \), the instantaneous profit function, \( \pi(p|t) = (p - c)[1 - F(p|t)] \), is single-peaked, i.e. first increases then decreases as the price \( p \) rises.\(^2\) (When the
capacity constraint is binding, the relevant value for \( c \) is higher than \( c_d \), see Section 4.1.)

**Assumption 1** (Nondecreasing price elasticity). **For all** \( t \), **the elasticity of demand, \( \varepsilon(p|t) \), is
nondecreasing in** \( p \).

Assumption 1 is technical (see Maskin and Riley (1984)) and implies that the instantaneous
profit is single-peaked. Indeed the derivative of \( \pi(p|t) \) with respect to \( p \) can be written as
\[ fp[1/\varepsilon - (p - c)/p] \]. Under Assumption 1, the bracketed term is decreasing in \( p \), and hence
the derivative is first positive then negative as \( p \) rises.

Assumption 1 holds in particular when consumer valuations at each time \( t \) are Pareto-
distributed: \( 1 - F(p|t) = (p/\underline{v})^{-\varepsilon(t)} \), for some \( \varepsilon(t) > 0 \). In this case, the elasticity of demand
does not depend on the level of price: \( \varepsilon(p|t) = \varepsilon(t) \), for all \( p \).

\(^2\)To simplify the presentation, we assume that for all cohorts \( t \), the profit function, \( \pi(p|t) \), attains its maximum
in the interior of the interval \((\underline{v}, \bar{v})\).
The variation of the elasticity $\varepsilon(p|t)$ with respect to the arrival time $t$ depends on the statistical link between the random variables $t$ and $v$, as stated in Lemma 1 below.

**Lemma 1.** The elasticity of demand, $\varepsilon(p|t)$, does not depend on the arrival time $t$ if and only if the random variables $v$ and $t$ are statistically independent.

If the elasticity increases (decreases) over time, the valuation $v$ first-order stochastically decreases (increases) with the arrival time $t$.

**Proof:** Since $\varepsilon(p|t) = p h(p|t)$, the elasticity of demand varies with $t$ in the same way as the hazard rate $h$ given by

$$h(v|t) = \frac{f(v|t)}{1 - F(v|t)}.$$ Integrating between $v$ and $v$ yields

$$\int_{v}^{v} h(x|t) \, dx = - \ln[1 - F(v|t)].$$

If the elasticity does not depend on (increases with, decreases with) $t$, the same is true for the hazard rate, and hence also for the cumulative distribution function $F(v|t)$, which yields the results. \footnote{The variable $v$ first-order stochastically decreases (increases) with $t$ if and only if $F(v|t)$ increases (decreases) with $t$.} \qed

### 4 The optimal time-dependent price schedule

In this section, we study the benchmark situation where the seller can only choose the price as a function of the purchase time. At the start of the selling period ($\tau = 0$), she commits to a price schedule, which we denote by $p(\tau)$, where $\tau$ is the purchase time, $0 \leq \tau \leq 1$.

In Section 4.1, we show that the consumer’s ability to delay his purchase translates into
a monotonicity constraint on the price schedule. In Section 4.2, we ignore the monotonicity constraint and solve the “relaxed problem”. In Section 4.3, we solve the complete problem.

4.1 The monotonicity constraint

A consumer arriving at time $t$, should he decide to buy the product, purchases when the price is minimum. We denote by $\hat{p}(t)$ the minimum of prices offered after date $t$:

$$\hat{p}(t) = \min_{\tau \geq t} p(\tau).$$  \hspace{1cm} (2)

The consumer decides to purchase the good if and only if his valuation for the good exceeds the minimum price, i.e. if and only if $\hat{p}(t) \leq v$, thus getting indirect utility $v - \hat{p}(t)$. The purchase time, $\tau$, depends on the arrival time, $t$, but not on the valuation $v$, so we denote it by $\tau(t)$. The monopolist maximizes her intertemporal profit

$$\Pi = N \int_0^1 [p(\tau(t)) - c_d] [1 - F(\hat{p}(t)|t)] g(t) \, dt$$  \hspace{1cm} (3)

subject to the capacity constraint

$$N \int_0^1 [1 - F(\hat{p}(t)|t)] g(t) \, dt \leq \bar{Q}.$$

The above inequality expresses the fact that the monopolist cannot sell more units than her initial inventory. Denoting by $\mu$ the Lagrange multiplier associated to the capacity constraint, we can rewrite the monopolist’s objective function as

$$\mathcal{L} = N \int_0^1 [p(\tau(t)) - c] [1 - F(\hat{p}(t)|t)] g(t) \, dt + \mu \bar{Q},$$  \hspace{1cm} (4)
with \( c = c_d + \mu \). The total cost \( c \) is the sum of the distribution cost, \( c_d \), and the shadow cost of an extra unit, \( \mu \).

Clearly, no consumer purchases during periods where \( p(t) \) is decreasing as it is then profitable to delay purchase. The next lemma introduces a transformation that removes such regions.

**Lemma 2.** Let \( p(t) \) be any intertemporal price schedule. Changing from schedule \( p \) to schedule \( \hat{p} \), where \( \hat{p} \) is given by (2), does not alter the seller’s profit. The price schedule \( \hat{p}(t) \) is nondecreasing over time.

**Proof:** The monotonicity of \( \hat{p} \) is obvious from (2). Under the schedule \( \hat{p} \), the consumers should purchase upon arrival as they will not get a lower price later.\(^4\) Hence, the seller’s profit under \( \hat{p} \) is given by

\[
N \int_0^1 [\hat{p}(t) - c_d] [1 - F(\hat{p}(t)|t)] g(t) \, dt,
\]

which is the same as expression (3) for the profit \( \Pi \) because \( \hat{p}(t) = p(\tau(t)) \). \( \square \)

It follows from Lemma 2 that the seller’s optimal profit, denoted hereafter by \( \Pi^* \), is obtained by maximizing

\[
\Pi = N \int_0^1 [p(t) - c_d] [1 - F(p(t)|t)] g(t) \, dt
\]

under the capacity constraint

\[
N \int_0^1 [1 - F(p(t)|t)] g(t) \, dt \leq \bar{Q}
\]

and the monotonicity constraint:

\[
p'(t) \geq 0
\]

\(^4\)In periods where the original schedule \( p \) is decreasing, the new schedule \( \hat{p} \) is constant. The consumers arriving at such times indifferently purchase upon arrival or later, which does not affect the price paid nor the seller’s profit.
for all \( t \in [0, 1] \). Hereafter, we refer to this problem as to the “complete problem”.

### 4.2 The relaxed problem

In this section, we first ignore the monotonicity constraint, maximizing the profit (5) under the sole capacity constraint (6). Then we give a sufficient condition for the monotonicity constraint to be satisfied at the relaxed solution.

The Lagrangian of the relaxed problem

\[
\mathcal{L} = N \int_0^1 [p(t) - c] [1 - F(p(t)|t)] g(t) \, dt + \mu \bar{Q}
\]

(8)

can be maximized separately at each time \( t \). As the instantaneous profit is single-peaked (see Section 3), the solution of the relaxed problem, denoted \( p^r(\cdot) \), is characterized by the first-order condition

\[
\frac{p^r(t) - c}{p^r(t)} = \frac{1}{\varepsilon(p^r(t)|t)}.
\]

(9)

The relaxed solution is solution to the complete problem if and only if

\[
\frac{dp^r}{dt} \geq 0,
\]

(10)

for all \( t \in [0, 1] \). A necessary condition for the relaxed solution to be nondecreasing over time is that the price elasticity of the instantaneous demand, evaluated at the relaxed solution, \( \varepsilon(p^r(t)|t) \), be non-increasing in \( t \). Proposition 1 provides a sufficient condition.

**Proposition 1.** Suppose that the elasticity of demand, \( \varepsilon(p|t) \), is nondecreasing in price (Assumption 1) and non-increasing in time. Then the relaxed solution, given by (9), is also solution to the complete problem.
Proof: Assume that the demand becomes less elastic as time passes, i.e. \( \varepsilon(p|t) \) is non-increasing in the arrival time, \( t \). This implies that

\[
\frac{p - c}{p} - \frac{1}{\varepsilon(p|t)}
\]

is non-increasing in \( t \). Under Assumption 1, this term increases with \( p \). We conclude that \( p^*(.) \) is nondecreasing over time, and hence is solution to the complete problem. \( \square \)

Under Assumption 1, the intertemporal price schedule increases over time when the consumers who arrive later are less price sensitive. Proposition 1 holds in particular when the elasticity of demand is constant over time, i.e. when the arrival time and the valuation are independent (see Lemma 1). In this case, the optimal price is constant over time: the monopolist uses a uniform price. In contrast, when demand becomes strictly less elastic as time passes, the optimal price increases over time: using a uniform price would be suboptimal.

We now provide two examples to illustrate the effects of price discrimination. We denote by \( p^u \) the best uniform price, which maximizes \( (p - c) D(p) \), where the aggregate demand function \( D(.) \) is given by (1). We compare the performances of \( p^u \) and \( p^r \) given by (9) under the assumptions of Proposition 1. In both examples, the elasticity is constant in price and non-increasing over time, the minimum valuation is \( v = 1 \), consumers arrive uniformly during the selling period, i.e. \( g = 1 \) on \([0, 1]\), the market size is \( N = 1000 \), and the cost is \( c = 1 \).

- Suppose that the demand elasticity decreases linearly from three to two over the selling period: \( \varepsilon(p|t) = 3 - 2t \), for all \( p \) and \( t \in [0, 1] \). Then price discrimination raises profits by 14.7\% and reduces consumer surplus by 2.5\% and welfare by 1.8\%; 199 units (resp. 225 units) are sold under uniform pricing (resp. at the optimum).

- Suppose that \( \varepsilon(t) = 5 \) for consumers arriving before \( t = .5 \) and \( \varepsilon(t) = 1.5 \) for consumers
arriving after $t = .5$. Then price discrimination raises profits, consumer surplus and welfare by respectively 18.1%, 1.4% and 5.4%; 122 units (resp. 260 units) are sold under uniform pricing (resp. at the optimum).

Intertemporal price discrimination always increases profits. It may reduce (first example) or raise (second example) aggregate consumer surplus and total welfare.

### 4.3 The complete problem

We now investigate the situation where the monotonicity constraint (7) is binding, i.e. the relaxed solution violates (10). In this case, the optimal price schedule remains constant on some time intervals, which we call bunching periods.

To derive the first-order conditions in this context, we employ the method of Mussa and Rosen (1978), considering infinitesimal variations of the intertemporal price schedule that are admissible, i.e. that respect the monotonicity constraint. Translating slightly the schedule on a bunching period $(t_0, t_1)$, we get

$$
\int_{t_0}^{t_1} \left[ \frac{p - c}{p} - \frac{1}{\varepsilon(p|t)} \right] f(p|t) g(t) \, dt = 0, \tag{11}
$$

where $p$ is the constant price during the bunching period. In other words, the first-order condition (9) does not hold pointwise as is the case under separation, but only in expectation on bunching intervals. The optimal margin rate, $(p - c)/p$, is equal to the average inverse elasticity during the bunching period.

Increasing slightly the price schedule on subinterval $(s, t_1)$ yields

$$
\int_{s}^{t_1} \left[ \frac{p - c}{p} - \frac{1}{\varepsilon(p|t)} \right] f(p|t) g(t) \, dt \geq 0, \tag{12}
$$
for all $s$ in $(t_0, t_1)$. Hence, the optimal markup is below (above) the inverse elasticity at the start (end) of bunching periods. As the bracketed term increases in $p$, the solution to the complete problem is below (above) the relaxed solution $p^r$ at the start (end) of bunching periods, as represented on Figure 1.

Finally, increasing (decreasing) slightly the price at the left (right) of $t_0$ ($t_1$), we get

$$\frac{1}{\varepsilon(p_0^+|t_0)} \leq \frac{p_0^- - c}{p_0^-} \quad \text{and} \quad \frac{p_1^+ - c}{p_1^+} \leq \frac{1}{\varepsilon(p_1^-|t_1)},$$

(13)

where $p_0^-$ and $p_1^+\!$ are the limits of $p(t)$ as $t$ tends to $t_0$ from below and $t$ tends to $t_1$ from above. Assuming that the densities $f(p|t)$ and $g(t)$ are continuous in the arrival time, $t$, we conclude from (12) and (13) that the price schedule is continuous at the extremities of the bunching periods, where it coincides with the relaxed solution. The first-order conditions at the extremities, together with (11), jointly determine $t_0$, $t_1$ and the level $p$ of the price during the bunching period.

Outside bunching intervals, the seller is able to discriminate consumers on the basis of their purchase times, and the price scheme is solution to the relaxed problem, given by (9): the earlier the consumers arrive, the more price sensitive they are (the corresponding price elasticity being evaluated at the offered price).

In contrast, during bunching periods, consumers buying at different times are offered the same price. Using a uniform price is optimal if and only if the solution of the complete problem exhibits full bunching. This occurs in particular when the elasticity $\varepsilon(p|t)$ monotonically increases over time, or equivalently, when valuations first-order stochastically decrease with arrival times (see Lemma 1), in which case the relaxed solution, $p^r(.)$, is monotonically de-

\[ \frac{p - c}{p} \geq \frac{1}{\varepsilon(p|t_1)}, \]

which, combined with the right inequality of (13), shows the continuity of the price schedule at $t_1$.\footnote{Letting $s$ tend to $t_1$ in (12), we get}
Figure 1: Mark-up for the relaxed solution (dashed line) and for the complete solution (solid line), with $c > 0$.

creasing over time.

4.4 Positive waiting costs

We have assumed, for simplicity, that buyers can delay their purchase at no cost. The results readily extend under positive waiting costs. Suppose that the net utility of a buyer arriving at time $t$ and purchasing at time $\tau \geq t$ is

$$v - p - \gamma(\tau - t),$$

where $\gamma \geq 0$ denotes the waiting cost per unit of time. Using dots to denote derivatives with respect to time, it is easy to show that the monotonicity constraint $\dot{p} \geq 0$ introduced in Section 4.1 should be replaced with $\dot{p} + \gamma \geq 0$. The latter constraint becomes less stringent as the waiting cost $\gamma$ increases; hence the optimal profit is nondecreasing in $\gamma$. The larger the waiting cost, the easier for the seller to price discriminate the buyers on the basis of their arrival time. The
relaxed problem studied in Section 4.2 corresponds to infinitely impatient buyers, \( \gamma = +\infty \).

**Proposition 2.** The optimal profit is nondecreasing in the waiting cost \( \gamma \). At the optimum, all consumers purchase upon arrival, and hence the waiting cost is not incurred.

*Proof.* See Appendix A.

The only change in the analysis of the complete problem is that the optimal price is not constant when the monotonicity constraint binds, but decreasing at rate \( \gamma \). Accordingly, \( p \) must be replaced with \( p(t) \) in equations (11) and (12). At the optimum, all consumers purchase upon arrival; those arriving later during a bunching period enjoy a lower price.

5 Do more sophisticated instruments enhance the profit?

So far, we have endowed the monopolist with a simple policy instrument, namely the price as a function of the purchase time. In this section, we ask whether more sophisticated instruments make it possible to extract a higher surplus from the consumers. First, we introduce quantity rationing and behavior-based pricing. Assuming consumer risk neutrality, we show that these extra instruments do not enhance the seller’s profit.\(^6\) Second, we argue that only an extreme and unrealistic form of commitment power would allow the seller to achieve the relaxed profit.

5.1 Quantity rationing and customer recognition

We now allow the seller to commit on quantities as well as on prices and to let her policy depend on the observed visiting history of each consumer. As explained in the introduction, observing and conditioning upon past consumer behavior is made possible by technological evolutions in online retailing. For instance, an airline company can easily place cookies in the web browser \(^6\)Liu and van Ryzin (2008) develop a two-period model with risk aversion in which all consumers are present from the start. They study strategic rationing, but do not consider behavior-based pricing.
of its online visitors and, in principle, might condition the price displayed on their screen upon their browsing behavior.

On the theoretical side, we consider the most general mechanism we can think of, assuming that the price and quantity available at any time for any consumer can be made conditional on the whole history of the consumer’s visiting behavior.

Formally, the seller commits to serve a customer who shows up at date $t_0$ with probability $\sigma_{t_0}$, to serve him with probability $\sigma_{t_1}$ should he be rationed at time $t_0$ and return at $t_1$, with probability $\sigma_{t_2}$ should he again be rationed at $t_1$ and return at $t_2$, etc. In other words, the seller is able to announce the complete sequence of events depending on the customer’s first arrival date, $t_0$, and on all the subsequent dates at which he unsuccessfully tried to purchase the good

$$H(t_0, t_1, \ldots, t_M) = \{(\sigma_{t_0}, p_{t_0}), (\sigma_{t_1}, p_{t_1}), \ldots, (\sigma_{t_M}, p_{t_M})\}.$$

The above sequence represents the consumer’s history of purchase attempts. Hence any customer knows the price and the probability of being served at time $t_M$ conditionally on having been rationed at times $t_0, t_1, \ldots, t_{M-1}$. The pricing is behavior-based when the values of $(\sigma_{t_i}, p_{t_i}), i > 0$, vary across histories. Otherwise, when these values depend only on the $t_i$’s, the seller uses mere quantity rationing. This is the case for instance when the seller is not able to recognize previously rationed customers.

We thus endow the seller with an apparently huge commitment power, and yet the next proposition shows that the seller thus equipped cannot do better than under the simple pricing policy studied in Sections 3 and 4.

**Proposition 3.** Suppose that consumers are risk-neutral and that the instantaneous profit function, $\pi(p|t) = (p - c)[1 - F(p|t)]$, is concave in $p$. 

19
Then the optimal profit under quantity rationing and behavior-based pricing is the same as under a simple time-dependent price schedule, namely the optimum of the complete problem (5), (6), (7).

The monopolist cannot improve upon the profits of the complete problem by introducing rationing and behavior-based pricing. This result extends Nocke and Peitz (2007)’s Proposition 1 to a framework with elastic demands, new inflows of consumers at each time and behavior-based pricing.

**Proof.** We assume here zero waiting costs, γ = 0. The case γ > 0 is relegated in Appendix B. Suppose that a consumer with valuation \( v \), arriving at time \( t \), attempts to purchase at times \( t_0, t_1, \ldots, t_M \), with \( t_0 \geq t \). The utility corresponding to history \( H_{t_0} \), is given by

\[
U(v; H_{t_0}) = \sigma_{t_0}(v - p_{t_0}) + \sigma_{t_1}(1 - \sigma_{t_0})(v - p_{t_1}) + \cdots + \sigma_{t_M}(v - p_{t_M}) \prod_{i=0}^{M-1} (1 - \sigma_{t_i}).
\]  
(14)

A consumer arriving at date \( t \) maximizes her expected utility over all possible histories \( H_{t_0} \) starting after \( t \) (including the option of no purchase attempt at all). His indirect utility is given by

\[
\tilde{U}(v|t) = \max_{t_0 \geq t} U(v; H_{t_0}).
\]  
(15)

According to (14), the utility \( U(v; H_{t_0}) \) is affine in \( v \). It follows from (15) that the indirect utility is the upper bound of a family of affine functions, and hence is convex in \( v \). By the envelope theorem, the derivative of \( \tilde{U}(v|t) \) with respect to the valuation \( v \) is equal to the probability that the consumer \( v \) arrived at date \( t \) eventually buys the product (evaluated on his optimal history):

\[
\tilde{U}'(v|t) = \sigma_{t_0} + \sigma_{t_1}(1 - \sigma_{t_0}) + \cdots + \sigma_{t_M} \prod_{i=0}^{M-1} (1 - \sigma_{t_i}).
\]  
(16)
The slope $\tilde{U}'(v|t)$ lies between zero and one. We assume that the consumer with the lowest valuation above the marginal cost never purchases the good, hence $\tilde{U}(\min(v,c)|t) = 0$ and $\tilde{U}'(\min(v,c)|t) = 0$. Finally, we see from (15) that $\tilde{U}$ is a non-increasing function of the arrival date, $t$.

The profit earned on a consumer with valuation $v$ arriving at time $t$ is the total surplus he generates, $(v - c)\tilde{U}'(v|t)$, minus his expected rent. The profit earned on the cohort arriving at time $t$ is therefore

$$\tilde{\Pi}(\tilde{U}|t) = \int_{\underline{v}}^{\bar{v}} \left[ (v - c)\tilde{U}'(v|t) - \tilde{U}(v|t) \right] f(v|t) \, dv.$$  

Using $\tilde{U}(\underline{v}|t) = \tilde{U}'(\underline{v}|t) = 0$ and integrating by part, we get

$$\tilde{\Pi}(\tilde{U}|t) = \int_{\underline{v}}^{\bar{v}} \tilde{U}''(v|t) \pi(v|t) \, dv,$$

A second integration by part, together with $\tilde{U}'(\bar{v}|t) = \pi(\bar{v}|t) = 0$, yields

$$\tilde{\Pi}(\tilde{U}|t) = -\int_{\underline{v}}^{\bar{v}} \tilde{U}'(v|t) \frac{d\pi(v|t)}{dv} \, dv.$$  

A last integration by parts yields:

$$\tilde{\Pi}(\tilde{U}|t) = \int_{\underline{v}}^{\bar{v}} \left[ \tilde{U}(v|t) - \tilde{U}(\bar{v}|t) \right] \frac{d^2\pi(v|t)}{dv^2} \, dv - \tilde{U}(\bar{v}|t) \frac{d\pi(v|t)}{dv}.$$  

It follows that the total profit earned by the seller is given by

$$\Pi(\tilde{U}) = \int_{0}^{1} \int_{\underline{v}}^{\bar{v}} \left[ \tilde{U}(v|t) - \tilde{U}(\bar{v}|t) \right] \frac{d^2\pi(v|t)}{dv^2} \, dv \, g(t) \, dt - \tilde{U}(\bar{v}|t) \frac{d\pi(\bar{v}|t)}{dv}.$$  

21
We have seen above that the indirect utility function $\tilde{U}(v|t)$ satisfies the following properties:

$\tilde{U}(v|t) = \tilde{U}'(v|t) = 0$ for all $t$; $\tilde{U}(v|t)$ is convex in $v$, with slope between zero and one; $\tilde{U}(v|t)$ is non-increasing in $t$.

Let $\hat{U}(v|t) = \max(0, \tilde{U}(\bar{v}|t) + v - \bar{v})$. It is easy to check that $\hat{U}$ satisfies all the above properties. Actually $\hat{U}$ is the lowest function satisfying these properties and coinciding with $\tilde{U}$ for $v = \bar{v}$. The function $\hat{U}$ is attained when the seller does not use rationing nor behavior-based price discrimination, and simply offers a single price at each time. The time-dependent price schedule corresponding to indirect utility $\hat{U}$ is given by $\hat{U}(\bar{v}|t) + p(t) - \bar{v} = 0$. We have: $\hat{U} \leq \tilde{U}$ with equality at $\bar{v}$. Using the concavity of the instantaneous profit $\pi$, we find that

$$\Pi(\hat{U}) \geq \Pi(\tilde{U}).$$

The derivative of $\hat{U}(v|t)$ with respect to $v$ is either zero or one: at each time, the seller offers a single price (that is nondecreasing in time), and all consumers whose valuation exceeds this price level are served. We conclude that any scheme that maximizes the seller’s profit is such that only one single price is offered at each date, with no rationing, as is the case in Section 4. Hence the seller profit is obtained by solving the complete problem studied there.

5.2 A policy based on observed aggregate sales

Suppose that the seller announces that she will charge the static monopoly price at each time and that she will once-for-all stop selling in the event that the observed sales are not in line with what she expected. If the consumers believe the seller’s commitment, they cannot do better than purchase upon arrival. Indeed, absent uncertainty, the seller is able to perfectly anticipate demand and hence would immediately detect any deviation, should some customers
delay their purchase during high price periods.

It follows that this policy makes it possible for the seller to earn the relaxed profit, which is higher than the complete profit if the monotonicity constraint is binding. We argue, however, that such a commitment is extreme because it involves giving up all sales after a deviation is observed—a very costly policy ex post, given that leftover units have zero values at the end of the selling period. For example, this commitment could involve letting a train depart with very few passengers. It seems unrealistic because it is based on aggregate data and it is unclear how the seller would let individual consumers know about her policy in practice.

Last, note that such a policy based on observed aggregate sales does not require that the monopolist observes the arrival dates of consumers. Had it been the case, the monopolist can then condition her price to the arrival date and forces a consumer who arrives at date \( t \) to be charged the corresponding monopoly price on his cohort, as is the case in third-degree price discrimination. Doing so, the monopolist also improves her complete profit.

6 First-come first-served policy

Ruling out extreme forms of commitment, such as that of Section 5.2, we have seen that the seller cannot earn more than the optimal profit of the complete problem, even when equipped with sophisticated instruments. It is therefore important to find selling mechanisms that yield this profit level and are easy to implement. In this section, we show that first-come first-served policies meet these requirements when consumers can delay their purchase at no cost.

A first-come first-served policy can be described by a pair \((Q, r(q))\), where \( Q \leq \bar{Q} \) is the total number of units on sale and \( r(q) \) is the price of the \( q \)-th cheapest unit, \( q \) being expressed as a percentage of the total number of units on sale. All units are made available at time zero and are allocated on a first-come first-served basis: consumers who arrive earlier get better price
By construction, the function $r(q)$, which represents the $q$-th percentile of the distribution of offered fares, is nondecreasing on $[0, 1]$. Its inverse function, denoted by $q(r)$, is the fraction of units on sale that are offered at a price lower than or equal to $r$. For a uniform pricing policy, the support of the distribution of offered prices is a single point $\{p^u\}$ and hence $q(.)$ is a step function that jumps from zero to one at $p^u$.

**Proposition 4.** When the consumers can delay their purchase at no cost, $\gamma = 0$, the optimal first-come first-served policy yields the same profit as the optimal time-dependent price schedule, $\Pi^*$, defined as the maximum of (5).

**Proof:** Under a first-come first-served policy, consumers who arrive earlier get better deals, so the induced intertemporal price schedule is nondecreasing: the monotonicity constraint (7) is satisfied.

Conversely, let $p(.)$ be the optimal intertemporal price profile, i.e. the solution to the complete problem derived in Section 4. The total number of units sold is

$$Q = N \int_0^1 [1 - F(p(t)|t)] g(t) dt. \quad (17)$$

Let $q(t)$ be the fraction of units purchased before time $t$:

$$q(t) = \frac{\int_0^t [1 - F(p(s)|s)] g(s) ds}{\int_0^1 [1 - F(p(s)|s)] g(s) ds}. \quad (18)$$

The function $q(.)$ is continuous and increasing, with $q(0) = 0$ and $q(1) = 1$. The nondecreasing functions $q(t)$ and $p(t)$ parameterize a curve $(q(t), p(t))$ in the $(q, r)$ space, implicitly defining a first-come first-served scheme $r(q)$ that satisfies $r(q(t)) = p(t)$ for all $t$. 
The cheapest units are sold at price \( r(0) = p(0) \) and the most expensive ones at price \( r(1) = p(1) \). Using (17) and differentiating (18), we get

\[
Q \, dq = N [1 - F(p(t)|t)] \, g(t) \, dt.
\]

Changing variables \( q \) and \( t \) in (5) and using \( r(q(t)) = p(t) \) shows that between times \( t \) and \( t + dt \), \( Q \, dq \) units are sold at the optimal price \( p(t) \), which yields

\[
\Pi = Q \int_{t_0}^{t_1} [r(q) - c_d] \, dq.
\]

The seller’s profit is equal to the average price of a sold unit minus the unit cost, \( \int_0^1 r(q) \, dq - c_d \), multiplied by the number of units sold. □

![Figure 2](image.png)

Figure 2: Left panel: Fraction of units purchased by early consumers (assuming that early consumers buy relatively less units). Right panel: Optimal intertemporal price schedule \( p(t) \) (dashed); first-come first-served scheme \( r(q) \) (solid).

As shown on the right panel of Figure 2, bunching at the optimum of the complete problem translates into flat parts of the first-come first-served scheme \( r(q) \). If the optimal price is constant and equal to \( p \) between times \( t_0 \) and \( t_1 \), the schedule \( r(q) \) is constant and equal to \( p \) on \((q_0, q_1)\), where \( q_0 = q(t_0) \) and \( q_1 = q(t_1) \) are given by equation (18). The number of units sold
at price $p$ is therefore

$$(q_1 - q_0)Q = N \int_{t_0}^{t_1} [1 - F(p|t)] g(t) \, dt.$$ 

Among consumers who arrive at time $t$, the fraction $\sigma(t) = 1 - F(p(t)|t)$ gets the good. If early consumers buy relatively less units, i.e. $\sigma(t)g(t)$ increases over time, then the function $q$ is convex in $t$ and hence the speed at which units are sold, $Q \frac{dq}{dt}$, increases over time. In this case, shown on Figure 2, we have: $q(t) \leq t$ and $p(t) = r(q(t)) \leq r(t)$ for all $t$. If early consumers buy relatively more units, i.e. $\sigma(t)g(t)$ decreases over time, then $q$ is concave and the speed at which units are sold decreases over time. In this case: $q(t) \geq t$ and $p(t) = r(q(t)) \geq r(t)$. Finally, if $\sigma(t)g(t)$ remains constant, the flow of units sold is constant over time, $q(t) = t$ and $p(t) = r(t)$.

A first-come first-served policy can only implement a nondecreasing price schedule, and hence does not allow to exploit positive waiting costs. Under such a policy, the highest profit corresponds to zero waiting costs.

7 Conclusion

We have examined optimal selling mechanisms when the seller does not observe the times at which her customers arrive on the market and how much they are willing to pay for the good. Assuming that the consumers are risk neutral, that they are equally patient, and that they simultaneously recognize their need for the good and their willingness to pay for it, we have demonstrated that quantity rationing and behavior-based price discrimination do not improve the seller’s profit compared to a simple time-dependent price schedule. We have also shown that when the consumers can delay their purchase at no cost, the optimal profit may be achieved with a first-come first-served policy, whereby the price depends only on the orders of arrival,
and not on the exact purchase times. Finally, we have explained how the level of the waiting costs and the variations of the instantaneous price elasticity over time affect the shape of the optimal price schedule.

Two directions for further research are worth mentioning. First, it would be interesting to allow for unobserved heterogeneity in waiting costs, on top of the heterogeneity in valuations; doing so would require solving a challenging multi-dimensional screening problem. Second, we have assumed that the consumers arrive in the market at exogenously given times. Yet one purpose of intertemporal pricing might be to encourage buyers to actively monitor the seller’s offers. Further research is needed to understand optimal pricing strategies under endogenous arrival times.
References


Appendix

In the appendix, we check the robustness of our results when the waiting cost per unit of time, $\gamma$, is positive.

A Time-dependent price schedule under positive waiting costs

We present here the proof of Proposition 2. Let $p(t)$ be any time-dependent price schedule. A consumer arriving at time $t$, should he decide to buy the product, chooses the purchase time, $\tau$, so as to minimize the total cost

$$C(t) = \min_{\tau \geq t} p(\tau) + \gamma(\tau - t). \quad (A.1)$$

The consumer indeed purchases the good if and only if his valuation for the good exceeds the total cost, i.e. if and only if $C(t) \leq v$. A buyer thus earns indirect utility $v - C(t)$. The purchase time, $\tau$, depends on the arrival time, $t$, but not on the valuation $v$, so we denote it by $\tau(t)$. The monopolist maximizes her intertemporal profit

$$\Pi = N \int_0^1 [w(\tau(t)) - c_d] [1 - F(C(t)|t)] g(t) \, dt$$

subject to the capacity constraint

$$N \int_0^1 [1 - F(C(t)|t)] g(t) \, dt \leq \bar{Q}.$$
The above inequality expresses that the monopolist cannot sell more units than her inventory. Denoting by $\mu$ the Lagrange multiplier associated to the capacity constraint, we can rewrite the monopolist’s objective function as

$$\mathcal{L} = N \int_0^1 [w(\tau(t)) - c] \left[ 1 - F(C(t)|t) \right] g(t) \, dt + \mu \hat{Q}, \quad (A.2)$$

with $c = c_d + \mu$. The total cost $c$ is the sum of the distribution cost, $c_d$, and the shadow cost of an extra unit, $\mu$.

No consumer purchases during periods where $w(t)$ decreases at a higher rate than $\gamma$, as it is then profitable to delay purchase. The next lemma introduces a transformation that removes such region (see also Figure 3).

**Lemma A.1.** Let $p(t)$ be any intertemporal price schedule. Changing from schedule $p$ to schedule $\hat{p} = C$, where $C$ is given by (A.1), does not alter the seller’s profit. The price schedule $\hat{p}(t)$ does not decrease at a higher rate than the waiting cost: $\hat{p}'(t) \geq -\gamma$.

**Proof.** Let $\hat{C}$ be the consumer’s total cost associated with the price profile $\hat{p}$. As $\hat{p} \leq p$, we have $\hat{C} \leq C$. Consider consumers arriving at some time $t$. Suppose they purchase at time $\hat{\tau} \geq t$ under tariff $\hat{p}$. By definition of $\hat{p}$, there exists $t' \geq \hat{\tau}$ such that $\hat{p}(\hat{\tau})$ is equal or infinitely close to $p(t') + \gamma(t' - \hat{\tau})$. It follows that

$$\hat{C}(t) = \hat{p}(\hat{\tau}) + \gamma(\hat{\tau} - t)$$

$$= p(t') + \gamma(t' - \hat{\tau}) + \gamma(\hat{\tau} - t)$$

$$= p(t') + \gamma(t' - t) \geq C(t),$$

implying that $\hat{C} = C$. Consumers arriving at time $t$ and choosing to purchase at time $\tau \geq t$
under tariff $p$ can still do so under the new tariff $\hat{p}$, as $\hat{p}(\tau) + \gamma(\tau - t) \leq p(\tau) + \gamma(\tau - t) = C(t) = \hat{C}(t)$. It follows that the last inequality is in fact an equality and hence $\hat{p}(\tau) = p(\tau)$. The change from $p$ to $\hat{p}$ does not alter the consumers incentives. Using equation (A.2), we conclude that the seller’s profit is the same under the two tariffs.

Finally, we observe that $C(t) = p(\tau) + \gamma(\tau - t) - \lambda(\tau - t)$, where $\lambda \geq 0$ is the Lagrange multiplier of the constraint $\tau \geq t$ in (A.1). The envelope theorem then yields $C'(t) = \lambda - \gamma \geq -\gamma$.

\[ \text{Figure 3: Transformation to a linear price schedule with slope } -\gamma. \]

In periods where the slope of the original schedule $p$ is smaller than $-\gamma$, the new schedule $\hat{p}$ is linear with slope $-\gamma$, see Figure 3. Consider such a period $[t_0, t_1]$. The consumers arriving at time $t$ between $t_0$ and $t_1$ indifferently purchase at any time $\tau$ between $t$ and $t_1$, while the seller prefers them to purchase as soon as they arrive (indeed, she then earns $\hat{p}(t)$ rather than $\hat{p}(\tau) < \hat{p}(t)$).

To avoid consumer indifference, we may apply the above construction with $\gamma'$ slightly below $\gamma$, rather than with $\gamma$. Then prices cannot fall at a rate higher than $\gamma'$, which is lower than the waiting cost, implying that consumers strictly prefer to purchase upon arrival and that the
seller’s revenue from these customers is higher. The consumer total cost, and hence the fraction of the demand that is served at each time, are changed only infinitesimally.\footnote{Formally, we define: $\tilde{p}(t) = \min_{\tau \geq t} p(\tau) + \gamma'(t - \tau)$, with $\gamma'$ slightly below $\gamma$. We have: $\tilde{p}' \geq -\gamma' > -\gamma$. Hence consumers purchase upon arrival under $\tilde{p}$. The consumer total cost $\tilde{C}$ under $\tilde{p}$ tends to $\tilde{C} = \hat{C}$ as $\gamma'$ goes to $\gamma$ from below.}

We denote by $\Pi^*(\gamma)$ the optimal profit when the level of the waiting costs is given by $\gamma$. It follows from the above analysis that $\Pi^*(\gamma)$ is obtained by maximizing

$$\Pi = N \int_0^1 [p(t) - c_d] [1 - F(p(t)|t)] g(t) \, dt$$

under the capacity constraint

$$N \int_0^1 [1 - F(p(t)|t)] g(t) \, dt \leq \bar{Q}.$$  \hspace{1cm} (A.4)

and the extra constraint that the price does not decrease too quickly:

$$p'(t) \geq -\gamma$$  \hspace{1cm} (A.5)

for all $t \in [0, 1]$. We call the latter constraint the “monotonicity constraint” and refer to this problem as to the “complete problem”. Equations (A.3), (A.4) and (A.5) assume that the customers purchase upon arrival, $\tau(t) = t$, and hence incur no waiting costs: $C(t) = p(t)$. As explained above, these assumptions entail no loss of the generality; the seller may attain a profit equal to, or arbitrarily close to, $\Pi^*(\gamma)$.\footnote{\textsuperscript{7}}
B  Rationing and behavior-based pricing under positive waiting costs

We now prove Proposition 3 in the case $\gamma > 0$. The consumers’ expected utility given by (14) must be adapted. Denoting by $U(v, t|H_{t_0}; \gamma)$ the utility of consumer with valuation $v$ and waiting cost $\gamma$, arriving at time $t$, when picking history $H_{t_0}$ starting at $t_0$, we have

$$U(v, t|H_{t_0}; \gamma) = U(v|H_{t_0}; 0) - \Gamma(H_{t_0}, t; \gamma)$$

where $U(v|H_{t_0}; 0)$ is given by (14) and $\Gamma(H_{t_0}, t; \gamma)$ represents the expected waiting cost:

$$\Gamma(H_{t_0}, t; \gamma) = \gamma \left[ \sigma_{t_0}(t_0 - t) + (1 - \sigma_{t_0})\sigma_{t_1}(t_1 - t) + \cdots + \sigma_{t_M}(t_M - t) \prod_{i=0}^{M-1} (1 - \sigma_{t_i}) \right],$$

which does not depend on $v$. The indirect utility $\tilde{U}(v, t; \gamma)$, equal to the maximum of $U$ over all histories starting after the arrival time $t$, is convex in $v$ and nondecreasing with slope between 0 and 1, as under zero waiting costs. The derivative of $\tilde{U}$ with respect to $v$ is still given by (16).

As regards the dependence on arrival time $t$, we observe that

$$\frac{\partial \Gamma(H_{t_0}, t; \gamma)}{\partial t} = -\gamma \left[ \sigma_{t_0} + \sigma_{t_1}(1 - \sigma_{t_0}) + \cdots + \sigma_{t_M} \prod_{i=0}^{M-1} (1 - \sigma_{t_i}) \right] \geq -\gamma,$$

Introducing $\tilde{\Gamma}(v, t; \gamma)$ the expected waiting cost incurred by consumer $(v, t)$ along her preferred history and applying the envelope theorem yields $\partial \tilde{U}/\partial t \leq \gamma$.

The profit earned on a consumer with valuation $v$ arriving at time $t$ is the total surplus he generates, $(v - c)\tilde{U}'(v|t) - \tilde{\Gamma}(v, t; \gamma)$, minus his expected rent. The profit earned on the cohort
arriving at time $t$ is therefore

$$
\Pi(\tilde{U}|t) = \int_\mathbb{U} \left[ (v - c)\tilde{U}'(v|t) - \tilde{\Gamma}(v, t; \gamma) - \tilde{U}(v|t) \right] f(v|t) \, dv.
$$

We apply the same integrations by parts as above, leaving aside the waiting cost term $\Gamma$. We introduce the same function $\tilde{U}$, corresponding to a single price at each time (and no rationing nor behavior-based pricing). Given that $\partial\tilde{U}/\partial t \leq \gamma$, the single price $p(t)$ satisfies $\dot{p}(t) \geq -\gamma$ as in Section 4.4. We conclude that it is optimal for the seller to use a single price and let the consumers purchase upon arrival, which ensures that the expected waiting costs are zero.